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# Bruhat order on plane posets and applications

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ABSTRACT. A plane poset is a finite set with two partial orders, satisfying a certain incompatibility condition. The set  $\mathcal{PP}$  of isoclasses of plane posets owns two products, and an infinitesimal Hopf algebra structure is defined on the vector space  $\mathcal{H}_{\mathcal{PP}}$  generated by  $\mathcal{PP}$ , using the notion of biideals of plane posets.

We here define a partial order on  $\mathcal{PP}$ , making it isomorphic to the set of partitions with the weak Bruhat order. We prove that this order is compatible with both products of  $\mathcal{PP}$ ; moreover, it encodes a non degenerate Hopf pairing on the infinitesimal Hopf algebra  $\mathcal{H}_{\mathcal{PP}}$ .

Keywords. Plane posets; weak Bruhat order; infinitesimal Hopf algebras.

AMS classification. 06A11, 16W30, 06A07.

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## Introduction

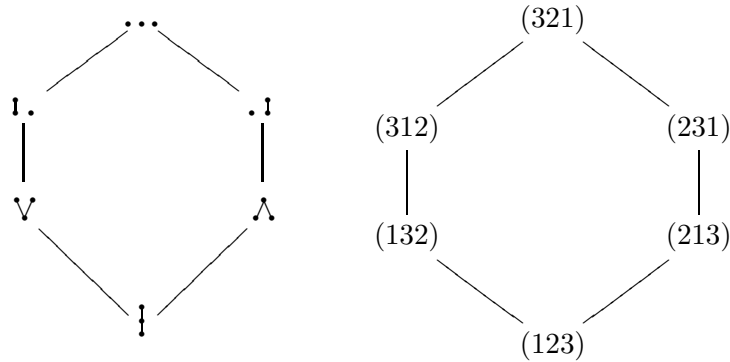
A *double poset* is a finite set with two partial orders. The set of (isoclasses of) double posets owns several algebraic structures, as:

- a product called *composition*; it corresponds, roughly speaking, to the concatenation of Hasse graphs.
- a coproduct, defined with the notion of *ideals* for the first partial order. One obtains in this way the Malvenuto-Reutenauer Hopf algebra of double posets [13].

- a pairing defined with the help of Zelevinsky *pictures*. It is shown in [13] that this pairing is Hopf; consequently, the Hopf algebra of double posets is free, cofree and self-dual.

This Hopf algebra also contains many interesting subobjects, as, for example, the Hopf algebra of *special posets*, that is to say double posets such that the second partial order is total, the Hopf algebra of plane posets [4, 8], that is to say double posets such that the two partial orders satisfy an incompatibility condition (see definition 1 below), or the noncommutative Hopf algebra of plane trees [5, 6, 9], also known as the noncommutative Connes-Kreimer Hopf algebra. In particular, the Hopf subalgebra of plane poset turns out to be isomorphic to the Hopf algebra of permutations introduced by Malvenuto and Reutenauer in [12], also known as the Hopf algebra of free quasi-symmetric functions [1, 2]. An explicit isomorphism can be defined with the help of a bijection  $\Psi_n$  between the set of plane posets on  $n$  vertices and the symmetric group on  $n$  letters, recalled here in theorem 3. This isomorphism and its applications are studied in [4].

We proceed here with the algebraic study of the links between permutations and plane posets. As the symmetric group  $\mathfrak{S}_n$  is partially ordered by the weak Bruhat order, via the bijection  $\Psi_n$  the set of plane posets is also partially ordered. This order has a nice combinatorial description, see definition 8. It admits a decreasing bijection  $\iota$ , given by the exchange of the two partial orders defining plane posets; on the permutation side, this bijection consists of reversing the words representing the permutations. For example, let us give the Hasse graph of this partial order restricted to plane posets of degree 3, and the Hasse graph of the weak Bruhat order on  $\mathfrak{S}_3$ :



This partial order is related to an infinitesimal Hopf algebra structure on plane posets. Recall that an infinitesimal Hopf algebra  $\mathcal{H}$  [10, 11] is both an algebra and a coalgebra, satisfying the following compatibility: if  $x, y \in \mathcal{H}$ ,

$$\Delta(xy) = \Delta(x)(1 \otimes y) + (x \otimes 1)\Delta(y) - x \otimes y.$$

For a certain coproduct  $\Delta_1$ , given by biideals, the space of plane posets  $\mathcal{H}_{\mathcal{PP}}$  becomes an infinitesimal Hopf algebra for two products, the composition  $m$  and the transformation  $\zeta$  of  $m$  by  $\iota$ . This coproduct is a special case of the four-parameters deformation of [3]. This structure is also self-dual, with an explicit Hopf pairing  $\langle -, - \rangle_1$  (theorem 23). This pairing is related to the partial Bruhat order in the following way: if  $P, Q$  are two plane posets,

$$\langle P, Q \rangle_1 = \begin{cases} 1 & \text{if } \iota(P) \leq Q, \\ 0 & \text{if not.} \end{cases}$$

All these results admit a one parameter deformation, which is given in this text.

The text is organised as follows: the first section deals with double and plane posets: after some recalls, we give the definition of the infinitesimal coproduct and its one-parameter deformation. The partial order on plane posets is defined in the second section; the isomorphism with the weak Bruhat order is also proved. In the last section, the infinitesimal Hopf algebra structure

and the partial order are related via the definition of a Hopf pairing.

**Notations.**  $K$  is a commutative field. All the vector spaces, algebras, coalgebras, ... of this text are taken over  $K$ .

## 1 Double and plane posets

### 1.1 Reminders

- Definition 1** 1. [13] A double poset is a finite set  $P$  with two partial orders  $\leq_h$  and  $\leq_r$ .
2. A plane poset is a double poset  $P$  such that for all  $x, y \in P$ , such that  $x \neq y$ ,  $x$  and  $y$  are comparable for  $\leq_h$  if, and only if,  $x$  and  $y$  are not comparable for  $\leq_r$ . The set of isoclasses of plane posets will be denoted by  $\mathcal{PP}$ . For all  $n \in \mathbb{N}$ , the set of isoclasses of plane posets of cardinality  $n$  will be denoted by  $\mathcal{PP}(n)$ .
3. Let  $P, Q \in \mathcal{PP}$ . We shall say that  $P$  is a plane subposet of  $Q$  if  $P \subseteq Q$  and if the two partial orders of  $P$  are the restriction of the two partial orders of  $Q$  to  $P$ .

**Examples.** Here are the plane posets of cardinal  $\leq 4$ . They are given by the Hasse graph of  $\leq_h$ ; if  $x$  and  $y$  are two vertices of this graph which are not comparable for  $\leq_h$ , then  $x \leq_r y$  if  $y$  is more on the right than  $x$ .

$$\begin{aligned} \mathcal{PP}(0) &= \{\emptyset\}, \\ \mathcal{PP}(1) &= \{.\}, \\ \mathcal{PP}(2) &= \{., 1\}, \\ \mathcal{PP}(3) &= \{\dots, .1, 1., \vee, \uparrow, \wedge\}, \\ \mathcal{PP}(4) &= \left\{ \begin{array}{l} \dots, .1, .1., 1\dots, \vee, \vee., .\uparrow, \uparrow., \wedge, \wedge., 11, \\ \Psi, \downarrow, \downarrow, \Upsilon, \uparrow, \blacktriangle, \uparrow, \blacktriangle, \blacktriangle, \mathcal{N}, \mathcal{N}, \mathcal{N}, \diamond \end{array} \right\}. \end{aligned}$$

The following proposition is proved in [8] (proposition 11):

**Proposition 2** Let  $P \in \mathcal{PP}$ . We define a relation  $\leq$  on  $P$  by:

$$(x \leq y) \text{ if } (x \leq_h y \text{ or } x \leq_r y).$$

Then  $\leq$  is a total order on  $P$ .

As a consequence, for any plane poset  $P \in \mathcal{PP}(n)$ , we shall assume that  $P = \{1, \dots, n\}$  as a totally ordered set.

The following theorem is proved in [4] (up to a passage to the inverse):

**Theorem 3** 1. Let  $\sigma \in \mathfrak{S}_n$ . We define a plane poset  $P_\sigma$  in the following way:

- $P_\sigma = \{1, \dots, n\}$  as a set.
- If  $i, j \in P_\sigma$ ,  $i \leq_h j$  if  $i \leq j$  and  $\sigma^{-1}(i) \leq \sigma^{-1}(j)$ .
- If  $i, j \in P_\sigma$ ,  $i \leq_r j$  if  $i \leq j$  and  $\sigma^{-1}(i) \geq \sigma^{-1}(j)$ .

The total order on  $\{1, \dots, n\}$  induced by this plane poset structure is the usual one.

