



HAL
open science

An approach of the Minimal Model Program for horospherical varieties via moment polytopes

Boris Pasquier

► **To cite this version:**

Boris Pasquier. An approach of the Minimal Model Program for horospherical varieties via moment polytopes. *Journal für die reine und angewandte Mathematik*, 2015, 708, pp.173-212. 10.1515/crelle-2013-0103 . hal-00757355

HAL Id: hal-00757355

<https://hal.science/hal-00757355>

Submitted on 26 Nov 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An approach of the Minimal Model Program for horospherical varieties via moment polytopes

Boris Pasquier*

November 27, 2012

Abstract

We describe the Minimal Model Program in the family of \mathbb{Q} -Gorenstein projective horospherical varieties, by studying a family of polytopes defined from the moment polytope of a Cartier divisor of the variety we begin with. In particular, we generalize the results on MMP in toric varieties due to M. Reid, and we complete the results on MMP in spherical varieties due to M. Brion in the case of horospherical varieties.

Mathematics Subject Classification. 14E30 14M25 52B20 14M17

Keywords. Minimal Model Program, Toric varieties, Horospherical varieties, Moment polytopes.

Contents

1	Introduction	2
2	Horospherical varieties and polytopes	9
2.1	Notations and horospherical embedding theory	9
2.2	Correspondence between projective horospherical varieties and polytopes	10
2.3	G -equivariant morphisms and polytopes	12
2.4	Curves in horospherical varieties	13
3	One-parameter families of polytopes	15
3.1	A first one-parameter family of polytopes: definitions and results	16
3.2	Proof of Theorem 11	17
3.3	A second one-parameter family of polytopes	22

*Boris PASQUIER, Université Montpellier 2, CC 051, Place Eugène Bataillon, 34095 Montpellier cedex, France.
E-mail: boris.pasquier@math.univ-montp2.fr

4	MMP via a one-parameter family of polytopes	23
4.1	The one-parameter family of polytopes	23
4.2	Construction of varieties and morphisms	24
4.3	Description of the contracted curves	25
4.4	\mathbb{Q} -Gorenstein singularities	26
4.5	\mathbb{Q} -factorial singularities	27
4.6	General fibers of Mori fibrations	28
5	Examples	30

1 Introduction

The Minimal Model Program (MMP) takes an important place in birational algebraic geometry in order to get a birational classification of algebraic varieties. A lot of progress has been done in the last three decades. We come back here to an original version of the MMP, summarized by Figure 1 where \mathcal{H} denote a family of \mathbb{Q} -factorial varieties (see [Mat02] to have a good overview of this theory). For any \mathbb{Q} -Gorenstein variety X , we denote by $NE(X)$ the nef cone of curves of X , by K_X a canonical divisor of X and by $NE(X)_{K_X < 0}$ (resp. $NE(X)_{K_X > 0}$) the intersection of the nef cone with the open half-space of curves negative (resp. positive) along the divisor K_X .

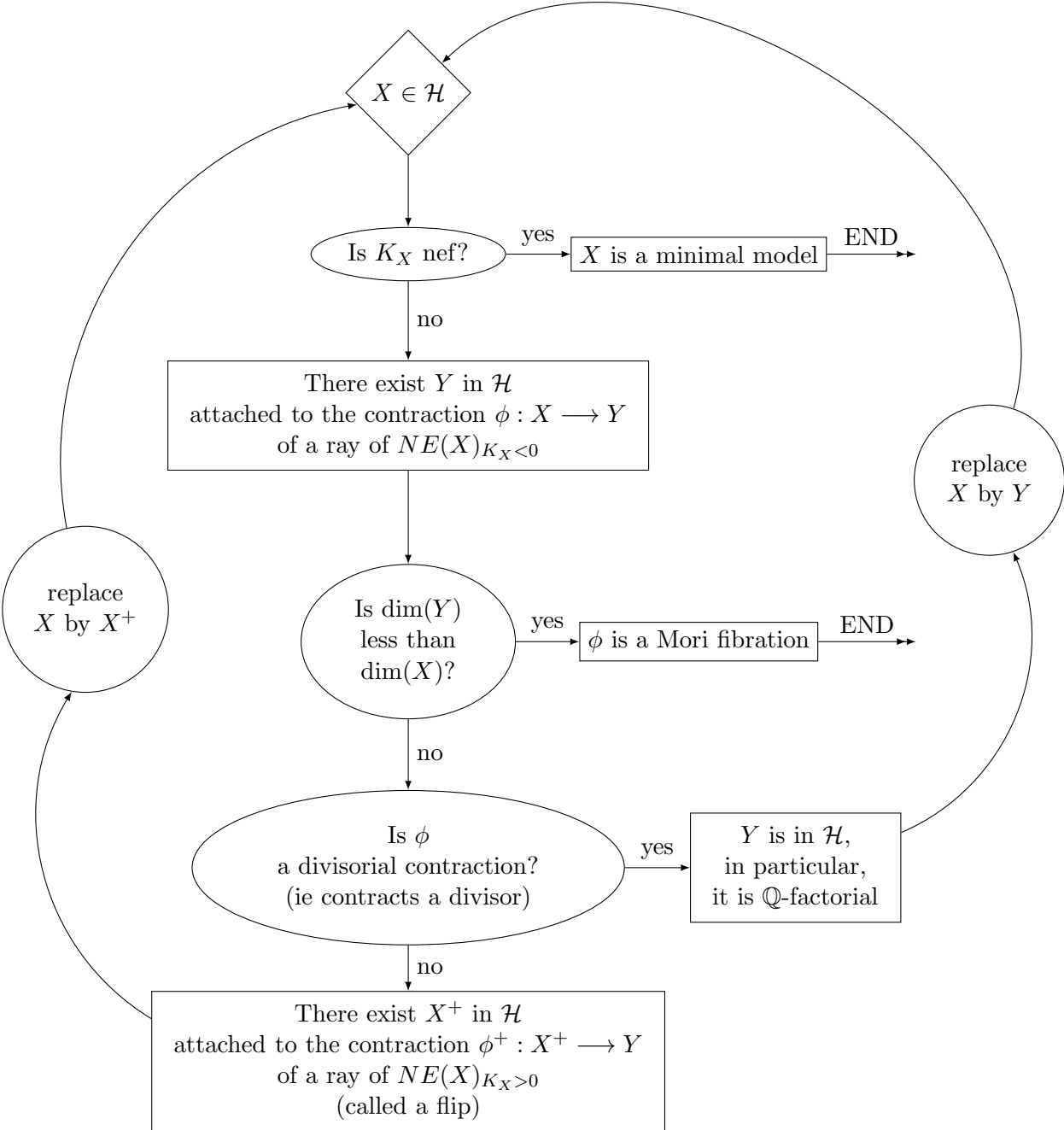
When \mathcal{H} is the family of \mathbb{Q} -factorial toric varieties, M. Reid proved in 1983 that the MMP works [Rei83]. In particular, the MMP ends (there is no infinite series of flips). Note that, for this family, the cone $NE(X)$ is polyhedral generated by finitely many rays, and is very well-understood. Note also that, since toric varieties are rational varieties, a minimal model is then a point. Moreover, M. Reid proved that the general fibers of the Mori fibrations, for \mathbb{Q} -factorial toric varieties, are weighted projective spaces.

When \mathcal{H} is the family of \mathbb{Q} -factorial spherical G -varieties, for any connected reductive algebraic group G , M. Brion proved in 1993 that the MMP works [Bri93]. For this family, the cone $NE(X)$ is still polyhedral generated by finitely many rays and described in [Bri93]. Note that spherical varieties are also rational varieties, so that minimal models are still points here. Nevertheless, it is very difficult to compute concretely $NE(X)$ and K_X , so that it makes difficult the application of the MMP to explicit examples of this family. That is why we reduce the study to horospherical varieties, for which a canonical divisor is well-known, and that is also why we present another approach that does not need the computation of $NE(X)$. Moreover, the general fibers of Mori fibrations for \mathbb{Q} -factorial spherical G -varieties are not known.

In this paper, we first consider the case where \mathcal{H} is the family of \mathbb{Q} -Gorenstein projective horospherical G -varieties. The family of horospherical varieties is contained in the family of spherical varieties and contains toric varieties. Using a different approach than the one used by M. Reid or M. Brion, we obtain the main result of this paper.

Theorem 1. *The MMP described by Figure 2 works if \mathcal{H} is the family of \mathbb{Q} -Gorenstein projective horospherical G -varieties for any connected reductive algebraic group G . Moreover, for any X in \mathcal{H} and from any choice of an ample Cartier divisor of X , we can concretely describe each step of this MMP until it ends.*

Figure 1: Original MMP for \mathbb{Q} -factorial varieties



Remark that the main difference with the original MMP is that a divisorial contraction can give a flip,. It comes from the fact that the varieties are here not necessarily \mathbb{Q} -factorial.

We use here convex geometry, and in particular continuous transformations of moment polytopes. That is why we need to restrict to projective varieties. To illustrate Theorem 1, we give two first examples, both from the same toric variety X . Note that the result of the MMP applied to a variety X is not unique: for the original MMP, it depends on the choices of a ray in the effective cone of all varieties appearing in the beginning of each loop of the program, and here it depends on the choice of a \mathbb{Q} -Cartier ample divisor of the variety X (only in the beginning of the program). To choose a \mathbb{Q} -Cartier ample divisor of a projective toric variety is equivalent to choose its moment polytope (see Section 2.2). We suppose here that the reader know the classification of toric varieties in terms of fans, see [Ful93] or [Oda88] if it is not the case.

Example 1. In \mathbb{Q}^3 consider the simple polytope Q defined by the following inequations, where (x, y, z) are the coordinates of a point in \mathbb{Q}^3 .

$$\begin{aligned} z &\geq -1 \\ -x - y - 2z &\geq -5 \\ 2x - z &\geq -3 \\ -2x - z &\geq -3 \\ 2y - z &\geq -3 \\ -2y - z &\geq -3. \end{aligned}$$

It is a pyramid whose summit is cut by a plane.

The edges of the fan \mathbb{F}_X of the toric variety X associated to Q are $x_1 := (0, 0, 1)$, $x_2 := (-1, -1, -2)$, $x_3 := (2, 0, -1)$, $x_4 := (-2, 0, -1)$, $x_5 := (0, 2, -1)$ and $x_6 := (0, -2, -1)$. And the maximal cones of \mathbb{F}_X are the cones respectively generated by (x_1, x_3, x_5) , (x_1, x_3, x_6) , (x_1, x_4, x_5) , (x_1, x_4, x_6) , (x_2, x_3, x_5) , (x_2, x_3, x_6) , (x_2, x_4, x_5) , (x_2, x_4, x_6) . (See Section 2.2, to have an explanation of the correspondence between moment polytopes and fans.)

Note that X is \mathbb{Q} -factorial, because Q is simple (and \mathbb{F}_X is simplicial).

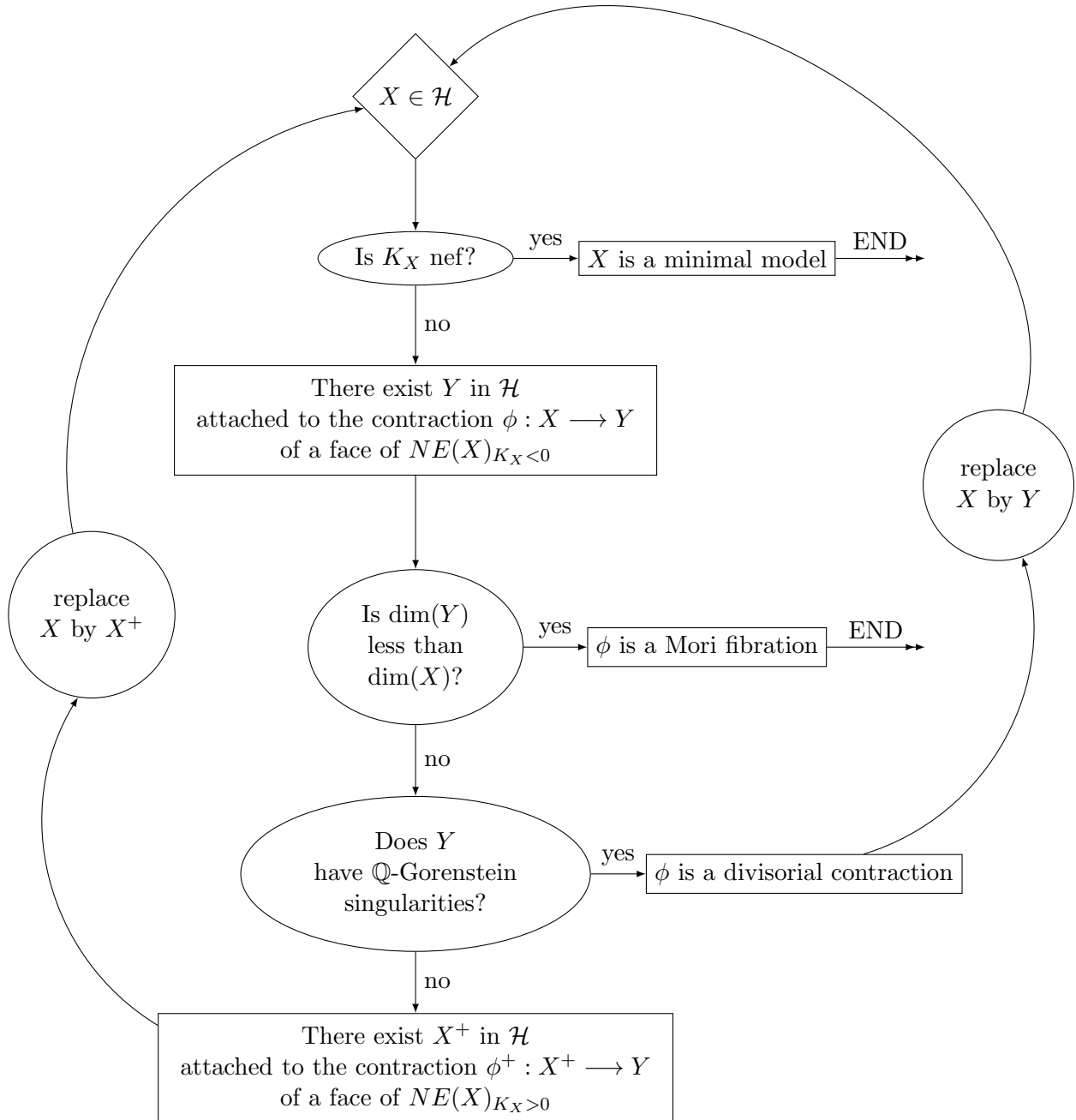
We consider the family of polytopes Q^ϵ defines by the following inequations:

$$\begin{aligned} z &\geq -1 + \epsilon \\ -x - y - 2z &\geq -5 + \epsilon \\ 2x - z &\geq -3 + \epsilon \\ -2x - z &\geq -3 + \epsilon \\ 2y - z &\geq -3 + \epsilon \\ -2y - z &\geq -3 + \epsilon. \end{aligned}$$

Note that, for $\epsilon > 0$ and small enough, it is the moment polytope of $D + \epsilon K_X$, where D is the divisor of X whose moment polytope is Q , and K_X is the canonical divisor.

For $\epsilon \in [0, 1[$, all polytopes Q^ϵ have the same structure so that they all correspond to the toric variety X . For any $\epsilon \in [1, 2[$, the polytopes Q^ϵ are pyramids. They all correspond to the toric variety Y whose fan \mathbb{F}_Y is described by the cones respectively generated by (x_1, x_3, x_5) ,

Figure 2: MMP for \mathbb{Q} -Gorenstein projective horospherical G -varieties



(x_1, x_3, x_6) , (x_1, x_4, x_5) , (x_1, x_4, x_6) , (x_3, x_4, x_5, x_6) . Remark that the last cone is not simplicial, so that Y is not \mathbb{Q} -factorial. The fact that Y is \mathbb{Q} -Gorenstein comes from the fact that x_3 , x_4 , x_5 and x_6 are in a common plane. Note also that x_2 is not an edge of \mathbb{F}_Y .

And for $\epsilon = 2$, the polytope Q^ϵ is a point. Then the family $(Q^\epsilon)_{\epsilon \in [0,2]}$ reveals a divisorial contraction $\phi : X \rightarrow Y$ and a Mori fibration from Y to a point.

We illustrate this example in Figure 3.

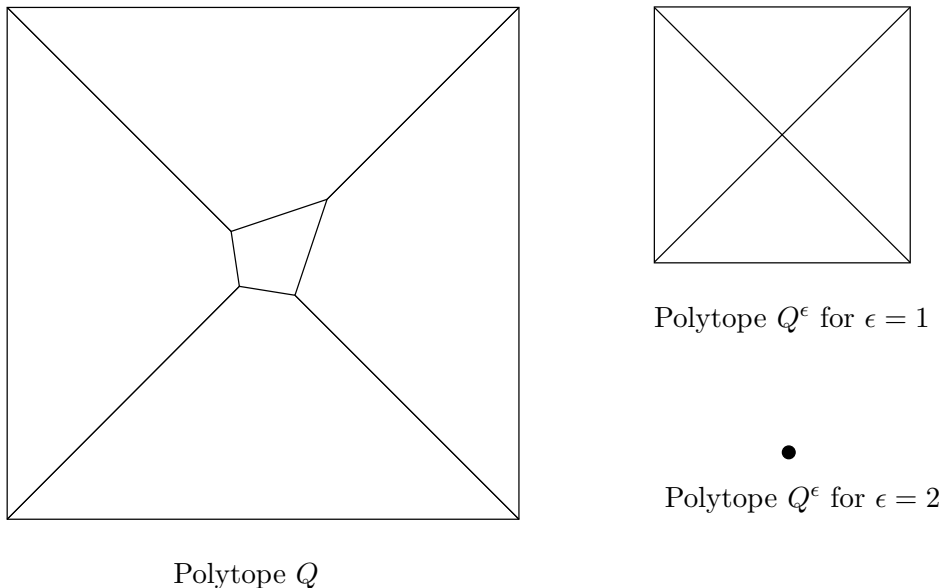


Figure 3: Evolution of Q^ϵ in Example 1 (view from the top of the pyramid)

Before to give the second example, we can notice that, in Example 1, the divisorial contraction ϕ goes from a \mathbb{Q} -factorial variety to a not \mathbb{Q} -factorial variety. It does not contradict the results of M. Reid, because ϕ is not the contraction of a ray of $NE(X)$ but of a 2-dimensional face of $NE(X)$. That gives a reason of considering first \mathbb{Q} -Gorenstein varieties instead of \mathbb{Q} -factorial varieties. Fortunately, what occurs in Example 1 can be observed only in very particular cases (see Theorem 2). Now, we consider a second example, with a more general divisor D .

