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Abstract

We study the discrete-time approximation for solutions of quadratic forward backward stochastic differential equations (FBSDEs) driven by a Brownian motion and a jump process which could be dependent. Assuming that the generator has a quadratic growth w.r.t. the variable z and the terminal condition is bounded, we prove the convergence of the scheme when the number of time steps n goes to infinity. Our approach is based on the companion paper [15] and allows to get a convergence rate similar to that of schemes of Brownian FBSDEs.

Keywords: discrete-time approximation, forward-backward SDE with a jump, generator of quadratic growth, progressive enlargement of filtrations, decomposition in the reference filtration.

MSC classification (2000): 65C99, 60J75, 60G57.

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Contents

1 Introduction

In this paper, we study a discrete-time approximation for the solution of a forward-backward stochastic differential equation (FBSDE) with a jump and taking the following form

$$\begin{cases} X_t &= x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s)dW_s + \int_0^t \beta(s, X_{s^-})dH_s \ , \\ Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T U_s dH_s \ , \end{cases} \quad 0 \leq t \leq T,$$

where $H_t = \mathbb{1}_{\tau \leq t}$ and τ is a jump time, depending on W, which can represent a default time in credit risk or counterparty risk. Such equations naturally appear in finance, see for example Bielecki and Jeanblanc [2], Lim and Quenez [19], Ankirchner *et al.* [1] for an application to the exponential utility maximization problem and Kharroubi and Lim [15] for the hedging problem in a complete market. The approximation of such equation is therefore of important interest for practical applications in finance.

This paper is the second part of a two sides work studying the discretization of such FBSDEs. The first part [16] deals with the case of a Lipschitz continuous generator f and this second part focuses on the case of a generator f with quadratic growth w.r.t. z.

In the literature, the problem of discretization of FBSDEs with Lipschitz generator has been widely studied in the Brownian framework, *i.e.* no jump, see e.g. [20, 7, 5, 3, 24, 6]. More recently, the case of quadratic generators w.r.t. z has been considered by Imkeller *et al.* [8] and Richou [23].

The discretization of BSDEs with jumps has been studied by Bouchard and Elie [4] in the case where the generator is Lipschitz continuous and the jumps are independent of the Brownian motion.

In this paper, we study the discrete-time approximation of FBSDEs with

– a generator satisfying a quadratic growth w.r.t. the variable z,

– a jump time τ that could be dependent on the Brownian motion via the density assumption.

Such FBSDEs are of interest in finance since utility maximization problems lead in general to quadratic generators and default time models generally impose the dependence of τ w.r.t. the Brownian motion.

To study the discretization of such equations, we can not directly work on the BSDE as done in [4] since their approach uses a regularity result for the process Z based on Malliavin calculus. In our situation, the dependence of τ w.r.t. W prevent us from using such an approach since no Malliavin calculus theory has been set for this framework.

To get a discrete-time approximation scheme we then use the results of [15], that allow to decompose the FBSDE with a jump in a recursive system of Brownian FBSDEs. We then provide estimates on the solutions to each Brownian FBSDE. These estimates allow to prove that the solution satisfies a Lipschitz FBSDE. Finally, we obtain a discrete-time approximation scheme by using the results of the previous part [16] on Lipschitz BSDEs.

The paper is organized as follows. The next section presents the FBSDE, the different assumptions on the coefficients of the FBSDE and recalls the result of [15]. In Section 3, we give some estimations on the solution of the FBSDE. In Section 4, we give a discrete-time approximation scheme for the FBSDE and provide a global error estimate.

2 Preliminaries

2.1 Notation

Throughout this paper, we let $(\Omega, \mathcal{G}, \mathbb{P})$ a complete probability space on which is defined a standard one dimensional Brownian motion W. We denote $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the natural filtration of W augmented by all the $\mathbb{P}$-null sets. We also consider on this space a random time τ, i.e. a nonnegative $\mathcal{F}$-measurable random variable, and we denote classically the associated jump process by H which is given by

$$H_t \ := \ \mathbb{1}_{\tau \leq t} \ , \quad t \geq 0 \ .$$

We denote by $\mathbb{D} := (\mathcal{D}_t)_{t\geq 0}$ the smallest right-continuous filtration for which τ is a stopping time. The global information is then defined by the progressive enlargement $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ of the initial filtration where

$$\mathcal{G}_t \ := \ \bigcap_{\varepsilon>0} \Big(\mathcal{F}_{t+\varepsilon} \vee \mathcal{D}_{t+\varepsilon}\Big)$$

for all $t \geq 0$. This kind of enlargement was introduced by Jacod, Jeulin and Yor in the 80s (see e.g. [10], [11] and [9]). We introduce some notations used throughout the paper

– $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) is the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable measurable subsets of $\Omega \times \mathbb{R}_+$, i.e. the σ-algebra generated by the left-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes,

– $\mathcal{PM}(\mathbb{F})$ (resp. $\mathcal{PM}(\mathbb{G})$) is the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-progressively measurable subsets of $\Omega \times \mathbb{R}_+$.

We shall make, throughout the sequel, the standing assumption in the progressive enlargement of filtrations known as density assumption (see e.g. [12, 13, 15]).

(DH) There exists a positive and bounded $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable process γ such that

$$\mathbb{P}\big[\tau \in d\theta \,\big|\, \mathcal{F}_t\big] \ = \ \gamma_t(\theta) d\theta \ , \quad t \geq 0 \ .$$

Using Proposition 2.1 in [15] we get that **(DH)** ensures that the process H admits an intensity.

Proposition 2.1. *The process H admits a compensator of the form $\lambda_t dt$, where the process λ is defined by*

$$\lambda_t \quad := \quad \frac{\gamma_t(t)}{\mathbb{P}\big[\tau > t \mid \mathcal{F}_t\big]} \mathbb{1}_{t\leq\tau} \ , \quad t \geq 0 \ .$$

We impose the following assumption to the process λ.

(HBI) The process λ is bounded.

We also introduce the martingale invariance assumption known as the **(H)**-hypothesis.

(H) Any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale.

We now introduce the following spaces, where $a, b \in \mathbb{R}_+$ with $a < b$, and $T < \infty$ is the terminal time.