2. For all  $n \geq 0$ , the following map is a bijection:

$$\Psi_n : \begin{cases} \mathfrak{S}_n & \longrightarrow \mathcal{PP}(n) \\ \sigma & \longrightarrow P_\sigma. \end{cases}$$

**Examples.**

$$\begin{array}{lll}
\Psi((1)) = \bullet, & \Psi_2((12)) = \mathbf{1}, & \Psi_2((21)) = \bullet\bullet, \\
\Psi_3((123)) = \mathbf{1}, & \Psi_3((132)) = \mathbf{V}, & \Psi_3((213)) = \mathbf{\Lambda}, \\
\Psi_3((231)) = \bullet\mathbf{1}, & \Psi_3((312)) = \mathbf{1}\bullet, & \Psi_3((321)) = \bullet\bullet\bullet, \\
\Psi_4((1234)) = \mathbf{1}, & \Psi_4((1243)) = \mathbf{Y}, & \Psi_4((1342)) = \mathbf{V}, \\
\Psi_4((1324)) = \mathbf{1}, & \Psi_4((1423)) = \mathbf{V}, & \Psi_4((1432)) = \mathbf{V}, \\
\Psi_4((2134)) = \mathbf{\Lambda}, & \Psi_4((2143)) = \mathbf{N}, & \Psi_4((2314)) = \mathbf{1}\mathbf{\Lambda}, \\
\Psi_4((2341)) = \mathbf{1}\bullet, & \Psi_4((2413)) = \mathbf{U}, & \Psi_4((2431)) = \mathbf{V}\bullet, \\
\Psi_4((3124)) = \mathbf{1}, & \Psi_4((3142)) = \mathbf{N}, & \Psi_4((3214)) = \mathbf{\Lambda}, \\
\Psi_4((3241)) = \mathbf{\Lambda}\bullet, & \Psi_4((3412)) = \mathbf{1}\mathbf{1}, & \Psi_4((3421)) = \mathbf{1}\bullet\bullet, \\
\Psi_4((4123)) = \bullet\mathbf{1}, & \Psi_4((4132)) = \bullet\mathbf{V}, & \Psi_4((4213)) = \bullet\mathbf{\Lambda}, \\
\Psi_4((4231)) = \bullet\mathbf{1}\bullet, & \Psi_4((4312)) = \bullet\bullet\mathbf{1}, & \Psi_4((4321)) = \bullet\bullet\bullet\bullet.
\end{array}$$

We shall use three particular families of plane posets:

**Definition 4** Let  $P \in \mathcal{PP}$ .

1. We shall say that  $P$  is a plane forest if it does not contain  $\mathbf{\Lambda}$  as a plane subposet. The set of plane forests is denoted by  $\mathcal{PF}$ .
2. We shall say that  $P$  is a plane forest if it does not contain  $\mathbf{\Lambda}$  as a plane subposet. The set of plane forests will be denoted by  $\mathcal{PF}$ , and the set of plane forests of cardinality  $n$  will be denoted by  $\mathcal{PF}(n)$ .

**Remark.** In other words, a plane poset is a plane forest if, and only if, its Hasse graph is a rooted forest.

## 1.2 Algebraic structures on plane posets

We define two products on  $\mathcal{PP}$ . The first is called *composition* in [13] and is denoted by  $\rightsquigarrow$  in [8]. We shall shortly denote it by  $m$  in this text.

**Definition 5** Let  $P, Q \in \mathcal{PP}$ .

1. The double poset  $PQ$  is defined as follows:
  - $PQ = P \sqcup Q$  as a set, and  $P, Q$  are plane subposets of  $PQ$ .
  - For all  $x \in P$ , for all  $y \in Q$ ,  $x \leq_r y$ .
2. The double poset  $P \dot{\lrcorner} Q$  is defined as follows:
  - $P \dot{\lrcorner} Q = P \sqcup Q$  as a set, and  $P, Q$  are plane subposets of  $PQ$ .
  - For all  $x \in P$ , for all  $y \in Q$ ,  $x \leq_h y$ .

**Examples.**

1. The Hasse graph of  $PQ$  is the concatenation of the Hasse graphs of  $P$  and  $Q$ .
2. Here are examples for  $\dot{\lrcorner}$ :  $\bullet \dot{\lrcorner} \mathbf{1} = \mathbf{1}$ ,  $\mathbf{1} \dot{\lrcorner} \bullet = \mathbf{1}$ ,  $\bullet \dot{\lrcorner} \bullet\bullet = \mathbf{V}$ ,  $\bullet\bullet \dot{\lrcorner} \bullet = \mathbf{\Lambda}$ .

The vector space generated by  $\mathcal{PP}$  is denoted by  $\mathcal{H}_{\mathcal{PP}}$ . These two products are linearly extended to  $\mathcal{H}_{\mathcal{PP}}$ ; then  $(\mathcal{H}_{\mathcal{PP}}, \cdot)$  and  $(\mathcal{H}_{\mathcal{PP}}, \dot{\lrcorner})$  are two associative, unitary algebras, sharing the same unit  $\mathbf{1}$ , which is the empty plane poset. Moreover, they are both graded by the cardinality of plane posets.

### 1.3 Infinitesimal coproducts

**Definition 6** [13]. Let  $P = (P, \leq_h, \leq_r)$  be a plane poset, and let  $I \subseteq P$ .

1. We shall say that  $I$  is a  $h$ -ideal of  $P$ , if, for all  $x, y \in P$ :

$$(x \in I, x \leq_h y) \implies (y \in I).$$

2. We shall say that  $I$  is a  $r$ -ideal of  $P$ , if, for all  $x, y \in P$ :

$$(x \in I, x \leq_r y) \implies (y \in I).$$

3. We shall say that  $I$  is a biideal of  $P$  if it both a  $h$ -ideal and a  $r$ -ideal.

**Remark.** If  $P$  is a plane poset and  $I \subseteq P$ ,  $I$  is a biideal of  $P$  if, for all  $x, y \in P$ :

$$(x \in I, x \leq y) \implies (y \in I).$$

**Theorem 7** Let  $q \in K$ . We define a coproduct on  $\mathcal{H}_{\mathcal{PP}}$  in the following way: for all  $P \in \mathcal{PP}$ ,

$$\Delta_q(P) = \sum_{I \text{ biideal of } P} q^{h_{P \setminus I}^I} (P \setminus I) \otimes I,$$

where, for all  $I, J \subseteq P$ ,  $h_I^J = \{(x, y) \in I \times J \mid x <_h y\}$ . Then  $\Delta_q$  is coassociative and for all  $x, y \in \mathcal{H}_{\mathcal{PP}}$ , using Sweedler notations:

$$\begin{aligned} \Delta_q(xy) &= \sum x^{(1)} \otimes x^{(2)}y + \sum xy^{(1)} \otimes y^{(2)} - x \otimes y, \\ \Delta_q(x \dot{\downarrow} y) &= \sum q^{|x^{(1)}||y|} x^{(1)} \otimes x^{(2)} \dot{\downarrow} y + \sum q^{|x||y^{(2)}|} x \dot{\downarrow} y^{(2)} \otimes y^{(1)} - q^{|x||y|} x \otimes y. \end{aligned}$$

Hence,  $(\mathcal{H}_{\mathcal{PP}}, m, \Delta_q)$  is an infinitesimal Hopf algebra, as well as  $(\mathcal{H}_{\mathcal{PP}}, \dot{\downarrow}, \Delta_1)$ .

**Notation.** For all nonempty  $P \in \mathcal{PP}$ , we put  $\tilde{\Delta}_q(P) = \Delta_q(P) - P \otimes 1 - 1 \otimes P$ .

**Proof.** Let  $P$  be a double poset. Let  $I$  be a biideal of  $P$  and let  $J$  be a biideal of  $I$ ; then  $J$  is a biideal of  $P$ . Let  $I$  be a biideal of  $P$  and let  $J$  be a biideal of  $P \setminus I$ ; then  $I \sqcup J$  is a biideal of  $P$ . Hence:

$$\begin{aligned} (Id \otimes \Delta_q) \circ \Delta_q(P) &= \sum_{J \subseteq I \text{ biideals of } P} q^{h_{P \setminus I}^I + h_{I \setminus J}^J} P \setminus (I \cup J) \otimes J \setminus I \otimes I, \\ (\Delta_q \otimes Id) \circ \Delta_q(P) &= \sum_{J \subseteq I \text{ biideals of } P} q^{h_{P \setminus J}^J + h_{P \setminus I}^I} P \setminus (I \cup J) \otimes J \setminus I \otimes I. \end{aligned}$$

Moreover:

$$h_{P \setminus I}^I + h_{I \setminus J}^J = h_{P \setminus I}^J + h_{P \setminus I}^{I \setminus J} + h_{I \setminus J}^J = h_{P \setminus J}^J + h_{P \setminus I}^{I \setminus J},$$

so  $\Delta_q$  is coassociative. Let us prove the compatibility of the products and the coproducts. We restrict ourselves to  $x = P, y = Q \in \mathcal{PP}$ . The result is obvious if  $P = 1$  or  $Q = 1$ . Let us assume that  $P, Q \neq 1$ . The nontrivial biideals of  $PQ$  are the nontrivial biideals of  $Q$ ,  $Q$ , and the biideals

$IQ$ , where  $I$  is a nontrivial biideal of  $P$ . Consequently:

$$\begin{aligned}
\tilde{\Delta}_q(PQ) &= \sum_{I \text{ nontrivial biideal of } Q} q^{h_{PQ \setminus I}^I} PQ \setminus I \otimes I \\
&+ \sum_{I \text{ nontrivial biideal of } P} q^{h_{P \setminus I}^{IQ}} P \setminus I \otimes IQ + q^{h_P^Q} P \otimes Q \\
&= \sum_{I \text{ nontrivial biideal of } Q} q^{0+h_{Q \setminus I}^I} PQ \setminus I \otimes I \\
&+ \sum_{I \text{ nontrivial biideal of } P} q^{h_{P \setminus I}^{I+0}} P \setminus I \otimes IQ + q^0 P \otimes Q \\
\Delta_q(PQ) &= PQ \otimes 1 + 1 \otimes PQ + (P \otimes 1)\tilde{\Delta}_q(Q) + \tilde{\Delta}_q(P)(1 \otimes Q) + P \otimes Q \\
&= PQ \otimes 1 + 1 \otimes PQ + (P \otimes 1)\Delta_q(Q) - P \otimes Q - PQ \otimes 1 \\
&+ \Delta_q(P)(1 \otimes Q) - P \otimes Q - 1 \otimes PQ + P \otimes Q \\
&= (P \otimes 1)\Delta_q(Q) - P \otimes Q + \Delta_q(P)(1 \otimes Q) - P \otimes Q.
\end{aligned}$$