Example 2. Let Q^ϵ be the polytopes defined by the following inequations,

$$\begin{aligned}
 z &\geq -1 + \epsilon \\
 -x - y - 2z &\geq -5 + \epsilon \\
 2x - z &\geq -4 + \epsilon \\
 -2x - z &\geq -4 + \epsilon \\
 2y - z &\geq -3 + \epsilon \\
 -2y - z &\geq -3 + \epsilon.
 \end{aligned}$$

The simple polytope Q^0 is still a pyramid whose summit is cut by a plane. It is the moment polytope of a \mathbb{Q} -Cartier ample divisor of the toric variety X of Example 1.

For $\epsilon \in [0, \frac{1}{2}[$, all polytopes Q^ϵ have the same structure so that they all correspond to the toric variety X .

Now, for $\epsilon = \frac{1}{2}$, Q^ϵ corresponds to the fan whose maximal cones are the cones respectively generated by (x_1, x_3, x_5) , (x_1, x_3, x_6) , (x_1, x_4, x_5) , (x_1, x_4, x_6) , (x_2, x_4, x_5) , (x_2, x_4, x_6) and (x_2, x_3, x_5, x_6) . The associated toric variety Y is not \mathbb{Q} -factorial. In fact, it is even not \mathbb{Q} -Gorenstein (because x_2, x_3, x_5 and x_6 are not in a common plane).

For $\epsilon \in]\frac{1}{2}, \frac{3}{2}[$, the polytope Q^ϵ is the moment polytope of a divisor of the toric variety X^+ whose fan has maximal cones respectively generated by (x_1, x_3, x_5) , (x_1, x_3, x_6) , (x_1, x_4, x_5) , (x_1, x_4, x_6) , (x_2, x_4, x_5) , (x_2, x_4, x_6) , (x_2, x_5, x_6) and (x_3, x_5, x_6) . The variety X^+ is clearly \mathbb{Q} -factorial, and defines a flip $X \rightarrow Y \leftarrow X^+$.

Now, for $\epsilon \in [\frac{3}{2}, 2[$, Q^ϵ is a simple polytope with 6 vertices and the moment polytope of a divisor of the toric variety Z whose fan has maximal cones respectively generated by (x_1, x_3, x_5) , (x_1, x_3, x_6) , (x_1, x_4, x_5) , (x_1, x_4, x_6) , (x_3, x_5, x_6) and (x_4, x_5, x_6) . And we get a divisorial contraction from X^+ to Z .

To finish, for $\epsilon = 2$, Q^ϵ is the segment whose extremities are $(-\frac{1}{2}, 0, 1)$ and $(\frac{1}{2}, 0, 1)$, corresponding to the toric variety \mathbb{P}^1 . It gives a Mori fibration from Z to the projective line.

We illustrate this example in Figure 4.

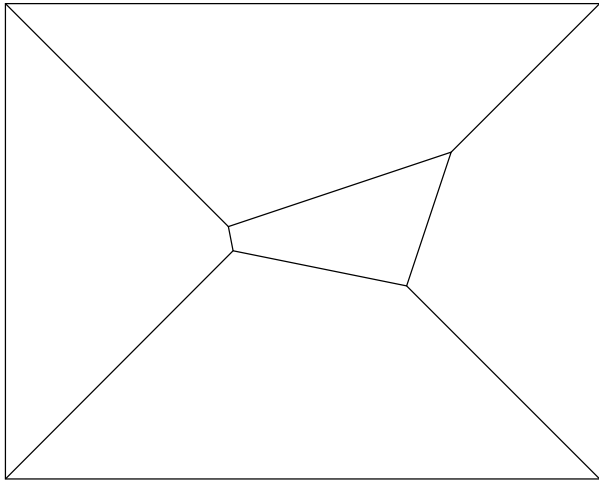
Moreover, the general fiber of the Mori fibration is the toric variety associated to the polytope in \mathbb{Q}^2 defined by the inequalities $z \geq -1$, $2y - z \geq -3$ and $-2y - z \geq -3$. Its fan is isomorphic to the complete fan in \mathbb{Q}^2 whose edges are $(0, 1)$, $(-2, 1)$ and $(-2, -1)$. It is the weight projective plane $\mathbb{P}(1, 1, 2)$.

Example 2 arises a natural question: can we recover the original MMP for \mathbb{Q} -factorial projective horospherical varieties, by choosing a good divisor D ? The answer is yes and given in the following result.

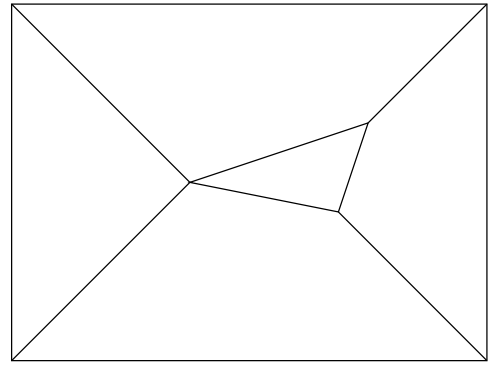
Theorem 2. *Suppose that X is \mathbb{Q} -factorial. Then, by taking a general divisor D of X , we have that*

- *at any loops of the MMP described in Figure 2 (until it ends), the morphisms ϕ and ϕ^+ are contractions of rays of $NE(X)$ and $NE(X^+)$;*
- *the MMP described in Figure 2 still works by replacing \mathbb{Q} -Gorenstein singularities with \mathbb{Q} -factorial ones everywhere;*
- *the general fibers of Mori fibrations are projective \mathbb{Q} -factorial horospherical varieties with Picard number 1 (whose moment polytopes are simplexes intersecting all walls of a dominant chamber along facets).*

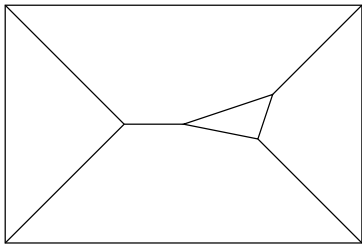
Remark 1. If X is smooth and D general, the general fibers of Mori fibrations are projective smooth horospherical varieties with Picard number 1. These varieties have been classified in [Pas09], in particular we get flag varieties and some two-orbit varieties. Hence, the general fibers of Mori fibration are not only weighted projective spaces as in the toric case.



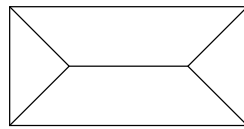
Polytope Q



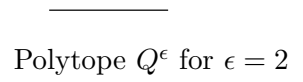
Polytope Q^ϵ for $\epsilon = \frac{1}{2}$



Polytope Q^ϵ for $\epsilon = 1$



Polytope Q^ϵ for $\epsilon = \frac{3}{2}$



Polytope Q^ϵ for $\epsilon = 2$

Figure 4: Evolution of Q^ϵ in Example 2 (view from the top of the pyramid)

The paper is organized as follows.

In Section 2, we recall the theory of horospherical varieties, we state the correspondence between polarized horospherical varieties and moment polytopes, we rewrite the existence criterion of an equivariant morphism between two horospherical varieties in terms of polytopes, and we describe the curves of horospherical varieties from a moment polytope.

In Section 3, we study particular (linear) one-parameter families of polytopes in \mathbb{Q}^n and we define some equivalence relation in these families.

In Section 4, we prove Theorems 1 and 2, by using the previous two sections. In particular, we construct a one-parameter families of moment polytopes whose equivalent classes describe all loops of the MMP given in Figure 2.

In Section 5, we give five examples to illustrate what can happen in the MMP for horospherical (and not toric) varieties.

2 Horospherical varieties and polytopes

2.1 Notations and horospherical embedding theory

We begin by recalling briefly the Luna-Vust theory of horospherical embeddings and by setting the notations used in the rest of the paper. For more details, the reader could have a look at [Pas08], or in more generalities at the Luna-Vust theory of spherical embeddings in [Kno91].

We fix a connected reductive algebraic group G . A closed subgroup H of G is said to be horospherical if it contains the unipotent radical U of a Borel subgroup B of G . It is equivalent to say that G/H is a torus fibration over a flag variety G/P . The parabolic subgroup P is the normalizer $N_G(H)$ of H in G and contains B . We fix a maximal torus T of B . Then we denote by S the set of simple roots of (G, B, T) . Also denote by R the subset of S of simple roots of P . Let $X(T)$ (respectively $X(T)^+$) be the lattice of characters of T (respectively the set of dominant characters). Similarly, we define $X(P)$ and $X(P)^+ = X(P) \cap X(T)^+$. Note that $X(P)$ is generated by the fundamental weights ϖ_α with $\alpha \in S \setminus R$. Let M be the sublattice of $X(P)$ consisting of characters of P vanishing on H . The rank of M is called the rank of G/H and denoted by n . Let $N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. For any free lattice \mathbb{L} , we denote by $\mathbb{L}_{\mathbb{Q}}$ the \mathbb{Q} -vector space $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$. For any simple root $\alpha \in S \setminus R$, the restriction of the coroot α^\vee to M is a point of N , which we denote by α_M^\vee . For any $\alpha \in S \setminus R$, we also denote by $W_{\alpha, P}$ the hyperplane defined by $\{m \in X(P)_{\mathbb{Q}} \mid \langle m, \alpha^\vee \rangle = 0\}$ (note that it corresponds to a wall of the dominant chamber of characters of P).

A G/H -embedding is couple (X, x) , where X is a normal algebraic G -variety and x a point of X such that $G \cdot x$ is open in X and isomorphic to G/H . By abuse of notation, we will forget the point x .

The G/H -embeddings are classified by colored fans in $N_{\mathbb{Q}}$. A colored fan is a set of colored cones in $N_{\mathbb{Q}}$ stable by taking colored subcones and without overlap (see for example [Pas08] for the complete definition of a colored fan). A colored cone is a couple $(\mathcal{C}, \mathcal{F})$, where \mathcal{F} is a subset of $S \setminus R$ such that $\alpha_M^\vee \neq 0$ for all $\alpha \in \mathcal{F}$, and \mathcal{C} is a strictly convex cone generated by the α_M^\vee with $\alpha \in \mathcal{F}$ and a finite set of points in N .

If $G = (\mathbb{C}^*)^n$ and $H = \{1\}$, we get the well-known classification of toric varieties.

The colored fans of complete G/H -embeddings are the complete ones (ie such that $N_{\mathbb{Q}}$ is the union of the colored cones of the colored fan).

If X is a G/H -embedding, we denote by \mathbb{F}_X the colored fan of X in $N_{\mathbb{Q}}$ and we denote by \mathcal{F}_X the set $\cup_{(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X} \mathcal{F} \subset S \setminus R$ of colors of X .

Moreover, the set of G -orbits of a G/H -embedding X is naturally in bijection with the set of colored cones of \mathbb{F}_X (reversing usual orders). In particular, the G -stable irreducible divisors correspond to the colored edges of \mathbb{F}_X of the form (\mathcal{C}, \emptyset) . We denote them by X_1, \dots, X_m , and for any $i \in \{1, \dots, m\}$, we denote by x_i the primitive element in N of the corresponding edge.

2.2 Correspondence between projective horospherical varieties and polytopes

In this section, we list results coming directly from the characterization of Cartier, \mathbb{Q} -Cartier and ample divisors of horospherical varieties due to M. Brion in the more general case of spherical varieties ([Bri89]). And we classify projective G/H -embeddings in terms of G/H -polytopes (defined below in Definition 1).

First, describe the B -stable irreducible divisors of a G/H -embedding X . We already defined the G -stable divisors X_1, \dots, X_m . The other ones are the closures in X of B -stable irreducible divisors of G/H , which are the inverse images by the torus fibration $G/H \rightarrow G/P$ of the Schubert divisors of the flag variety G/P . We denote these divisors of X by D_{α} for any $\alpha \in S \setminus R$.

In all the paper, a divisor of a horospherical variety is always supposed to be B -stable.

Let X be a projective G/H -embedding and let $D = \sum_{i=1}^m a_i X_i + \sum_{\alpha \in S \setminus R} a_{\alpha} D_{\alpha}$ be an ample Cartier divisor of X . Then the set

$$\tilde{Q}_{X,D} = \tilde{Q}_D := \{m \in M_{\mathbb{Q}} \mid \langle m, x_i \rangle \geq -a_i, \forall i \in \{1, \dots, m\} \text{ and } \langle m, \alpha_M^{\vee} \rangle \geq -a_{\alpha}, \forall \alpha \in \mathcal{D}_X\}$$

is a lattice polytope in $M_{\mathbb{Q}}$ (ie with vertices in M) of maximal dimension. If D is only \mathbb{Q} -Cartier, then \tilde{Q}_D is a polytope in $M_{\mathbb{Q}}$ of maximal dimension. The polytope $Q_{X,D} = Q_D := v^0 + \tilde{Q}_D$ in $X(P)_{\mathbb{Q}}$, where $v^0 = \sum_{\alpha \in S \setminus R} a_{\alpha} \varpi_{\alpha}$, is called the moment polytope of the polarized variety (X, D) (or of D in X). We call \tilde{Q}_D the pseudo-moment polytope of (X, D) .

Moreover, the moment polytope of an ample \mathbb{Q} -Cartier divisor D in a G/H -embedding X is always contained in the dominant chamber $X(P)^+$, and $Q_D \cap W_{\alpha,P} \neq \emptyset$ if and only if $\alpha \in \mathcal{F}_X$. We note also that, since any $\alpha \in \mathcal{F}_X$ satisfies $\alpha_M^{\vee} \neq 0$, a moment polytope is contained in no wall $W_{\alpha,P}$, with $\alpha \in S \setminus R$.

An important tool of the paper is the fact that the colored fan of X is reconstructible from the moment polytope Q_D of (X, D) for any ample \mathbb{Q} -Cartier (B -stable) divisor D . Indeed, any maximal colored cone of the complete colored fan \mathbb{F}_X can be defined from a vertex of Q_D as follows. Let v be a vertex of Q_D . We define \mathcal{C}_v to be the cone of $N_{\mathbb{Q}}$ generated by inward-pointing normal vectors of the facets of Q_D containing v . And we set $\mathcal{F}_v = \{\alpha \in S \setminus R \mid v \in W_{\alpha,P}\}$. Then $(\mathcal{C}_v, \mathcal{F}_v)$ is a maximal colored cone of \mathbb{F}_X .

Moreover, the divisor D can also be computed from the pair (Q, \tilde{Q}) . Indeed, the coefficients a_{α} with $\alpha \in S \setminus R$ are given by the translation vector in $X(P)^+$ that maps \tilde{Q} to Q . And, for any $i \in \{1, \dots, m\}$, the coefficient a_i is given by $\langle v_i, x_i \rangle$ for any element $v_i \in M_{\mathbb{Q}}$ in the facet of \tilde{Q} for which x_i is an inward-pointing normal vector.

In order to classify projective G/H -embeddings in terms of polytopes, we give the following definition.

Definition 1. Let Q be a polytope in $X(P)_{\mathbb{Q}}^+$. We say that Q is a G/H -polytope, if its direction is $M_{\mathbb{Q}}$ and if it is contained in no wall $W_{\alpha,P}$ with $\alpha \in S \setminus R$.

Let Q and Q' be two G/H -polytopes in $X(P)_{\mathbb{Q}}^+$. Consider any polytopes \tilde{Q} and \tilde{Q}' in $M_{\mathbb{Q}}$ obtained by translations of Q and Q' respectively. We say that Q and Q' are equivalent G/H -polytopes if the following conditions are satisfied.

1. There exist an integer j , $2j$ affine half-spaces $\mathcal{H}_1^+, \dots, \mathcal{H}_j^+$ and $\mathcal{H}'_1^+, \dots, \mathcal{H}'_j^+$ of $M_{\mathbb{Q}}$ (respectively delimited by the affine hyperplanes $\mathcal{H}_1, \dots, \mathcal{H}_j$ and $\mathcal{H}'_1, \dots, \mathcal{H}'_j$) such that \tilde{Q} is the intersection of the \mathcal{H}_i^+ , \tilde{Q}' is the intersection of the \mathcal{H}'_i^+ , and for all $i \in \{1, \dots, j\}$, \mathcal{H}_i^+ is the image of \mathcal{H}'_i^+ by a translation.
2. With notations of the previous item, for all subset J of $\{1, \dots, j\}$, the intersections $\cap_{i \in J} \mathcal{H}_i \cap Q$ and $\cap_{i \in J} \mathcal{H}'_i \cap Q'$ have the same dimension.
3. Q and Q' intersect exactly the same walls $W_{\alpha,P}$ of $X(P)^+$ (with $\alpha \in S \setminus R$).

