Modelling a vehicle-sharing station as a dual waiting system: stochastic framework and stationary analysis

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Abstract

A waiting system with two kinds of resources, say the vehicles and the docks in a vehicle-sharing service, is considered. Two arrival flows of customers are assumed, access customers who require a vehicle versus egress customers that bring back their vehicle and require a dock at the station. The total number of docks sets a limit capacity for the service. A stochastic, markovian, state-transition model is defined, which constitutes a bi-sided capacitated queuing system. The balance equations are stated and solved, yielding a stationary distribution under two conditions of compatibility. Indicators of service quality and system performance are defined and formulated under steady state.

Keywords

Vehicle-sharing. Stochastic model. Customer flow. Docking capacity. Waiting behavior

1. Introduction

Recently, vehicle-sharing systems (VSS) of transportation have enjoyed rapid diffusion in many cities throughout the world, at first for bikes and then for cars. They can be classified in several kinds, depending on not only the type of vehicle but also the operational protocol: does it involve prior booking or not, are there stations of vehicle depot and, if so, is there a return constraint to bring the vehicle back to the station of take-off? The location of the stations in the urban area, the number of them and their respective docking capacities, as well as the size of the vehicle fleet, constitute the main variables of system design. Pioneering implementations can be traced back to the 1960s for bicycles, in Amsterdam (the “White Bike” program, 1965) or in La Rochelle, France (the “yellow bike” system, 1974), and even to 1948 for cars (the Sefage program in Zurich). In 2007, the Velib system of shared bikes in Paris, though not the first one of its kind (Cf. Velo’v in Lyons, 2005), has broken the path to large systems involving several thousands of vehicles and docks. In 2011, an analogous system of shared cars, dubbed Autolib, has been deployed in Paris also and with similar dimensional parameters.

As such systems could be useful in a number of cities, simulation models are important to evaluate the performance of a service scheme and to aid decision-making in transportation planning. Several streams of scientific literature have developped rapidly, in pace with the surging programs of VSS. The first and most ancient stream pertains to OR models of system design and/or management: a VSS is modeled as a network of stations, possibly with an underlying network of streets, with an objective function involving at least the costs of operations and maybe also the set-up costs and the customer costs, to be optimized with

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respect to a set of design variables by using typical or innovative OR algorithms (e.g. Lin and Yang, 2011, Contardo *et al*, 2012, Martinez *et al*, 2012). Within this stream, some models are focused on the real-time operations and the optimization of, e.g., vehicle redistribution between stations. The stream is focused on the supply side and deals with the demand side by crude assumptions: the trip demand is represented basically as an origin-destination matrix of trip flows between pairs of stations, with no customer behavior other than diverting to alternative modes (not modelled explicitly) if no vehicle is available at the station at the instant of the trip-maker's arrival.

A second stream belongs to the traditional field of travel demand analysis. Some works are devoted to analyze VSS customer databases to yield trip matrices by period and other patterns of usage such as the distribution of trip duration (e.g. Nair *et al*, 2013). Customer surveys yield additional patterns of usage, by type of person and trip purpose (e.g. Kumar and Bierlaire, 2012). Bordagaray *et al* (2012) have surveyed a sample of bike-sharing users by an ad-hoc experiment of stated preferences in order to model the determinants of the quality of service. However, there has been little attempt to integrate VSS as a modal option in the setting of multimodal travel demand models. In our opinion, this integration is of primary importance because, in real-world conditions, the VSS is a minor travel mode within its area of coverage, where it competes with the major modes including walking, the private car and transit modes. Conversely, the sharing service may be used as a trip leg complementary to other modal legs, notably transit legs. For both reasons, the issue of modal choice should be addressed to yield a relevant origin-destination matrix of trip flows for the vehicle-sharing service. To that end, a model of route choice on a multimodal network should be developped. Dynamic microsimulation might be a solution – but a costly one to deal with a large urban area with millions of inhabitants, all of whom must be simulated to deliver a relevant picture of, e.g., station occupancy in real time. Ciari *et al* (2011) have modeled a VSS as an abstract mode in the MatSim-T multi-agent model of mobility: VSS legs can be included in trip tours that are considered and possibly selected by the modeled trip-makers. The agent framework could also enable one to model the system operator as a specific agent with its own objectives and resources. But in this implementation the VSS mode is not modeled physically, meaning that it is assumed to be available at any station with no capacity constraint and that the quality of service is derived from the travel conditions by private car. This raises the issue of modeling the availability of trip resources (vehicles and docks) with other components of service quality that are essential to the trip-makers as candidate customers. Resource availability depends on not only the system setting in terms of stations, vehicles and operating processes, but also on the pattern of traffic, which involves the various trips of the whole population of trip-makers within the studied area. So, in the framework of dynamic simulation, a realistic model of VSS quality of service requires to simulate all the eligible trips in the area; it cannot be restricted to a sample of trip-makers. Furthermore, for recurrent travel purposes such as home to work, the trip-maker evaluates quality of service on an average basis out of a set of reiterated occurrences. Thus, both the issue of resource availability and that of average quality of service call for some macroscopic modeling of any VSS within the multimodal transportation system. The development of a macroscopic multimodal model will involve three steps: first, a local model of station occupancy, second, a macroscopic model of traffic assignment to a monomodal network of vehicle sharing, third, the multimodal network traffic assignment model.

This brings us to the third and last stream in the scientific literature on VSS models: that of stochastic models. In OR models there may be some flavor of stochasticity: demand variation is identified and discussed, but most of the models are focused on average flows (e.g. Shu *et al*, 2010). Improvements have been provided by Lin and Yang (2011) who integrate the variance of vehicule inventory in the cost function of the system, and by Nair *et al* (2013) who deal with each demand flow by a quantile of in its hypothesized distribution function, in order to address the risk of local resource shortage. More advanced, markovian models of a closed queuing network have been developped to focus on demand stochasticity across space and timpe in system operations. One trend is purported to establish structural properties in a coverage area assumedly homogeneous, with stations that are assumed identical in capacity as well as in spatial context even if they are spread over the area; furthermore, flows from one station to the other stations are assumed to be distributed according to a single pattern or at most two patterns, so that in essence the model deals with one or two typical stations only. This results in a traffic pattern that is uniform between stations if they are homogenous (Pricker and Gast, 2012). In another trend, station heterogeneity in size, location and trade of customers is modeled explicitly in order to study yield management policies (Waserhole and Jost, 2012). Whatever the trend, the physical representation is limited with no capacity constraint (George and Xia, 2010) or only one type of waiting: excess pedestrians are dismissed to avoid vehicle shortage so that waiting is restricted to excess vehicles,, or excess vehicles are immediately diverted to neighboring stations (Fricker and Gast, 2012). So, to the physical limitation are added a number of gross behavioral restrictions. Furthermore, the trip duration between stations is modelled as an exponential variable, which would fit to return trips better than to one-way trips.

Here the objective is to provide a stochastic model of a station, as the basic modeling brick that could be integrated in a wider, network model. Macroscopic properties are derived from probabilistic assumptions about the arrival flows of the access and egress customers. As the customers on egress bring the vehicles to the station, they may be considered as suppliers to those on access. Conversely, if all the docks are occupied, then a customer on access will take a vehicle and also vacate a dock, thus providing a resource and being a supplier to the customers on egress. In other words, the station is a dual system of service and waiting. It is modeled here as a pair of waiting systems, an access one for the vehicles and an egress one for the docks, which are so strongly coupled that the both of them make a bi-sided queuing system. In a markovian setting, the state variable of interest is the number of present vehicles, extended artificially to negative values so as to account for waiting customers on the access side. A state-transition model is formulated. The balance equations are provided and solved for stochastic equilibrium. The stationary distribution enables us to derive quality or performance indicators on the basis of simple, closed-form formulae.