The nontrivial ideals of  $P \wr Q$  are:

- nontrivial ideals  $I$  of  $Q$ . In this case,

$$h_{P \wr Q \setminus I}^I = h_{Q \setminus I}^I + h_P^I = h_{Q \setminus I}^+ + |I| \cdot |P|.$$

- $Q$ . In this case:

$$h_P^Q = |P| \cdot |Q|.$$

- ideals of the form  $I \wr Q$ , where  $I$  is a nontrivial ideal of  $P$ . In this case:

$$h_{P \setminus I}^{I \wr Q} = h_{P \setminus I}^Q + h_{P \setminus I}^Q = h_{P \setminus I}^Q + |P \setminus I| \cdot |Q|.$$

$$\begin{aligned}
\tilde{\Delta}_q(P \wr Q) &= \sum_{I \text{ nontrivial biideal of } Q} q^{|I| \cdot |P|} q^{h_{PQ \setminus I}^I} P \wr Q \setminus I \otimes I \\
&+ \sum_{I \text{ nontrivial biideal of } P} q^{|P \setminus I| \cdot |Q|} q^{h_{P \setminus I}^{IQ}} P \setminus I \otimes I \wr Q + q^{|P| \cdot |Q|} q^{h_P^Q} P \otimes Q \\
\Delta_q(PQ) &= PQ \otimes 1 + 1 \otimes PQ + \sum q^{|P| \cdot |Q^{(1)}|} P \wr Q^{(1)} \otimes Q^{(2)} - P \wr Q \otimes 1 - q^{|P| \cdot |Q|} P \otimes Q \\
&+ \sum q^{|P^{(1)}| \cdot |Q|} P^{(1)} \otimes P^{(2)} \wr Q - 1 \otimes P \wr Q - q^{|P| \cdot |Q|} P \otimes Q + q^{|P| \cdot |Q|} P \otimes Q \\
&= \sum q^{|P| \cdot |Q^{(1)}|} P \wr Q^{(1)} \otimes Q^{(2)} + \sum q^{|P^{(1)}| \cdot |Q|} P^{(1)} \otimes P^{(2)} \wr Q - q^{|P| \cdot |Q|} P \otimes Q.
\end{aligned}$$

In particular, if  $q = 1$ , we recover the axiom of an infinitesimal Hopf algebra.  $\square$

### Remarks.

1. Obviously, both  $(\mathcal{H}_{\mathcal{P}\mathcal{P}}, m, \Delta_q)$  and  $(\mathcal{H}_{\mathcal{P}\mathcal{P}}, \wr, \tilde{\Delta}_q)$  are graded by the cardinality of the double posets.
2. The coproduct  $\Delta_q$  restricted to  $\mathcal{H}_{\mathcal{P}\mathcal{P}}$  is the coproduct  $\Delta_{(q,0,1,0)}$  of [3].

### Examples.

$$\begin{aligned}
\tilde{\Delta}_q(\mathbf{1}) &= \cdot \otimes \cdot, & \tilde{\Delta}_q(\cdot) &= q \cdot \otimes \cdot, \\
\tilde{\Delta}_q(\mathbf{\dot{1}}) &= \cdot \otimes \mathbf{1} + \mathbf{1} \otimes \cdot, & \tilde{\Delta}_q(\mathbf{V}) &= \cdot \otimes \cdot + q \mathbf{1} \otimes \cdot, \\
\tilde{\Delta}_q(\mathbf{\wedge}) &= q \cdot \otimes \mathbf{1} + \cdot \otimes \cdot, & \tilde{\Delta}_q(\mathbf{1} \cdot) &= q \cdot \otimes \cdot + q^2 \mathbf{1} \otimes \cdot, \\
\tilde{\Delta}_q(\cdot \mathbf{\dot{1}}) &= q^2 \cdot \otimes \mathbf{1} + q \cdot \otimes \cdot, & \tilde{\Delta}_q(\dots) &= q^2 \cdot \otimes \cdot + q^2 \cdot \otimes \cdot.
\end{aligned}$$

## 2 Bruhat order on plane posets

### 2.1 Definition of the partial order

**Definition 8** Let  $P, Q$  be two plane posets of the same cardinality. We denote by  $\theta_{P,Q}$  the unique increasing bijection (for the total order) from  $P$  to  $Q$ .

**Remark.** If  $P, Q, R$  are plane posets of the same cardinality, then obviously,  $\theta_{Q,P} = \theta_{P,Q}^{-1}$  and  $\theta_{P,R} = \theta_{Q,R} \circ \theta_{P,Q}$ .

**Lemma 9** Let  $P, Q \in \mathcal{PP}(n)$ . The following assertions are equivalent:

1.  $\forall x, y \in P, (\theta_{P,Q}(x) \leq_h \theta_{P,Q}(y) \text{ in } Q) \implies (x \leq_h y \text{ in } P)$ .
2.  $\forall x, y \in P, (x \leq_r y \text{ in } P) \implies (\theta_{P,Q}(x) \leq_r \theta_{P,Q}(y) \text{ in } Q)$ .

If this holds, we shall say that  $P \leq Q$ .

**Proof.**  $1 \implies 2$ . Let us assume that  $x \leq_r y$  in  $P$ . As  $\theta_{P,Q}$  is increasing,  $\theta_{P,Q}(x) \leq_h \theta_{P,Q}(y)$  or  $\theta_{P,Q}(x) \leq_r \theta_{P,Q}(y)$  in  $Q$ . If  $\theta_{P,Q}(x) \leq_h \theta_{P,Q}(y)$ , by hypothesis,  $x \leq_h y$  in  $P$ . As  $P$  is a plane poset,  $x = y$ , so in both cases  $\theta_{P,Q}(x) \leq_r \theta_{P,Q}(y)$ .

$2 \implies 1$ . Similar proof. □

**Proposition 10** For all  $n \geq 1$ , the relation  $\leq$  is a partial order on  $\mathcal{PP}(n)$ .

**Proof.** Let us assume that  $P \leq Q$  and  $Q \leq P$ . As  $\theta_{Q,P} = \theta_{P,Q}^{-1}$ , it satisfies the following assertion:

$$\forall x, y \in P, (x \leq_h y \text{ in } P) \iff (\theta_{P,Q}(x) \leq_h \theta_{P,Q}(y) \text{ in } Q).$$

Moreover, if  $x \leq_r y$  in  $P$ , then, as  $\theta_{P,Q}$  is increasing,  $\theta_{P,Q}(x) \leq_h \theta_{P,Q}(y)$  or  $\theta_{P,Q}(x) \leq_r \theta_{P,Q}(y)$  in  $Q$ . If  $\theta_{P,Q}(x) \leq_h \theta_{P,Q}(y)$ , then  $x \leq_h y$  in  $P$ . By the incompatibility condition between  $\leq_r$  and  $\leq_h$ ,  $x = y$ , so  $\theta_{P,Q}(x) = \theta_{P,Q}(y)$ , and  $\theta_{P,Q}(x) \leq_r \theta_{P,Q}(y)$ . In any case,  $\theta_{P,Q}(x) \leq_r \theta_{P,Q}(y)$ . Working with  $\theta_{P,Q}^{-1}$ , we obtain:

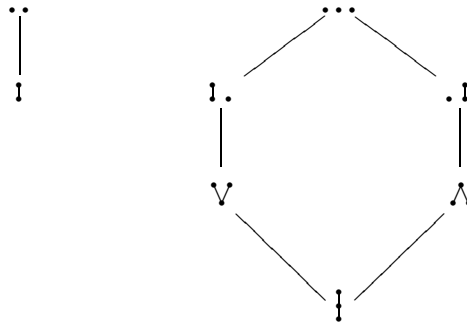
$$\forall x, y \in P, (x \leq_r y \text{ in } P) \iff (\theta_{P,Q}(x) \leq_r \theta_{P,Q}(y) \text{ in } Q).$$

So  $\theta_{P,Q}$  is an isomorphism of plane posets. As a consequence,  $P = Q$ .