Remark that this definition does not depend on the choice of \tilde{Q} and \tilde{Q}' . As a corollary of what we say just above, we obtain the following result.

Proposition 3. *The correspondence between moment polytopes and colored fans gives a bijection between the set of classes of G/H -polytopes and (isomorphic classes of) projective G/H -embeddings.*

Moreover, the set of G -orbits of a projective G/H -embedding is in bijection with the set of faces of one of its moment polytope (preserving the respective orders).

In section 4.6, we need precise description of the G/H -embedding associated to a G/H -polytope. For any dominant weight χ , we denote by $V(\chi)$ the irreducible G -module of highest weight χ , and we fix a highest weight vector v_{χ} in $V(\chi)$. The Borel subgroup of G opposite to B is denoted by B^- .

Proposition 4. *Suppose D is Cartier and very ample, then X is isomorphic to the closure of $G \cdot [\sum_{\chi \in (v^0+M) \cap Q} v_{\chi}]$ in $\mathbb{P}(\oplus_{\chi \in (v^0+M) \cap Q} V(\chi))$.*

Remark 2. If D is Cartier and only ample, then $(n-1)D$ is very ample ([Pas06, Theorem 0.3]). In particular, the assumptions Cartier and very ample, instead of \mathbb{Q} -Cartier and ample, are not really restrictive. Indeed, if D is \mathbb{Q} -Cartier and ample, then there exists a non-zero integer p such that pD is Cartier and very ample, and X is isomorphic to the closure of $G \cdot [\sum_{\chi \in (pv^0+M) \cap pQ} v_{\chi}]$ in $\mathbb{P}(\oplus_{\chi \in (pv^0+M) \cap pQ} V(\chi))$.

Proof. First, since \tilde{Q} is lattice polytope of maximal dimension, the G -orbit $G \cdot [\sum_{\chi \in (v^0+M) \cap Q} v_{\chi}]$ is isomorphic to G/H . By [Pas06, Lemma 5.1], the closure X' of this G -orbit is normal, and then a G/H -embedding. Now, if χ_0 is a vertex of Q , the intersection of the closure of the G -orbit with the affine space $\cap \mathbb{P}(\oplus_{\chi \in (pv^0+M) \cap pQ} V(\chi))_{v_{\chi_0}^* \neq 0}$ is the B^- -stable affine variety whose

structure ring is the B^- -module generated by the rational functions $\frac{v^*}{v^{\chi_0}}$ of $\mathbb{P}(\oplus_{\chi \in (v^0 + M) \cap Q} V(\chi))$, with $\chi_0 \neq \chi \in (v^0 + M) \cap Q$. It is also the B^- -stable affine variety constructed from the maximal colored cone of \mathbb{F}_X associated to the vertex χ_0 of Q . Hence, the colored fan of X is the same as the colored fan of X' and X is isomorphic to X' . \square

2.3 G -equivariant morphisms and polytopes

The existence of G -equivariant morphisms between horospherical varieties can be characterized in terms of colored fans [Kno91]. In this section, we rewrite this characterization in terms of moment polytopes.

Let (X, D) be a polarized G/H -embedding, let (X', D') be a polarized G/H' -embedding and denote by Q and Q' the corresponding moment polytopes respectively. (We denote all data corresponding to G/H' and X' with primes: $H', P', R', M', N', \dots$). We denote by \tilde{Q} and \tilde{Q}' the pseudo-moment polytopes.

A first necessary condition to the existence of a dominant G -equivariant morphism from X to X' , is that there exists a projection π from G/H to G/H' . In particular $H' \supset H, P' \supset P, R' \supset R$. The projection π induces an injective morphism π^* from M' to M and a surjective morphism π_* from N to N' . We suppose that this necessary condition is satisfied in all the rest of the section and we identify M' with $\pi_*(M')$.

Now, we define an application ψ from the set of facets of \tilde{Q} to the set of faces of \tilde{Q}' . First, note a general fact on polytopes: if \mathcal{P} is a polytope in \mathbb{Q}^r , then for any affine half-space \mathcal{H}^+ delimited by the affine hyperplane \mathcal{H} in \mathbb{Q}^r , there exists a unique face F of \mathcal{P} such that there exists $x \in \mathbb{Q}^r$ such that F is defined by $x + \mathcal{H}$ (ie $F = \mathcal{P} \cap (x + \mathcal{H})$ and $\mathcal{P} \subset v + \mathcal{H}^+$). Then, for any facet F of \tilde{Q} , let \mathcal{H}^+ be the affine half-space in $M_{\mathbb{Q}}$ containing F . If $\mathcal{H}^+ \cap M'_{\mathbb{Q}} \neq M'_{\mathbb{Q}}$, it is an affine half-space in $M'_{\mathbb{Q}}$ and, applying the fact above to $\mathcal{P} = \tilde{Q}'$, it gives a unique face F' of \tilde{Q}' . We set $\psi(F) = F'$. And if $\mathcal{H}^+ \cap M'_{\mathbb{Q}} = M'_{\mathbb{Q}}$, we set $\psi(F) = \tilde{Q}'$.

Proposition 5. *Under the above conditions, there exists a dominant G -equivariant morphism from X to X' , if and only if*

1. for any subset \mathcal{G} of facets of \tilde{Q} , $\cap_{F \in \mathcal{G}} F \neq \emptyset$ implies $\cap_{F \in \mathcal{G}} \psi(F) \neq \emptyset$, and
2. for any $\alpha \in S \setminus R$ such that $Q \cap W_{\alpha, P} \neq \emptyset$, we have $Q' \cap W_{\alpha, P} \neq \emptyset$.

Remark that, in Proposition 5, we can replace \tilde{Q} by Q (by extending the definition of ψ).

Proof. For the second condition, remark that if $\alpha \in S \setminus R$ is in R' , then Q' is contained in $W_{\alpha, P}$. And, if $\alpha \in S \setminus R'$, $Q' \cap W_{\alpha, P} \neq \emptyset$ is equivalent to $Q' \cap W_{\alpha, P'} \neq \emptyset$ because $Q' \subset X(P')$ and $W_{\alpha, P'} = X(P') \cap W_{\alpha, P}$. Recall also that $Q \cap W_{\alpha, P} \neq \emptyset$ is equivalent to $\alpha \in \mathcal{F}_X$ (and $Q' \cap W_{\alpha, P'} \neq \emptyset$ is equivalent to $\alpha \in \mathcal{F}_{X'}$).

By [Kno91], there exists a dominant G -equivariant morphism from X to X' if and only if, for all colored cone $(\mathcal{C}, \mathcal{F})$ of X , there exists a colored cone $(\mathcal{C}', \mathcal{F}')$ of X' such that $\pi_*(\mathcal{C}) \subset \mathcal{C}'$ and any element $\alpha \in \mathcal{F}$, either $\alpha \in R'$ or $\alpha \in \mathcal{F}'$.

In our case, since X and X' are complete, we can rewrite this characterization as follows. Denote by y_i with $i \in \{1, \dots, k\}$ (resp. y'_i with $i \in \{1, \dots, k'\}$) the primitive elements of the edges of the colored fan of X (resp. X'). For all $j \in \{1, \dots, k\}$, let J'_j be the minimal subset of $\{1, \dots, k'\}$ such that $\pi_*(y_j)$ is in the cone \mathcal{C}'_j generated by the y'_i with $i \in J'_j$ (it is never empty because X' is complete). We prove now that there exists a dominant G -equivariant morphism from X to X' if and only if, for all (colored) cone \mathcal{C} of X , generated by the y_j with $j \in J_{\mathcal{C}}$, the cone generated by the y'_i with $i \in \cup_{j \in J_{\mathcal{C}}} J'_j$ is contained in a (colored) cone of X' ; and \mathcal{F}_X is contained in $R' \cup \mathcal{F}_{X'}$.

Indeed, let \mathcal{C} be a cone of X , generated by the y_j with $j \in J_{\mathcal{C}}$. If there exists a cone \mathcal{C}' of X' such that $\pi_*\mathcal{C} \subset \mathcal{C}'$, then \mathcal{C}' contains all y_j with $j \in J_{\mathcal{C}}$. Then by minimality of the J'_j , it contains also cones \mathcal{C}'_j with $j \in J_{\mathcal{C}}$ and also the cone generated by the y'_i with $i \in \cup_{j \in J_{\mathcal{C}}} J'_j$. Conversely, if the cone generated by the y'_i with $i \in \cup_{j \in J_{\mathcal{C}}} J'_j$ is contained in a cone \mathcal{C}' of X' , then it is obvious that $\pi_*(\mathcal{C}) \subset \mathcal{C}'$. And the condition on colors is the same in both cases.

Now, the proposition comes from the bijective correspondence between colored cones of X (resp. X') and faces of Q (resp. Q'), the first paragraph of the proof, and the following fact: the intersection of some facets of Q is not empty if and only if the cone generated by the inward-pointing normal vectors corresponding to these facets is included in a cone of X (and this latter cone corresponds to the face defined as the intersection of these facets). \square

Corollary 6. *Suppose there exists a dominant G -equivariant morphism ϕ from X to X' . Let \mathcal{O} be the G -orbit in X associated to a face $\cap_{F \in \mathcal{G}} F$. Then $\phi(\mathcal{O})$ is the G -orbit in X' associated to a face $\cap_{F \in \mathcal{G}} \psi(F)$.*

Proof. Rewrite, in terms of polytopes, the fact that if \mathcal{O} is the G -orbit in X associated to a colored cone $(\mathcal{C}, \mathcal{F})$ then $\phi(\mathcal{O})$ is the G -orbit in X' associated to the minimal colored cone $(\mathcal{C}, \mathcal{F})$ such that $\pi_*(\mathcal{C}) \subset \mathcal{C}'$. \square

2.4 Curves in horospherical varieties

We begin this section by collecting, in the following Theorem, some results on curves in spherical varieties due to M. Brion [Bri93]. We denote by $N_1(X)$ the group of numerical classes of 1-cycles of the variety X . Recall that $NE(X)$ is the convex cone in $N_1(X)$ generated by effective 1-cycles. We denote by $N^1(X)$ the group of numerical classes of \mathbb{Q} -Cartier divisors of X , it is the dual of $N_1(X)$.

Theorem 7 (M. Brion). *Let X be a complete spherical variety. Let Y and Z be two distinct closed G -orbits in X . Note that Y and Z are flag varieties, so that they have exactly one point fixed by B .*

1. *There exists a B -stable curve C containing the B -fixed points of Y and Z if and only if the colored cones in the colored fan \mathbb{F}_X corresponding to the two closed G -orbits Y and Z intersect along a one-codimensional colored cone μ . And, in that case, C is unique, isomorphic to \mathbb{P}^1 , we denote it by C_μ .*

2. The space $N_1(X)_{\mathbb{Q}}$ is generated by the classes of the curves C_{μ} where μ is any one-codimensional colored cone in \mathbb{F}_X , and some classes $[C_{D,Y}]$ where Y is any closed G -orbit of X and D is any irreducible B -stable divisor of X not containing Y .
3. Any classes $[C_{D,Y}]$ that is not in a ray generated by the class of a curve C_{μ} is represented by a (not unique in general) B -stable curve $C_{D,Y}$ admitting the B -fixed point of Y as unique B -fixed point.
4. The cone $NE(X)$ is generated by the classes $[C_{\mu}]$ and $[C_{D,Y}]$.

Now, in the particular case of horospherical varieties, we complete this collection of results by the following proposition that describes an explicit curve in all classes of the form $[C_{D,Y}]$. Note that a homogeneous G -space has a B -fixed point if and only if it is complete (projective), so that the B -fixed points of a complete G -variety are the (unique) B -fixed points of its closed G -orbits. Denote by s_{α} the simple reflection associated to a simple root α , and denote by w_0 the longest element of the Weyl group of (G, T) .

Proposition 8. 1. Let X be a complete horospherical variety. If a B -stable curve C in X contains a unique B -fixed point y , then C is contained in the closed G -orbit $Y := G \cdot y$. In particular, it is a Schubert subvariety of Y .

2. For any closed G -orbit Y and any divisor D_{α} that does not contain Y , the class of $[C_{D_{\alpha},Y}]$ is represented by the Schubert variety of Y given by the simple root α (ie $\overline{Bs_{\alpha} \cdot y}$, where y is the B -fixed point of Y). We denote it by $C_{\alpha,Y}$.

Proof. 1. Let C be a B -stable curve in X . Suppose that C is not contained in Y . By replacing X by the closure of the biggest G -orbit of X that intersects C , we can assume that C intersect the open G -orbit G/H that is at least of rank one. And then the intersection $C \cap G/H$ is an open set of C . Recall that G/H is a G -equivariant torus fibration over the flag variety G/P . Then $C \cap G/H$ is the fiber P/H of this fibration over P/P . Indeed, the B -orbits of G/H are the inverse image of the Schubert cells of G/P . The (unique) smallest one is P/H , but by hypothesis, P/H has positive dimension. Then, since $C \cap G/H$ is B -stable, it is P/H . Moreover P/H has to be one-dimensional.

Prove now that C has two B -fixed points. By the previous paragraph, P/H is one-dimensional, ie X is of rank one. Then, there exists a \mathbb{P}^1 -bundle \tilde{X} over G/P and a G -equivariant birational morphism $\phi : \tilde{X} \rightarrow X$ (\tilde{X} is the toroidal variety over X , see [Pas06, Example 1.13 (2)]). The closure of the P -orbit P/H in \tilde{X} is the fiber \mathbb{P}^1 over P/P . Then it has exactly two B -fixed points corresponding to the two \mathbb{C}^* -fixed points of the toric variety \mathbb{P}^1 . Moreover, the two-closed G -orbits of \tilde{X} are sent, by ϕ , respectively to the two-closed G -orbits of X (here we use that X is horospherical of rank one). Then $C = \phi(\tilde{C})$ has also two B -fixed points (and is isomorphic to \mathbb{P}^1).

2. Let Y be a G -closed orbit of X and let $\alpha \in S \setminus R$ such that D_{α} does not contain Y . Let $C_{\alpha,Y} := \overline{Bs_{\alpha} \cdot y}$, where y is the B -fixed point of Y . Let $D = \sum_{i=1}^m b_i X_i + \sum_{\beta \in S \setminus R} b_{\beta} D_{\beta}$ be a Cartier divisor of X . We want to compute $D \cdot C_{\alpha,Y}$. Since D is Cartier, there exists

an eigenvector $f_Y(D)$ of B in the set of rational function on X such that the support of $D - \text{div}(f_Y(D))$ is in the union of irreducible B -stable divisors that do not contain Y . Denote by $\chi_Y(D)$ the weight of the eigenvector $f_Y(D)$. But for any divisor X_i with $i \in \{1, \dots, m\}$ such that $Y \not\subset X_i$, we clearly have $X_i \cdot C_{\alpha, Y} = 0$ because Y and X_i are disjoint. Hence

$$D \cdot C_{\alpha, Y} = (D - \text{div}(f_Y(D))) \cdot C_{\alpha, Y} = \sum_{\beta, Y \not\subset D_\beta} (b_\beta - \langle \chi_Y(D), \beta_M^\vee \rangle) D_\beta \cdot C_{\alpha, Y}.$$

We conclude, by the following Lemma 9 and the formula given in [Bri93, 3.2]. □

Lemma 9. *Let Y be a G -closed orbit of X and let $\alpha, \beta \in S \setminus R$ such that D_α and D_β do not contain Y . Then $D_\beta \cdot C_{\alpha, Y} = \delta_{\alpha\beta}$, where δ is the Kronecker delta.*

Proof. Denote by y the B -fixed point of Y . Remark that y is also fixed by P . By [Pas06, Lemma 2.8], there exists a unique G -equivariant morphism ϕ from the open G -stable set $X_Y := \{x \in X \mid \overline{G \cdot x} \supset Y\}$ to the closed G -orbit Y . Note that $y = \phi(H/H)$ because y is the only point of Y fixed by $U \subset H$ and, similarly, ϕ is the identity on Y .

Recall that D_β is the closure in X of the B -orbit $Bs_\beta w_0 P/H$. Then $D_\beta \cap Y$ is contained in (and then equals) the closure of $\phi(Bs_\beta w_0 P/H)$ in Y , which is the Schubert variety $\overline{Bs_\beta w_0} \cdot y$ of Y . Moreover, since D_β does not contain Y , $D_\beta \cap Y$ is a divisor of Y . Hence $D_\beta \cdot C_{\alpha, Y} = (D_\beta \cap Y) \cdot C_{\alpha, Y} = \delta_{\alpha\beta}$. □

With the correspondence between colored fans and moment polytopes, if X is a horospherical variety and D is a \mathbb{Q} -Cartier divisor, we denote by C_μ for any edge μ of Q_D , and by $C_{\alpha, v}$ for any $\alpha \in S \setminus R$ and any vertex v not contained in $W_{\alpha, P}$, the curves defined above in terms of the colored fans (one-codimensional colored cones correspond to edges of moment polytopes, and closed G -orbits correspond to maximal colored cones and then to vertices of moment polytopes).

The following results is a direct consequence of the formula of [Bri93, section 3.2], already used in the proof of Proposition 8.

Proposition 10. *Let X be a horospherical variety and D an ample \mathbb{Q} -Cartier divisor.*

Then, for any edge μ of the moment polytope Q_D , the intersection number $D \cdot C_\mu$ is the integral length of μ , ie the length of μ divided by the length of the primitive element in the direction of μ .