The rest of the paper is in four parts. Section 2 brings about the model assumptions and state-transition framework. Then, Section 3 provides the balance equations and the stationary distribution of probability. Next, Section 4 deals with indicators of service quality, on the customer side, and of system performance, on the operator side. Lastly, Section 5 is devoted to sensivity analysis and illustration.

Table of notation

κ docking capacity

λ time intensity of arrival flow of customers on system access

μ time intensity of arrival flow of customers on system egress

r probability of acceptance to wait for access customers

s probability of acceptance to wait for egress customers

α^- stationary probability of vehicle shortage

α^+ stationary probability of dock saturation

2. Model framework

2.1 Station assumptions

Let us define a station as a limited space with a given number, say κ, of docks to accommodate idle vehicles. Two distinct flows of customers arrive for service at the station, either for service access or egress. On the access side, customers without vehicle – who are pedestrians at that stage – arrive at rate λ to get a vehicle. On the egress side, customers holding a vehicle arrive at rate μ to get a dock so as to leave their vehicle.

The main state variable is the number of idle vehicles, say N with current value denoted by n. When $N > 0$ there is some idle vehicle available for a customer at access. When $N < \kappa$ there is some dock available to egress customers.

Let us adapt the state variable by extending variable N to values less than zero, on the access side: a value $n < 0$ means that $|n|$ customers are waiting to get a vehicle. On the egress side, a value $n > \kappa$ means that $n - \kappa$ customers holding a vehicle are waiting for a vacant dock to leave their vehicle.

The number of available docks, D, is equal to $|\kappa - N|$ when $N \in [0, \kappa]$, or zero when $N > \kappa$, or κ when $N < 0$. So in general

$$D = \min\{\kappa, (\kappa - N)^+\}. \tag{1}$$

2.2 Customer behavior

Access customers arrive at the station as a stochastic, markovian process with rate λ. If $N > 0$ then no such customer is waiting for a vehicle so the next access customer gets a vehicle with no delay save for the transaction time. If $N \leq 0$ then $|N|$ such customers are already waiting for a vehicle at the instant of arrival of the next one. Let us assume that there is a given acceptance rate, denoted r, to join the stock and wait for a vehicle.

On the egress side, exiting customers arrive at the station as a stochastic, markovian process with rate μ. If $N < \kappa$ then no such customer is waiting for a dock so the next egress customer leaves his vehicle with no delay save for the transaction time. If $N \geq \kappa$ then $|N - \kappa|$ egress customers are already waiting for an available dock at the instant of arrival of the next one. Let us assume that there is a given acceptance rate, denoted s, to join this stock and wait for a dock at the station.

2.3 State-transition model

The station has been defined as a stochastic system subject to two random processes of pedestrian arrivals and vehicle arrivals, respectively, with a state variable N that summarizes the issues of vehicle availability, dock availability, waiting pedestrians and waiting couples of egress customer with his vehicle.

From a given state value n, system transitions of state occur to the neighboring values $n-1$ or $n+1$ according to the following rules:

- from n to $n+1$, due to the flow of egress customers and their behavior, the transition rate is μ if $n < \kappa$ or μs if $n \geq \kappa$.

- from n to $n-1$, due to the flow of access customers and their behavior, the transition rate is λ if $n > 0$ or λr if $n \leq 0$.
- Any transition between n and $n+i$ for $i \notin \{-1,1\}$ has null rate.

Figure 1 gives the state-transition diagram for that stochastic system, which may be called a bi-sided waiting system where each side has a specific significance and two different stocks are involved in a way which seems complementary in our setting, but which is mutually exclusive at any instant of system performance and of potential customer arrival.

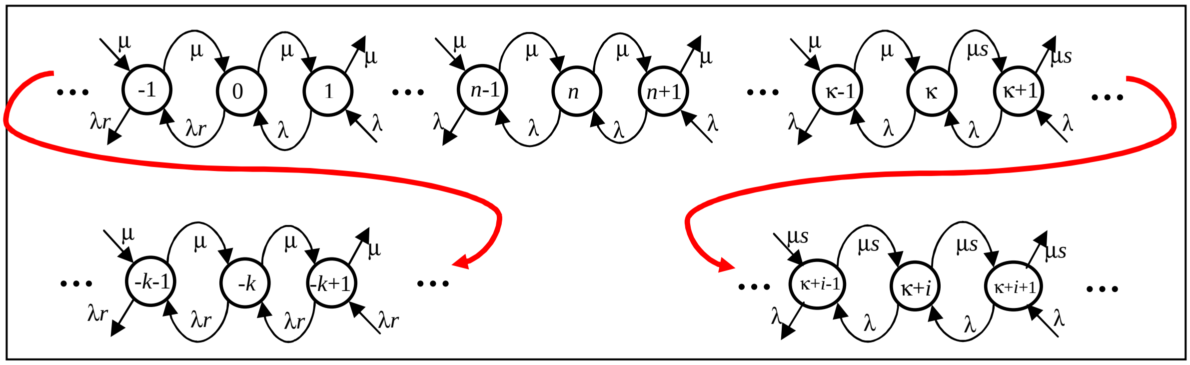

Fig. 1. State-transition diagram.

3. Stationary state

3.1 Balance equations

Stochastic equilibrium is achieved when the system state evolves according to a stationary distribution of probability. Let us denote $\mathbf{p} = [p_n : n \in \mathbb{Z}]$ such a distribution in the form of a line vector.

Denoting by $\mathbf{Q}$ the matrix of transition rates, with q_{nm} defined above from n to $m \neq n$ and $q_{nn} = -\sum_{m \neq n} q_{nm}$, the stationary distribution satisfies the conservation equation $\mathbf{pQ} = 0$. Precisely, the local balance equations fall into five kinds as follows:

$$p_n(\lambda + \mu) = p_{n-1}\mu + p_{n+1}\lambda \text{ if } n \in]0, \kappa[, \tag{2a}$$

$$p_n(\lambda + \mu s) = p_{n-1}\mu s + p_{n+1}\lambda \text{ if } n > \kappa, \tag{2b}$$

$$p_n(\lambda + \mu s) = p_{n-1}\mu + p_{n+1}\lambda \text{ if } n = \kappa, \tag{2c}$$

$$p_n(\lambda r + \mu) = p_{n-1}\mu + p_{n+1}\lambda r \text{ if } n < 0, \tag{2d}$$

$$p_n(\lambda r + \mu) = p_{n-1}\mu + p_{n+1}\lambda \text{ if } n = 0. \tag{2e}$$

Let us derive necessary conditions on the stationary distribution from the balance equations, by focusing on each subdomain in turn: first on the left hand side of excess waiting pedestrians (i.e. vehicle shortage), then on the right hand side of excess waiting vehicles (i.e. dock saturation) and last on the intermediate subdomain where both vehicles and docks are available to the customers, meaning unrestrained availability.