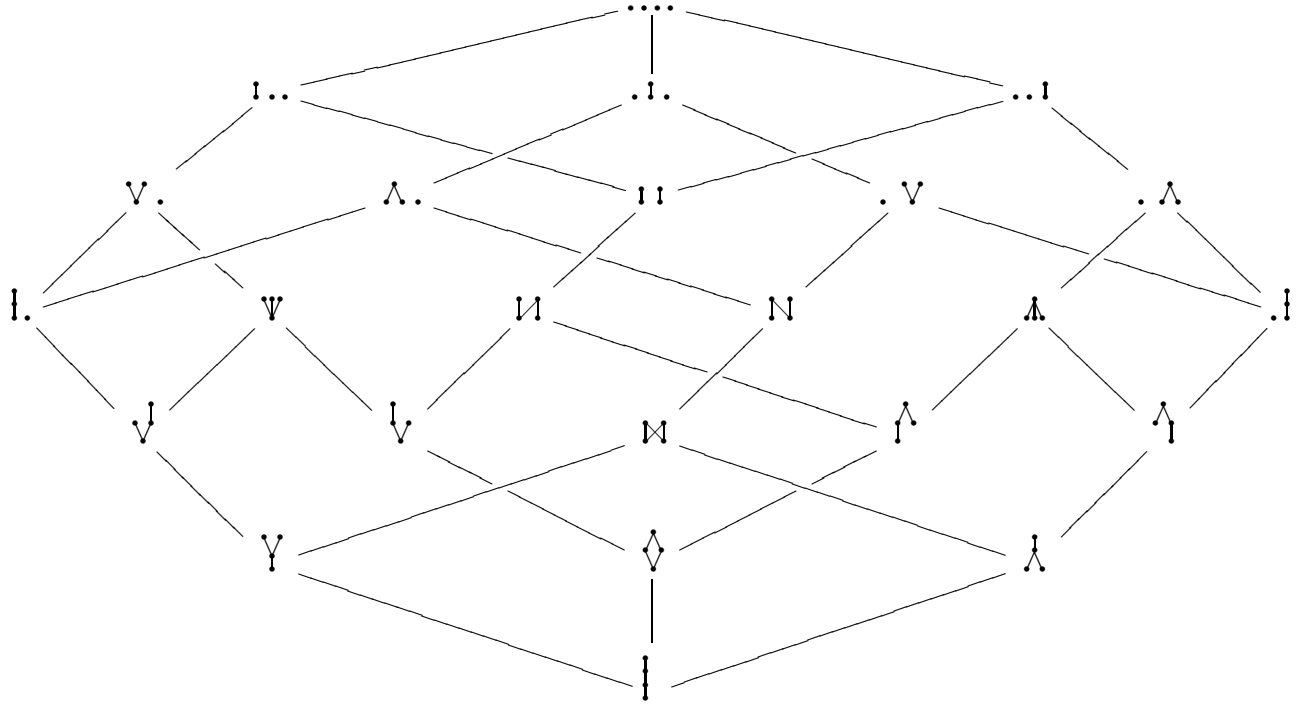
Let us assume that  $P \leq Q$  and  $Q \leq R$ . As  $\theta_{P,R} = \theta_{Q,R} \circ \theta_{P,Q}$ , if  $\theta_{P,R}(x) \leq_h \theta_{P,R}(y)$  in  $R$ , then  $\theta_{P,Q}(x) \leq_h \theta_{P,Q}(y)$  in  $Q$ , so  $x \leq_h y$  in  $P$ . So  $P \leq R$ .

Let  $P \in \mathcal{PP}(n)$ . The unique increasing bijection from  $P$  to  $P$  is  $Id_P$ , so, clearly,  $P \leq P$ . □

**Examples.** Here are the Hasse diagrams of  $(\mathcal{PP}(2), \leq)$ ,  $(\mathcal{PP}(3), \leq)$  and  $(\mathcal{PP}(4), \leq)$ :







## 2.2 Isomorphism with the weak Bruhat order on permutations

**Lemma 11** *Let  $\sigma \in \mathfrak{S}_n$  and let  $P = \Psi_n(\sigma)$ . Let  $1 \leq i < j \leq n$ . Then  $\sigma$  is of the form  $(\dots ij \dots)$  if, and only if, the three following conditions are satisfied:*

- $i <_h j$  in  $P$ .
- If  $x <_h j$  in  $P$ , then  $x \leq_h i$  or  $x \geq_r i$ .
- If  $x >_h i$  in  $P$ , then  $x \geq_h j$  or  $x \leq_r j$ .

**Proof.**  $\implies$ . By definition of  $P$ , indeed  $i <_h j$  in  $P$ .

If  $x <_h j$  in  $P$ , then  $x < j$  and  $\sigma^{-1}(x) < \sigma^{-1}(j)$ . If  $x = i$ , then  $x \leq_h i$  and  $x \geq_r i$ . If  $x \neq i$ , then  $x$  appears before  $j$  in the word representing  $\sigma$ , so it appears before  $i$ . So if  $x < i$ , then  $x \leq_h i$  and if  $x > i$ ,  $x \geq_r i$ .

If  $x >_h i$ , then  $x > i$  and  $\sigma^{-1}(x) > \sigma^{-1}(i)$ . If  $x = j$ , then  $x \geq_h j$  and  $x \leq_r j$ . If  $x \neq j$ , then  $x$  appears after  $i$  in the word representing  $\sigma$ , so it appears after  $j$ . If  $x > j$ , then  $x \geq_h j$  and if  $x < j$ , then  $x \leq_r j$ .

$\impliedby$ . As  $i <_h j$ ,  $i$  appears before  $j$  in the word representing  $\sigma$ . If there is a letter  $x$  between  $i$  and  $j$  in this word, three cases are possible.

- $x < i < j$ . Then  $x <_h j$  and  $x <_r i$ , so we do not have  $x \leq_h i$  nor  $x \geq_r i$ . This contradicts the second condition.
- $i < x < j$ . Then  $x <_h j$  and  $x >_h i$ , so we do not have  $x \leq_h i$  nor  $x \geq_r i$ . This contradicts the second condition.
- $i < j < x$ . Then  $x >_h i$  and  $x >_r j$ , so we do not have  $x \geq_h j$  nor  $x \leq_r j$ . This contradicts the third condition.

As a consequence,  $i$  and  $j$  are two consecutive letters of the word representing  $\sigma$ . □

**Notation.** Let  $P$  be a double poset. We put  $E(P) = \{(i, j) \in P^2 \mid i <_h j\}$ .

**Remark.** By definition of the partial order on  $\mathcal{PP}(n)$ ,  $P \leq Q$  if, and only if,  $E(Q) \subseteq E(P)$ . Consequently,  $P = Q$  if, and only if,  $E(P) = E(Q)$ .

**Lemma 12** *Let  $1 \leq i < j \leq n$  and  $\sigma$  be a permutation of the form  $(\dots ij \dots)$ . We put  $\tau = (ij) \circ \sigma = (\dots ji \dots)$  (the other letters being unchanged). Then  $E(\Psi_n(\tau)) = E(\Psi_n(\sigma)) - \{(i, j)\}$ .*

**Proof.** We put  $P = \Psi_n(\sigma)$  and  $Q = \Psi_n(\tau)$ .

$E(Q) \subseteq E(P) - \{(i, j)\}$ . If  $(k, l) \in E(Q)$ , then  $k < l$  and  $\tau^{-1}(k) < \tau^{-1}(l)$ , so  $(k, l) \neq (i, j)$ . If  $k, l \neq i, j$ , then  $\tau^{-1}(k) = \sigma^{-1}(k)$  and  $\tau^{-1}(l) = \sigma^{-1}(l)$ , so  $(k, l) \in E(P)$ . If  $k = i$  or  $j$ , then  $l$  appears in the word representing  $\tau$  after  $i$  or  $j$ , so after  $i$  and  $j$ , so it also appears in the word representing  $\sigma$  after  $i$  and  $j$ . As a consequence,  $(k, l) \in E(P)$ . If  $l = i$  or  $j$ , we can prove in the same way that  $(k, l) \in E(P)$ .

$E(P) - \{(i, j)\} \subseteq E(Q)$ . Similar proof.  $\square$

**Notation.** Let  $P, Q \in \mathcal{PP}(n)$ . We assume that  $P = Q = \{1, \dots, n\}$  as totally ordered sets. We shall say that  $P \preceq Q$  if there exists  $(i, j) \in E(P)$  such that  $E(Q) = E(P) - \{(i, j)\}$ .

**Lemma 13** *Let  $P \leq Q$  in  $\mathcal{PP}(n)$ . There exists  $P_0 = P, \dots, P_k = Q$ , such that  $P = P_0 \preceq P_1 \preceq \dots \preceq P_k = Q$ .*

**Proof.** By definition of the partial order of  $\mathcal{PP}(n)$ , if  $i <_h j$  in  $Q$ , then  $i <_h j$  in  $P$ . So  $E(Q) \subseteq E(P)$ . We proceed by induction on  $k = E(P) - E(Q)$ . If  $k = 0$ , then  $P = Q$  and the result is obvious. Let us assume that  $k \geq 1$ . We put  $\sigma = \Psi_n^{-1}(P)$  and  $\tau = \Psi_n^{-1}(Q)$ . We choose  $(i, j) \in E(P) - E(Q)$ , such that the distance  $d$  between the letters  $i$  and  $j$  in the word representing  $\sigma$  is minimal. Let us assume that  $d \geq 2$ . As  $i <_h j$  in  $P$ , there exists a letter  $x$  such that  $\sigma = (\dots i \dots x \dots j \dots)$ . Three cases are possible.