And for any $\alpha \in S \setminus R$ and for any vertex v of Q_D not in the wall $W_{\alpha, P}$, we have $D \cdot C_{\alpha, v} = \langle v, \alpha^\vee \rangle$.

3 One-parameter families of polytopes

In this section, we study particular one-parameter families of polytopes. This section can be read independently from the rest of the paper. Corollary 21 is an essential tool in the proofs of Theorems 1 and 2.

3.1 A first one-parameter family of polytopes: definitions and results

First, we fix notations. Let n and m be two positive integers. Consider three matrices A , B and C respectively in $M_{m \times 1}(\mathbb{Q})$, $M_{m \times 1}(\mathbb{Q})$ and $M_{m \times n}(\mathbb{Q})$. Then we define a first family of polyhedrons indexed by $\epsilon \in \mathbb{Q}$ as follows:

$$P^\epsilon := \{x \in \mathbb{Q}^n \mid Ax \geq B + \epsilon C\}.$$

We do not exclude the case where some lines of A are zero. Note that P^ϵ can be empty (even for all $\epsilon \in \mathbb{Q}$).

If there exists $\epsilon \in \mathbb{Q}$ such that P^ϵ is a (not empty) polytope (ie is bounded), then there is no non-zero $x \in \mathbb{Q}^n$ satisfying $Ax \geq 0$. Inversely, if there exists $\epsilon \in \mathbb{Q}$ such that P^ϵ is not bounded then there exists a non-zero $x \in \mathbb{Q}^n$ satisfying $Ax \geq 0$ (because P^ϵ contains at least an affine half-line and x can be taken to be a generator of the direction of this half-line).

From now on, we suppose that there is no non-zero $x \in \mathbb{Q}^n$ satisfying $Ax \geq 0$.

Let $I_0 := \{1, \dots, m\}$. For any matrix \mathcal{M} and any $i \in I_0$, we denote by \mathcal{M}_i the matrix consisting of the line i of \mathcal{M} . And more generally, for any subset I of I_0 we denote by \mathcal{M}_I the matrix consisting of the lines $i \in I$ of \mathcal{M} .

Let $\epsilon \in \mathbb{Q}$. We denote by \mathcal{H}_i^ϵ

the hyperplane $\{x \in \mathbb{Q}^n \mid A_i x = B + \epsilon C\}$. For any $I \subset I_0$, denote by F_I^ϵ the face of P^ϵ defined by

$$F_I^\epsilon := \left(\bigcap_{i \in I} \mathcal{H}_i^\epsilon \right) \cap P^\epsilon.$$

Note that for any face F^ϵ of P^ϵ there exists a unique maximal $I \subset I_0$ such that $F^\epsilon = F_I^\epsilon$ (we include the empty face and P^ϵ itself).

Let $I \subset I_0$. Define Ω_{I, I_0}^0 to be the set of $\epsilon \in \mathbb{Q}$ such that F_I^ϵ is not empty; define Ω_{I, I_0}^1 to be the set of $\epsilon \in \mathbb{Q}$ such that, if $I' \subset I_0$ satisfies $F_{I'}^\epsilon = F_I^\epsilon$, then $I' \subset I$. In other words, $\epsilon \in \Omega_{I, I_0}^0$ if and only if there exists $x \in \mathbb{Q}^n$ such that $Ax \geq B + \epsilon C$, and $\epsilon \in \Omega_{I, I_0}^1$ if and only if there exists $x \in \mathbb{Q}^n$ such that $A_I = B_I + \epsilon C_I$ and $A_{I_0 \setminus I} > B_{I_0 \setminus I} + \epsilon C_{I_0 \setminus I}$.

To make the notations not too heavy, we often write i instead of $\{i\}$, for any $i \in I_0$. Remark that if $\epsilon \in \Omega_{\emptyset, I_0}^1$, the polytope P^ϵ is of dimension n (ie has a non-empty interior). And, for any $i \in I_0$, if $\epsilon \in \Omega_{i, I_0}^1$ and $A_i \neq 0$, F_i^ϵ is a facet of P^ϵ .

Now, we define an equivalence on subfamily of $(P^\epsilon)_{\epsilon \in \mathbb{Q}}$, that we extend to another family $(Q^\epsilon)_{\epsilon \in \mathbb{Q}}$ of polytopes constructed later (see Definition 3), and corresponding to the equivalence of G/H -polytopes given in Definition 1 restricted to the family used in Section 4 (see Proposition 22).

Let K_0 be the subset of I_0 consisting of indices of zero lines of A .

Definition 2. Let $K_0 \subset K \subset I_0$.

We first define $\Omega_{K, I_0}^{max} := \Omega_{\emptyset, I_0}^1 \cap \bigcap_{i \in I_0 \setminus K} \Omega_{i, I_0}^1$

Let ϵ and η both in Ω_{K, I_0}^{max} . We say that the polytopes P^ϵ and P^η are equivalent if, for any $I \subset I_0$, ϵ and η are either both in Ω_{I, I_0}^1 , either both in $\mathbb{Q} \setminus \Omega_{I, I_0}^0$, or both in $\Omega_{I, I_0}^0 \setminus \Omega_{I, I_0}^1$. (In other

words, F_I^e and F_I^c are either both not empty faces with I maximal, either both empty, or both not empty with I not maximal.)

Some properties of the family $(P^\epsilon)_{\epsilon \in \mathbb{Q}}$ are listed in the following result.

Theorem 11. *With the notations above, let $K_0 \subset K \subset I_0$.*

1. *The set Ω_{K,I_0}^{max} is an open segment of \mathbb{Q} (with extremities in $\mathbb{Q} \cup \{\pm\infty\}$, or empty).*
2. *If Ω_{K,I_0}^{max} is not empty, there exist an integer k and $\alpha_0 < \dots < \alpha_k$ in $\mathbb{Q} \cup \{\pm\infty\}$, such that the equivalent classes of polytopes in the family $(P^\epsilon)_{\epsilon \in \Omega_{K,I_0}^{max}}$ correspond exactly to the open segments $] \alpha_i, \alpha_{i+1}[$ for any $i \in \{0, \dots, k-1\}$ and the singletons $\{\alpha_i\}$ for any $i \in \{0, \dots, k\}$ with $\alpha_i \in \mathbb{Q}$. In particular, $\Omega_{K,I_0}^{max} =] \alpha_0, \alpha_k[$.*
3. *Suppose that $\alpha_k \in \mathbb{Q}$. If $\alpha_k \in \Omega_{\emptyset, I_0}$, there exists $I_1 \subset I_0$ containing K such that $P^{\alpha_k} = P_{I_1}^{\alpha_k} := \{x \in \mathbb{Q}^n \mid A_{I_1}x \geq B_{I_1} + \alpha_k C_{I_1}\}$ and $\alpha_k \in \Omega_{K,I_1}^{max}$ (defined for the family $(P_{I_1}^\epsilon)_{\epsilon \in \mathbb{Q}}$). If $\alpha_k \notin \Omega_{\emptyset, I_0}$, there exist subsets J_1 and I_1 of I_0 such that $J_1 \cap I_1 = \emptyset$, $P^{\alpha_k} = P_{I_1, J_1}^{\alpha_k} := \{x \in \bigcap_{j \in J_1} \mathcal{H}_j^{\alpha_k} \mid A_{I_1}x \geq B_{I_1} + \alpha_k C_{I_1}\}$ and α_k is in $\Omega_{I_1 \cap K, I_1}^{max}$ (defined for the family $(P_{I_1, J_1}^\epsilon)_{\epsilon \in \mathbb{Q}}$ of polytopes in $\bigcap_{j \in J_1} \mathcal{H}_j^{\alpha_k}$).*

We have a similar result on the other extremity of Ω_{K,I_0}^{max} if $\alpha_0 \in \mathbb{Q}$.

3.2 Proof of Theorem 11

We study all the sets Ω_{I,I_0}^0 and Ω_{I,I_0}^1 .

For any $I \subset I_0$, denote by \bar{I} the complementary of I in I_0 .

Lemma 12. *For any $I \subset I_0$, the sets Ω_{I,I_0}^0 and Ω_{I,I_0}^1 are convex subsets of \mathbb{Q} (may be empty).*

Proof. Let ϵ_1 and ϵ_2 be in Ω_{I,I_0}^0 . Then there exist x_1 and x_2 in \mathbb{Q}^n such that $A_I x_1 = B_I + \epsilon_1 C_I$, $A_I x_2 = B_I + \epsilon_2 C_I$, $A_{\bar{I}} x_1 \geq B_{\bar{I}} + \epsilon_1 C_{\bar{I}}$ and $A_{\bar{I}} x_2 \geq B_{\bar{I}} + \epsilon_2 C_{\bar{I}}$. Then, for any rational number $\lambda \in [0, 1]$, we get easily by adding equalities and inequalities that $A_I(\lambda x_1 + (1-\lambda)x_2) = B_I + (\lambda\epsilon_1 + (1-\lambda)\epsilon_2)C_I$ and $A_{\bar{I}}(\lambda x_1 + (1-\lambda)x_2) \geq B_{\bar{I}} + (\lambda\epsilon_1 + (1-\lambda)\epsilon_2)C_{\bar{I}}$, so that $\lambda\epsilon_1 + (1-\lambda)\epsilon_2$ is in Ω_{I,I_0}^0 .

By replacing \geq by $>$, we similarly prove the convexity of Ω_{I,I_0}^1 . □

Lemma 13. *For any $I \subset I_0$, the set Ω_{I,I_0}^1 is either reduced to a point, or an open subset of \mathbb{Q} (may be empty). Moreover, if Ω_{I,I_0}^1 is reduced to a point ϵ_0 , the system $A_I X = B_I + \epsilon C_I$ has at least a solution only when $\epsilon = \epsilon_0$; in particular, in that case, Ω_{I,I_0}^0 is also reduced to ϵ_0 .*

Proof. Suppose that Ω_{I,I_0}^1 is neither empty nor reduced to a point. Let ϵ_1 and ϵ_2 two different points of Ω_{I,I_0}^1 . Then there exist x_1 and x_2 in \mathbb{Q}^n such that $A_I x_1 = B_I + \epsilon_1 C_I$, $A_I x_2 = B_I + \epsilon_2 C_I$, $A_{\bar{I}} x_1 > B_{\bar{I}} + \epsilon_1 C_{\bar{I}}$ and $A_{\bar{I}} x_2 > B_{\bar{I}} + \epsilon_2 C_{\bar{I}}$. In particular both linear systems $A_I X = B_I$ and $A_I X = C_I$ have solutions, hence for any $\epsilon \in \mathbb{Q}$, the linear system $A_I X = B_I + \epsilon C_I$ has a set of solutions, depending continuously on ϵ . More precisely, if x_B and x_C are some solutions of $A_I X = B_I$ and $A_I X = C_I$ respectively, then the set of solutions of $A_I X = B_I + \epsilon C_I$ is $\ker(A_I) + x_B + \epsilon x_C$.

Now, let $\epsilon \in \Omega_{I,I_0}^1$, and let $y := x_0 + x_B + \epsilon x_C$ with $x_0 \in \ker(A_I)$ such that $A_{\bar{I}}y > B_{\bar{I}} + \epsilon C_{\bar{I}}$. Then there clearly exists a neighborhood \mathcal{V} of ϵ in \mathbb{Q} such that $A_{\bar{I}}(y + (\eta - \epsilon)x_C) > B_{\bar{I}} + \eta C_{\bar{I}}$ for all $\eta \in \mathcal{V}$. We also check easily that $A_I(y + (\eta - \epsilon)x_C) = B_I + \eta C_I$ so that $\mathcal{V} \subset \Omega_{I,I_0}^1$.

To prove the last statement, we suppose that the system $A_I X = B_I + \epsilon C_I$ has a solution for at least two values of ϵ , and then by the same arguments as above, we deduce that Ω_{I,I_0}^1 is open, which gives a contradiction. The result follows immediately. \square

Lemma 14. *For any $I \subset I_0$ such that Ω_{I,I_0}^1 is not empty, we have $\Omega_{I,I_0}^1 \subset \Omega_{I,I_0}^0 \subset \overline{\Omega_{I,I_0}^1}$.*

Proof. The first inclusion is obvious.

To prove the second one, remark that for all $\epsilon_1 \in \Omega_{I,I_0}^1$ and $\epsilon_2 \in \Omega_{I,I_0}^0$, the segment $[\epsilon_1, \epsilon_2[$ (or $]\epsilon_2, \epsilon_1]$) is contained in Ω_{I,I_0}^1 . Indeed, if there exists x_1 and x_2 such that $A_I x_1 = B_I + \epsilon_1 C_I$, $A_I x_2 = B_I + \epsilon_2 C_I$, $A_{\bar{I}} x_1 > B_{\bar{I}} + \epsilon_1 C_{\bar{I}}$ and $A_{\bar{I}} x_2 \geq B_{\bar{I}} + \epsilon_2 C_{\bar{I}}$, then for any $\lambda \in]0, 1]$, $x_\lambda := \lambda x_1 + (1 - \lambda)x_2$ and $\epsilon_\lambda := \lambda \epsilon_1 + (1 - \lambda)\epsilon_2$ clearly satisfy $A_I x_\lambda = B_I + \epsilon_\lambda C_I$, $A_{\bar{I}} x_\lambda > B_{\bar{I}} + \epsilon_\lambda C_{\bar{I}}$, so that $\epsilon_\lambda \in \Omega_{I,I_0}^1$. This remark implies directly that any $\epsilon_2 \in \Omega_{I,I_0}^0$ is in $\overline{\Omega_{I,I_0}^1}$. \square

Lemma 15. *Let $I \subset I_0$ be such that Ω_{I,I_0}^1 is not empty. Then the supremum and the infimum (well-defined in \mathbb{R}) of Ω_{I,I_0}^1 are either infinite or rational numbers. Moreover, $\Omega_{I,I_0}^0 = \overline{\Omega_{I,I_0}^1}$.*

Proof. If Ω_{I,I_0}^1 is reduced to a point, by Lemma 14 we have nothing to prove, so we suppose that Ω_{I,I_0}^1 contains at least two points.

Here (and only here), we need to consider the family of polytopes $(P^\epsilon)_{\epsilon \in \mathbb{R}}$. Note that all definitions and all results given until now are still available replacing \mathbb{Q} by \mathbb{R} . When it is necessary, we denote by $\Omega(\mathbb{Q})$ and by $\Omega(\mathbb{R})$ the sets Ω defined at the beginning of the section respectively in \mathbb{Q} and in \mathbb{R} . Now, by Lemmas 12 and 14, it is enough to prove that, if they are finite, the supremum and the infimum of $\Omega_{I,I_0}^1(\mathbb{R})$ are rational numbers contained in the set $\Omega_{I,I_0}^0(\mathbb{Q})$.

Suppose that the supremum $\epsilon_0 \in \mathbb{R}$ of $\Omega_{I,I_0}^1(\mathbb{R})$ is finite. Let $\epsilon_1 \in \Omega_{I,I_0}^1(\mathbb{R})$. Then, for any $\epsilon \in [\epsilon_1, \epsilon_0]$, the polytope P^ϵ is contained in the polytope

$$P^{[\epsilon_1, \epsilon_0]} := \{x \in \mathbb{R}^n \mid Ax \geq \text{Min}(B + \epsilon_1 C, B + \epsilon_0 C)\},$$

where the minimum Min is taken line by line. Now for all $\epsilon \in [\epsilon_1, \epsilon_0]$, let $x^\epsilon \in F_I^\epsilon$ such that $A_{\bar{I}} x^\epsilon > B_{\bar{I}} + \epsilon C_{\bar{I}}$ (and $A_I x^\epsilon = B_I + \epsilon C_I$). Since the points x^ϵ are in the compact set $P^{[\epsilon_1, \epsilon_0]}$ of \mathbb{R}^n , there exists $x_0 \in \mathbb{R}^n$ such that $A_I x_0 = B_I + \epsilon_0 C_I$ and $A_{\bar{I}} x_0 \geq B_{\bar{I}} + \epsilon_0 C_{\bar{I}}$, ie $x_0 \in F_I^{\epsilon_0}$. It means that ϵ_0 is in $\Omega_{I,I_0}^0(\mathbb{R})$.

Define the maximal subset J of I_0 containing I such that for any $x \in F_J^{\epsilon_0}$ we have $A_J x = B_J + \epsilon_0 C_J$. In particular, $A_{\bar{J}} x_0 > B_{\bar{J}} + \epsilon_0 C_{\bar{J}}$. We now prove that, for any $\epsilon_1 \in \Omega_{I,I_0}^1$, the face $F_J^{\epsilon_1}$ is empty. Indeed, if it is not empty, there exists $x_1 \in \mathbb{R}^n$ such that $A_J x_1 = B_J + \epsilon_1 C_J$ and $A_{\bar{J}} x_1 \geq B_{\bar{J}} + \epsilon_1 C_{\bar{J}}$. Let $\eta > 0$, let $x_2 := x_0 + \eta(x_0 - x_1)$ and let $\epsilon_2 := \epsilon_0 + \eta(\epsilon_0 - \epsilon_1)$. Then we have $A_J x_2 = B_J + \epsilon_2 C_J$ and, for η small enough, we have $A_{\bar{J}} x_2 > B_{\bar{J}} + \epsilon_2 C_{\bar{J}}$. Hence $F_J^{\epsilon_2} \supset F_J^{\epsilon_1}$ is not empty, which contradicts the fact that $\epsilon_2 > \epsilon_0$ is not in Ω_{I,I_0}^0 (see Lemma 14).