3.2 Vehicle shortage subdomain

This subdomain pertains to system states $-k$ for any positive integer k. The related balance equation, (2d), yields that

$\mu(p_{-k} - p_{-k-1}) = \lambda r(p_{-k+1} - p_{-k})$, or

$$\delta_{-k} = \rho\delta_{-k+1} \text{ wherein } \delta_{-k} \equiv p_{-k} - p_{-k-1} \text{ and } \rho \equiv \lambda r / \mu\,. \tag{3}$$

By recursion, it comes out that

$$\delta_{-k} = \rho^k \delta_0\,. \tag{4}$$

Furthermore, $p_{-k} = p_0 - \sum_{i=0}^{k-1} \delta_{-i}$ so that

$$p_{-k} = p_0 - \delta_0 \frac{1-\rho^k}{1-\rho}\,. \tag{5}$$

As the total probability of the states $-k$ must be less than one, condition (5) requires that $p_0 - \delta_0 /(1-\rho) = 0$, so $\delta_0 = (1-\rho) p_0$. By substitution into (5), it comes out that

$$p_{-k} = \rho^k p_0\,,\ \forall k \geq 0\,. \tag{6}$$

Furthermore,
$$\sum_{k\geq 0} p_{-k} = p_0 /(1-\rho)\,. \tag{7}$$

Another requirement of total probability is that $\rho < 1$, which is a condition of compatibility between the macroscopic parameters: with respect to the basic parameters, it is expressed as

$$\lambda r < \mu\,. \tag{8}$$

3.3 Dock shortage subdomain

This subdomain pertains to states $k = \kappa + i$ with $i \geq 0$: no dock is available to an arriving customer willing to leave his vehicle. The related balance equation, (2b), yields that $\mu s(p_k - p_{k-1}) = \lambda(p_{k+1} - p_k)$ so that, letting $\delta_k \equiv p_k - p_{k-1}$ as previously and $\sigma \equiv s\mu / \lambda$,

$$\delta_{k+1} = \sigma\delta_k\,. \tag{9}$$

By recursion from state $\kappa+1$, we get that

$$\delta_{\kappa+1+i} = \sigma^i \delta_{\kappa+1}\,,\ \forall i \geq 0\,. \tag{10}$$

Furthermore, $p_{\kappa+1+i} = p_\kappa + \sum_{n=0}^{i} \delta_{\kappa+1+n}$ so that

$$p_{\kappa+1+i} = p_\kappa + \delta_{\kappa+1} \frac{1-\sigma^{i+1}}{1-\sigma}\,. \tag{11}$$

As the total probability of the states $\kappa+1+i$ must be less than one, condition (11) requires that $p_\kappa + \delta_{\kappa+1} /(1-\sigma) = 0$, so $\delta_{\kappa+1} = (\sigma-1) p_\kappa$ hence $p_{\kappa+1} = \sigma p_\kappa$.

Combining with (11), we get that

$$p_{\kappa+m} = \sigma^m p_\kappa\,,\ \forall m \geq 0\,. \tag{12}$$

As the total probability must be less than one, it must hold that $\sigma < 1$, which is another condition of compatibility between the macroscopic parameters: with respect to the basic parameters, its basic expression is

$$s\mu < \lambda\,. \tag{13}$$

Furthermore,
$$\sum_{m\geq 0} p_{\kappa+m} = p_\kappa /(1-\sigma)\,. \tag{14}$$

3.4 Availability subdomain

This subdomain pertains to states $k \in [1, \kappa-1]$ at which $k > 0$ vehicles are available if as well as $\kappa - k > 0$ docks, if $\kappa > 1$. At $\kappa = 1$ there is no subdomain of unrestrained availability.

The related balance equation, (2a), yields that $\mu(p_n - p_{n-1}) = \lambda(p_{n+1} - p_n)$ so

$$\delta_{k+1} = \varphi\delta_k \text{ wherein } \delta_k \equiv p_k - p_{k-1} \text{ as previously and } \varphi \equiv \mu/\lambda. \tag{15}$$

Then, $\delta_{k+1} = \varphi^k \delta_1$, so that, as $p_k = p_0 + \sum_{n=0}^{k-1} \delta_{n+1}$ it comes out that

$$p_k = p_0 + \delta_1 \frac{1-\varphi^k}{1-\varphi} \text{ if } \varphi \neq 1, \tag{16a}$$

$$p_k = p_0 + k\delta_1 \text{ if } \varphi = 1. \tag{16b}$$

From this stems the total probability of the subdomain,

$$\sum_{k=1}^{\kappa-1} p_k = (\kappa-1)(p_0 + \frac{\delta_1}{1-\varphi}) - \delta_1 \frac{\varphi - \varphi^\kappa}{(1-\varphi)^2} \text{ if } \varphi \neq 1, \tag{17a}$$

$$\sum_{k=1}^{\kappa-1} p_k = (\kappa-1) p_0 - \tfrac{1}{2}\kappa(\kappa-1)\delta_1 \text{ if } \varphi = 1. \tag{17b}$$

The balance equation at $n = 0$, (2e), yields that $p_1\lambda = \lambda r\, p_0 + \mu\delta_0$. As $\delta_0 = (1-\rho) p_0$ due to (6), we get that $p_1\lambda = p_0\mu$ hence $p_1 = \varphi\, p_0$ and $\delta_1 = (\varphi - 1) p_0$. Thus, (16) is reduced to

$$p_k = p_0\, \varphi^k \quad \forall k \in [0, \kappa-1] \text{ and whatever } \varphi \geq 0. \tag{18}$$

Then the subdomain probability amounts to

$$\sum_{k=1}^{\kappa-1} p_k = p_0 \frac{\varphi - \varphi^\kappa}{1-\varphi} \text{ if } \varphi \neq 1, \tag{19a}$$

$$\sum_{k=1}^{\kappa-1} p_k = (\kappa-1) p_0 \text{ if } \varphi = 1. \tag{19b}$$

3.5 Overall stationary distribution

At the frontier state $k = \kappa$ between availability and dock shortage, the balance equation (2c) combined to (18) at $k = \kappa - 1$ and to (12) at $m = 1$ yields that $p_\kappa(\lambda + \mu s) = \mu\varphi^{\kappa-1} p_0 + p_\kappa \sigma\lambda$, so that $p_\kappa = \varphi^\kappa p_0$ which extends the domain of validity of (18) to every $k \in [0, \kappa]$.

By joining the three subdomains, the total probability amounts to

$$\sum_{k\in Z} p_k = \left(\sum_{k\geq 0} p_{-k}\right) + \left(\sum_{k=1}^{\kappa-1} p_k\right) + \left(\sum_{k\geq 1} p_{\kappa+k}\right) = \frac{p_0}{1-\rho} + p_0 \frac{\varphi-\varphi^\kappa}{1-\varphi} + \frac{p_0\, \varphi^\kappa}{1-\sigma}.$$

From this and the law of total probability stems the pivot probability p_0 (under the convention that $\frac{\varphi-\varphi^\kappa}{1-\varphi} = \kappa - 1$ if $\varphi = 1$):

$$p_0 = 1/[\frac{1}{1-\rho} + \frac{\varphi - \varphi^\kappa}{1-\varphi} + \frac{\varphi^\kappa}{1-\sigma}]. \tag{20}$$

This ends up the derivation of necessary conditions on the stationary distribution. When the parameters do not satisfy the compatibility requirements (8) and (14), then no stationary distribution can exist. Conversely, if the parameters check the compatibility requirements, then let us consider the system of quantities defined as follows from (20), (18), (12) and (6):

$$p_0 = 1/[\frac{1}{1-\rho} + \frac{\varphi - \varphi^\kappa}{1-\varphi} + \frac{\varphi^\kappa}{1-\sigma}], \tag{21a}$$

$$p_k = p_0\, \varphi^k \quad \forall k \in [0, \kappa], \tag{21b}$$

$$p_{\kappa+m} = \sigma^m p_\kappa, \ \forall m \geq 0, \tag{21c}$$

$$p_{-k} = \rho^k p_0, \ \forall k \geq 0. \tag{21d}$$

By construction, this is a probability distribution which solves the balance equations, so under the compatibility conditions a stationary state exists for the dual waiting system and it is unique.