- $\mathcal{S}^\infty_{\mathbb{G}}[a,b]$ (resp. $\mathcal{S}^\infty_{\mathbb{F}}[a,b]$) is the set of $\mathcal{PM}(\mathbb{G})$ (resp. $\mathcal{PM}(\mathbb{F})$)-measurable processes $(Y_t)_{t\in[a,b]}$ essentially bounded

$$\|Y\|_{\mathcal{S}^\infty[a,b]} \quad := \quad \operatorname*{ess\,sup}_{t\in[a,b]} |Y_t| \ < \ \infty \ .$$

- $\mathcal{S}^p_{\mathbb{G}}[a,b]$ (resp. $\mathcal{S}^p_{\mathbb{F}}[a,b]$), with $p \geq 2$, is the set of $\mathcal{PM}(\mathbb{G})$ (resp. $\mathcal{PM}(\mathbb{F})$)-measurable processes $(Y_t)_{t\in[a,b]}$ such that

$$\|Y\|_{\mathcal{S}^p[a,b]} \quad := \quad \Big(\mathbb{E}\Big[\sup_{t\in[a,b]} |Y_t|^p\Big]\Big)^{\frac{1}{p}} \ < \ \infty \ .$$

- $H^p_{\mathbb{G}}[a,b]$ (resp. $H^p_{\mathbb{F}}[a,b]$), with $p \geq 2$, is the set of $\mathcal{P}(\mathbb{G})$ (resp. $\mathcal{P}(\mathbb{F})$)-measurable processes $(Z_t)_{t\in[a,b]}$ such that

$$\|Z\|_{H^p[a,b]} \quad := \quad \mathbb{E}\Big[\Big(\int_a^b |Z_t|^2 dt\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \ < \ \infty \ .$$

- $L^2(\lambda)$ is the set of $\mathcal{P}(\mathbb{G})$-measurable processes $(U_t)_{t\in[0,T]}$ such that

$$\|U\|_{L^2(\mu)} \quad := \quad \Big(\mathbb{E}\Big[\int_0^T \lambda_s |U_s|^2 ds\Big]\Big)^{\frac{1}{2}} \ < \ \infty \ .$$

2.2 Quadratic growth Forward-Backward SDE with a jump

Given measurable functions $b : [0,T]\times\mathbb{R} \to \mathbb{R}$, $\sigma : [0,T] \to \mathbb{R}$, $\beta : [0,T]\times\mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $f : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, and an initial condition $x \in \mathbb{R}$, we study the discrete-time approximation of the solution (X, Y, Z, U) in $\mathcal{S}^2_{\mathbb{G}}[0,T] \times \mathcal{S}^\infty_{\mathbb{G}}[0,T] \times H^2_{\mathbb{G}}[0,T] \times L^2(\lambda)$ to the following forward-backward stochastic differential equation

$$X_t \quad = \quad x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s) dW_s + \int_0^t \beta(s, X_{s^-}) dH_s \ , \quad 0 \leq t \leq T \ , \tag{2.1}$$

$$\begin{aligned} Y_t \quad &= \quad g(X_T) + \int_t^T f\big(s, X_s, Y_s, Z_s, U_s(1 - H_s)\big) ds \\ &\quad - \int_t^T Z_s dW_s - \int_t^T U_s dH_s \ , \quad 0 \leq t \leq T \ , \end{aligned} \tag{2.2}$$

when the generator of the BSDE (2.2) has a quadratic growth w.r.t. z.

Remark 2.1. In the BSDE (2.2), the jump component U of the unknown (Y, Z, U) appears in the generator f with an additional multiplicative term $1 - H$. This ensures the equation to be well posed in $\mathcal{S}^\infty_{\mathbb{G}}[0,T] \times H^2_{\mathbb{G}}[0,T] \times L^2(\lambda)$. Indeed, the component U lives in $L^2(\lambda)$, thus its value on $(\tau \wedge T, T]$ is not defined, since the intensity λ vanishes on $(\tau \wedge T, T]$. We therefore introduce the term $1 - H$ to kill the value of U on $(\tau \wedge T, T]$ and hence to avoid making the equation depends on it.

We first prove that the decoupled system (2.1)-(2.2) admits a solution. To this end, we introduce several assumptions on the coefficients b, σ, β, g and f. We consider the following assumptions for the forward coefficients.

(HF) There exist two constants K_a and L_a such that the functions b, σ and β satisfy

$$|b(t,0)| + |\sigma(t)| + |\beta(t,0)| \leq K_a ,$$

and

$$|b(t,x) - b(t,x')| + |\beta(t,x) - \beta(t,x')| \leq L_a|x - x'| ,$$

for all $(t, x, x') \in [0,T] \times \mathbb{R} \times \mathbb{R}$.

For the backward coefficients g and f, we consider the following assumptions.

(HBQ)

– There exist three constants M_g, K_g and K_q such that the functions g and f satisfy

$$\begin{aligned} |g(x)| &\leq M_g , \\ |g(x) - g(x')| &\leq K_g|x - x'| , \\ |f(t,x,y,z,u) - f(t,y',z,u)| &\leq K_q|y - y'| , \\ |f(t,x,y,z,u)| &\leq K_q\big(1 + |y| + |z|^2 + |u|\big) , \end{aligned}$$

for all $(t, x, x', y, y', z, u) \in [0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$.

– For any $R > 0$ there exists a function mc^f_R such that $\lim_{\epsilon\to 0} mc^f_R(\epsilon) = 0$ and

$$|f(t,x,y,z,u-y) - f(t,x,y',z',u-y')| \leq mc^f_R(\epsilon)$$

for all $(t, x, y, y', z, z', u) \in [0,T] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ s.t. $|y|$, $|z|$, $|y'|$, $|z'| \leq R$ and $|y - y'| + |z - z'| \leq \epsilon$.

– $f(t,.,u) = f(t,.,0)$ for all $u \in \mathbb{R}$ and all $t \in (\tau \wedge T, T]$.

– The function $f(t,x,y,.,u)$ is convexe (or concave) uniformly in $(t,x,y,u) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

In the sequel K denotes a generic constant appearing in **(HBQ)** and **(HF)** and which may vary from line to line.

In the purpose to prove the existence of a solution to the FBSDE (2.1)-(2.2) we follow the decomposition approach initiated by [15] and for that we introduce the recursive system of FBSDEs associated with (2.1)-(2.2).

• Find $(X^1(\theta), Y^1(\theta), Z^1(\theta)) \in \mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[\theta,T] \times H^2_{\mathbb{F}}[\theta,T]$ such that

$$X^1_t(\theta) = x + \int_0^t b\big(s, X^1_s(\theta)\big)ds + \int_0^t \sigma(s)dW_s + \beta\big(\theta, X^1_{\theta^-}(\theta)\big)\mathbb{1}_{\theta\leq t}\ , \quad 0 \leq t \leq T\ , \tag{2.3}$$

$$Y^1_t(\theta) = g\big(X^1_T(\theta)\big) + \int_t^T f\big(s, X^1_s(\theta), Y^1_s(\theta), Z^1_s(\theta), 0\big)ds - \int_t^T Z^1_s(\theta)dW_s\ , \quad \theta \leq t \leq T\ , \tag{2.4}$$

for all $\theta \in [0,T]$.