We now claim that the intersection of the vector space $\text{Im}(A_J)$ with the affine line $\{B_J + \epsilon C_J \mid \epsilon \in \mathbb{R}\}$ is reduced to the point ϵ_0 . Indeed, ϵ_0 clearly belongs to this intersection, which can be either reduced to a point or the affine line. But, if some $\epsilon \in \Omega_{I,I_0}^1$ is in this intersection, then

there exists $x_3 \in \mathbb{R}^n$ such that $A_J x_3 = B_J + \epsilon C_J$. Then, for $\eta > 0$ small enough, we can prove that $\eta x_3 + (1 - \eta)x_0$ is in $F_J^{\epsilon_1}$ with $\epsilon_1 = \eta\epsilon + (1 - \eta)\epsilon_0$, which is not possible because $F_J^{\epsilon_1}$ is necessarily empty. To conclude that $\epsilon_0 \in \mathbb{Q}$, it is now enough to recall that A , B and C have rational coefficients.

It remains to prove that x_0 can be chosen in \mathbb{Q}^n so that ϵ_0 is in $\Omega_{I,I_0}^0(\mathbb{Q})$. But x_0 can be taken in an open set of the set of solutions of $A_J x = B_J + \epsilon_0 C_J$. We conclude by noticing that $B_J + \epsilon_0 C_J$ have rational coefficients.

For the infimum of Ω_{I,I_0}^1 , it is the same proof (for example by replacing ϵ and C by their opposites). \square

Corollary 16. *Let $I \subset I_0$. We have the following cases:*

1. Ω_{I,I_0}^1 is an open segment of \mathbb{Q} (with extremities in $\mathbb{Q} \cup \{\pm\infty\}$) and Ω_{I,I_0}^0 is the closed segment $\overline{\Omega_{I,I_0}^1}$ (may both be empty);
2. Ω_{I,I_0}^1 is reduced to a point ϵ_0 , then it equals Ω_{I,I_0}^0 and the system $A_I X = B_I + \epsilon C_I$ has at least a solution only when $\epsilon = \epsilon_0$;
3. Ω_{I,I_0}^1 is empty and Ω_{I,I_0}^0 is the closure of some Ω_{J,I_0}^1 with $I \subsetneq J \subset I_0$.

Moreover, if $I = \emptyset$ or reduced to an index $i \in I_0 \setminus K_0$, then only Cases 1 and 3 are possible, so that Ω_{I,I_0}^1 is open (or empty).

Proof. The two first items, and the fact that we cannot have more cases, can be deduced directly from Lemmas 13 and 15.

Suppose now that Ω_{I,I_0}^1 is empty, but not Ω_{I,I_0}^0 . Let $J \subset I_0$ be the maximal set containing I such that, for all $\epsilon \in \mathbb{Q}$, $F_I^\epsilon = F_J^\epsilon$. Then $J \neq I$ by hypothesis. Indeed, if $I = J$ then there exist subsets J_1, \dots, J_k of I_0 whose intersection is I , and also rational numbers $\epsilon_1, \dots, \epsilon_k$ respectively in $\Omega_{J_1,I_0}^1, \dots, \Omega_{J_k,I_0}^1$. Hence $\frac{\epsilon_1 + \dots + \epsilon_k}{k}$ is in Ω_{I,I_0}^1 that is not possible.

Then $\Omega_{I,I_0}^0 = \Omega_{J,I_0}^0$ and by maximality of J , Ω_{J,I_0}^1 is not empty. We conclude by using the first two items.

Now, for the last statement remark that, if $I = \emptyset$ or reduced to an index $i \in I_0 \setminus K_0$, the system $A_I X = B_I + \epsilon C_I$ has solutions for all ϵ and then, we cannot be in Case 2. \square

We now prove three lemmas to get the connexity of K -equivalent classes of polytopes.

Lemma 17. *Let $I \subset I_0$ such that Ω_{I,I_0}^0 is bounded, but neither empty nor reduced to a point. Denote by ϵ_1 and ϵ_2 the extremities of Ω_{I,I_0}^0 . Then, there exist J_1 and J_2 containing I such that $\{\epsilon_1\} = \Omega_{J_1,I_0}^1$ and $\{\epsilon_2\} = \Omega_{J_2,I_0}^1$. In particular, if ϵ_1 and ϵ_2 are in Ω_{K,I_0}^{max} , P^{ϵ_1} is not equivalent to P^{ϵ_2} .*

Proof. By Corollary 16, we can suppose that $\Omega_{I,I_0}^0 = \overline{\Omega_{I,I_0}^1} = [\epsilon_1, \epsilon_2]$. Then, there exists $j \in \bar{I}$ such that $x \in Q^{\epsilon_1}$ and $A_I x = B_I + \epsilon_1 C_I$ imply that $A_j x = B_j + \epsilon_1 C_j$. Let J_1 be the union of I with the set of all such indices j . In particular, $\epsilon_1 \in \Omega_{J_1,I_0}^1$. Moreover, since Ω_{J_1,I_0}^1 is contained in $\Omega_{I,I_0}^0 = [\epsilon_1, \epsilon_2]$, it cannot be open, and then by Corollary 16, Ω_{J_1,I_0}^1 is reduced to ϵ_1 .

Similarly, we prove the existence of J_2 .

In particular, if ϵ_1 and ϵ_2 are in Ω_{K,I_0}^{max} , by definition, the classes of P^{ϵ_1} and P^{ϵ_2} are both reduced to one polytope and then distinct. \square

Lemma 18. *Let $I \subset I_0$ such that $\Omega_{I,I_0}^0 = \{\epsilon_0\}$, with $\epsilon_0 \in \Omega_{K,I_0}^{max}$. There exists $I_1 \subset I_0$ such that the upper extremity of Ω_{I_1,I_0}^0 is ϵ_0 .*

Proof. First, we can suppose that I is the maximal subset of I_0 such that $\Omega_{I,I_0}^0 = \{\epsilon_0\}$, (ie such that $\Omega_{I,I_0}^1 = \{\epsilon_0\}$). Then there exists $x_0 \in \mathbb{Q}^n$ satisfying $A_I x_0 = B_I + \epsilon_0 C_I$ and $A_{\bar{I}} x_0 > B_{\bar{I}} + \epsilon_0 C_{\bar{I}}$.

Consider a subset I_1 of I such that there exists $\epsilon_1 < \epsilon_0$ in Ω_{I_1,I_0}^0 . Choose I_1 maximal in I with this property. It exists (but can be empty) because ϵ_0 is in the open set Ω_{K,I_0}^{max} . Moreover $I_1 \neq I$. Then there exist $\epsilon_1 < \epsilon_0$ and $x_1 \in \mathbb{Q}^n$ such that $A_{I_1} x_1 = B_{I_1} + \epsilon_1 C_{I_1}$ and $A_{\bar{I}_1} x_1 \geq B_{\bar{I}_1} + \epsilon_1 C_{\bar{I}_1}$. Hence, for any $\epsilon \in]\epsilon_1, \epsilon_0[$, the element $x^\epsilon := \frac{(\epsilon_0 - \epsilon)x_1 + (\epsilon - \epsilon_1)x_0}{\epsilon_0 - \epsilon_1}$ satisfies $A_{I_1} x^\epsilon = B_{I_1} + \epsilon C_{I_1}$ and $A_{\bar{I}_1} x^\epsilon > B_{\bar{I}_1} + \epsilon C_{\bar{I}_1}$ by maximality of I_1 . It proves that $] \epsilon_1, \epsilon_0 [\subset \Omega_{I_1,I_0}^1$. In particular, with Corollary 16, Ω_{I_1,I_0}^1 is open with closure equal to Ω_{I_1,I_0}^0 .

It is now enough to prove that the upper extremity of Ω_{I_1,I_0}^0 is ϵ_0 . Suppose the contrary, then there exists $\epsilon_2 > \epsilon_0$ and $x_2 \in \mathbb{Q}^n$ such that $A_{I_1} x_2 = B_{I_1} + \epsilon_2 C_{I_1}$ and $A_{\bar{I}_1} x_2 \geq B_{\bar{I}_1} + \epsilon_2 C_{\bar{I}_1}$. For any $\epsilon \in]\epsilon_0, \epsilon_2[$, the element $y^\epsilon := \frac{(\epsilon_2 - \epsilon)x_0 + (\epsilon - \epsilon_0)x_2}{\epsilon_2 - \epsilon_0}$ satisfies $A_{I_1} y^\epsilon = B_{I_1} + \epsilon C_{I_1}$ and $A_{\bar{I}_1} y^\epsilon > B_{\bar{I}_1} + \epsilon C_{\bar{I}_1}$. Moreover, for $\epsilon = \epsilon_0$ (ie $y^\epsilon = x_0$), we have $A_I y^\epsilon = B_I + \epsilon C_I$ and $A_{\bar{I}} y^\epsilon > B_{\bar{I}} + \epsilon C_{\bar{I}}$. Then by continuity, for any $\epsilon < \epsilon_0$ big enough, we have $A_{I_1} y^\epsilon = B_{I_1} + \epsilon C_{I_1}$, $A_{\bar{I}} y^\epsilon > B_{\bar{I}} + \epsilon C_{\bar{I}}$ and $A_{I \setminus I_1} y^\epsilon < B_{I \setminus I_1} + \epsilon C_{I \setminus I_1}$. Choose such an ϵ in $] \epsilon_1, \epsilon_0 [$. Hence, there exist z^ϵ in the segment $]x^\epsilon, y^\epsilon[$ and some $j \in I \setminus I_1$ such that $A_{I_1} z^\epsilon = B_{I_1} + \epsilon C_{I_1}$, $A_{\bar{I}_1} z^\epsilon \geq B_{\bar{I}_1} + \epsilon C_{\bar{I}_1}$ and $A_j z^\epsilon = B_j + \epsilon C_j$. This contradicts the maximality of I_1 , so that we have proved that the upper extremity of Ω_{I_1,I_0}^0 is ϵ_0 . \square

Lemma 19. *Let $\eta < \epsilon$ in Ω_{K,I_0}^{max} such that there exists $I \subset I_0$ satisfying $\Omega_{I,I_0}^0 \subset]\eta, \epsilon[$. Then P^η is not equivalent to P^ϵ .*

Proof. Among the set of subsets I of I_0 satisfying $\Omega_{I,I_0}^0 \subset]\eta, \epsilon[$, choose the one whose lower extremity is minimal.

There are two cases: in the first one, $\Omega_{I,I_0}^0 = \{\epsilon_0\}$. By the Lemma 18, there exist $I_1 \subset I_0$ and $\epsilon_1 < \epsilon_0$ (may be $-\infty$) such that $\Omega_{I_1,I_0}^0 = [\epsilon_1, \epsilon_0]$. By hypothesis of minimality, η is in Ω_{I_1,I_0}^0 . But ϵ is not in Ω_{I_1,I_0}^0 , so that P^η is not equivalent to P^ϵ .

Consider the second case: $\Omega_{I,I_0}^0 = [\epsilon_1, \epsilon_2]$ with $\epsilon_1 < \epsilon_2$. Let J be the maximal subset of I_0 such that $x \in Q^{\epsilon_1}$ and $A_I x = B_I + \epsilon_1 C_I$ imply that $A_J x = B_J + \epsilon_1 C_J$. In particular, J strictly contains I and $\epsilon_1 \in \Omega_{J,I_0}^1$. Then $\Omega_{J,I_0}^1 \subset \Omega_{I,I_0}^0$ cannot be open, hence it is reduced to a point and we conclude by the first case. \square

Proof of Theorem 11. 1. It comes directly from the last statement of Corollary 16.

2. Now, let $\epsilon_0 \in \Omega_{K,I_0}^{max}$. Prove that the equivalent class of P^{ϵ_0} corresponds to a segment of Ω_{K,I_0}^{max} or is reduced to P^{ϵ_0} .

If there exists $I \subset I_0$ such that $\Omega_{I,I_0}^1 = \{\epsilon_0\}$, then the equivalent class of P^{ϵ_0} is clearly reduced to P^{ϵ_0} . By Lemma 17, we get the same conclusion if there exists $I \subset I_0$ such that Ω_{I,I_0}^1 is not empty and $\epsilon_0 \in \Omega_{I,I_0}^0 \setminus \Omega_{I,I_0}^1$ (ie if there exists $I \subset I_0$ such that ϵ_0 is an extremity of Ω_{I,I_0}^0 , by Corollary 16 Case 3).

Suppose now that for all subsets I of I_0 , $\Omega_{I,I_0}^1 \neq \{\epsilon_0\}$, and ϵ_0 is not an extremity of Ω_{I,I_0}^0 . Then the set of $\epsilon \in \Omega_{K,I_0}^{max}$ such that P^ϵ is equivalent to P^{ϵ_0} is the intersection of open segment of \mathbb{Q} . Indeed, it is the intersection of open segments of type Ω_{I,I_0}^1 and connected components of some $\mathbb{Q} \setminus \Omega_{I,I_0}^0$ (by Lemma 19).

3. Now, let ϵ_1 be an extremity of Ω_{K,I_0}^{max} . Suppose that $\epsilon_1 \in \Omega_{\emptyset,I_0}^1$. Define I_2 to be the set of indices $i \in I_0 \setminus K$ such that ϵ_1 is not in Ω_{i,I_0}^1 . Denote by I_1 the complementary of I_2 in I_0 . Then by definition, I_1 contains K and ϵ_1 is in the set Ω_{K,I_1}^{max} . Denote by $P_{I_1}^{\epsilon_1}$ the polyhedron $\{x \in \mathbb{Q}^n \mid A_{I_1}x \geq B_{I_1} + \epsilon_1 C_{I_1}\}$, and prove that $P^{\epsilon_1} = P_{I_1}^{\epsilon_1}$. We clearly have $P^{\epsilon_1} \subset P_{I_1}^{\epsilon_1}$, so that we have to prove that $P_{I_1}^{\epsilon_1}$ is contained in all closed half-spaces $\mathcal{H}_i^{+\epsilon_1} := \{x \in \mathbb{Q}^n \mid A_i x \geq B_i + \epsilon_1 C_i\}$ with $i \in I_2$, or equivalently that the interior $\{x \in \mathbb{Q}^n \mid A_{I_1}x > B_{I_1} + \epsilon_1 C_{I_1}\}$ of $P_{I_1}^{\epsilon_1}$ (that is not empty because $\epsilon_1 \in \Omega_{\emptyset,I_0}^1$) is contained in all open half-spaces $\mathcal{H}_i^{++\epsilon_1} := \{x \in \mathbb{Q}^n \mid A_i x > B_i + \epsilon_1 C_i\}$ with $i \in I_2$. Since the interior of $P_{I_1}^{\epsilon_1}$ intersects the connected component $\bigcap_{i \in I_2} \mathcal{H}_i^{++\epsilon_1}$ of $\mathbb{Q}^n \setminus \bigcup_{i \in I_2} \mathcal{H}_i^{\epsilon_1}$, it is enough to prove that the interior of $P_{I_1}^{\epsilon_1}$ does not intersect the union of hyperplanes $\bigcup_{i \in I_2} \mathcal{H}_i^{\epsilon_1}$.

Let $i \in I_2$. Suppose that the interior of $P_{I_1}^{\epsilon_1}$ intersects the hyperplane $\mathcal{H}_i^{\epsilon_1}$. There exists $x_1 \in \mathbb{Q}^n$ such that $A_{I_1}x_1 > B_{I_1} + \epsilon_1 C_{I_1}$ and $A_i x_1 = B_i + \epsilon_1 C_i$. Since there also exists $x_0 \in \mathbb{Q}^n$ such that $Ax_0 > B + \epsilon_1 C$, an element x_2 of the segment $[x_1, x_0[$ is in the interior of $P_{I_1}^{\epsilon_1}$, in P^{ϵ_1} and in $\mathcal{H}_j^{\epsilon_1}$ for at least one $j \in I_2$. Let $J \subset I_2$ the maximal set such that $A_J x_2 = B_J + \epsilon_1 C_J$. Then the set $\mathcal{C} := \{x \in \mathbb{Q}^n \mid A_J x \geq B_J + \epsilon_1 C_J\}$ is a cone (of apex $x_2 + \ker(A_J)$) with non-empty interior. Let $j \in J$ such that $\mathcal{H}_j^{\epsilon_1} \cap \mathcal{C}$ is a facet of \mathcal{C} . By Lemma 20 below, all hyperplanes $\mathcal{H}_j^{\epsilon_1}$ with $j \in J$ are different, then there exists $y \in \mathcal{H}_j^{\epsilon_1} \cap \mathcal{C}$ that is not in the other hyperplanes $\mathcal{H}_i^{\epsilon_1}$ with $j \neq i \in J$. Since x_2 is in the interior of $P_{I_0 \setminus J}^{\epsilon_1}$, for any rational number $\lambda > 0$ small enough, the point $x_2 + \lambda(y - x_2)$ is in the interior of $P_{I_0 \setminus J}^{\epsilon_1}$, also in $\mathcal{H}_j^{\epsilon_1} \cap \mathcal{C}$ and not in $\mathcal{H}_i^{\epsilon_1}$ for all $i \in J$ different from j . This contradicts the fact that $\epsilon_1 \notin \Omega_{j,I_0}^1$.