- If  $x < i < j$ , then, in  $P$ ,  $x <_r i$ ,  $x <_h j$  and  $i <_h j$ . Hence,  $(x, i) \notin E(P)$ , so  $(x, i) \notin E(Q)$  and  $x <_r i$  in  $Q$ ; as  $(i, j) \notin E(Q)$ ,  $i <_r j$  in  $Q$ ; finally,  $x <_r j$  in  $Q$ . Consequently,  $(x, j) \in E(P) - E(Q)$ : contradicts the minimality of  $d$ .
- If  $i < x < j$ , then, in  $P$ ,  $i <_h x$ ,  $x <_h j$  and  $i <_h j$ . By minimality of  $d$ ,  $i <_h x$  and  $x <_h j$  in  $Q$ , so  $i <_h j$  in  $Q$ : contradicts that  $(i, j) \in E(P) - E(Q)$ .
- If  $i < j < x$ , then, in  $P$ ,  $i <_h x$ ,  $j <_r x$  and  $i <_h j$ . By minimality of  $d$ ,  $i <_h x$  in  $Q$ . As  $(j, x) \notin E(P)$ ,  $(j, x) \notin E(Q)$ , so  $j <_r x$  in  $Q$ ; as  $(i, j) \notin E(Q)$ ,  $i <_r j$  in  $Q$ , and finally  $i <_r x$  in  $Q$ : contradicts  $i <_h x$  in  $Q$ .

We deduce that  $d = 1$ :  $i$  and  $j$  are two consecutive letters in  $\sigma$ . We then take  $P_1 = \Psi_n^{-1}((ij) \circ \sigma)$ . By lemma 12,  $E(P_1) = E(P) - \{(i, j)\}$ . We then apply the induction hypothesis to the couple  $(P_1, Q)$ .  $\square$

**Lemma 14** *Let  $P, Q \in \mathcal{PP}(n)$ , such that  $P \preceq Q$ . We put  $E(Q) = E(P) - \{(i, j)\}$ . Then  $i, j$  are two consecutive letters in  $\Psi_n^{-1}(P)$ . Moreover,  $\Psi_n^{-1}(Q)$  is obtained by permuting the two consecutive letters  $ij$  in  $\Psi_n^{-1}(P)$ .*

**Proof.** Let us prove that  $i, j$  satisfy the three conditions of lemma 11. As  $(i, j) \in E(P)$ ,  $i <_h j$  in  $P$ . If  $x <_h j$  in  $P$ , three cases are possible.

- If  $x = i$ , then  $x \leq_h i$  and  $x \geq_r i$  in  $P$ .
- If  $x < i$ , let us assume that  $x <_r i$  in  $P$ . Hence,  $(x, i) \notin E(P)$ , so  $(x, i) \notin E(Q)$  and  $x <_r i$  in  $Q$ . As  $i <_r j$  in  $Q$ ,  $x <_r j$  in  $Q$ . As  $E(P) = E(Q) \cup \{(i, j)\}$ ,  $(x, j) \notin E(P)$  and  $x <_r j$  in  $P$ : contradiction. So  $x <_h j$ .

- If  $x > i$ , let us assume that  $x >_h i$  in  $P$ . As  $E(Q) = E(P) - \{(i, j)\}$ ,  $i <_h x$  and  $x <_h j$  in  $Q$ , so  $i <_h j$  in  $Q$ : contradiction,  $(i, j) \notin E(Q)$ . So  $x >_r i$ .

Let us now prove the third condition. If  $x >_h i$  in  $P$ , three cases are possible.

- If  $x = j$ , then  $x \geq_h j$  and  $x \leq_r j$  in  $P$ .
- If  $x < j$ , let us assume that  $x <_h j$  in  $P$ . As  $E(Q) = E(P) - \{(i, j)\}$ ,  $x <_h j$  and  $i <_h x$  in  $Q$ , so  $i <_h j$  in  $Q$ : contradiction,  $(i, j) \notin E(Q)$ . So  $x <_r j$  in  $Q$ .
- If  $x > j$ , let us assume that  $x >_r j$  in  $P$ . As  $E(Q) = E(P) - \{(i, j)\}$ ,  $x >_r j$  and  $j >_r i$  in  $Q$ , so  $x >_r i$  in  $Q$ . As  $E(P) = E(Q) \cup \{(i, j)\}$ ,  $x >_r i$  in  $P$ : contradicts  $x >_h i$  in  $P$ . So  $x >_h j$  in  $P$ .

Finally, the three conditions of lemma 11 are satisfied.

We put  $\Psi_n^{-1}(P) = \sigma$  and  $\Psi_n^{-1}(Q) = \tau$ . By lemma 11,  $ij$  are two consecutive letters in the word representing  $\sigma$ . Let  $\varsigma$  be the permutation obtained by permuting these two letters. By lemma 12,  $E(\Psi_n(\varsigma)) = E(P) - \{(i, j)\} = E(Q)$ , so  $\Psi_n(\varsigma) = Q$  and  $\tau = \varsigma$ .  $\square$

**Theorem 15** *We partially order  $\mathfrak{S}_n$  by the weak Bruhat order [14, 15]. For all  $n \geq 0$ , the bijection  $\Psi_n$  is an isomorphism of posets, that is to say: for all  $\sigma, \tau \in \mathfrak{S}_n$ ,  $\sigma \leq \tau$  if, and only if,  $\Psi_n(\sigma) \leq \Psi_n(\tau)$  in  $\mathcal{PP}(n)$ .*

**Proof.** We consider  $\sigma, \tau \in \mathfrak{S}_n$ . We put  $\Psi_n(\sigma) = P$  and  $\Psi_n(\tau) = Q$ .

Let us assume that  $\sigma \leq \tau$  in  $\mathfrak{S}_n$ . By definition of the weak Bruhat order, there exists  $\sigma_0 = \sigma, \dots, \sigma_k = \tau$ , such that  $\sigma_{p+1}$  is obtained from  $\sigma_p$  by permuting two consecutive letters  $ij$ , with  $i < j$ , in the word representing  $\sigma_p$ . By lemma 12, for all  $p$ ,  $E(\Psi_n(\sigma_{p+1})) \subseteq E(\Psi_n(\sigma_p))$ . Consequently,  $E(Q) \subseteq E(P)$ , so  $P \leq Q$ .

Let us assume that  $P \leq Q$ . From lemma 13, there exists  $P = P_0 \preceq \dots \preceq P_k = Q$ . We put  $\sigma_p = \Psi_n^{-1}(P_p)$  for all  $0 \leq p \leq k$ . From lemma 14, we obtain  $\sigma_{p+1}$  from  $\sigma_p$  by permuting two consecutive letters  $ij$ , with  $i < j$ , in the word representing  $\sigma_p$ . By definition of the weak Bruhat order,  $\sigma \leq \tau$ .  $\square$

### 2.3 Properties of the partial order

**Proposition 16** *Let  $P_1, Q_1 \in \mathcal{PP}(k)$ ,  $P_2, Q_2 \in \mathcal{PP}(l)$ . The following conditions are equivalent:*

1.  $P_1 P_2 \leq Q_1 Q_2$ .
2.  $P_1 \not\leq P_2 \leq Q_2 \not\leq Q_1$ .
3.  $P_1 \leq Q_1$  and  $P_2 \leq Q_2$ .

**Proof.** We put  $\theta_i = \theta_{P_i, Q_i}$ ,  $i = 1, 2$ . It is clear that the unique increasing bijection from  $P_1 P_2$  to  $Q_1 Q_2$  and the unique increasing bijection from  $P_1 \not\leq P_2$  to  $Q_1 \not\leq Q_2$  are both  $\theta = \theta_1 \otimes \theta_2$ . As an immediate consequence, 1 or 2 implies 3.

3  $\implies$  1. Let us assume that  $\theta(i) <_h \theta(j)$  in  $Q_1 Q_2$ . Two cases are possible.

- $i, j \in P_1$ . Then  $\theta_1(i) <_h \theta_1(j)$  in  $Q_1$ , so  $i <_h j$  in  $P_1$ , so  $i <_h j$  in  $P_1 P_2$ .
- $i, j \in P_2$ . Then  $\theta_2(i) <_h \theta_2(j)$  in  $Q_2$ , so  $i <_h j$  in  $P_2$ , so  $i <_h j$  in  $P_1 P_2$ .

So  $P_1 P_2 \leq Q_1 Q_2$ .

3  $\implies$  2. Let us assume that  $\theta(i) <_h \theta(j)$  in  $Q_1 \not\leq Q_2$ . Three cases are possible.

- $i, j \in P_1$ . Then  $\theta_1(i) <_h \theta_1(j)$  in  $Q_1$ , so  $i <_h j$  in  $P_1$ , so  $i <_h j$  in  $P_1 \not\leq P_2$ .
- $i, j \in P_2$ . Then  $\theta_2(i) <_h \theta_2(j)$  in  $Q_2$ , so  $i <_h j$  in  $P_2$ , so  $i <_h j$  in  $P_1 \not\leq P_2$ .
- $i \in P_1$  and  $j \in P_2$ . Then  $i <_h j$  in  $P_1 \not\leq P_2$ .

So  $P_1 \not\leq P_2 \leq Q_1 \not\leq Q_2$ . □

**Definition 17** Let  $P = (P, \leq_h, \leq_r)$  be a plane poset. We put  $\iota(P) = (P, \leq_r, \leq_h)$ . Note that  $\iota$  is an involution of  $\mathcal{PP}$ .