• Find $(X^0, Y^0, Z^0) \in \mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[0,T] \times H^2_{\mathbb{F}}[0,T]$ such that

$$X^0_t = x + \int_0^t b\big(s, X^0_s\big)ds + \int_0^t \sigma(s)dW_s\ , \quad 0 \leq t \leq T\ , \tag{2.5}$$

$$Y^0_t = g\big(X^0_T\big) + \int_t^T f\big(s, X^0_s, Y^0_s, Z^0_s, Y^1_s(s) - Y^0_s\big)ds - \int_t^T Z^0_s dW_s\ , \quad 0 \leq t \leq T\ . \tag{2.6}$$

Then, the link between the FBSDE (2.1)-(2.2) and the recursive system of FBSDEs (2.3)-(2.4) and (2.5)-(2.6) is given by the following result.

Theorem 2.1. *Assume that* ***(DH)***, ***(HBI)***, ***(H)***, ***(HF)*** *and* ***(HBQ)*** *hold true. Then, the FBSDE* (2.1)-(2.2) *admits a unique solution* $(X, Y, Z, U) \in \mathcal{S}^2_{\mathbb{G}}[0,T] \times \mathcal{S}^\infty_{\mathbb{G}}[0,T] \times H^2_{\mathbb{G}}[0,T] \times L^2(\lambda)$ *given by*

$$\begin{cases} X_t = X^0_t\mathbb{1}_{t<\tau} + X^1_t(\tau)\mathbb{1}_{\tau\leq t}\ , \\ Y_t = Y^0_t\mathbb{1}_{t<\tau} + Y^1_t(\tau)\mathbb{1}_{\tau\leq t}\ , \\ Z_t = Z^0_t\mathbb{1}_{t\leq\tau} + Z^1_t(\tau)\mathbb{1}_{\tau<t}\ , \\ U_t = \big(Y^1_t(t) - Y^0_t\big)\mathbb{1}_{t\leq\tau}\ , \end{cases} \tag{2.7}$$

where $(X^1(\theta), Y^1(\theta), Z^1(\theta))$ *is the unique solution to the FBSDE* (2.3)-(2.4) *in* $\mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[\theta,T] \times H^2_{\mathbb{F}}[\theta,T]$, *for* $\theta \in [0,T]$, *and* (X^0, Y^0, Z^0) *is the unique solution to the FBSDE* (2.5)-(2.6) *in* $\mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[0,T] \times H^2_{\mathbb{F}}[0,T]$.

Proof. The existence and uniqueness of the forward process X and its link with X^0 and X^1 have already been proved in the first part of this work [16]. We now concentrate on the backward equation.
To follow the decomposition approach initiated by the authors in [15], we need the generator to be predictable. To this end, we notice that in the BSDE (2.2), we can replace the generator $(t, y, z, u) \mapsto f(t, X_t, y, z, u(1-H_t))$ by the predictable map $(t, y, z, u) \mapsto f(t, X_{t^-}, y, z, u(1-H_{t^-}))$.

Suppose that **(DH)**, **(HBI)**, **(H)** and **(HBQ)** hold true. The existence of a solution $(Y, Z, U) \in \mathcal{S}^2_{\mathbb{G}}[0,T] \times H^2_{\mathbb{G}}[0,T] \times L^2(\lambda)$ is then a direct consequence of Proposition 3.1 in [15]. We then notice that from the definition of H we have $f(t,x,y,z,u(1-H_t)) = f(t,x,y,z,0)$ for all $t \in (\tau \wedge T, T]$. This property and **(DH)**, **(HBI)**, **(H)** and **(HBQ)** allow to apply Theorem 4.2 in [15], which gives the uniqueness of a solution to BSDE (2.2). □

Throughout the sequel, we give an approximation of the solution to the FBSDE (2.1)-(2.2) by studying the approximation of the solutions to the recursive system of FBSDEs (2.3)-(2.4) and (2.5)-(2.6).

3 A priori estimation on the gain process

Before giving the discrete-time scheme for the FBSDE (2.1)-(2.2) we give a uniform bound for the processes Z^0 and Z^1 which allows to prove that the BSDE (2.2) is Lipschitz and thus we can use the discrete-time scheme given in [16]. For that we introduce the BMO-martingales class, and we also give some bounds for the processes X^0, X^1, Y^0 and Y^1.

3.1 BMO property for the solution of the BSDE

To obtain a uniform bound for the processes Z^0 and Z^1 we need the following assumption.

(HBQD) There exists a constant K_f such that the function f satisfies

$$\begin{aligned} |f(t,x,y,z,u) - f(t',x',y',z',u')| &\leq K_f\big[|x-x'| + |y-y'| + |u-u'| + |t-t'|^{\frac{1}{2}}\big] \\ &\quad + L_{f,z}(1+|z|+|z'|)|z-z'| , \end{aligned}$$

for all $(t,x,x',y,y',z,z',u,u') \in [0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$.

In the sequel of this section, the space of BMO martingales plays a key role for the a priori estimates of processes Z^0 and Z^1. We refer to [14] for the theory of BMO martingales. Here, we just give the definition of a BMO martingale and recall a property that we use in the sequel.

Definition 3.1. *A process M is said to be a $BMO_{\mathbb{F}}[0,T]$-martingale if M is a square integrable $\mathbb{F}$-martingale s.t.*

$$\|M\|_{BMO_{\mathbb{F}}[0,T]} \;:=\; \sup_{\tau \in \mathcal{T}_{\mathbb{F}}[0,T]} \mathbb{E}\Big[\big|M_T - M_\tau\big|^2 \Big| \mathcal{F}_\tau\Big]^{1/2} \;<\; \infty ,$$

where $\mathcal{T}_{\mathbb{F}}[0,T]$ denotes the set of $\mathbb{F}$-stopping times valued in $[0,T]$.

The BMO condition provides a property on the Dolean-Dade exponential of the process M.

Lemma 3.1. *Let M be a $BMO_{\mathbb{F}}[0,T]$-martingale. Then the stochastic exponential $\mathcal{E}(M)$ defined by*

$$\mathcal{E}(M)_t \;=\; \exp\Big(M_t - \frac{1}{2}\langle M, M\rangle\Big)_t\,, \quad 0 \le t \le T\,,$$

is a uniformly integrable $\mathbb{F}$-martingale.

We refer to [14] for the proof of this result.

We first state a BMO property for the processes Z^0 and Z^1, which will be used in the sequel to provide an estimate for these processes.