Suppose now that $\epsilon_1 \notin \Omega_{\emptyset,I_0}^1$. Let J_1 the maximal subset of I_0 such that P^{ϵ_1} is contained in the subspace $\mathcal{H}_{J_1} := \bigcap_{j \in J_1} \mathcal{H}_j^{\epsilon_1}$ of \mathbb{Q}^n . Then P^{ϵ_1} is clearly the polytope $\{x \in \bigcap_{j \in J_1} \mathcal{H}_j^{\epsilon_1} \mid A_{I_0 \setminus J_1}x \geq B_{I_0 \setminus J_1} + \epsilon_1 C_{I_0 \setminus J_1}\}$. Then $\epsilon_1 \in \Omega_{\emptyset, I_0 \setminus J_1}^1$ (defined in for the family of polytopes in \mathcal{H}_{J_1}). If $\epsilon_1 \in \Omega_{K \cap (I_0 \setminus J_1), I_0 \setminus J_1}^{max}$, then $I_1 = I_0 \setminus J_1$ gives the result. If not, ϵ_1 is an extremity of $\Omega_{K \cap (I_0 \setminus J_1), I_0 \setminus J_1}^{max}$ and $\epsilon_1 \in \Omega_{\emptyset, I_0 \setminus J_1}^1$. Then we apply that we prove in the previous paragraph, in order to find $I_1 \subset I_0 \setminus J_1$ that gives the result. \square

Lemma 20. *If Ω_{K,I_0}^{max} is not empty, then for any i and j in $I_0 \setminus K$, the existence of $\lambda \in \mathbb{Q}_{>0}$ such that $A_i = \lambda A_j$, implies that $i = j$.*

Proof. Let $\epsilon \in \Omega_{K,I_1}^{max}$. Suppose that there exists $i \neq j$ in $I_0 \setminus K$ such that $A_i = \lambda A_j$ for some $\lambda \in \mathbb{Q}_{>0}$. Then, since $\epsilon \in \Omega_{i,I_0}^1 \cap \Omega_{j,I_0}^1$, there exist x and y in \mathbb{Q}^n , such that $A_i x = B_i + \epsilon C_i$, $A_j x > B_j + \epsilon C_j$, $A_j y = B_j + \epsilon C_j$ and $A_i y > B_i + \epsilon C_i$. And we have $\lambda(B_j + \epsilon C_j) < \lambda A_j x = A_i x = B_i \epsilon C_i < A_i y = \lambda A_j y = \lambda(B_j + \epsilon C_j)$, which gives a contradiction. \square

3.3 A second one-parameter family of polytopes

In section 4, we apply Theorem 11 to a family $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ constructed by iteration from the family $(P^\epsilon)_{\epsilon \in \mathbb{Q}}$, using the last statement of Theorem 11 in the following way.

Definition 3. Let A, B and C be as in Theorem 11. Let $K \subset I_0$ containing K_0 . Suppose that $0 \in \Omega_{K,I_0}^{max}$. For any $\epsilon \in \mathbb{Q}_{\geq 0} \cap \Omega_{K,I_0}^{max}$, define \tilde{Q}^ϵ to be P^ϵ . If Ω_{K,I_0}^{max} has a finite supremum ϵ_1 and if $\epsilon_1 \in \Omega_{\emptyset,I_0}$ consider I_1 as in Theorem 11. Then define for any $\epsilon \in [\epsilon_1, +\infty[\cap \Omega_{K,I_1}^{max}$, \tilde{Q}^ϵ to be the polytope $P_{I_1}^\epsilon$.

If $\epsilon_1 \notin \Omega_{\emptyset,I_0}$, we stop the construction and we define $\tilde{Q}^\epsilon = \emptyset$ for any $\epsilon > \epsilon_1$ by convention.

Iterating the construction, we obtain a family of polytopes $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$. Remark that the construction depends on K .

We say that \tilde{Q}^ϵ is K -equivalent to \tilde{Q}^η in the family $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ if they are both defined at the same step of the construction above, ie if they correspond to P_I^ϵ and P_I^η , with ϵ and η both in $\Omega_{K,I}^{max}$ for some $I \subset I_0$ containing K , and P_I^ϵ and P_I^η are equivalent (according to Definition 2).

Applying Theorem 11 to each subfamilies $(P_{I_i}^\epsilon)$ of the family $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$, we obtain immediately the following result.

Corollary 21 (of Theorem 11). *Suppose that \tilde{Q}^0 and \tilde{Q}^ϵ are equivalent for $\epsilon > 0$ small enough.*

There exist non-negative integers k, j_0, \dots, j_k , rational numbers $\alpha_{i,j}$ for $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, j_i\}$ and $\alpha_{k,j_k+1} \in \mathbb{Q}_{>0} \cup \{+\infty\}$ ordered as follows with the convention that $\alpha_{i,j_i+1} = \alpha_{i+1,0}$ for any $i \in \{0, \dots, k-1\}$:

1. $\alpha_{0,0} = 0$;
2. for any $i \in \{0, \dots, k\}$, and for any $j < j'$ in $\{0, \dots, j_i + 1\}$ we have $\alpha_{i,j} < \alpha_{i,j'}$;

and such that the K -equivalent classes in the family of polytopes $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ is given by the following segments:

1. $[\alpha_{i,0}, \alpha_{i,1}[$, with $i \in \{0, \dots, k\}$;
2. $]\alpha_{i,j}, \alpha_{i,j+1}[$, with $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$;
3. $\{\alpha_{i,j}\}$ with $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$;
4. if $\alpha_{k,j_k+1} \neq +\infty$, $\{\alpha_{k,j_k+1}\}$ and $]\alpha_{k,j_k}, +\infty[$.

Remarks 3. For any $i \in \{0, \dots, k\}$, the rational numbers $\alpha_{i,0}, \dots, \alpha_{i,j_i}$ can come from several consecutive steps of the construction of $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$. Indeed, for example, α_{0,j_0} is not necessarily the supremum ϵ_1 of Ω_{K,I_0}^{max} because $\{\tilde{Q}^{\epsilon_1}\}$ may be an equivalent class.

We have $\alpha_{k,j_{k+1}} = +\infty$ if, in the construction of $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$, some set $\Omega_{K,I}^{max}$ has no upper bound. And if $\alpha_{k,j_{k+1}} \neq +\infty$, it is the upper extremity of some $\Omega_{K,I}^{max}$ that is not an element of $\Omega_{\emptyset,I}^1$. In this latter case, $]\alpha_{k,j_{k+1}}, +\infty[$ is the class of empty polytopes and $\alpha_{k,j_{k+1}}$ corresponds to the last not empty polytope. If $C > 0$, we are always in that case.

4 MMP via a one-parameter family of polytopes

Let X be a projective horospherical G -variety, with open G -orbit isomorphic to G/H . Let D be an ample \mathbb{Q} -Cartier divisor. Suppose that X is \mathbb{Q} -Gorenstein (ie the canonical divisor K_X of X is \mathbb{Q} -Cartier). We keep the notations given in Sections 2.

4.1 The one-parameter family of polytopes

To construct the one-parameter family of polytopes that permits to run the MMP from X , we start the same way as in the classical approach of the MMP. Indeed, for $\epsilon > 0$ small enough, the divisor $D + \epsilon K_X$ is still ample (and \mathbb{Q} -Cartier by hypothesis), so that it defines a moment polytope Q^ϵ and a pseudo-moment polytope \tilde{Q}^ϵ . More precisely, (for ϵ small enough) $Q^\epsilon := \{x \in v^\epsilon + M_{\mathbb{Q}} \mid Ax \geq B + \epsilon C\}$ and $\tilde{Q}^\epsilon := \{x \in M_{\mathbb{Q}} \mid Ax \geq \tilde{B} + \epsilon \tilde{C}\}$ where the matrices $A, B, C, \tilde{B}, \tilde{C}$ and the vector v^ϵ are defined below.

Recall that x_1, \dots, x_m denote the primitive elements of the rays of the colored fan of X that are not generated by a vector α_M^\vee with $\alpha \in \mathcal{F}_X$. We choose an order in $S \setminus R$ and we then denote by $\alpha_1, \dots, \alpha_r$ its elements. We fix a basis \mathcal{B} of M . Now define $A \in \mathcal{M}_{m+r,n}(\mathbb{Q})$ whose first m lines are the coordinates of the vectors x_i in the basis \mathcal{B} with $i \in \{1, \dots, m\}$ and whose last r lines are the coordinates of the vectors α_j^\vee in \mathcal{B} with $j \in \{1, \dots, r\}$. Let \tilde{B} be the column vector such that the pseudo-moment polytope of D is defined by $\{x \in M_{\mathbb{Q}} \mid Ax \geq \tilde{B}\}$. In fact, if $D = \sum_{i=1}^m b_i X_i + \sum_{\alpha \in S \setminus R} b_\alpha D_\alpha$, then \tilde{B} is the column matrix associated to the vector $(-b_1, \dots, -b_m, -b_{\alpha_1}, \dots, -b_{\alpha_r})$. Similarly, the column matrix \tilde{C} corresponds to the vector $(1, \dots, 1, c_{\alpha_1}, \dots, c_{\alpha_r})$, where $-K_X = \sum_{i=1}^m X_i + \sum_{\alpha \in S \setminus R} c_\alpha D_\alpha$ (the coefficients are explicitly defined with $c_\alpha = \langle 2\rho_P, \alpha^\vee \rangle$, where ρ_P is the sum of positive roots of G that are not roots of P). Now, define $v^\epsilon := \sum_{\alpha \in S \setminus R} (b_\alpha - \epsilon c_\alpha) \varpi_\alpha$. Since $Q^\epsilon = v^\epsilon + \tilde{Q}^\epsilon$, we compute that B and C are respectively the column matrices associated to $(-b_1 + \langle x_1, \sum_{\alpha \in S \setminus R} b_\alpha \varpi_\alpha \rangle, \dots, -b_m + \langle x_m, \sum_{\alpha \in S \setminus R} b_\alpha \varpi_\alpha \rangle, 0, \dots, 0)$ and $(1 - \langle x_1, \sum_{\alpha \in S \setminus R} c_\alpha \varpi_\alpha \rangle, \dots, 1 - \langle x_m, \sum_{\alpha \in S \setminus R} c_\alpha \varpi_\alpha \rangle, 0, \dots, 0)$.

From the matrices A, \tilde{B} and \tilde{C} we construct the family $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ of polytopes in $M_{\mathbb{Q}}$ as in Section 3. And for any $\epsilon \geq 0$ we define $Q^\epsilon = v^\epsilon + \tilde{Q}^\epsilon$. Since $\tilde{C} > 0$, by Remark 3, there exists $\epsilon_{max} \in \mathbb{Q}_{>0}$ (it is the number $\alpha_{k,j_{k+1}}$ given in Corollary 21) such that for all $\epsilon \in [0, \epsilon_{max}[$, Q^ϵ is a G/H -polytope and for all $\epsilon > \epsilon_{max}$, $Q^\epsilon = \emptyset$. The polytope $Q^{\epsilon_{max}}$ is neither empty nor a G/H -polytope, but it is a G/H^1 -polytope with some subgroup H^1 of G containing H (see later).

Proposition 22. *The two partitions of $[0, \epsilon_{max}[$ given by equivalence classes of G/H -polytopes (Definition 1) in the family $(Q^\epsilon)_{\epsilon \in [0, \epsilon_{max}[}$ and by equivalent classes in the family $(\tilde{Q}^\epsilon)_{\epsilon \in [0, \epsilon_{max}[}$ according to Definition 3 with $K = \{m+1, \dots, m+r\}$ are the same.*

Proof. Let ϵ and η be two rational numbers.

Suppose that Q^ϵ and Q^η are equivalent G/H -polytopes. In Definition 1, take j to be minimal, so that the hyperplanes $\mathcal{H}_1, \dots, \mathcal{H}_j$ and $\mathcal{H}'_1, \dots, \mathcal{H}'_j$ corresponds respectively to $\mathcal{H}_{i_1}^\epsilon, \dots, \mathcal{H}_{i_j}^\epsilon$ and $\mathcal{H}_{i_1}^\eta, \dots, \mathcal{H}_{i_j}^\eta$ for $J := \{i_1, \dots, i_j\} \subset \{1, \dots, m+r\}$. Then, if we denote by I the union $J \cup K$, the polytopes \tilde{Q}^ϵ and \tilde{Q}^η respectively equal P_I^ϵ and P_I^η such that ϵ and η are in $\Omega_{K,I}^{max}$. Then we have to prove that P_I^ϵ and P_I^η are equivalent according to Definition 2, which comes directly from the definition of equivalence of G/H -polytopes.

Suppose now that \tilde{Q}^ϵ and \tilde{Q}^η are equivalent according to Definition 3. They are constructed at the same step so that there exists $I \subset \{1, \dots, m+r\}$ containing K such that \tilde{Q}^ϵ and \tilde{Q}^η respectively equal P_I^ϵ and P_I^η , with ϵ and η in $\Omega_{K,I}^{max}$. Then $\tilde{Q}^\epsilon = \{x \in M_{\mathbb{Q}} \mid A_I \geq \tilde{B}_I + \epsilon \tilde{C}_I\}$ and $\tilde{Q}^\eta = \{x \in M_{\mathbb{Q}} \mid A_I \geq \tilde{B}_I + \eta \tilde{C}_I\}$. This directly gives the first item of Definition 1. The second item is also clear from Definition 2. And the third item comes from the fact that $\Omega_{i,I}^0$ contains both ϵ and η or no of the two, for any $i \in K$. \square

4.2 Construction of varieties and morphisms

We apply Corollary 21 to the family $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$. Then the family $(Q^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ gives a list of G/H -embeddings:

1. $X_{i,j}$ for any $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, j_i\}$, respectively associated to moment polytopes Q^ϵ with $\epsilon \in]\alpha_{i,j}, \alpha_{i,j+1}[$;
2. $Y_{i,j}$ for any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$, respectively associated to moment polytopes $Q^{\alpha_{i,j}}$;

It also gives a projective horospherical G -variety Z associated to the moment polytope $Q^{\alpha_{k,j_k+1}} = Q^{\alpha_{max}}$. Indeed, let $M_{\mathbb{Q}}^1$ be the minimal vector subspace containing $\tilde{Q}^{\alpha_{max}}$ and let $M^1 := M_{\mathbb{Q}}^1 \cap M$. Let R^1 be the union of R with the set of $\alpha \in S \setminus R$ such that $Q^{\alpha_{max}}$ is contained in the wall $W_{\alpha,P}$. Then we define the subgroup H^1 of $P^1 := P_{R^1}$ to be the intersection of kernels of characters of P^1 in M^1 . Then $Q^{\alpha_{max}}$ is a G/H^1 -polytope and corresponds to a G/H^1 -embedding Z . Remark that, by definition $M^1 \subset M$ and $R \subset R^1$ so that we have a projection $\pi : G/H \longrightarrow G/H^1$.

Now, by Proposition 5, we get G -equivariant morphisms:

1. $\phi_{i,j} : X_{i,j-1} \longrightarrow Y_{i,j}$ for any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$;
2. $\phi_{i,j}^+ : X_{i,j} \longrightarrow Y_{i,j}$ for any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$;
3. $\phi_i : X_{i,j_i} \longrightarrow X_{i+1,0}$ for any $i \in \{0, \dots, k-1\}$;
4. and $\phi : X_{k,j_k} \longrightarrow Z$.

In the next section, we prove that the morphisms $\phi_{i,j}$ and $\phi_{i,j}^+$ give flips (may be divisorial, see Remark 3 and Example 7), the morphisms ϕ_i are divisorial contractions and the morphism ϕ is a Mori fibration.

4.3 Description of the contracted curves

Proposition 23. 1. For any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$, the curves C contracted by the morphism $\phi_{i,j}$ satisfy $K_{X_{i,j-1}} \cdot C < 0$; for any $i \in \{0, \dots, k-1\}$, the curves C contracted by the morphism ϕ_i satisfy $K_{X_{i,j_i}} \cdot C < 0$; and the the curves C contracted by the morphism ϕ satisfy $K_{X_{k,j_k}} \cdot C < 0$.

2. For any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$, the curves C contracted by the morphism $\phi_{i,j}^+$ satisfy $K_{X_{i,j}} \cdot C > 0$.

3. For any $i \in \{0, \dots, k-1\}$, the morphism ϕ_i contracts (at least) a G -stable divisor of X_{i,j_i} .

Proof. 1. Let $]a, b[$ or $[a, b[$ corresponding to an equivalent class in the family of polytopes $(Q^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$. In particular, there exists $I \subset \{1, \dots, m+r\}$ containing K such that for any $\epsilon \in]a, b[$, $\tilde{Q}^\epsilon = P_I^\epsilon$ and $\epsilon \in \Omega_{K,I}^{max}$.

Denote by X the variety given by the G/H -polytopes Q^ϵ with $\epsilon \in]a, b[$, denote by Y^b the variety given by the G/H -polytope Q^b or to the G/H' -polytope Q^b (as we define Z in the previous section). And denote by ϕ the projective G -equivariant morphism from X to Y^b .

Fix $c \in]a, b[$. Recall that a G -orbit of X corresponding to a face F_J^c of Q^c with $J \subset I$, is sent to the G -orbit of Y^b corresponding to the face F_I^b of Q^b . Recall also that we describe the curves of horospherical varieties in Section 2.4. Hence, a curve C_μ of X is contracted by ϕ if and only if the G -orbit of X intersecting C_μ in an open set (ie the G -orbit corresponding to the edge μ of Q^c) is sent to a closed G -orbit of Y^b ; and a curve $C_{\alpha,v}$ of X is contracted by ϕ if the closed G -orbit corresponding to v is sent to a closed G -orbit isomorphic to G/P' where α is a root of the parabolic subgroup P' . In other words, C_μ is contracted by ϕ if and only if for any $J \subset I$ such that $\mu = F_J^c$, the face F_J^b of Q^b is in fact a vertex of Q^b . And $C_{\alpha,v}$ is contracted by ϕ if and only if for any $J \subset I$ such that $v = F_J^c$ (which is not in $W_{\alpha,P}$), the vertex F_J^b of Q^b is in $W_{\alpha,P}$.