It is easy to evaluate the mean and variance of the state variable in steady regime: letting $\alpha^- \equiv p_0/(1-\rho)$, $\overline{\alpha} \equiv p_0(\varphi - \varphi^\kappa)/(1-\varphi)$ and $\alpha^+ \equiv p_0 \varphi^\kappa/(1-\sigma)$ be the probabilities of the subdomains, then

$$\mathrm{E}[N] = -\alpha^- \frac{\rho}{1-\rho} + \overline{\alpha}\,\varphi \frac{(\kappa-1)\varphi^\kappa - \kappa\varphi^{\kappa-1} + 1}{(1-\varphi)^2} + \alpha^-(\kappa + \frac{\sigma}{1-\sigma}).$$

The variance can be established similarly by composition. However, such quantities are meaningless since variable N is an artificial construct, with negative values associated to excess pedestrians and values larger than κ to excess vehicles. The major outcomes of the model pertain to the distributions of the physically-meaningful variables, respectively those of waiting pedestrians $(-N)^+$, of busy docks $\tilde{D} = \min\{(N)^+, \kappa\}$ and of undocked vehicles, $(N-\kappa)^+$.

For instance, undocked vehicles have been neglected in some previous models in which immediate diversion to other stations was assumed in case of dock saturation (Fricker and Gast, 2012). Then, letting $p'_0 = (1-\varphi)/(1-\varphi^{\kappa+1})$, the probability of dock saturation, $\Pr\{\tilde{D} = \kappa\}$, would be evaluated as $p'_0 \varphi^\kappa$ instead of α^+. Similarly, neglecting the waiting pedestrians would lead to evaluate $\Pr\{\tilde{D} = 0\}$ as p'_0 instead of α^-.

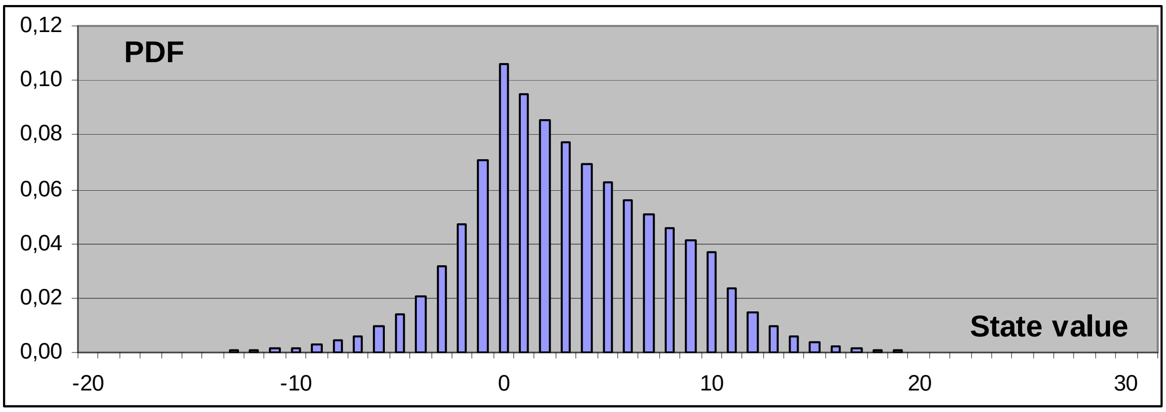

Fig. 2. PDF of stationary state ($\kappa = 10$, $\lambda = 1$, $\mu = .9$, $r = .6$, $s = .7$).

4. Indicators of quality and performance

Let us state the system characteristics of interest to the customer and/or the operator, by defining indicators of quality or performance and providing formulae to evaluate them under stochastic equilibrium. In practice, the system operator will be eager to manage it in an active way by contributing to station refill in vehicles or discharge: this will determine the stationary state but will not alter the significance of the performance indicators.

4.1 Demand side

On the demand side, access customers are interested in the immediate availability of a vehicle. The relevant indicator is the probability of immediate vehicle availability (PIVA) at any instant. Denoting by α^- the probability of vehicle shortage, $\text{PIVA} = 1 - \alpha^-$. In the stationary regime,

$$\alpha^- \equiv \sum_{k \geq 0} p_{-k} = p_0 /(1-\rho)\,. \tag{22}$$

When no vehicle is available, the access customers are interested in the wait time to get a vehicle. Assuming a priority queue, a customer arriving at state value $-k$ (for $k \geq 0$) has to wait for the $k+1$-th vehicle to arrive, so his wait time is a random variable as follows:

$$w_k^- \equiv \sum_{i=0}^{k} w_{\text{veh}}^{(i)}\,, \tag{23}$$

Wherein $w_{\text{veh}}^{(i)}$ denotes the random time for a vehicle arrival. Under the markovian assumption, $w_{\text{veh}} \approx \text{Exp}(\mu)$ so that

$$\text{E}[w_k^-] = (k+1)/\mu\,, \tag{24a}$$

$$\text{V}[w_k^-] = (k+1)/\mu^2 \text{ due to the independency of markovian arrivals.} \tag{24b}$$

This wait time variable is conditional on one state $-k \leq 0$.

Conditional on all states $-k \leq 0$ i.e. on vehicle shortage, the wait time is a probabilistic mixture of the elemental wait times, which yields its mean and variance as follows:

$$\text{E}[w_{N \leq 0}^-] = \frac{\sum_{k \geq 0} p_{-k}\, \text{E}[w_k^-]}{\sum_{k \geq 0} p_{-k}} = \frac{p_0 (\sum_{k \geq 0} (k+1)\rho^k)/\mu}{p_0/(1-\rho)} = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu - \lambda r}\,, \tag{25a}$$

$$\begin{aligned} \text{V}[w_{N \leq 0}^-] &= \left(\frac{\sum_{k \geq 0} p_{-k}\, \text{V}[w_k^-]}{\sum_{k \geq 0} p_{-k}} \right) + \left(\frac{\sum_{k \geq 0} p_{-k}\, (\text{E}[w_k^-] - \text{E}[w_{N \leq 0}^-])^2}{\sum_{k \geq 0} p_{-k}} \right) \text{ by the law of total variance} \\ &= \frac{1}{\mu^2(1-\rho)} + \frac{\rho}{\mu^2 (1-\rho)^2} = (\mu - \lambda r)^{-2} \end{aligned}\,, \tag{25b}$$

In the appendix, it is shown that the wait time conditional on vehicle shortage has an exponential distribution of parameter $\mu - \lambda r$.

Egress customers are interested in the immediate availability of a dock: the probability of immediate dock availability (PIDA), satisfies $\text{PIDA} = 1 - \alpha^+$ wherein α^+ denotes the probability of dock saturation. In the stationary regime,

$$\alpha^+ \equiv \sum_{k \geq \kappa} p_k = p_0 \varphi^\kappa /(1-\sigma)\,. \tag{26}$$

When no dock is available, the egress customers are interested in the wait time to get a dock. Assuming a priority queue, a customer arriving at state value $\kappa+k$ ($k \geq 0$) has to wait for the $k+1$-th pedestrian to arrive, so his wait time is a random variable as follows:

$$w_k^+ \equiv \sum_{i=0}^{k} w_{\text{ped}}^{(i)}, \tag{27}$$

Wherein $w_{\text{ped}}^{(i)}$ denotes the random time for a pedestrian arrival. Under the markovian assumption, $w_{\text{ped}} \approx \text{Exp}(\lambda)$ so that, as in the previous case,

$$\text{E}[w_k^+] = (k+1)/\lambda, \text{ and } \text{V}[w_k^+] = (k+1)/\lambda^2. \tag{28a, b}$$

This wait time variable is conditional on one state $\kappa+k$. Conditional on all states $\kappa+k$ for $k \geq 0$ i.e. on dock shortage, the wait time is a probabilistic mixture of the elemental wait times. As previously, it has an exponential distribution but with parameter $\lambda-\mu s$, yielding that

$$\text{E}[w_{N\geq\kappa}^+] = (\lambda-\mu s)^{-1} \text{ and } \text{V}[w_{N\geq\kappa}^+] = (\lambda-\mu s)^{-2}. \tag{29a, b}$$

4.2 Supply side

On the supply side, the operator is interested in the numbers of serviced and lost customers, of waiting customers, of idle docks and idle vehicles, all of which are random variables so that both the expected value and the standard deviation are important. Furthermore, in a yield management setting, the above mentioned customer indicators characterize the quality of service so they interest the service operator in an indirect way. A synthetic indicator of quality is the probability of unrestrained availability, denoted $\overline{\alpha} \equiv 1-\alpha^- - \alpha^+$. In the stationary regime,

$$\overline{\alpha} = \frac{\varphi-\varphi^\kappa}{(1-\varphi)[\frac{1}{1-\rho}+\frac{\varphi-\varphi^\kappa}{1-\varphi}+\frac{\varphi^\kappa}{1-\sigma}]}. \tag{30}$$

By time period of length H, the average numbers of accessing customers are of $\lambda H(\alpha^- r+1-\alpha^-)$ serviced customers and of $\lambda H\alpha^-(1-r)$ lost ones: these do depend explicitly on the quality of service at the local level of the station, and they would do so even more at the level of the network as the users would choose their stations. As the arrival flow is Markovian, making the next arrival independent from the current system state, both flows are Poisson-distributed stochastic processes, which yields their respective variance that is equal to their respective average. So their covariance is zero, as the halved difference between the variance of the sum and the sum of the respective variances.