Lemma 3.2. *Under **(HF)**, **(HBQ)** and **(HBQD)**, the martingales $\int_0^\cdot Z_s^0 dW_s$ and $\int_0^\cdot Z_s^1(\theta)\mathbb{1}_{s\ge\theta}dW_s$, $\theta \in [0,T]$ are $BMO_{\mathbb{F}}[0,T]$-martingales and there exists a constant K which is independent from θ such that*

$$\Big\|\int_0^\cdot Z_s^0 dW_s\Big\|_{BMO_{\mathbb{F}}[0,T]} \;\le\; K\,,$$

$$\sup_{\theta\in[0,T]}\Big\|\int_0^\cdot Z_s^1(\theta)\mathbb{1}_{s\ge\theta}dW_s\Big\|_{BMO_{\mathbb{F}}[0,T]} \;\le\; K\,.$$

Proof. Define the function $\phi:\ \mathbb{R}\to\mathbb{R}$ by

$$\phi(x) \;=\; (e^{2K_q x} - 2K_q x - 1)/2K_q^2\,, \quad x\in\mathbb{R}\,. \tag{3.1}$$

We notice that ϕ satisfies

$$\phi'(x) \;\ge\; 0 \;\text{ and }\; \frac{1}{2}\phi''(x) - K_q\phi'(x) \;=\; 1\,,$$

for $x \ge 0$. Since Y^0 and $Y^1(.)$ are solutions to quadratic BSDEs with bounded terminal conditions, we get from Proposition 2.1 in [18] the existence of a constant m such that

$$\|Y^0\|_{\mathcal{S}^\infty[0,T]} \;\le\; m \quad\text{and}\quad \sup_{\theta\in[0,T]}\|Y^1(\theta)\|_{\mathcal{S}^\infty[\theta,T]} \;\le\; m\,. \tag{3.2}$$

Applying Itô's formula we get

$$\phi(Y_\nu^0+m) + \mathbb{E}\Big(\int_\nu^T \frac{1}{2}\phi''(Y_s^0+m)|Z_s^0|^2 ds\Big|\mathcal{F}_\nu\Big) =$$
$$\mathbb{E}(\phi(Y_T^0+m)|\mathcal{F}_\nu) + \mathbb{E}\Big(\int_\nu^T \phi'(Y_s^0+m)f(s,X_s^0,Y_s^0,Z_s^0,Y_s^1(s)-Y_s^0)ds\Big|\mathcal{F}_\nu\Big)\,.$$

for any $\mathbb{F}$-stopping time ν valued in $[0,T]$. From the growth assumption on the generator f in **(HBD)**, (3.1) and (3.2), we obtain

$$\phi(Y_\nu^0+m) + \mathbb{E}\Big(\int_\nu^T |Z_s^0|^2 ds\Big|\mathcal{F}_\nu\Big) \le$$
$$\mathbb{E}(\phi(Y_T^0+m)|\mathcal{F}_\nu) + \mathbb{E}\Big(\int_\nu^T \phi'(Y_s^0+m)K_q(1+2\|Y^0\|_{\mathcal{S}^\infty} + \sup_{\theta\in[0,T]}\|Y^1(\theta)\|_{\mathcal{S}^\infty[\theta,T]})ds\Big|\mathcal{F}_\nu\Big)\,.$$

This last inequality and (3.2) imply that there exists a constant K which depends only on m, T and K_q such that for all $\mathbb{F}$-stopping times $\nu \in [0,T]$

$$\mathbb{E}\Big(\int_\nu^T |Z_s^0|^2 ds \Big| \mathcal{F}_\nu\Big) \;\; \leq \;\; K \; .$$

For the process Z^1, we use the same technics. Let us fix $\theta \in [0,T]$. Applying Itô's fomula we get

$$\phi(Y^1_{\nu\vee\theta}(\theta)+m) + \mathbb{E}\Big(\int_{\nu\vee\theta}^T \frac{1}{2}\phi''(Y_s^1(\theta)+m)|Z_s^1(\theta)|^2 ds\Big|\mathcal{F}_{\nu\vee\theta}\Big) =$$
$$\mathbb{E}(\phi(Y_T^1(\theta)+m)|\mathcal{F}_\nu) + \mathbb{E}\Big(\int_{\nu\vee\theta}^T \phi'(Y_s^1(\theta)+m) f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0) ds \Big| \mathcal{F}_{\nu\vee\theta}\Big) \; ,$$

for any $\mathbb{F}$-stopping time ν valued in $[0,T]$. From the growth assumption on the generator f in **(HBQ)**, (3.1) and (3.2), we obtain

$$\phi(Y^1_{\nu\vee\theta}(\theta)+m) + \mathbb{E}\Big(\int_{\nu\vee\theta}^T |Z_s^1(\theta)|^2 ds\Big|\mathcal{F}_\nu\Big) \leq \mathbb{E}(\phi(Y_T^1(\theta)+m)|\mathcal{F}_\nu)$$
$$+\mathbb{E}\Big(\int_{\nu\vee\theta}^T \phi'(Y_s^1(\theta)+m)K_q(1+\|Y^1(\theta)\|_{\mathcal{S}^\infty[\theta,T]})ds\Big|\mathcal{F}_\nu\Big) \; .$$

This last inequality and (3.2) imply that there exists a constant K which depends only on m, T and K_q, such that for all $\mathbb{F}$-stopping times ν valued in $[0,T]$

$$\mathbb{E}\Big(\int_\nu^T |Z_s^1(\theta)|^2 \mathbb{1}_{s\geq\theta} ds\Big|\mathcal{F}_\nu\Big) \;\; \leq \;\; K \; .$$

$\square$

3.2 Some bounds about X^0 and X^1

In this part, we give some bounds about the processes X^0 and X^1 which are used to get a uniform bound for the processes Z^0 and Z^1.

Proposition 3.1. *Suppose that **(HF)** holds. Then, we have*

$$|\nabla X_t^0| \;:=\; \Big|\frac{\partial X_t^0}{\partial x}\Big| \;\;\leq\;\; e^{L_a T} \; , \quad 0 \leq t \leq T \; , \tag{3.3}$$

and for any $\theta \in [0,T]$ we have

$$|\nabla^\theta X_t^1(\theta)| \;:=\; \Big|\frac{\partial X_t^1(\theta)}{\partial X_\theta^1(\theta)}\Big| \;\;\leq\;\; e^{L_a T} \; , \quad \theta \leq t \leq T \; , \tag{3.4}$$

$$|\nabla X_t^1(\theta)| \;:=\; \Big|\frac{\partial X_t^1(\theta)}{\partial x}\Big| \;\;\leq\;\; (1+L_a e^{L_a T})e^{L_a T} \; , \quad \theta \leq t \leq T \; . \tag{3.5}$$

Proof. We first suppose that b and β are C_b^1 w.r.t. x. By definition we have

$$\nabla X_t^0 \;=\; 1+\int_0^t \partial_x b(s,X_s^0)\nabla X_s^0 ds\;, \quad 0\leq t\leq T\;.$$