**Proposition 18** For any  $P, Q \in \mathcal{PP}(n)$ ,  $P \leq Q$  if, and only if,  $\iota(Q) \leq \iota(P)$ .

**Proof.** Let  $P, Q \in \mathcal{PP}(n)$ . As the total orders on  $R$  and  $\iota(R)$  are identical for any  $R \in \mathcal{PP}(n)$ , the unique increasing bijection from  $\iota(Q)$  to  $\iota(P)$  is  $\theta_{Q,P}$ . Hence:

$$\begin{aligned}
P \leq Q &\iff \forall x, y \in P, (\theta_{P,Q}(x) \leq_h \theta_{P,Q}(y) \text{ in } Q) \implies (x \leq_h y \text{ in } P) \\
&\iff \forall x, y \in \iota(P), (\theta_{P,Q}(x) \leq_r \theta_{P,Q}(y) \text{ in } \iota(Q)) \implies (x \leq_r y \text{ in } \iota(P)) \\
&\iff \forall x', y' \in \iota(Q), (x' \leq_r y' \text{ in } \iota(Q)) \implies (\theta_{Q,P}(x') \leq_r \theta_{Q,P}(y') \text{ in } \iota(P)) \\
&\iff \iota(Q) \leq \iota(P).
\end{aligned}$$

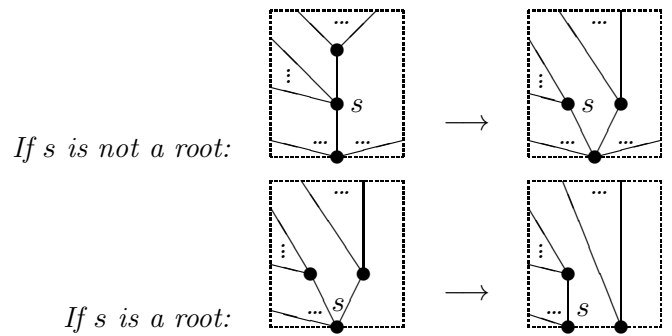
□

**Remark.** Let  $P \in \mathcal{PP}(n)$ . We put  $\sigma = \Psi_n^{-1}(P)$ . Then  $\Psi_n^{-1}(\iota(P)) = \sigma \circ (n \dots 1)$ .

## 2.4 Restriction to plane forests

Let us consider the restriction the partial order to the set of plane forests.

**Definition 19** Let  $F$  be a plane forest and let  $s$  be a vertex of  $F$  which is not a leaf. The transformation of  $F$  at vertex  $s$  is the plane forest obtained in one of the following way:



the part of the forest outside the frame being unchanged.

**Remark.** Up to a vertical symmetry, these transformation are used in [7] in order to define a partial order on the set of plane forests, making it isomorphic to the Tamari poset.

**Proposition 20** Let  $F, G$  be two plane forests of degree  $n$ . Then  $F \leq G$  if, and only if,  $G$  is obtained from  $F$  by a finite number of transformations of definition 19.

**Proof.**  $\implies$ . By induction on  $n$ . If  $n = 1$ , then  $F = G = \cdot$  and the result is obvious. Let us assume that  $n \geq 2$ .

*First case.* Let us assume that  $F$  is a plane tree  $F = \cdot \downarrow F'$ . We put  $G = (\cdot \downarrow G')G''$ , where  $F', G', G''$  are plane forests. The increasing bijection from  $F'$  to  $G'G''$  is  $(\Theta_{F,G})|_{F'}$ , as the root of  $F$  and the root of  $\cdot \downarrow F'$  are the smallest elements for their total orders. By the induction hypothesis, one can obtain  $G'G''$  from  $F'$  by a finite number of transformations. So  $\cdot \downarrow (G'G'')$  can be obtained from  $F$  by a finite number of transformations. Applying transformations to the left root of  $\cdot \downarrow (G'G'')$ , one can obtain  $G$  from  $\cdot \downarrow (G'G'')$  by a finite number of transformations.

*Second case.* Let us assume that  $F$  is not a tree. We put  $F = (\cdot \downarrow F')F''$ , where  $F', F''$  are plane forests,  $F'' \neq 1$ . We put  $\Theta_{F,G}(\cdot \downarrow F') = G'$  and  $\Theta_{F,G}(F'') = G''$ . Let us consider  $x \in G'$  and  $y \in G''$ . Then  $\Theta_{F,G}^{-1}(x) \leq_r \Theta_{F,G}^{-1}(y)$  in  $F$ , so  $x \leq y$  in  $G$ . Moreover, if  $x <_h y$  in  $G$ , then  $(x, y) \in E(G)$ . As  $F \leq G$ ,  $(\Theta_{F,G}^{-1}(x), \Theta_{F,G}^{-1}(y)) \in E(F)$ , so  $\Theta_{F,G}^{-1}(x) <_h \Theta_{F,G}^{-1}(y)$  in  $F$ : contradiction. So  $x <_r y$  in  $G$ . Consequently,  $G = G'G''$ . By proposition 16,  $\cdot \downarrow F' \leq G'$  and  $F'' \leq G''$ . By the induction hypothesis, one can obtain  $G'$  from  $\cdot \downarrow F'$  and  $G''$  from  $F''$  by a certain number of transformations. Hence, we can obtain  $G = G'G''$  from  $F = (\cdot \downarrow F')F''$  by a certain number of transformations.

$\Leftarrow$ . By transitivity, It is enough to prove it if  $G$  is obtained from  $F$  by a transformation. It is clear that this transformation does not affect the total order on the vertices of  $F$ , so the increasing bijection from  $F$  to  $G$  is the identity. Obviously, if  $x \leq_h y$  in  $G$ , then  $x \leq_h y$  in  $F$ , so  $F \leq G$ .  $\square$

From [7], we recover the injection from the Tamari poset into the Bruhat poset:

**Corollary 21** *The poset of plane forests of degree  $n$  is isomorphic to the Tamari poset on plane binary trees with  $n + 1$  leaves (or  $n$  internal vertices).*

### 3 Link with the infinitesimal structure

#### 3.1 A lemma on the Bruhat order

**Lemma 22** *Let  $P, Q, R$  be three double posets.*

1.  $P \downarrow Q \leq R$  if, and only if, there exists a biideal  $I_0$  of  $R$ , such that  $P \leq R \setminus I_0$  and  $Q \leq I_0$ . Moreover, if this holds,  $I_0$  is unique and  $I_0 = \theta_{PQ,R}(Q)$ .
2.  $PQ \leq R$  if, and only if, there exists a plane subposet  $I_0$  of  $R$ , such that  $R = (R \setminus I_0)I_0$ ,  $P \leq R \setminus I_0$  and  $Q \leq I_0$ . Moreover, if this holds,  $I_0$  is unique and  $I_0 = \theta_{P \downarrow Q,R}(Q)$ .
3.  $\iota(PQ) \leq R$  if, and only if, there exists a biideal  $I_0$  of  $R$ , such that  $\iota(R \setminus I_0) \leq P$  and  $\iota(I_0) \leq Q$ . Moreover, if this holds,  $I_0$  is unique and  $I_0 = \theta_{PQ,R}(Q)$ .
4.  $\iota(P \downarrow Q) \leq R$  if, and only if, there exists a plane subposet  $I_0$  of  $R$ , such that  $R = (R \setminus I_0)I_0$ ,  $\iota(R \setminus I_0) \leq P$  and  $\iota(I_0) \leq Q$ . Moreover, if this holds,  $I_0$  is unique and  $I_0 = \theta_{P \downarrow Q,R}(Q)$ .

**Proof.** 1.  $\implies$ . We consider  $I_0 = \theta_{P \downarrow Q,R}(Q)$ . If  $x \in I_0$  and  $y \in R$  satisfy  $x \leq y$ , then  $\theta_{P \downarrow Q,R}^{-1}(x) \in Q$  and  $\theta_{P \downarrow Q,R}^{-1}(x) \leq \theta_{P \downarrow Q,R}^{-1}(y)$ , so  $\theta_{P \downarrow Q,R}^{-1}(y) \in Q$  and  $y \in I_0$ : hence,  $I_0$  is a biideal. Moreover,  $\theta_{P,R \setminus I_0}$  is the restriction of  $\Theta_{P \downarrow Q,R}$  to  $P$  and  $\theta_{Q,I_0}$  is the restriction of  $\Theta_{P \downarrow Q,R}$  to  $Q$ ; as  $P \downarrow Q \leq R$ ,  $P \leq R \setminus I_0$  and  $Q \leq I_0$ .

1.  $\Leftarrow$ . Let  $I$  be a such a biideal. As  $\iota(Q) \leq I$ ,  $|I| = |Q| = |I_0|$ . As  $I$  is a biideal,  $|I|$  is made of the  $|Q|$  greatest elements of  $R$ . As  $\theta_{P \downarrow Q,R}$  is increasing and as the  $|Q|$  greatest elements of  $P \downarrow Q$  are the elements of  $Q$ ,  $I = \theta_{P \downarrow Q,R}(Q) = I_0$ . Let  $x, y \in P \downarrow Q$ , such that  $\theta_{P \downarrow Q,R}(x) \leq_h \theta_{P \downarrow Q,R}(y)$ . As  $I = I_0$  is a biideal, three cases are possible:

- $x, y \in P$ . As  $P \leq R \setminus I_0$ , then  $x \leq_h y$  in  $P$ , hence in  $P \downarrow Q$ .

•  $x, y \in Q$ . As  $Q \leq I_0$ , then  $x \leq_h y$  in  $Q$ , hence in  $P \dot{\downarrow} Q$ .

•  $x \in P, y \in Q$ . Then  $x \leq_h y$  in  $P \dot{\downarrow} Q$ .

So  $P \dot{\downarrow} Q \leq R$ .