Similarly, egressing customers either serviced or lost are Poisson random variables with respective parameter $\mu H(1-\alpha^+ + \alpha^+ s)$ and $\mu H\alpha^+(1-s)$.

Waiting customers on access are present only in the vehicle shortage domain. Their number, $C_{\overline{N}\leq 0}^-$, is a geometric random variable with parameter ρ. Thus the average and the variance are the following:

$$\text{E}[C_{\overline{N}\leq 0}^-] = \rho/(1-\rho) \text{ and } \text{V}[C_{\overline{N}\leq 0}^-] = \rho/(1-\rho)^2. \tag{31a, b}$$

Similarly, waiting customers on egress are present only in the case of dock saturation. Thus their number, $C_{N\geq\kappa}^+ \equiv (N-\kappa)^+$, is a geometric random variable with parameter σ, yielding

$$\mathrm{E}[C^{+}_{N\geq\kappa}]=\sigma/(1-\sigma) \text{ and } \mathrm{V}[C^{+}_{N\geq\kappa}]=\sigma/(1-\sigma)^{2}. \qquad (32a, b)$$

About dock utilization or idling, the number of busy docks is $\widetilde{D}\equiv\min\{(N)^{+},\kappa\}$, with average value of

$$\mathrm{E}[\widetilde{D}]=\alpha^{-}.0+\overline{\alpha}.\mathrm{E}[N_{\in]1,\kappa[}]+\alpha^{+}\kappa, \qquad (33)$$

Wherein the average number of busy docks conditional on availability is (Cf. appendix)

$$\mathrm{E}[N_{\in]1,\kappa[}]=\frac{\varphi}{1-\varphi}\,\frac{(\kappa-1)\varphi^{\kappa}-\kappa\varphi^{\kappa-1}+1}{\varphi-\varphi^{\kappa}}. \qquad (34)$$

The number of idle docks is $D\equiv\max\{\kappa-(n)^{+},0\}$, with average value of

$$\mathrm{E}[D]=\alpha^{-}.\kappa+\overline{\alpha}.(\kappa-\mathrm{E}[N_{\in]1,\kappa[}]). \qquad (35)$$

Vehicle utilization depends primarily on their trips between stations. In the station setting, the number of idle vehicles is $(N)^{+}$, with average value as follows:

$$\mathrm{E}[(N)^{+}]=\overline{\alpha}.\mathrm{E}[N_{\in]1,\kappa[}]+\alpha^{+}(\kappa+\mathrm{E}[C^{+}_{N\geq\kappa}]). \qquad (36)$$

However it may be relevant to distinguish between docked idle vehicles, in number of $\min\{(n)^{+},\kappa\}$, and undocked vehicles still utilized by a waiting customer, in number of $(n-\kappa)^{+}$, since the former type can be refilled in the case of an electric vehicle with a charging infrastructure at the dock, whereas the latter may give rise to additional revenue depending on the tariff policy.

Variances can be derived easily from the definition of the random variables.

5. Sensitivity analysis and illustration

Let us now investigate the sensitivity of the main model outcomes to the model parameters by qualitative assessment on the basis of the analytical formulae as well as by numerical application. The parameters of interest include the docking capacity which indicates the station size, the arrival intensities (λ and μ) and the acceptance rates (r and s).

5.1 Influences of the docking capacity

It is shown in Appendix that, given all of the other parameters hence given ρ, φ and σ, factor κ has a decreasing effect on the probability of vehicle shortage, α^{-}, as well as on the probability of dock saturation, α^{+}. So the availability probability, $\overline{\alpha}=1-\alpha^{-}-\alpha^{+}$, increases with κ. This is illustrated in Figure 3 for two sets of couples (r,s).

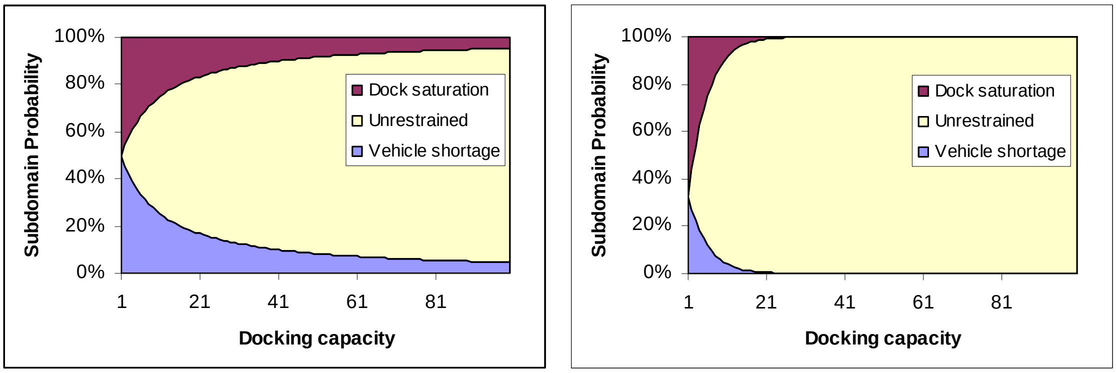

Fig. 3. Probability vs. capacity, with (a) $\lambda=\mu=1$, $r=s=.8$, *(b)* $\lambda=.8$, $\mu=1$, $r=s=.6$.

The customer flows, either for access or egress, do not depend on factor κ. Neither do the conditional average wait times for either a vehicle or a dock. The unconditional average wait times, however, do depend on κ through the subdomain probabilities: the wait time for a vehicle, $\alpha^{-}\,\mathrm{E}[w^{-}_{N\le 0}]$, decreases with κ and so does the wait time for a dock, $\alpha^{+}\,\mathrm{E}[w^{+}_{N\ge\kappa}]$.

The same applies to the average stocks of waiting customers, $\alpha^{-}\,\mathrm{E}[C^{-}_{N\le 0}]$ on access and $\alpha^{+}\,\mathrm{E}[C^{+}_{N\ge\kappa}]$ on egress, both of which decrease with κ owing only to its influence on the subdomain probabilities.

The influences on idle or busy resources are as follows (cf. Appendix):

- factor κ increases the average number of busy docks conditional on the availability domain, and also that conditional on dock saturation which is κ itself. Its effect on the overall average is increasing at least if $\varphi<1$.
- The conditional average numbers of idle vehicles, conditional either on availability or dock saturation, increase with κ. The overall average increases with κ at least if $\varphi<1$.
- Factor κ decreases the average numbers of idle docks, both that conditional on availability and the unconditional one.

Figure 4 depicts the effect on the average numbers of busy docks and idle vehicles.