We get from Gronwall's lemma

$$|\nabla X_t^0| \;\leq\; e^{L_a T}\;, \qquad 0\leq t\leq T\;.$$

In the same way, we have

$$\nabla^\theta X_t^1(\theta) \;=\; 1+\int_\theta^t \partial_x b(s,X_s^1(\theta))\nabla^\theta X_s^1(\theta) ds\;, \qquad \theta\leq t\leq T\;,$$

and from Gronwall's lemma we get

$$|\nabla^\theta X_t^1| \;\leq\; e^{L_a T}\;, \quad \theta\leq t\leq T\;.$$

Finally we prove the last inequality. By definition

$$\nabla X_t^1(\theta) \;=\; 1+\int_0^t \partial_x b(s,X_s^1(\theta))\nabla X_s^1(\theta) ds + \partial_x\beta(\theta,X_\theta^0)\nabla X_\theta^0\;, \qquad \theta\leq t\leq T\;.$$

Using the inequality (3.3), we get

$$|\nabla X_t^1(\theta)| \;\leq\; 1+L_a e^{L_a T}+\int_0^t L_a|\nabla X_s^1(\theta)| ds\;, \qquad \theta\leq t\leq T\;,$$

from Gronwall's lemma we get

$$|\nabla X_t^1(\theta)| \;\leq\; (1+L_a e^{L_a T})e^{L_a T}\;, \quad \theta\leq t\leq T\;.$$

When b and β are not differentiable, we can also prove the result by regularization. We consider a density q which is C_b^∞ on $\mathbb{R}$ with a compact support, and we define an approximation $(b^\epsilon,\beta^\epsilon)$ of (b,β) in C_b^1 by

$$(b^\epsilon,\beta^\epsilon)(t,x) \;=\; \frac{1}{\varepsilon}\int_{\mathbb{R}}(b,\beta)(t,x')q\Big(\frac{x-x'}{\epsilon}\Big)dx'\;, \quad (t,x)\in[0,T]\times\mathbb{R}\;.$$

We then use the convergence of $(X^{0,\epsilon},X^{1,\epsilon}(\theta))$ to $(X^0,X^1(\theta))$ and we get the result. □

3.3 Some bounds about Y^0 and Y^1

In this part, we give some bounds about the processes Y^0 and Y^1 which are used to get a uniform bound for the processes Z^0 and Z^1.

Lemma 3.3. *Suppose that **(HF)**, **(HBQ)** and **(HBQD)** hold. Then, for any $\theta\in[0,T]$*

$$|\nabla^\theta Y_t^1(\theta)| \;:=\; \Big|\frac{\partial Y_t^1(\theta)}{\partial X_\theta^1(\theta)}\Big| \;\leq\; e^{(L_a+K_f)T}(K_g+TK_f)\;, \quad \theta\leq t\leq T\;. \qquad (3.6)$$

Proof. We first suppose that b, f and g are C_b^1 w.r.t. x, y and z. In this case $(X^1(\theta), Y^1(\theta), Z^1(\theta))$ is also differentiable w.r.t. $X_\theta^1(\theta)$ and we have

$$\begin{aligned}\nabla^\theta Y_t^1(\theta) &= \nabla g(X_T^1(\theta))\nabla^\theta X_T^1(\theta) - \int_t^T \nabla^\theta Z_s^1(\theta)dW_s \\ &\quad + \int_t^T \nabla f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\big(\nabla^\theta X_s^1(\theta), \nabla^\theta Y_s^1(\theta), \nabla^\theta Z_s^1(\theta)\big)ds\ ,\end{aligned} \tag{3.7}$$

for $t \in [\theta, T]$. Define the process $R(\theta)$ by

$$R_t(\theta) \ := \ \exp\Big(\int_0^t \partial_y f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\mathbb{1}_{s\geq\theta}ds\Big)\ , \quad 0 \leq t \leq T\ .$$

Applying Itô's formula, we get

$$\begin{aligned}R_t(\theta)\nabla^\theta Y_t^1(\theta) &= R_T(\theta)\nabla g(X_T^1(\theta))\nabla^\theta X_T^1(\theta) \\ &\quad + \int_t^T R_s(\theta)\partial_x f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\nabla^\theta X_s^1(\theta)ds \\ &\quad - \int_t^T R_s(\theta)\nabla^\theta Z_s^1(\theta)dW_s^1(\theta)\ , \quad \theta \leq t \leq T\ ,\end{aligned} \tag{3.8}$$

where the process $W^1(\theta)$ is defined by

$$W_t^1(\theta) := W_t - \int_0^t \partial_z f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\mathbb{1}_{s\geq\theta}ds \tag{3.9}$$

for $t \in [0, T]$. From **(HBQD)**, there exists a constant $K > 0$ such that we have

$$\begin{aligned}\Big\|\int_0^\cdot \partial_z f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\mathbb{1}_{s\geq\theta}dW_s\Big\|^2_{BMO_{\mathbb{F}}[0,T]} &\leq \\ K\Big(1 + \sup_{\vartheta\in\mathcal{T}_{\mathbb{F}}[0,T]} \mathbb{E}\Big[\int_\vartheta^T |Z_s^1(\theta)|^2\mathbb{1}_{s\geq\theta}ds\Big|\mathcal{F}_\vartheta\Big]\Big) &\leq \\ K\Big(1 + \Big\|\int_0^\cdot Z_s^1(\theta)\mathbb{1}_{s\geq\theta}dW_s\Big\|^2_{BMO_{\mathbb{F}}[0,T]}\Big) &< \infty\ ,\end{aligned}$$

where the last inequality comes from Lemma 3.2.

Hence by Lemma 3.1 the process $\mathcal{E}(\int_0^\cdot \partial_z f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\mathbb{1}_{s\geq\theta}dW_s)$ is a uniformly integrable martingale. Therefore, under the probability measure $\mathbb{Q}^1(\theta)$ defined by

$$\frac{d\mathbb{Q}^1(\theta)}{d\mathbb{P}}\Big|_{\mathcal{F}_t} \ := \ \mathcal{E}\Big(\int_0^\cdot \partial_z f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\mathbb{1}_{s\geq\theta}dW_s\Big)_t\ , \quad 0 \leq t \leq T\ ,$$

we can apply Girsanov's theorem and $W^1(\theta)$ is a Brownian motion under the probability measure $\mathbb{Q}^1(\theta)$. We then get from (3.8)

$$R_t(\theta)\nabla^\theta Y_t^1 \ = \ \mathbb{E}_{\mathbb{Q}^1(\theta)}\Big[R_T(\theta)\nabla g(X_T^1)\nabla^\theta X_T^1 + \int_t^T R_s(\theta)\partial_x f(s, X_s^1, Y_s^1, Z_s^1, 0)\nabla^\theta X_s^1 ds\Big|\mathcal{F}_t\Big]\ .$$

This last equality, **(HBQD)** and (3.4) give

$$|\nabla^\theta Y_t^1(\theta)| \;\leq\; e^{(L_a+K_f)T}(K_g+TK_f)\ , \quad \theta\leq t\leq T\ . \tag{3.10}$$