2.  $\implies$ . We consider  $I_0 = \theta_{PQ,R}(Q)$ . If  $x \in R \setminus I_0$  and  $y \in I_0$ , then  $\theta_{PQ,R}^{-1}(x) \in P$  and  $\theta_{PQ,R}^{-1}(y) \in Q$ , so  $\theta_{PQ,R}^{-1}(x) \leq_r \theta_{PQ,R}^{-1}(y)$ . As  $PQ \leq R$ ,  $x \leq_r y$ , so  $R = (R \setminus I_0)I_0$ . Moreover,  $\theta_{P,R \setminus I_0}$  is the restriction of  $\Theta_{PQ,R}$  to  $P$  and  $\theta_{Q,I_0}$  is the restriction of  $\Theta_{PQ,R}$  to  $Q$ ; as  $PQ \leq R$ ,  $P \leq R \setminus I_0$  and  $Q \leq I_0$ .

2.  $\impliedby$ . Let  $I$  be such a subposet. As  $Q \leq I$ ,  $|Q| = |I| = |I_0|$ . Moreover,  $R = (R \setminus I)I$ , so the elements of  $I$  are the  $|Q|$  greatest elements of  $R$ . Hence,  $I = I_0 = \theta_{PQ,R}(Q)$ . Let  $x, y \in PQ$ , such that  $\theta_{PQ,R}(x) \leq_h \theta_{PQ,R}(y)$ . As  $R = (R \setminus I)I$ ,  $x, y$  are both in  $P$  or both in  $Q$ . As  $P \leq R \setminus I$  and  $Q \leq I$ ,  $x \leq_h y$  in  $P$  or in  $Q$ , hence in  $PQ$ . So  $PQ \leq R$ .

3 and 4. Reformulations of the first two points, with the observations that  $\iota(PQ) = \iota(P) \dot{\downarrow} \iota(Q)$ ,  $\iota(P \dot{\downarrow} Q) = \iota(P)\iota(Q)$  and  $\iota$  is decreasing for the Bruhat order  $\leq$ .  $\square$

### 3.2 Construction of the Hopf pairing

**Notations.** Let  $P, Q \in \mathcal{PP}(n)$ . We put:

$$\begin{aligned} \phi(P, Q) &= \#\{(x, y) \in P^2 \mid x <_r y \text{ and } \theta_{P,Q}(x) <_h \theta_{P,Q}(y)\} \\ &\quad + \#\{(x, y) \in P^2 \mid x <_h y \text{ and } \theta_{P,Q}(x) <_r \theta_{P,Q}(y)\}. \end{aligned}$$

**Theorem 23** *Let  $P, Q \in \mathcal{PP}$ . We put:*

$$\langle P, Q \rangle_q = \begin{cases} q^{\phi(P,Q)} & \text{if } \iota(P) \leq Q, \\ 0 & \text{if not.} \end{cases}$$

*This pairing is bilinearly extended to  $\mathcal{HPP}$ . Then  $\langle -, - \rangle_q$  is a symmetric Hopf pairing on  $(\mathcal{HPP}, m, \Delta_q)$ . It is nondegenerate if, and only if,  $q \neq 0$ .*

**Examples.**


So  $\phi(P, Q) = \phi(Q, P)$ , and this pairing is symmetric.

Let  $P, Q, R \in \mathcal{PP}$ . Let us prove that  $\langle PQ, R \rangle_q = \langle P \otimes Q, \Delta_q(R) \rangle_q$ .

*First case.* Let us assume that  $\iota(PQ) \leq R$ . By lemma 22, there exists a unique biideal  $I_0$  of  $R$ , such that  $\iota(P) \leq R \setminus I_0$  and  $\iota(Q) \leq I_0$ . Hence:

$$\begin{aligned} \langle P \otimes Q, \Delta_q(R) \rangle_q &= \sum_{I \text{ biideal of } R} q^{h_{R \setminus I}^I} \langle P, R \setminus I \rangle_q \langle Q, I \rangle_q \\ &= q^{h_{R \setminus I_0}^{I_0}} \langle P, R \setminus I_0 \rangle_q \langle Q, I_0 \rangle_q + 0 \\ &= q^{h_{R \setminus I_0}^{I_0} + \phi(P, R \setminus I_0) + \phi(Q, I_0)}. \end{aligned}$$

Moreover:

$$\begin{aligned} \Phi(PQ, R) &= \#\{(x, y) \in P^2 \mid x <_r y \text{ and } \phi_{PQ, R}(x) <_h \phi_{PQ, R}(y)\} \\ &\quad + \#\{(x, y) \in P^2 \mid x <_h y \text{ and } \phi_{PQ, R}(x) <_r \phi_{PQ, R}(y)\} \\ &\quad + \#\{(x, y) \in Q^2 \mid x <_r y \text{ and } \phi_{PQ, R}(x) <_h \phi_{PQ, R}(y)\} \\ &\quad + \#\{(x, y) \in Q^2 \mid x <_h y \text{ and } \phi_{PQ, R}(x) <_r \phi_{PQ, R}(y)\} \\ &\quad + \#\{(x, y) \in P \times Q \mid x <_r y \text{ and } \phi_{PQ, R}(x) <_h \phi_{PQ, R}(y)\} \\ &\quad + \#\{(x, y) \in P \times Q \mid x <_h y \text{ and } \phi_{PQ, R}(x) <_r \phi_{PQ, R}(y)\} \\ &\quad + \#\{(x, y) \in Q \times P \mid x <_r y \text{ and } \phi_{PQ, R}(x) <_h \phi_{PQ, R}(y)\} \\ &\quad + \#\{(x, y) \in Q \times P \mid x <_h y \text{ and } \phi_{PQ, R}(x) <_r \phi_{PQ, R}(y)\}. \end{aligned}$$

As  $I_0 = \theta_{PQ, R}(Q)$ ,  $\theta_{P, R \setminus I_0}$  is the restriction to  $P$  of  $\theta_{P \not\leq Q, R}$  and  $\theta_{Q, I_0}$  is the restriction to  $Q$  of  $\theta_{P \not\leq Q, R}$ , so:

$$\begin{aligned} \phi(P, R \setminus I_0) &= \#\{(x, y) \in P^2 \mid x <_r y \text{ and } \phi_{PQ, R}(x) <_h \phi_{PQ, R}(y)\} \\ &\quad + \#\{(x, y) \in P^2 \mid x <_h y \text{ and } \phi_{P, R \setminus I_0}(x) <_r \phi_{P, R \setminus I_0}(y)\}, \\ \phi(Q, I_0) &= \#\{(x, y) \in Q^2 \mid x <_r y \text{ and } \phi_{PQ, R}(x) <_h \phi_{PQ, R}(y)\} \\ &\quad + \#\{(x, y) \in Q^2 \mid x <_h y \text{ and } \phi_{P, R \setminus I_0}(x) <_r \phi_{P, R \setminus I_0}(y)\}. \end{aligned}$$

If  $x \in P$  and  $y \in Q$ , then  $x <_r y$ . So:

$$\begin{aligned} 0 &= \#\{(x, y) \in P \times Q \mid x <_h y \text{ and } \phi_{PQ, R}(x) <_r \phi_{PQ, R}(y)\} \\ &= \#\{(x, y) \in Q \times P \mid x <_r y \text{ and } \phi_{PQ, R}(x) <_h \phi_{PQ, R}(y)\} \\ &= \#\{(x, y) \in Q \times P \mid x <_h y \text{ and } \phi_{PQ, R}(x) <_r \phi_{PQ, R}(y)\}, \end{aligned}$$

and:

$$\begin{aligned} h_{R \setminus I_0}^{I_0} &= \#\{(x, y) \in P \times Q \mid \phi_{PQ, R}(x) <_h \phi_{PQ, R}(y)\} \\ &= \#\{(x, y) \in P \times Q \mid x <_r y \text{ and } \phi_{PQ, R}(x) <_h \phi_{PQ, R}(y)\} \end{aligned}$$

Finally,  $\phi(PQ, R) = \phi(P, R \setminus I_0) + \phi(Q, I_0) + h_{R \setminus I_0}^{I_0} + 0$ . Hence:

$$\langle P \otimes Q, \Delta_q(R) \rangle_q = q^{\phi(PQ, R)} = \langle PQ, R \rangle_q.$$

*Second case.* Let us assume that we do not have  $\iota(PQ) \leq R$ . By lemma 22, for any biideal  $I$  of  $R$ ,  $\langle P, R \setminus I \rangle_q \langle Q, I \rangle_q = 0$ , So  $\langle P \otimes Q, \Delta_q(R) \rangle_q = \langle PQ, R \rangle_q = 0$ .

Let us now study the degeneracy of this pairing. If  $q = 0$ , the examples below show that  $\mathbf{1}$  is in the orthogonal of the pairing  $\langle -, - \rangle_0$ , so this pairing is degenerate. Let us assume that  $q \neq 0$ .

*First step.* Let  $P \in \mathcal{PP}(n)$ . By definition of the pairing,  $\langle \iota(P), P \rangle_q = q^{\phi(P, \iota(P))}$ . Moreover, as  $\theta_{P, \iota(P)} = Id_P$ :

$$\begin{aligned} \phi(P, \iota(P)) &= \#\{(x, y) \in P^2 \mid x <_r y \text{ and } x <_r y\} + \#\{(x, y) \in P^2 \mid x <_h y \text{ and } x <_h y\} \\ &= \#\{(x, y) \in P^2 \mid x <_r y\} + \#\{(x, y) \in P^2 \mid x <_h y\} \\ &= \#\{(x, y) \in P^2 \mid x < y\} \\ &= \frac{n(n-1)}{2}. \end{aligned}$$

So  $\langle P, \iota(P) \rangle_q = q^{\frac{n(n-1)}{2}} \neq 0$ .