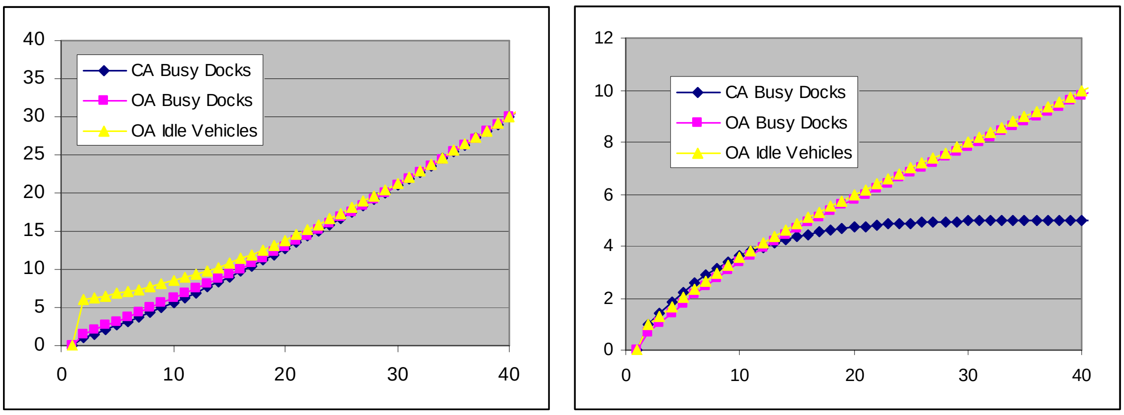

Fig. 4. Busy docks and idle vehicles w.r.t. capacity, with $\lambda/\mu=$ *10/11 (a) or 5/4 (b).*

5.2 Influences of the customer flow intensities

About the intensities of the customer flows, λ and μ, two kinds of influences should be distinguished: first there is a size effect because some conditional outcomes are proportional to these factors; second, the ratio between λ and μ, hence their relative magnitude, determines the derived parameters ρ, φ and σ, hence the stationary probabilities.

The size effect is proportional on each customer flow either serviced or lost, and inversely proportional onto the conditional average wait times for vehicles or docks (i.e. $1/(\mu-\lambda r)$ and $1/(\lambda-\mu s)$, respectively). It is neutral on conditional average numbers of waiting customers as well as on subdomain probabilities.

Relative magnitude exerts more profound effects. First, factor λ/μ must satisfy the compatibility conditions of vehicle shortage boundedness (i.e. $\mu-\lambda r>0$ hence $\lambda/\mu<1/r$) and of dock saturation boundedness (i.e. $\lambda-\mu s>0$ hence $\lambda/\mu>s$). This yields a compatibility interval of $]1/r, s[$ which is non-empty if both r and s are strictly less than one (so ensuring that the unit value belongs to the interval).

Second, relative magnitude influences the subdomain probabilities through the derived parameters $\varphi=\mu/\lambda$, $\rho=r\lambda/\mu$ and $s=s\mu/\lambda$. In Appendix it is shown that α^- increases with σ but decreases with ρ and φ, so the relative magnitude has two increasing influences and a decreasing one on α^-. As regards α^+, it decreases with σ and increases with φ and ρ. The availability probability $\overline{\alpha}$ increases with ρ, φ and σ; however this yields a controversial overall effect of λ/μ, too. Figure 5 suggests that the overall influence of λ/μ on α^- is increasing, whereas those on $\overline{\alpha}$ and α^+ are varying: but due to the symmetry between the two sides, in fact even the former influence is likely to vary.

Third, given μ, the influence of λ/μ on conditional average wait time is increasing for vehicles (as $\mu-\lambda r=\mu(1-r\lambda/\mu)$) but decreasing for docks. As concerns the conditional average numbers of waiting customers, the effect is decreasing on the waiting pedestrians (since $\rho/(1-\rho)$ increases with ρ) but increasing on the customers waiting for egress: Cf. figures 6 and 7. Fourth, about the influence on the average number of busy docks (ANBD) conditional on availability: a larger φ makes larger state values more likely and an increase in φ is even more favorable to them, so the conditional ANBD increases with φ and decreases with λ/μ.

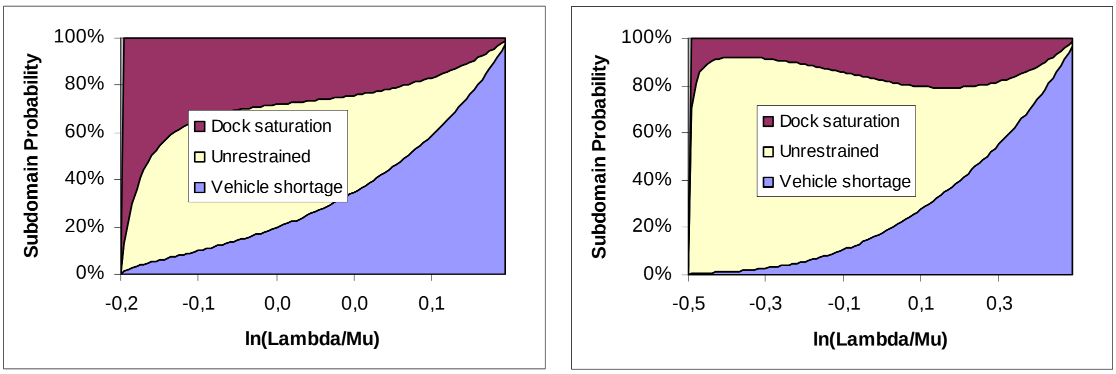

Fig. 5. Probability vs. λ/μ, *with (a)* $r=s=.8$, *(b)* $r=s=.6$.

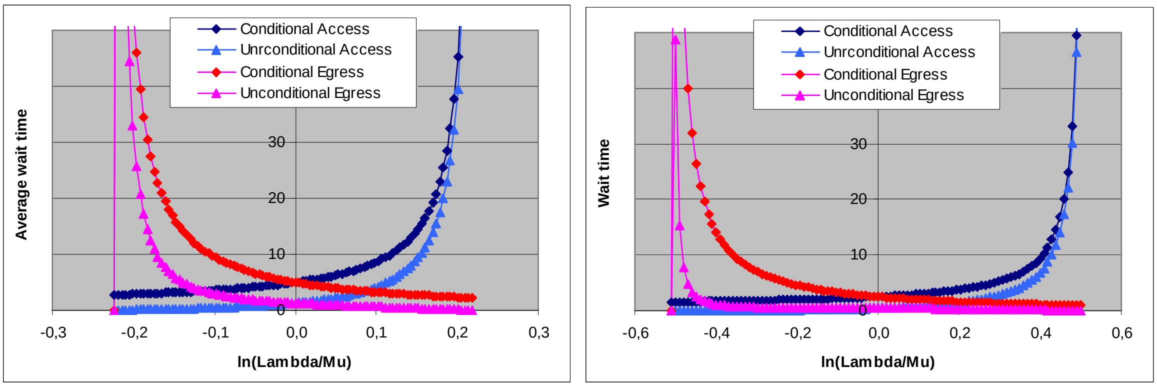

Fig. 6. Wait Time vs. λ / μ, *with (a)* $r = s = .8$, *(b)* $r = s = .6$.