When b, f and g are not C_b^1, we can also prove the result by regularization as for Proposition 3.1. □

Lemma 3.4. *Suppose that **(HF)**, **(HBQ)** and **(HBQD)** hold. Then,*

$$|\nabla Y_t^1(t)| \;\leq\; (1+L_a e^{L_aT})e^{(L_a+K_f)T}(K_g+TK_f)\ , \quad 0\leq t\leq T\ . \tag{3.11}$$

Proof. Firstly, we suppose that b, β, g and f are C_b^1 w.r.t. x, y and z. Then, for any $t\in[0,T]$

$$\begin{aligned}\nabla Y_t^1(t) \;:=\; \frac{\partial Y_t^1(t)}{\partial x} \;&=\; \nabla g(X_T^1(t))\nabla X_T^1(t) \\ &\quad+\int_t^T \nabla f(s,X_s^1(t),Y_s^1(t),Z_s^1(t),0)(\nabla X_s^1(t),\nabla Y_s^1(t),\nabla Z_s^1(t))ds \\ &\quad-\int_t^T \nabla Z_s^1(t)dW_s\ .\end{aligned} \tag{3.12}$$

Applying Itô's formula, we get

$$\begin{aligned}R_t(t)\nabla Y_t^1(t) \;&=\; R_T(t)\nabla g(X_T^1(t))\nabla X_T^1(t) \\ &\quad+\int_t^T R_s(t)\partial_x f(s,X_s^1(t),Y_s^1(t),Z_s^1(t),0)\nabla X_s^1(t)ds \\ &\quad-\int_t^T R_s(t)\nabla Z_s^1(t)dW_s^1(t)\ , \quad 0\leq t\leq T\ ,\end{aligned} \tag{3.13}$$

where the process $W^1(.)$ is defined in (3.9). We have proved previously that $W^1(t)$ is a Brownian motion under the probability measure $\mathbb{Q}^1(t)$. We then get

$$R_t(t)\nabla Y_t^1(t) \;=\; \mathbb{E}_{\mathbb{Q}^1(t)}\Big[R_T(t)\nabla g(X_T^1(t))\nabla X_T^1(t)+\int_t^T R_s(t)\partial_x f(s,X_s^1(t),Y_s^1(t),Z_s^1(t),0)\nabla X_s^1(t)ds\Big|\mathcal{F}_t\Big]\ .$$

This last inequality, **(HBQD)** and (3.4) give

$$|\nabla Y_t^1(t)| \;\leq\; (1+L_a e^{L_aT})e^{(L_a+K_f)T}(K_g+TK_f)\ , \quad 0\leq t\leq T\ .$$

When b, f and g are not C_b^1, we can also prove the result by regularization as for Proposition 3.1. □

Lemma 3.5. *Suppose that **(HF)**, **(HBQ)** and **(HBQD)** hold. Then,*

$$|\nabla Y_t^0| \;\leq\; e^{(2K_f+L_a)T}(K_g+K_fT)\big(1+TK_fe^{K_fT}(1+L_ae^{L_aT})\big)\ , \quad 0\leq t\leq T\ .$$

Proof. We first suppose that b, β, g and f are C_b^1 w.r.t. x, y, z and u, then (X^0, Y^0, Z^0) is differentiable w.r.t. x and we have

$$\begin{aligned}
\nabla Y_t^0 &= \nabla g(X_T^0)\nabla X_T^0 \\
&\quad + \int_t^T \Big(\nabla f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)(\nabla X_s^0, \nabla Y_s^0, \nabla Z_s^0, \nabla Y_s^1(s) - \nabla Y_s^0)ds \\
&\quad - \int_t^T \nabla Z_s^0 dW_s\ , \quad 0 \le t \le T\ .
\end{aligned} \tag{3.14}$$

Define the process R^0 by

$$R_t^0 \ := \ \exp\Big(\int_0^t (\partial_y - \partial_u)f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)ds\Big)\ , \quad 0 \le t \le T\ .$$

Applying Itô's fomula we have

$$\begin{aligned}
R_t^0\nabla Y_t^0 &= R_T^0\nabla g(X_T^0)\nabla X_T^0 \\
&\quad + \int_t^T R_s^0\partial_x f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)\nabla X_s^0 ds \\
&\quad + \int_t^T R_s^0\partial_u f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)\nabla Y_s^1(s) ds \\
&\quad - \int_t^T R_s^0\nabla Z_s^0 dW_s^0
\end{aligned}$$

where $dW_s^0 := dW_s - \partial_z f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)ds$. From **(HBQD)**, there exists a constant $K > 0$ such that we have

$$\begin{aligned}
\Big\| \int_0^\cdot \partial_z f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)dW_s\Big\|^2_{BMO_{\mathbb{F}}[0,T]} &\le \\
K\Big(1 + \sup_{\vartheta\in\mathcal{T}_{\mathbb{F}}[0,T]} \mathbb{E}\Big[\int_\vartheta^T |Z_s^0|^2 ds\Big|\mathcal{F}_\vartheta\Big]\Big) &\le \\
K\Big(1 + \Big\|\int_0^\cdot Z_s^0 dW_s\Big\|^2_{BMO_{\mathbb{F}}[0,T]}\Big) &< \infty\ ,
\end{aligned}$$

where the last inequality comes from Lemma 3.2.

Hence by Lemma 3.1 the process $\mathcal{E}(\int_0^\cdot \partial_z f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)dW_s)$ is a uniformly integrable martingale. Therefore, under the probability measure $\mathbb{Q}^0$ defined by

$$\frac{d\mathbb{Q}^0}{d\mathbb{P}}\Big|_{\mathcal{F}_t} \ := \ \mathcal{E}\Big(\int_0^\cdot \partial_z f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)dW_s\Big)_t$$

we can apply Girsanov's theorem and W^0 is a Brownian motion under the probability measure $\mathbb{Q}^0$. Then, we get

$$\begin{aligned}
R_t^0\nabla Y_t^0 &= \mathbb{E}_{\mathbb{Q}^0}\Big[R_T^0\nabla g(X_T^0)\nabla X_T^0 \\
&\quad + \int_t^T R_s^0\partial_x f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)\nabla X_s^0 ds \\
&\quad + \int_t^T R_s^0\partial_u f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)\nabla Y_s^1(s) ds\Big|\mathcal{F}_t\Big]\ .
\end{aligned}$$

Using inequalities (3.3) and (3.11) we get

$$|\nabla Y_t^0| \leq e^{(2K_f+L_a)T}(K_g + K_f T)\big(1 + K_f T e^{K_f T}(1 + L_a e^{L_a T})\big) , \quad 0 \leq t \leq T .$$

When b, β, f and g are not C_b^1, we can also prove the result by regularization as for Proposition 3.1. □

3.4 A uniform bound for Z^0 and Z^1

Using the previous bounds, we obtain a uniform bound for the processes Z^0 and Z^1.