Let D^c be the \mathbb{Q} -Cartier divisor defined by the moment polytope Q^c and the pseudo-moment polytope \tilde{Q}^c . Then, for any $\epsilon \in [c, b[$ the \mathbb{Q} -Cartier (and ample) divisor defined by $(Q^\epsilon, \tilde{Q}^\epsilon)$ is $D^c + (\epsilon - c)K_X$. But, by Proposition 10, $(D + (\epsilon - c)K_X) \cdot C_\mu$ is the integral length of the edge μ in Q^ϵ for any $\epsilon \in [c, b[$, we get by continuity that C_μ is contracted by ϕ if and only if $(D + (b - c)K_X) \cdot C_\mu = 0$. Similarly, since $(D + (\epsilon - \text{gamma})K_X) \cdot C_{\alpha,v} = \langle v, \alpha^\vee \rangle$ for any $\epsilon \in [c, b[$, we get that $C_{v,\alpha}$ is contracted by ϕ if and only if $(D + (b - c)K_X) \cdot C_{\alpha,v} = 0$. In particular, for any curve C contracted by ϕ , we have $K_X \cdot C < 0$.

2. By very similar arguments, we prove that if $]a, b[$ is an equivalent class in the family of polytopes $(Q^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$, if Y^a is the G/H -embedding associated to the G/H -polytope Q^a , and if $\phi^+ : X \rightarrow Y^a$ denote the G -equivariant morphism, then for any curve C contracted by ϕ^+ , we have $K_X \cdot C > 0$.

3. Now consider the case where $]a, b[$ (or $[a, b[$) and $[b, b'[$ correspond to two successive equivalent classes in the family of polytopes $(Q^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$. The subset I of $\{1, \dots, m+r\}$, the varieties X and Y^b , and the morphism ϕ are defined as above. Here, by hypothesis, $b \notin \Omega_{K,I}^{max}$ and there exists a proper subset I' of I containing K such that $\tilde{Q}^b = P_{I'}^b$. Then for any $c \in]a, b[$ and for any $i \in I \setminus I' \subset \{1, \dots, m\}$, F_i^c is a facet of Q^c (and corresponds to a G -stable divisor of X), but F_i^b is not a facet of Q^b (and corresponds to a G -stable divisor of Y^b of codimension at least 2). Hence, ϕ contracts (at least) a G -stable divisor of X . □

4.4 \mathbb{Q} -Gorenstein singularities

In this section, we prove in particular that all the varieties $X_{i,j}$ defined in Section 4.2 are \mathbb{Q} -Gorenstein. We begin by giving a \mathbb{Q} -Gorenstein criterion in terms of moment polytopes.

Proposition 24. *Let X be a projective G/H -embedding and let D be an ample \mathbb{Q} -Cartier divisor. Denote by \tilde{Q} the pseudo-moment polytope of (X, D) . Let A and \tilde{C} be the matrices defined in Section 4.1. For any vertex v of \tilde{Q} , we denote by I_v the maximal subset of $\{1, \dots, m+r\}$ such that $v = F_{I_v}$.*

Then X is \mathbb{Q} -Gorenstein if and only if, for any vertex v of \tilde{Q} , the linear system $A_{I_v}X = \tilde{C}_{I_v}$ have (at least) a solution.

Proof. The proposition is just a translation, in terms of moment polytopes, of the criterion of \mathbb{Q} -Cartier divisor, applied to the divisor K_X . □

By applying this criterion to the family $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$, we easily get the following result.

Corollary 25. *Let $\epsilon \geq 0$ such that Q^ϵ is a G/H -polytope. Let X^ϵ be the G/H -embedding corresponding to the G/H -polytope Q^ϵ . Denote by I the subset of I_0 containing K , such that $\tilde{Q}^\epsilon = P_I^\epsilon$ and $\epsilon \in \Omega_{K,I}^{max}$. For any vertex v^ϵ of \tilde{Q} , we denote by I_{v^ϵ} the maximal subset of I such that $v = F_{I_{v^\epsilon}}$.*

Then X^ϵ is \mathbb{Q} -Gorenstein if and only if for any vertex v^ϵ of \tilde{Q}^ϵ , the linear system $A_{I_{v^\epsilon}}X = \tilde{C}_{I_{v^\epsilon}}$ have (at least) a solution.

Now, using Section 3, we can know exactly when X^ϵ is \mathbb{Q} -Gorenstein (except for Z).

Proposition 26. *The varieties $X_{i,j}$ with $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, j_i\}$ are \mathbb{Q} -Gorenstein. And the varieties $Y_{i,j}$ with $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, j_i\}$ are not \mathbb{Q} -Gorenstein.*

Proof. Let $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, j_i\}$. The variety $X_{i,j}$ is defined by G/H -polytopes Q^ϵ with ϵ in an open segment of $\mathbb{Q}_{\geq 0}$. In particular, for all these rational numbers ϵ , for any vertex v^ϵ of \tilde{Q}^ϵ , the linear system $A_{I_{v^\epsilon}}X = \tilde{B}_{I_{v^\epsilon}} + \epsilon \tilde{C}_{I_{v^\epsilon}}$ have a solution. Hence, $A_{I_{v^\epsilon}}X = \tilde{C}_{I_{v^\epsilon}}$ has also a solution. It proves that $X_{i,j}$ is \mathbb{Q} -Gorenstein.

Now, let $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, j_i\}$. The variety $Y_{i,j}$ is defined by the G/H -polytope $Q^{\alpha_{i,j}}$. By Corollary 16, there exist J and I subsets of $\{1, \dots, m+r\}$ such that $J \subset I$ and $\alpha_{i,j}$ is the extremity of the segment $\Omega_{J,I}^0$ or $\{\alpha_{i,j}\} = \Omega_{J,I}^1$. But by Lemma 17, we can always choose

J and I such that $\{\alpha_{i,j}\} = \Omega_{J,I}^1$. Moreover, by taking J maximal with this property, we get a vertex v of $Q^{\alpha_{i,j}}$ satisfying $\{\alpha_{i,j}\} = \Omega_{I_v,I}^1$. In particular, by Corollary 16, the system $A_{I_v}X = \tilde{C}_{I_v}$ have no solution. It proves that $Y_{i,j}$ is not \mathbb{Q} -Gorenstein. \square

4.5 \mathbb{Q} -factorial singularities

In this section, we prove that, for D general, the MMP works in the family of projective \mathbb{Q} -factorial horospherical varieties.

Proposition 27. *Suppose that X is \mathbb{Q} -factorial. Choose D such that the vector \tilde{B} is in the open set*

$$\bigcup_{I \subset \{1, \dots, m+r\}, |I| > n} \pi_I^{-1}(\mathbb{Q}^{|I|} \setminus \text{Im}(A_I)),$$

where π_I is the canonical projection of \mathbb{Q}^{m+r} to its vector subspace of \mathbb{Q}^{m+r} corresponding to the coordinates in I .

Then, for any $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, j_i\}$, the variety $X_{i,j}$ is \mathbb{Q} -factorial.

Proof. A horospherical variety X is \mathbb{Q} -factorial if and only if the colored cones of \mathbb{F}_X are all simplicial and any color of X is the unique color of a colored edge of \mathbb{F}_X . In terms of moment polytopes, if Q is a moment polytope of X , then X is \mathbb{Q} -factorial if and only if Q is simple (ie each vertex belongs exactly to n facets, where n is the dimension of Q), Q intersects a wall $W_{\alpha,P}$, with $\alpha \in S \setminus R$, only along one of its facets and a facet is never in 2 walls $W_{\alpha,P}$ and $W_{\beta,P}$ with $\alpha \neq \beta \in S \setminus R$.

Let $i \in \{0, \dots, k-1\}$ and $j \in \{0, \dots, j_i\}$. Let $\epsilon \in]\alpha_{i,j}, \alpha_{i,j+1}[$ so that Q^ϵ is a moment polytope of $X_{i,j}$. Let $I \subset I_0$ containing K such that $\tilde{Q}^\epsilon = P_I^\epsilon$ and $\epsilon \in \Omega_{K,I}^{max}$.

Let v be a vertex of Q^ϵ . Denote by I_v the maximal subset of I such that $v = F_{I_v}^\epsilon$. Note that $|I_v| \geq n$. We want to prove that $|I_v| = n$. In particular, ϵ is a point of $\Omega_{I_v}^1$, which is open (because contains $]\alpha_{i,j}, \alpha_{i,j+1}[$). It implies that, for any $\eta \in \mathbb{Q}$, $\tilde{B}_{I_v} + \eta \tilde{C}_{I_v}$ is in the image of A_{I_v} . In particular, \tilde{B}_{I_v} is in the image of A_{I_v} . By hypothesis on D , the cardinality of I_v has to be n . This proves that $X_{i,j}$ is \mathbb{Q} -factorial. \square

Remark 4. The open set where \tilde{B} is chosen, is clearly not empty and dense in \mathbb{Q}^{m+r} , because for any I of cardinality greater than n , the image of A_I is of codimension at least one. But, since X is \mathbb{Q} -factorial, any vector $\tilde{B} \in \mathbb{Q}^{m+r}$ gives a \mathbb{Q} -Cartier divisor.

Taking D general, we also get the following result.

Proposition 28. *Suppose that X is \mathbb{Q} -factorial. If D is general in the set of ample \mathbb{Q} -Cartier divisors, all morphisms $\phi_{i,j}$, $\phi_{i,j}^+$, ϕ_i and ϕ defined in Section 4.2 are contractions of rays of the corresponding effective cones $NE(X_{i,j})$.*

Proof. Let Y be a projective \mathbb{Q} -factorial G/H -embedding whose colored fan is made from some edges generated by x_i with $i \in \{1, \dots, m\}$ and α_M^\vee with $\alpha \in S \setminus R$. In particular, the vector space $\text{Cartier}(X)_\mathbb{Q}$ of \mathbb{Q} -Cartier divisors of X projects on the vector space $\text{Cartier}(Y)_\mathbb{Q}$ of \mathbb{Q} -Cartier

divisors of Y . Denote by p_Y this projection. Let V be a vector subspace of $N_1(Y)_{\mathbb{Q}}$ of dimension at least 2. Denote by V^* the dual of V in $N^1(X)_{\mathbb{Q}}$. Then $V^* + \mathbb{Q}K_Y$ is a proper vector subspace of $N^1(X)_{\mathbb{Q}}$. Hence, the intersection \mathcal{I}_Y , for all faces of $NE(X)$ of dimension at least 2 of direction the vector subspace V , of $N^1(Y) \setminus (V^* + \mathbb{Q}K_Y)$ is a dense open set of $N^1(Y)_{\mathbb{Q}}$. If we denote by q_Y the projection of $\text{Cartier}(Y)_{\mathbb{Q}}$ on $N^1(Y)_{\mathbb{Q}}$, then $(q_Y \circ p_Y)^{-1}(\mathcal{I}_Y)$ is open and dense in $\text{Cartier}(X)_{\mathbb{Q}}$.

Consider now, an ample \mathbb{Q} -Cartier divisor D in the dense open intersection of the set given in Proposition 27 with the sets $(q_Y \circ p_Y)^{-1}(\mathcal{I}_Y)$ for all projective \mathbb{Q} -factorial G/H -embeddings Y whose colored fan is made from some edges generated by x_i with $i \in \{1, \dots, m\}$ and α_M^{\vee} with $\alpha \in S \setminus R$. Then, for any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$, the divisor $p_{X_{i,j-1}}(D) + \alpha_{i,j}K_{X_{i,j-1}}$ vanishes at most on a ray of $NE(X_{i,j-1})$, so that $\phi_{i,j}$ is the contraction of a ray of $NE(X_{i,j-1})$. Similarly, for any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$, the divisor $p_{X_{i,j}}(D) + \alpha_{i,j}K_{X_{i,j}}$ vanishes at most on a ray of $NE(X_{i,j})$, so that $\phi_{i,j}^+$ is the contraction of a ray of $NE(X_{i,j})$. And for any $i \in \{0, \dots, k\}$, the divisor $p_{X_{i,j_i}}(D) + \alpha_{i,j_i+1}K_{X_{i,j_i}}$ vanishes at most on a ray of $NE(X_{i,j_i})$, so that ϕ_i (ϕ if $i = k$) is the contraction of a ray of $NE(X_{i,j_i})$. \square

This result seems to be true without the hypothesis of \mathbb{Q} -factoriality, nevertheless the proof above really uses this hypothesis.

4.6 General fibers of Mori fibrations

In this section, we study the general fibers of the morphism $\phi : X_{k,j_k} \rightarrow Z$ defined in Section 4.2 and prove the last statement of Theorem 2. To simplify the notations, we suppose that $k = 1$ and $j_0 = 0$, in particular $X_{k-1,j_{k-1}} = X$. We also denote by ϵ_1 the rational number $\alpha_{1,0}$. We distinguish two cases.

First suppose that Q^{ϵ_1} and Q have the same dimension. It implies that Q^{ϵ_1} is in some wall $W_{\alpha,P}$ with $\alpha \in S \setminus R$, ie $R^1 \neq R$. By Proposition 28, if X is \mathbb{Q} -factorial, then for D general, we have $|R^1 \setminus R| = 1$ and then P^1/P of Picard number 1. Indeed, if $|R^1 \setminus R| > 1$, ϕ is not a contraction of an extremal ray, because $G/P \rightarrow G/P^1$ is clearly not a contraction of an extremal ray. In that case, the fibers of the morphism ϕ is a flag variety of Picard number one.

Secondly, suppose that the dimension of Q^{ϵ_1} is less than the dimension of Q .

Let x_0 (resp. x_0^1) be the unique point of the open G -orbit of X (resp. Z) fixed by H (resp. H^1). We claim that the general fiber of ϕ is the closure in X of the H^1 -orbit $H^1 \cdot x_0$. Indeed, since $G \cdot x_0^1$ is open in Z , the fiber $\phi^{-1}(x_0^1)$ is a general fiber of ϕ . This fiber intersected by the open G -orbit $G \cdot x_0$ of X is $H^1 \cdot x_0$. We conclude the claim by the fact that, since X is irreducible, a general fiber of ϕ is also irreducible. Denote by F_{ϕ} the H^1 -variety $\phi^{-1}(x_0^1)$.

Since an open set of X is isomorphic to the bundle $G \times^{H^1} F_{\phi}$, we get that F_{ϕ} has the same singularities as X (normal and \mathbb{Q} -Gorenstein, and \mathbb{Q} -factorial or smooth according to the hypothesis done on X). Moreover, the unipotent radical $R_u(H^1)$ of H^1 (which is also the unipotent radical of P^1) acts trivially on x_0 , then it also acts trivially on $H^1 \cdot x_0$ and F_{ϕ} . Hence, F_{ϕ} is a L^1 -variety, where $L^1 = H^1/R_u(H^1)$ is reductive. Let $H^2 := H/R_u(H^1)$. Then the open L^1 -orbit of F_{ϕ} is isomorphic to L^1/H^2 and is horospherical, because H^2 contains the unipotent radical

$U/R_u(H^1)$ of the Borel subgroup $B/R_u(H^1)$ of L^1 . In the rest of the section, we describe the L^1/H^2 -embedding F_ϕ .

First describe the combinatorial data associated to the horospherical homogeneous space L^1/H^2 . The simple roots of L^1 are those of P^1 , ie the simple roots in the set R^1 . Then the normalizer P^2 of H^2 in L^1 is the parabolic subgroup of L^1 whose set of simple roots is R . Moreover, the set of characters of P^2 trivial on H^2 is isomorphic to the quotient of the set of characters of P trivial on H with M^1 . Then we set $M^2 := M/M^1$. The set of colors of the horospherical homogeneous space L^1/H^2 is $R^1 \setminus R$.

Let I be the maximal subset of $\{1, \dots, m+r\}$ such that $\tilde{Q}^{\epsilon_1} \subset \{x \in M \mid A_I x = \tilde{B}_I + \epsilon_1 \tilde{C}_I\}$. In particular, $M_{\mathbb{Q}}^1$ equals the kernel of A_I . Now, we prove that the projection of \tilde{Q} on $M_{\mathbb{Q}}^2 := M_{\mathbb{Q}}/M_{\mathbb{Q}}^1$ is the polytope $\tilde{Q}^2 := \{x \in M_{\mathbb{Q}}/\text{Ker}(A_I) \mid A_I x \geq \tilde{B}_I\}$. One inclusion is obvious. To prove the second one, let $x + \text{Ker}(A_I) \in \tilde{Q}^2$ be a vertex F_J^2 of \tilde{Q}^2 where $J \subset I$. Then $A_J x = \tilde{B}_J$ and $A_I x \geq \tilde{B}_I$. Moreover, by maximality of I , there exists $x' \in M_{\mathbb{Q}}$ such that $A_I x' = \tilde{B}_I + \epsilon_1 \tilde{C}_I$ and $A_{\bar{I}} x' > \tilde{B}_{\bar{I}} + \epsilon_1 \tilde{C}_{\bar{I}}$ (where $\bar{I} = \{1, \dots, m+r\} \setminus I$). Hence, for $\lambda > 0$ small enough, we have $x'' := \lambda x + (1-\lambda)x'$ satisfies $A_J x'' = \tilde{B}_J + \epsilon'' \tilde{C}_J$, $A_I x'' \geq \tilde{B}_I + \epsilon'' \tilde{C}_I$ and $A_{\bar{I}} x'' > \tilde{B}_{\bar{I}} + \epsilon'' \tilde{C}_{\bar{I}}$, with $\epsilon'' = (1-\lambda)\epsilon_1$, so that x'' is a point of the face $F_J^{\epsilon''}$ of $\tilde{Q}^{\epsilon''}$. Since $0 \leq \epsilon'' < \epsilon_1$, the face F_J of \tilde{Q} is not empty. But every point of F_J projects on F_J^2 , which is a point. Then, every vertex of \tilde{Q}^2 is the image of a point of \tilde{Q} by the projection on $M_{\mathbb{Q}}^2$, that proves the second inclusion.