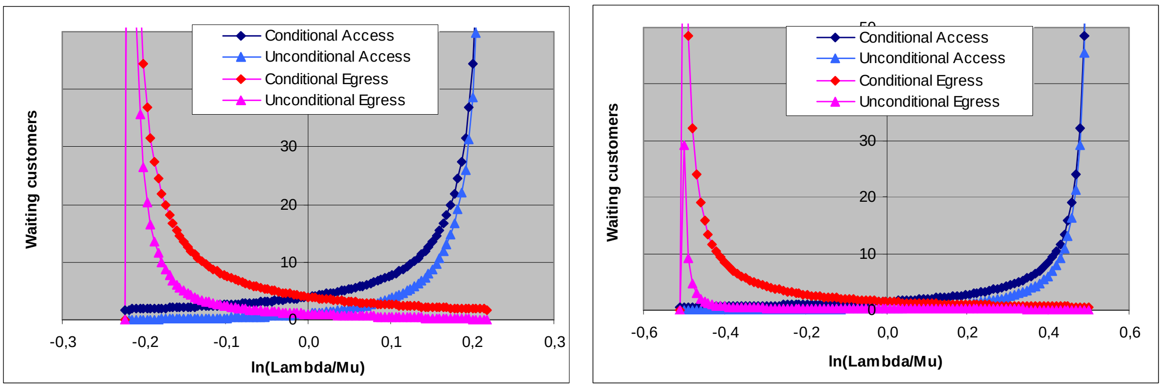

Fig. 7. Waiting Customers vs. λ / μ, *with (a)* $r = s = .8$, *(b)* $r = s = .6$.

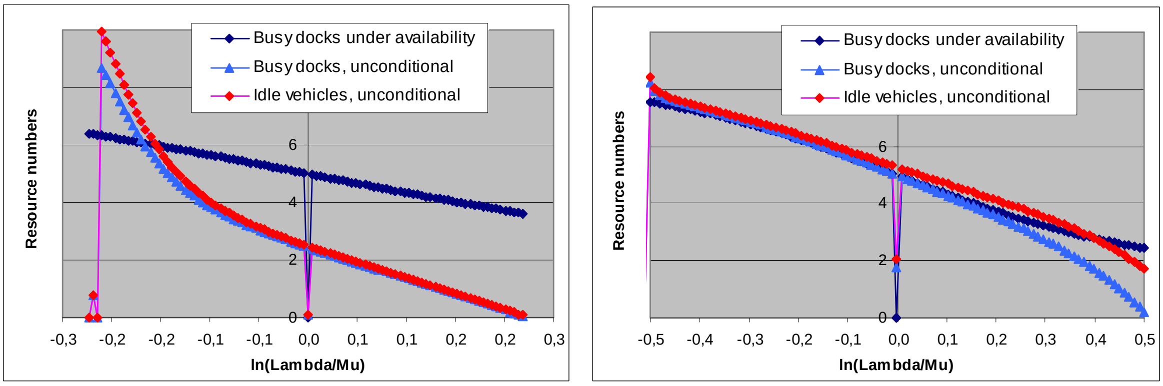

Fig. 8. Resource numbers vs. λ / μ, *with* $\kappa = 10$ *and (a)* $r = s = .8$, *(b)* $r = s = .6$.

5.3 Influences of the acceptance rates

The acceptance rates r and s shape the compatibility domain of the (λ, μ) pair. Within that domain, access acceptance r (resp. egress acceptance s) exerts a decreasing effect on $\mu - \lambda r$ (resp. on $\lambda - \mu s$), hence an increasing influence on the conditional average wait time for a vehicle, $1/(\mu - \lambda r)$ (resp. wait time for a dock, $1/(\lambda - \mu s)$).

Furthermore, through ρ that is proportional to it, factor r has an increasing effect on the conditional average number of customers waiting for access and a neutral one on that of customers waiting for egress. There is no effect on resource stocks conditional on availability. The effects on the subdomain probabilities are deduced from those of ρ, namely increasing onto α^{+} and $\overline{\alpha}$ but decreasing onto α^{-}.

All of these arguments can be transposed to s through σ, yielding an increasing influence on the conditional average number of customers waiting on egress, increasing influences onto α^- and $\overline{\alpha}$ but decreasing onto α^+.

6. Conclusion

A waiting system with two kinds of resources, vehicles versus docks at a station of vehicle-sharing, and two kinds of customers, each kind contributing to supply the other one with the required resource, has been modelled in a markovian setting. The state variable is the number of waiting vehicles, extended to negative values to account for waiting customers that require a vehicle.

The state-transition model has been specified, its balance equations formulated and solved for the stationary probability distribution, in other words for stochastic equilibrium. Three sub-domains are meaningful for the value of the state variable: non-positive values are associated to vehicle shortage, values greater than docking capacity stand for dock saturation, whereas the sub-domain of unrestrained availability lies between the two constrained sub-domains. The stationary distribution has a simple, geometric form in each sub-domain, thus yielding simple formulae for the conditional outcomes of numbers of waiting customers, numbers of idle resources and average waiting time.

These macroscopic properties may enable one to use the station model as a sub-model in a network model of the system that would address the origin-destination flows between stations. Further research may also be targeted to the refinement of two underlying, simplifying assumptions. First, on the customer side, the acceptance of waiting has been modelled by neglecting the influence of the waiting stock, although it is likely to determine the length of waiting; furthermore, a first in – first out service discipline has been assumed, which is realistic enough for systems with booking but less so for a system where waiting customers are mingling rather than queuing. Second, on the operator side the effects of operational policy, including notably station balancing, could be integrated in the state-transition model.

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8. Appendix on model outcomes

8.1 Decomposition of total variance

The variance of the conditional wait time $w_{\bar{N}\leq 0}$ is composed of two parts respectively intraclass and interclass. The intraclass part amounts to

$$\begin{aligned} \mathrm{V}_{\mathrm{intra}}[w_{\bar{N}\leq 0}] &= \frac{\sum_{k\geq 0} p_{-k}\,\mathrm{V}[w_{\bar{k}}]}{\sum_{k\geq 0} p_{-k}} = (1-\rho)\sum_{k\geq 0}\rho^k(k+1)\mu^{-2} \\ &= (1-\rho)\mu^{-2}/(1-\rho)^2 = \mu^{-2}/(1-\rho) \end{aligned}$$

Now, about the interclass part,

$$\begin{aligned} \mathrm{V}_{\mathrm{inter}}[w_{\bar{N}\leq 0}] &= \frac{\sum_{k\geq 0} p_{-k}(\mathrm{E}[w_{\bar{k}}]-\mathrm{E}[w_{\bar{N}\leq 0}])^2}{\sum_{k\geq 0} p_{-k}} \\ &= (1-\rho)\left(\sum_{k\geq 0}\rho^k(k+1)^2\mu^{-2}\right) - \mathrm{E}[w_{\bar{N}\leq 0}]^2 \\ &= \mu^{-2}(1-\rho)\left(\rho\sum_{k\geq 0}\rho^{k-1}(k+1)k + \sum_{k\geq 0}\rho^k(k+1)\right) - \mathrm{E}[w_{\bar{N}\leq 0}]^2 \\ &= \mu^{-2}(1-\rho)\left(\frac{2\rho}{(1-\rho)^3}+\frac{1}{(1-\rho)^2}\right) - \mathrm{E}[w_{\bar{N}\leq 0}]^2 \\ &= \mu^{-2}\left(\frac{2\rho}{(1-\rho)^2}+\frac{1}{1-\rho}\right) - (\mu(1-\rho))^{-2} \end{aligned}$$

So the total variance is

$$\begin{aligned}\mathrm{V_{intra}}[w_{\bar{N}\le 0}]+\mathrm{V_{inter}}[w_{\bar{N}\le 0}] &= \mu^{-2}\frac{1}{(1-\rho)}+\mu^{-2}\left(\frac{2\rho}{(1-\rho)^2}+\frac{1}{1-\rho}\right)-(\mu(1-\rho))^{-2} \\ &= \mu^{-2}(1-\rho)^{-2}[2(1-\rho)+2\rho-1] \\ &= [\mu(1-\rho)]^{-2}\end{aligned}$$