Proposition 3.2. *Suppose that **(HF)**, **(HBQ)** and **(HBQD)** hold. Then, for any $\theta \in [0, T]$, there exists a version of $Z^1(\theta)$ such that*

$$|Z_t^1(\theta)| \leq e^{(2L_a+K_f)T}(K_g + TK_f)K_a , \quad \theta \leq t \leq T .$$

Proof. Using Malliavin calculus, we have the classical representation of the process $Z^1(\theta)$ given by $\nabla^\theta Y^1(\theta)(\nabla^\theta X^1(\theta))^{-1}\sigma(.)$ (see [16]). In the case where b, f and g are C_b^1 w.r.t. x, y and z, we obtain from (3.10)

$$|Z_t^1(\theta)| \leq e^{(2L_a+K_f)T}(K_g + TK_f)K_a \quad a.s.$$

since $|(\nabla^\theta X^1(\theta))^{-1}| \leq e^{L_a T}$ (the proof of this inequality is similar to the one of (3.4)).
When b, f and g are not differentiable, we can also prove the result by a standard approximation and stability results for BSDEs with linear growth. □

Proposition 3.3. *Suppose that **(HF)**, **(HBQ)** and **(HBQD)** hold. Then, there exists a version of Z^0 such that*

$$|Z_t^0| \leq e^{2(K_f+L_a)T}(K_g + K_f T)\big(1 + TK_f e^{K_f T}(1 + L_a e^{L_a T})\big)K_a , \quad 0 \leq t \leq T .$$

Proof. Thanks to the Malliavin calculus, it is classical to show that a version of Z^0 is given by $\nabla Y^0(\nabla X^0)^{-1}\sigma(.)$ (see [16]). So, in the case where b, β, g and f are C_b^1 w.r.t. x, y, z and u, we obtain from (3.3) and Lemma 3.5

$$|Z_t^0| \leq e^{2(K_f+L_a)T}(K_g + K_f T)\big(1 + TK_f e^{K_f T}(1 + L_a e^{L_a T})\big)M_\sigma \quad a.s.$$

since $|(\nabla X_t^0)^{-1}| \leq e^{L_a T}$ (the proof of this inequality is similar to the one of (3.3)).

When b, β, g and f are not differentiable, we can also prove the result by a standard approximation and stability results for BSDEs with linear growth. □

4 Discrete-time approximation for the FBSDE

4.1 Discrete-time scheme for the FBSDE

Throughout the sequel, we consider a discretization grid $\pi := \{t_0, \dots, t_n\}$ of $[0,T]$ with $0 = t_0 < t_1 < \dots < t_n = T$. For $t \in [0,T]$, we denote by $\pi(t)$ the largest element of π smaller than t

$$\pi(t) \quad := \quad \max\left\{ t_i \ , \ i = 0, \dots, n \mid t_i \leq t \right\} .$$

We also denote by $|\pi|$ the mesh of π

$$|\pi| \quad := \quad \max\left\{ t_{i+1} - t_i \ , \ i = 0, \dots, n-1 \right\} ,$$

that we suppose satisfying $|\pi| \leq 1$, and by ΔW_i^π (resp. Δt_i^π) the increment of W (resp. the difference) between t_i and t_{i-1}: $\Delta W_i^\pi := W_{t_i} - W_{t_{i-1}}$ (resp. $\Delta t_i^\pi := t_i - t_{i-1}$), for $1 \leq i \leq n$.

We introduce an approximation of the process X based on the discretization of the processes X^0 and X^1.

• *Euler scheme for X^0.* We consider the classical scheme $X^{0,\pi}$ defined by

$$\begin{cases} X_{t_0}^{0,\pi} \ = \ x \ , \\ X_{t_i}^{0,\pi} \ = \ X_{t_{i-1}}^{0,\pi} + b(t_{i-1}, X_{t_{i-1}}^{0,\pi})\Delta t_i^\pi + \sigma(t_{i-1})\Delta W_i^\pi \ , \quad 1 \leq i \leq n \ . \end{cases} \tag{4.15}$$

• *Euler scheme for X^1.* Since the process X^1 depends on two parameters t and θ, we introduce a discretization of X^1 in these two variables. We then consider the following scheme[1]

$$\begin{cases} X_{t_0}^{1,\pi}(t_k) \ = \ x + \beta(t_0, x)\delta_{k=0} \ , \quad 0 \leq k \leq N \ , \\ X_{t_i}^{1,\pi}(t_k) \ = \ X_{t_{i-1}}^{1,\pi}(t_k) + b(t_{i-1}, X_{t_{i-1}}^{1,\pi}(t_k))\Delta t_i^\pi + \sigma(t_{i-1})\Delta W_i^\pi \\ \qquad\qquad + \beta(t_{i-1}, X_{t_{i-1}}^{1,\pi}(t_k))\delta_{i=k} \ , \quad 1 \leq i,k \leq n \ . \end{cases} \tag{4.16}$$

We are now able to provide an approximation of the process X solution to the FSDE (2.1). We consider the scheme X^π defined by

$$X_t^\pi \quad = \quad X_{\pi(t)}^{0,\pi}\mathbb{1}_{t<\tau} + X_{\pi(t)}^{1,\pi}(\pi(\tau))\mathbb{1}_{t\geq\tau} \ , \qquad 0 \leq t \leq T \ . \tag{4.17}$$

We shall denote by $\{\mathcal{F}_i^{0,\pi}\}_{0\leq i\leq n}$ (resp. $\{\mathcal{F}_i^{1,\pi}(\theta)\}_{0\leq i\leq n}$) the discrete-time filtration associated with $X^{0,\pi}$ (resp. $X^{1,\pi}(\theta)$)

$$\begin{aligned} \mathcal{F}_i^{0,\pi} \quad &:= \quad \sigma(X_{t_j}^{0,\pi}, \ j \leq i) \\ \text{(resp. } \mathcal{F}_i^{1,\pi}(\theta) \quad &:= \quad \sigma(X_{t_j}^{1,\pi}(\theta), \ j \leq i)) \ . \end{aligned}$$

We introduce an approximation of (Y, Z) based on the discretization of (Y^0, Z^0) and (Y^1, Z^1). To this end we introduce the backward implicit schemes on π associated with

[1] $\delta_{i=k} = 0$ if $i \neq k$ and $\delta_{i=k} = 1$ if $i = k$.

the BSDEs (2.4) and (2.6). Since the system is recursively coupled, we first introduce the scheme associated with (2.4). We then use it to define the scheme associated with (2.6).