By translation, we get that the projection of Q on $X(P)_{\mathbb{Q}}/M_{\mathbb{Q}}^1$ is the polytope $Q^2 := \{x \in \bar{v}^0 + M_{\mathbb{Q}}/\text{Ker}(A_I) \mid A_I x \geq B_I\}$, where \bar{v}^0 is the image of v^0 in $X(P)_{\mathbb{Q}}/M_{\mathbb{Q}}^1$.

Suppose now that D is Cartier and very ample (or replace D by a multiple of D , see Remark 2). Then, by Proposition 4, X is isomorphic to the closure of $G \cdot [\sum_{\chi \in (v^0+M) \cap Q} v_\chi]$ in $\mathbb{P}(\oplus_{\chi \in (v^0+M) \cap Q} V(\chi))$. Then F_ϕ is isomorphic to $L^1 \cdot [\sum_{\chi \in (v^0+M) \cap Q} v_\chi]$ in $\mathbb{P}(\oplus_{\chi \in (v^0+M) \cap Q} V_{L^1}(\chi))$. But, for any $\chi \in X(P)$ and any $\chi' \in M^1$, the L^1 -modules $V_{L^1}(\chi)$ and $V_{L^1}(\chi + \chi')$ are isomorphic, so that Y is isomorphic to $L^1 \cdot [\sum_{\chi \in (\bar{v}^0+M^2) \cap Q^2} v_\chi]$ in $\mathbb{P}(\oplus_{\chi \in (\bar{v}^0+M^2) \cap Q^2} V_{L^1}(\chi))$. It proves that F_ϕ is the L^1/H^2 -embedding associated to the polytope Q^2 . Remark that Q^2 is of maximal dimension in $M_{\mathbb{Q}}^2$ because Q is of maximal dimension in $M_{\mathbb{Q}}$.

Suppose now that X is \mathbb{Q} -factorial. We prove that, for D general, I is of cardinality $\text{codim}(Q^{\epsilon_1}) + 1 = \dim(Q^2) + 1$, and that Q^2 is a simplex of $M_{\mathbb{Q}}^2$ such that all wall W_α^2 in $X(P^2)_{\mathbb{Q}}$ with $\alpha \in R^1 \setminus R$ gives a facet of Q^2 . Suppose that D (and then \tilde{B}) satisfies the following condition: for any subset J of $\{1, \dots, m+r\}$ with $|J| \geq \dim(\text{Im}(A_J)) + 2$, \tilde{B}_J is not in the (proper) vector subspace of \mathbb{Q}^{m+r} generated by $\text{Im}(A_J)$ and \tilde{C}_J . Such a D is general, since X is \mathbb{Q} -factorial. Note that $\text{codim}(Q^{\epsilon_1}) = \text{codim}(\text{Ker}(A_I)) = \dim(\text{Im}(A_I))$. Now, if $|I| > \text{codim}(Q^{\epsilon_1}) + 1$ then by hypothesis on D , $\tilde{B}_I + \epsilon \tilde{C}_I$ is never in $\text{Im}(A_I)$, which contradicts that $F_I^{\epsilon_1} = Q^{\epsilon_1}$ is not empty. Moreover, Q^2 is a polytope (and then bounded) of maximal dimension in $M_{\mathbb{Q}}^2$, so the number $|I|$ of inequations defining Q^2 must be at least $\dim(Q^2) + 1$. Hence, $|I| = \dim(Q^2) + 1$ and then Q^2 is a simplex of $M_{\mathbb{Q}}^2$. The set $R^1 \setminus R$ of colors of L^1/H^2 is clearly contained in I by definition of I and R^1 , so that, for any $\alpha \in R^1 \setminus R$, $W_\alpha^2 \cap Q^2$ is a face of Q^2 . But, since each $i \in I$ correspond necessarily to a facet of Q^2 because $|I| = \dim(Q^2) + 1$, this latter face is necessarily a facet.

In that case, F_ϕ is a projective \mathbb{Q} -factorial L^1/H^2 -embedding of Picard number 1 (cf [Pas06, Eq. (4.5.1)] for an explicit formula of the Picard number of \mathbb{Q} -factorial horospherical varieties).

5 Examples

In this section we give 5 examples with $G = \mathrm{SL}_3$. The first three ones give the MMP for the same horospherical smooth variety but with a different ample Cartier divisor. The fourth one gives a flip consisting of exchanging colors (see [Bri93, Section 4.5]). And the last one gives the MMP for a \mathbb{Q} -Gorenstein (not \mathbb{Q} -factorial) variety, we observe in particular a flip from a divisorial contraction.

Fix a Borel subgroup B of G . Denote by α and β the two simple roots of G .

Example 3. Consider the horospherical subgroup H defined as the kernel of the character $\varpi_\alpha + \varpi_\beta$ of B . In that case we have N and M isomorphic to \mathbb{Z} . The horospherical homogeneous space has two colors α and β whose image in N are respectively $\alpha_M^\vee = \beta_M^\vee = 1$.

If χ is a character of B , we denote by \mathbb{C}_χ the line \mathbb{C} where B acts by $b \cdot z = \chi(b)z$ for any $b \in B$ and $z \in \mathbb{C}$.

Let X be the \mathbb{P}^1 -bundle $X = G \times^B \mathbb{P}(\mathbb{C}_0 \oplus \mathbb{C}_{\varpi_\alpha + \varpi_\beta})$ over G/B , it is a smooth G/H -embedding. Its colored fan is the unique complete fan (of dimension 1) without color. Denote by X_1 and X_2 the two irreducible G -stable divisors of X , respectively corresponding to the primitive elements $x_1 = 1$ and $x_2 = -1$ of N . Here $-K_X = X_1 + X_2 + 2D_\alpha + 2D_\beta$.

Choose $D = X_1 + 2X_2 + 2D_\alpha + 2D_\beta$. Then the moment polytope Q is the segment $[\varpi_\alpha + \varpi_\beta, 4(\varpi_\alpha + \varpi_\beta)]$ in the dominant chamber of (G, B) .

The family $(Q^\epsilon)_{\epsilon \geq 0}$ is given by:

- for any $\epsilon \in [0, 1[$, Q^ϵ is the segment $[(1 - \epsilon)(\varpi_\alpha + \varpi_\beta), (4 - 3\epsilon)(\varpi_\alpha + \varpi_\beta)]$;
- for any $\epsilon \in [1, \frac{4}{3}]$, Q^ϵ is the segment $[0, (4 - 3\epsilon)(\varpi_\alpha + \varpi_\beta)]$;
- $Q^{\frac{4}{3}}$ is the point 0.

Hence, the MMP from (X, D) gives a divisorial contraction from X to the projective G/H -embedding with the two colors α and β , which is not \mathbb{Q} -factorial but \mathbb{Q} -Gorenstein, and Fano. It finishes by a Mori fibration from this Fano variety to a point.

Note that, here, the divisorial contraction contracts 2 divisors: the zero and infinite section of the \mathbb{P}^1 -bundle X . In particular, it is not the contraction of a ray of $NE(X)$.

Example 4. We keep the same G/H -embedding X but we choose another ample divisor $D = X_1 + 2X_2 + 3D_\alpha + 2D_\beta$. Then the moment polytope Q is the segment $[2\varpi_\alpha + \varpi_\beta, 5\varpi_\alpha + 4\varpi_\beta]$.

The family $(Q^\epsilon)_{\epsilon \geq 0}$ is given by:

- for any $\epsilon \in [0, 1[$, Q^ϵ is the segment $[2\varpi_\alpha + \varpi_\beta - \epsilon(\varpi_\alpha + \varpi_\beta), 5\varpi_\alpha + 4\varpi_\beta - 3\epsilon(\varpi_\alpha + \varpi_\beta)]$;
- for any $\epsilon \in [1, \frac{4}{3}]$, Q^ϵ is the segment $[\varpi_\alpha, 5\varpi_\alpha + 4\varpi_\beta - 3\epsilon(\varpi_\alpha + \varpi_\beta)]$;
- $Q^{\frac{4}{3}}$ is the point ϖ_α .

Hence, the MMP from (X, D) gives a divisorial contraction from X to the projective G/H -embedding with the color β . This is a contraction of the ray of $NE(X)$ generated by C_{β, X_1} . It finishes by a Mori fibration from this \mathbb{Q} -factorial variety to the flag variety G/P_α .

Example 5. We still keep the same G/H -embedding X and we choose now the ample divisor $D = X_2 + 2D_\alpha + 2D_\beta$. Then the moment polytope Q is the segment $[2(\varpi_\alpha + \varpi_\beta), 3(\varpi_\alpha + \varpi_\beta)]$.

The family $(Q^\epsilon)_{\epsilon \geq 0}$ is given by:

- for any $\epsilon \in [0, \frac{1}{2}[$, Q^ϵ is the segment $[2(\varpi_\alpha + \varpi_\beta) + \epsilon(-\varpi_\alpha + \varpi_\beta), 3(\varpi_\alpha + \varpi_\beta) - 3\epsilon(\varpi_\alpha + \varpi_\beta)]$;
- $Q^{\frac{1}{2}}$ is the point $\frac{3}{2}(\varpi_\alpha + \varpi_\beta)$.

Hence, the MMP from (X, D) gives the Mori fibration $X \rightarrow G/B$, whose fibers are projective lines.

In the following example, we illustrate a flip consisting of exchanging colors.

Example 6. Consider the horospherical subgroup H defined as the kernel of the character $\varpi_\alpha + 2\varpi_\beta$ of B . In that case we have N and M isomorphic to \mathbb{Z} . The horospherical homogeneous space has two colors α and β whose image in N are respectively $\alpha_M^\vee = 1$ and $\beta_M^\vee = 2$.

Let X be the \mathbb{Q} -factorial G/H -embedding whose colored fan is the complete fan with color β . Denote by X_1 the irreducible G -stable divisor of X , corresponding to the primitive elements $x_1 = -1$. Here $-K_X = X_1 + 2D_\alpha + 2D_\beta$.

Consider $D = 3X_1 + 2D_\alpha + 2D_\beta$. Then the moment polytope Q is the segment $[\varpi_\alpha, 5\varpi_\alpha + 8\varpi_\beta]$.

The family $(Q^\epsilon)_{\epsilon \geq 0}$ is given by:

- for any $\epsilon \in [0, 1[$, Q^ϵ is the segment $[(1 - \epsilon)\varpi_\alpha, 5\varpi_\alpha + 8\varpi_\beta - \epsilon(3\varpi_\alpha + 4\varpi_\beta)]$;
- Q^1 is the segment $[0, 2\varpi_\alpha + 4\varpi_\beta]$;
- for any $\epsilon \in]1, \frac{5}{3}[$, Q^ϵ is the segment $[2(\epsilon - 1)\varpi_\beta, 5\varpi_\alpha + 8\varpi_\beta - \epsilon(3\varpi_\alpha + 4\varpi_\beta)]$;
- $Q^{\frac{5}{3}}$ is the point $\frac{4}{3}\varpi_\beta$.

Hence, the MMP from (X, D) first gives a flip $X \rightarrow Y \leftarrow X^+$, where Y is the G/H -embedding corresponding to the complete colored fan with the two colors α and β and X^+ is the G/H -embedding corresponding to the complete colored fan with the color α . It finishes by a Mori fibration from X^+ to the flag variety G/P_β .

Example 7. Consider the case where the horospherical subgroup H is the maximal unipotent U subgroup of B . Then the lattice M is the lattice of characters of B with basis $(\varpi_\alpha, \varpi_\beta)$, and N is the coroot lattice with basis $(\alpha^\vee, \beta^\vee)$. Here $\alpha^\vee = \alpha_M^\vee$ and $\beta^\vee = \beta_M^\vee$.

Let X be the G/H -embedding whose colored fan is the complete colored fan with color β , and edges generated by $x_1 := -\beta^\vee$, $x_2 := \alpha^\vee$ and $x_3 := \beta^\vee - \alpha^\vee$. Note that X is not \mathbb{Q} -factorial because β^\vee is not in an edge of \mathbb{F}_X . An anticanonical divisor of X is $X_1 + X_2 + X_3 + 2D_\alpha + 2D_\beta$, and it is Cartier so that X is \mathbb{Q} -Gorenstein.

Consider $D = 3X_1 + X_3 + D_\alpha + D_\beta$. Then D is an ample Cartier divisor whose moment polytope Q is the triangle with vertices ϖ_α , $\varpi_\alpha + 4\varpi_\beta$ and $5\varpi_\alpha + 4\varpi_\beta$. The matrices defining

the family $(Q^\epsilon)_{\epsilon \geq 0}$ are

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -4 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then the family $(Q^\epsilon)_{\epsilon \geq 0}$ is given by:

- for any $\epsilon \in [0, 1[$, Q^ϵ is the triangle with vertices $(1 - \epsilon)\varpi_\alpha$, $\varpi_\alpha + 4\varpi_\beta - \epsilon(\varpi_\alpha + 3\varpi_\beta)$ and $5\varpi_\alpha + 4\varpi_\beta - \epsilon(4\varpi_\alpha + 3\varpi_\beta)$;
- Q^1 is the triangle with vertices 0 , ϖ_β and $\varpi_\alpha + \varpi_\beta$;
- for any $\epsilon \in]1, \frac{5}{4}[$, Q^ϵ is the triangle with vertices $(\epsilon - 1)\varpi_\beta$, $(4 - 3\epsilon)\varpi_\beta$ and $5\varpi_\alpha + 4\varpi_\beta - \epsilon(4\varpi_\alpha + 3\varpi_\beta)$;
- $Q^{\frac{5}{4}}$ is the point $\frac{1}{4}\varpi_\beta$.

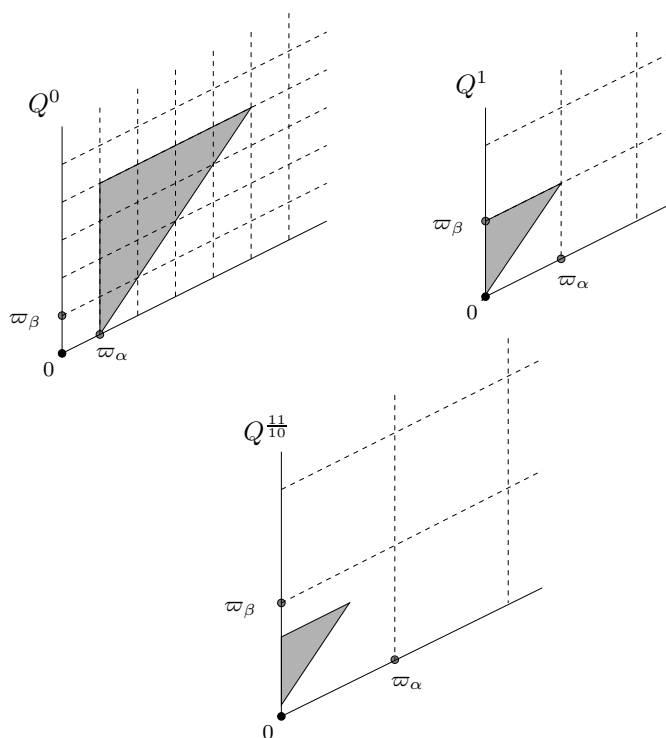


Figure 5: Evolution of Q^ϵ in Example 7

We illustrate the first three classes of the family $(Q^\epsilon)_{\epsilon \geq 0}$ in Figure 5.

Hence, the MMP from (X, D) first gives a flip $X \rightarrow Y \leftarrow X^+$, where Y is the G/H -embedding corresponding to the same complete colored fan as X but with the two colors α and β and X^+ is the G/H -embedding corresponding to the same complete colored fan as X but with the color α (instead of β). Note that the map $X \rightarrow Y$ contracts the divisor X_2 and that the map $X^+ \rightarrow Y$ does not contract a divisor. It finishes by a Mori fibration from X^+ to the flag variety G/P_β .

References

- [Bri89] Michel Brion, *Groupe de picard et nombres caractéristiques des variétés sphériques*, Duke Math. J. **58** (1989), no. 2, 397–424.
- [Bri93] ———, *Variétés sphériques et théorie de Mori*, Duke Math. J. **72** (1993), no. 2, 369–404.
- [Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.
- [Kno91] Friedrich Knop, *The Luna-Vust theory of spherical embeddings*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989) (Madras), Manoj Prakashan, 1991, pp. 225–249.
- [Mat02] Kenji Matsuki, *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002.
- [Oda88] Tadao Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 15, Springer-Verlag, Berlin, 1988, An introduction to the theory of toric varieties, Translated from the Japanese.
- [Pas06] Boris Pasquier, *Variétés horosphériques de Fano*, Ph.D. thesis, Université Joseph Fourier, Grenoble 1, Available at <http://tel.archives-ouvertes.fr/tel-00111912>, 2006.
- [Pas08] ———, *Variétés horosphériques de Fano*, Bull. Soc. Math. France **136** (2008), no. 2, 195–225.
- [Pas09] ———, *On some smooth projective two-orbit varieties with Picard number 1*, Math. Ann. **344** (2009), no. 4, 963–987.
- [Rei83] Miles Reid, *Decomposition of toric morphisms*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 395–418.