8.2 Exponential distribution of conditional wait time

Knowing that $N \le 0$, the conditional probability of state $-k$ (for $k \ge 0$) is $\pi_k = (1-\rho)\rho^k$. Knowing k, the wait time is the sum of $(k+1)$ exponential variables that are independent and identically distributed with parameter μ. So the Laplace transform of w_k^- is

$$W_k^-(\xi) = (1+\frac{\xi}{\mu})^{-k-1}.$$

The wait time conditional on $N \le 0$, $w_{\bar{N}\le 0}$, is the probabilistic mixture of the state-conditional wait time: so its Laplace transform is

$$\begin{aligned}W_{\bar{N}\le 0}(\xi) &= \textstyle\sum_{k\ge 0}\pi_k W_k^-(\xi) = \frac{1-\rho}{1+\xi/\mu}\sum_{k\ge 0}\rho^k(1+\xi/\mu)^{-k} \\ &= \frac{1-\rho}{1+\xi/\mu}\frac{1}{1-\rho(1+\xi/\mu)} = [1+\frac{\xi}{\mu(1-\rho)}]^{-1}\end{aligned}.$$

By identification, the last expression shows that $w_{\bar{N}\le 0}$ is an exponential random variable of parameter $\mu(1-\rho)$. This comes as no surprise, since the geometric law is a memoryless discrete distribution whereas the exponential is a memoryless continuous distribution: a memoryless mixture of memoryless continuous components yields a memoryless continuous distribution, thus an exponential one.

8.3 Resource number within availability domain

A system state $k \in]1,\kappa[$ has stationary probability $p_0\varphi^k$. The subdomain probability is $\bar{\alpha} = p_0\frac{\varphi-\varphi^\kappa}{1-\varphi}$.

The average number of available vehicles or, equivalently in that domain, of busy docks is

$$\begin{aligned}\mathrm{E}[N_{\in]1,\kappa[}] &= \bar{\alpha}^{-1}\textstyle\sum_{k=1}^{\kappa-1}p_0\varphi^k k = p_0\bar{\alpha}^{-1}\varphi\sum_{k=1}^{\kappa-1}\varphi^{k-1}k \\ &= p_0\bar{\alpha}^{-1}\varphi\frac{\mathrm{d}}{\mathrm{d}\varphi}\frac{1-\varphi^\kappa}{1-\varphi} \\ &= \frac{1-\varphi}{\varphi-\varphi^\kappa}\varphi\frac{(\kappa-1)\varphi^\kappa-\kappa\varphi^{\kappa-1}+1}{(1-\varphi)^2} \\ &= \frac{\varphi}{1-\varphi}\,\frac{(\kappa-1)\varphi^\kappa-\kappa\varphi^{\kappa-1}+1}{\varphi-\varphi^\kappa}\end{aligned}.$$

From this stems the number of idle docks, $\mathrm{E}[D_{\in]1,\kappa[}] = N - \mathrm{E}[N_{\in]1,\kappa[}]$.

When $\varphi < 1$, the limit value for large κ is $\mathrm{E}[N_{\in]1,\kappa[}] = 1/(1-\varphi) + \mathrm{o}(1)$.

When $\varphi > 1$, a series development for large κ is $\mathrm{E}[N_{\in]1,\kappa[}] = \kappa - \varphi/(\varphi - 1) + \mathrm{o}(1)$.

9. Appendix on sensitivity analysis

9.1 Subdomain probabilities

Let us define $A = 1/(1-\rho)$, $B = (\varphi - \varphi^{\kappa})/(1-\varphi)$ and $C = \varphi^{\kappa}/(1-\sigma)$, so that the main stationary probabilities are constituted in the following way:

$p_0 = 1/(A+B+C)$,

$\alpha^- = A/(A+B+C)$,

$\bar{\alpha} = B/(A+B+C)$,

$\alpha^+ = C/(A+B+C)$.

Factor ρ influences A in a decreasing way and is neutral on B and C. So $A+B+C$ decreases with ρ but relatively less than A alone (the three terms being positive). So α^- decreases with ρ whereas $\bar{\alpha}$ and α^+ increase with ρ.

Similarly, factor σ influences only C so that α^+ decreases with ρ whereas α^- and $\bar{\alpha}$ increase with it.

Factor φ exerts a twofold effect through B and C. Term B amounts to $\sum_{k=1}^{\kappa-1} \varphi^k$ so it increases with φ, as well as with κ. Also C increases with φ. So does $A+B+C$. The total effect on α^- is decreasing. Those on $\bar{\alpha}$ and α^+ are less obvious because they involve the relative values of B and C.

9.2 Dock saturation versus dock capacity

Let us study α^+ by restating that

$$\frac{1}{(1-\sigma)\alpha^+} = F + E\varphi^{-\kappa}, \text{ where } F \equiv \frac{1}{1-\sigma} - \frac{1}{1-\varphi} \text{ and } E \equiv \frac{\rho}{1-\rho} + \frac{1}{1-\varphi}.$$

At $\varphi > 1$, $F > 0$ and $E < 0$ as it is the quotient of $\rho(1-\varphi) + 1 - \rho = 1 - r$, which is positive, by $(1-\varphi)(1-\rho)$ which is negative. As $\varphi^{-\kappa}$ decreases with κ, $F + E\varphi^{-\kappa}$ increases with κ and is positive, making α^+ a decreasing function of κ at $\varphi > 1$.

At $\varphi < 1$, $F < 0$ as it is the quotient of $(1-\varphi) - (1-\sigma) = (\sigma - 1)\varphi$, which is negative, by $(1-\varphi)(1-\sigma)$ which is positive. Conversely, $E > 0$ and $\varphi^{-\kappa}$ increases with κ, making $F + E\varphi^{-\kappa}$ increase with κ at $\varphi < 1$, as well as at $\varphi > 1$. So α^+ is a decreasing function of κ overall.

In the previous subsection, a decreasing influence of κ onto B has been pointed out ; the influence onto C is decreasing also if $\varphi < 1$ but increasing if $\varphi > 1$. Thus, if $\varphi < 1$, α^- increases with κ and $\bar{\alpha} = 1 - \alpha^- - \alpha^+$ decreases with κ.

9.3 Busy docks versus dock capacity

Within the availability domain, the conditional average number of busy docks (hence of idle vehicles) is formulated as

$$\mathrm{E}[N_{\in]1,\kappa[}] = (\textstyle\sum_{k=1}^{\kappa-1} \varphi^k k) / (\sum_{k=1}^{\kappa-1} \varphi^k).$$

It is a weighted average of the state values between 1 and $\kappa - 1$. An increment in κ will assign some probability to the rightwing extreme value, which is larger than the other state values involved previously, yielding a larger average value.

Let us turn to the overall number of busy vehicles, $(N)^+$, with average formulated as

$$\mathrm{E}[(N)^+] = \overline{\alpha}.\mathrm{E}[N_{\in]1,\kappa[}] + \alpha^+(\kappa + \mathrm{E}[C^+_{N \geq \kappa}]).$$

It is comprised of two parts. The part related to availability increases with κ since both $\overline{\alpha}$ and the conditional average number of busy docks increase with κ. About the other part, as α^+ decreases with κ the behaviour must be studied in detail. Let

$\dfrac{1}{\kappa\alpha^+(1-\sigma)} = \dfrac{F}{\kappa} + \dfrac{E}{\kappa}\varphi^{-\kappa}$, wherein $F \equiv \dfrac{1}{1-\sigma} - \dfrac{1}{1-\varphi}$ and $E \equiv \dfrac{\rho}{1-\rho} + \dfrac{1}{1-\varphi}$ as previously.

The partial derivative with respect to κ is

$$\frac{\partial}{\partial\kappa}\frac{1}{\kappa\alpha^+(1-\sigma)} = -\frac{1}{\kappa}\left(\frac{1-\sigma}{\kappa\alpha^+}\right) - E\varphi^{-\kappa-1}.$$

At $\varphi < 1$, $E > 0$ so the derivative is negative, which implies that $\kappa\alpha^+$ increases with κ.