• *Backward Euler scheme for* (Y^1, Z^1). We consider the classical implicit scheme $(Y^{1,\pi}, Z^{1,\pi})$ defined by

$$\begin{cases} Y_T^{1,\pi}(\pi(\theta)) &= g(X_T^{1,\pi}(\pi(\theta)))\ , \\ Y_{t_{i-1}}^{1,\pi}(\pi(\theta)) &= \mathbb{E}\big[Y_{t_i}^{1,\pi}(\pi(\theta))\big|\mathcal{F}_{i-1}^{1,\pi}(\pi(\theta))\big] \\ &\quad + f(t_{i-1}, X_{t_{i-1}}^{1,\pi}(\pi(\theta)), Y_{t_{i-1}}^{1,\pi}(\pi(\theta)), Z_{t_{i-1}}^{1,\pi}(\pi(\theta)), 0)\Delta t_i^\pi\ , \\ Z_{t_{i-1}}^{1,\pi}(\pi(\theta)) &= \dfrac{1}{\Delta t_i^\pi}\mathbb{E}\big[Y_{t_i}^{1,\pi}(\pi(\theta))\Delta W_i^\pi\big|\mathcal{F}_{i-1}^{1,\pi}(\pi(\theta))\big]\ , \quad t_{i-1} \geq \pi(\theta)\ . \end{cases} \tag{4.18}$$

• *Backward Euler scheme for* (Y^0, Z^0). Since the generator of (2.6) involves the process $(Y_t^1(t))_{t\in[0,T]}$, we consider a discretization based on $Y^{1,\pi}$. We therefore consider the scheme $(Y^{0,\pi}, Z^{0,\pi})$ defined by

$$\begin{cases} Y_T^{0,\pi} &= g(X_T^{0,\pi})\ , \\ Y_{t_{i-1}}^{0,\pi} &= \mathbb{E}\big[Y_{t_i}^{0,\pi}\big|\mathcal{F}_{i-1}^{0,\pi}\big] + \bar{f}^\pi(t_{i-1}, X_{t_{i-1}}^{0,\pi}, Y_{t_{i-1}}^{0,\pi}, Z_{t_{i-1}}^{0,\pi})\Delta t_i^\pi\ , \\ Z_{t_{i-1}}^{0,\pi} &= \dfrac{1}{\Delta t_i^\pi}\mathbb{E}\big[Y_{t_i}^{0,\pi}\Delta W_i^\pi\big|\mathcal{F}_{i-1}^{0,\pi}\big]\ , \quad 1 \leq i \leq n\ , \end{cases} \tag{4.19}$$

where $\bar{f}^\pi(t,x,y,z) := f\big(t,x,y,z,Y_{\pi(t)}^{1,\pi}(\pi(t)) - y\big)$ for all $(t,x,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. We then consider the following scheme defined by

$$\begin{cases} Y_t^\pi &= Y_{\pi(t)}^{0,\pi}\mathbb{1}_{t<\tau} + Y_{\pi(t)}^{1,\pi}(\pi(\tau))\mathbb{1}_{t\geq\tau}\ , \\ Z_t^\pi &= Z_{\pi(t)}^{0,\pi}\mathbb{1}_{t\leq\tau} + Z_{\pi(t)}^{1,\pi}(\pi(\tau))\mathbb{1}_{t>\tau}\ , \\ U_t^\pi &= \big(Y_{\pi(t)}^{1,\pi}(\pi(t)) - Y_{\pi(t)}^{0,\pi}\big)\mathbb{1}_{t\leq\tau}\ , \end{cases} \tag{4.20}$$

for $t \in [0,T]$.

4.2 Convergence of the scheme for the FBSDE

We now concentrate on the error approximation of the processes X, Y, Z and U by their scheme X^π, Y^π, Z^π and U^π. To this end we introduce extra assumptions on the regularity of the forward coefficients w.r.t. the time variable t.

(HFD) There exists a constant K_t such that the functions b, σ and β satisfy

$$\begin{aligned} \big|b(t,x) - b(t',x)\big| + \big|\sigma(t) - \sigma(t')\big| &\leq K_t|t-t'|^{\frac{1}{2}}\ , \\ \big|\beta(t,x) - \beta(t',x)\big| + \big|\sigma(t) - \sigma(t')\big| &\leq K_t|t-t'|\ , \end{aligned}$$

for all $(t,t',x) \in [0,T]^2\times\mathbb{R}$.

We can now state the main result of the paper.

Theorem 4.1. *Under **(HF)**, **(HFD)**, **(HBQ)** and **(HBQD)** we have the following estimate*

$$\begin{aligned} \mathbb{E}\Big[\sup_{t\in[0,T]}\big|X_t - X_t^\pi\big|^2\Big] + \sup_{t\in[0,T]}\mathbb{E}\Big[\big|Y_t - Y_t^\pi\big|^2\Big] & \\ +\mathbb{E}\Big[\int_0^T \big|Z_t - Z_t^\pi\big|^2 dt\Big] + \mathbb{E}\Big[\int_0^T \lambda_t\big|U_t - U_t^\pi\big|^2 dt\Big] & \leq K|\pi| \, , \end{aligned}$$

for a constant K which does not depend on π.

Proof. Fix $M \in \mathbb{R}$ such that

$$\begin{aligned} M \quad \geq \quad \max\Big\{ & e^{(2L_a+K_f)T}(K_g + TK_f)K_a \; ; \\ & e^{2(K_f+L_a)T}(K_g + K_f T)\big(1 + TK_f e^{K_f T}(1 + L_a e^{L_a T})\big)K_a\Big\} \, , \end{aligned}$$

and define the function $\tilde{f}$ by

$$\tilde{f}(t,x,y,z,u) \quad = \quad f(t,x,y,\varphi_M(z),u) \, , \quad (t,x,y,z,u) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \, ,$$

where

$$\varphi_M(z) := \begin{cases} z & \text{if} \quad |z| \leq M \\ M\frac{z}{|z|} & \text{if} \quad |z| > M \end{cases} , \quad z \in \mathbb{R} \, .$$

We notice that φ_M is Lipschitz continuous and bounded. Therefore we obtain from **(HBQD)** that $\tilde{f}$ is Lipschitz continuous.

Moreover, using Propositions 3.2 and 3.3, we get that under **(HF)**, **(HBQ)** and **(HBQD)**, (X,Y,Z) is also solution to the Lipschitz FBSDE

$$\begin{aligned} X_t &= x + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s)dW_s + \int_0^t \beta(s,X_{s^-})dH_s \, , \quad 0 \leq t \leq T \, , \\ Y_t &= g(X_T) + \int_t^T \tilde{f}\big(s,X_s,Y_s,Z_s,U_s(1-H_s)\big)ds \\ & \qquad - \int_t^T Z_s dW_s - \int_t^T U_s dH_s \, , \quad 0 \leq t \leq T \, . \end{aligned}$$

Applying Theorem 5.1 in [16], we get the result. □

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