*Second step.* Let us fix  $n \geq 0$ . We index the elements of  $\mathcal{PP}(n)$  in such a way that if  $\iota(P_i) < \iota(P_j)$  for the Bruhat order, then  $i < j$ . Let  $x \in \mathcal{PP}(n)$ , nonzero. Let  $i$  be the smallest integer such that  $P_i$  appears in  $x$ . Let  $a$  be the coefficient of  $P_i$  in  $x$ . If  $j > i$ , then it is not possible to have  $\iota(P_j) \leq \iota(P_i)$ , so  $\langle P_j, \iota(P_i) \rangle_q = 0$ . Consequently:

$$\langle x, \iota(P_i) \rangle_q = \langle aP_i, \iota(P_i) \rangle_q + 0 = aq^{\frac{n(n-1)}{2}} \neq 0.$$

So  $x$  is not in the orthogonal of  $\mathcal{H}_{\mathcal{PP}}$ : the pairing is nondegenerate.  $\square$

**Remark.** This pairing is the pairing  $\langle -, - \rangle_{q,0,1,0}$  of [3].

**Proposition 24** We define a coproduct  $\Delta'_q$  on  $\mathcal{H}_{\mathcal{PP}}$  in the following way: for all  $P \in \mathcal{PP}$ ,

$$\Delta'_q(P) = \sum_{P_1 P_2 = P} q^{|P_1| |P_2|} P_1 \otimes P_2.$$

Then for all  $x, y, z \in \mathcal{H}_{\mathcal{PP}}$ ,  $\langle x \dot{\smile} y, z \rangle_q = \langle x \otimes y, \Delta'_q(z) \rangle_q$ .

**Proof.** Let  $P, Q, R \in \mathcal{PP}$ . Let us prove that  $\langle P \dot{\smile} Q, R \rangle_q = \langle P \otimes Q, \Delta'_q(R) \rangle_q$ .

*First case.* Let us assume that  $\iota(P \dot{\smile} Q) \leq R$ . By lemma 22, there exists a unique  $I_0 \subseteq R$ , such that  $R = (R \setminus I_0)I_0$ ,  $\iota(P) \leq R \setminus I_0$  and  $\iota(Q) \leq I_0$ . Hence:

$$\begin{aligned} \langle P \otimes Q, \Delta'_q(R) \rangle_q &= \sum_{R=R_1 R_2} q^{|R_1| |R_2|} \langle P, R \setminus I \rangle_q \langle Q, I \rangle_q \\ &= q^{|R \setminus I_0| |I_0|} \langle P, R \setminus I_0 \rangle_q \langle Q, I_0 \rangle_q + 0 \\ &= q^{|R \setminus I_0| |I_0| + \phi(P, R \setminus I_0) + \phi(Q, I_0)}. \end{aligned}$$

Moreover:

$$\begin{aligned} \Phi(P \dot{\smile} Q, R) &= \#\{(x, y) \in P^2 \mid x <_r y \text{ and } \phi_{P \dot{\smile} Q, R}(x) <_h \phi_{P \dot{\smile} Q, R}(y)\} \\ &\quad + \#\{(x, y) \in P^2 \mid x <_h y \text{ and } \phi_{P \dot{\smile} Q, R}(x) <_r \phi_{P \dot{\smile} Q, R}(y)\} \\ &\quad + \#\{(x, y) \in Q^2 \mid x <_r y \text{ and } \phi_{P \dot{\smile} Q, R}(x) <_h \phi_{P \dot{\smile} Q, R}(y)\} \\ &\quad + \#\{(x, y) \in Q^2 \mid x <_h y \text{ and } \phi_{P \dot{\smile} Q, R}(x) <_r \phi_{P \dot{\smile} Q, R}(y)\} \\ &\quad + \#\{(x, y) \in P \times Q \mid x <_r y \text{ and } \phi_{P \dot{\smile} Q, R}(x) <_h \phi_{P \dot{\smile} Q, R}(y)\} \\ &\quad + \#\{(x, y) \in P \times Q \mid x <_h y \text{ and } \phi_{P \dot{\smile} Q, R}(x) <_r \phi_{P \dot{\smile} Q, R}(y)\} \\ &\quad + \#\{(x, y) \in Q \times P \mid x <_r y \text{ and } \phi_{P \dot{\smile} Q, R}(x) <_h \phi_{P \dot{\smile} Q, R}(y)\} \\ &\quad + \#\{(x, y) \in Q \times P \mid x <_h y \text{ and } \phi_{P \dot{\smile} Q, R}(x) <_r \phi_{P \dot{\smile} Q, R}(y)\}. \end{aligned}$$



As  $I_0 = \theta_{P \dot{\downarrow} Q, R}(Q)$ ,  $\theta_{P, R \setminus I_0}$  is the restriction to  $P$  of  $\theta_{P \dot{\downarrow} Q, R}$  and  $\theta_{Q, I_0}$  is the restriction to  $Q$  of  $\theta_{P \dot{\downarrow} Q, R}$ , so:

$$\begin{aligned} \phi(P, R \setminus I_0) &= \#\{(x, y) \in P^2 \mid x <_r y \text{ and } \phi_{P \dot{\downarrow} Q, R}(x) <_h \phi_{P \dot{\downarrow} Q, R}(y)\} \\ &\quad + \#\{(x, y) \in P^2 \mid x <_h y \text{ and } \phi_{P, R \setminus I_0}(x) <_r \phi_{P, R \setminus I_0}(y)\}, \end{aligned}$$

$$\begin{aligned} \phi(Q, I_0) &= \#\{(x, y) \in Q^2 \mid x <_r y \text{ and } \phi_{P \dot{\downarrow} Q, R}(x) <_h \phi_{P \dot{\downarrow} Q, R}(y)\} \\ &\quad + \#\{(x, y) \in Q^2 \mid x <_h y \text{ and } \phi_{P, R \setminus I_0}(x) <_r \phi_{P, R \setminus I_0}(y)\}. \end{aligned}$$

If  $x \in P$  and  $y \in Q$ , then  $x <_h y$ . So:

$$\begin{aligned} 0 &= \#\{(x, y) \in P \times Q \mid x <_r y \text{ and } \phi_{P \dot{\downarrow} Q, R}(x) <_r \phi_{P \dot{\downarrow} Q, R}(y)\} \\ &= \#\{(x, y) \in Q \times P \mid x <_r y \text{ and } \phi_{P \dot{\downarrow} Q, R}(x) <_h \phi_{P \dot{\downarrow} Q, R}(y)\} \\ &= \#\{(x, y) \in Q \times P \mid x <_h y \text{ and } \phi_{P, R \setminus I_0}(x) <_r \phi_{P, R \setminus I_0}(y)\}. \end{aligned}$$

Moreover:

$$\begin{aligned} |R \setminus I_0| |I_0| &= \#\{(x', y') \in (R \setminus I_0) \times I_0 \mid x' <_r y'\} \\ &= \#\{(x, y) \in P \times Q \mid \phi_{P \dot{\downarrow} Q, R}(x) <_r \phi_{P \dot{\downarrow} Q, R}(y)\} \\ &= \#\{(x, y) \in P \times Q \mid x <_h y \text{ and } \phi_{P, R \setminus I_0}(x) <_r \phi_{P, R \setminus I_0}(y)\}. \end{aligned}$$

Finally,  $\phi(P \dot{\downarrow} Q, R) = \phi(P, R \setminus I_0) + \phi(Q, I_0) + |R \setminus I_0| |I_0| + 0$ . Hence:

$$\langle P \otimes Q, \Delta'_q(R) \rangle_q = q^{\phi(P \dot{\downarrow} Q, R)} = \langle P \dot{\downarrow} Q, R \rangle_q.$$

*Second case.* Let us assume that we don't have  $\iota(P \dot{\downarrow} Q) \leq R$ . By lemma 22, if  $R = (R \setminus I)I$ , then  $\langle P, R \setminus I \rangle_q \langle Q, I \rangle_q = 0$ , So  $\langle P \otimes Q, \Delta'_q(R) \rangle_q = \langle P \dot{\downarrow} Q, R \rangle_q = 0$ .  $\square$

**Remark.** The coproduct  $\Delta'_q$  is the coproduct  $\Delta_{(0,0,q,0)}$  of [3].

Let us conclude this section by the case  $q = 0$ .

**Proposition 25** 1. For all plane poset  $P$ :

$$\Delta_0(P) = \sum_{P_1 P_2 = P} P_1 \otimes P_2.$$

2. For any plane posets  $P, Q$ ,  $\langle P, Q \rangle_0 \neq 0$  if, and only if, there exists  $n \in \mathbb{N}$  such that  $P = Q = \cdot^n$ . Consequently, the kernel of the pairing  $\langle -, - \rangle_0$  is the ideal generated by plane posets which are not equal to  $\cdot$ .

**Proof.** 1. By definition of  $\Delta_0$  :

$$\Delta_0(P) = \sum_{\substack{I \text{ biideal of } P, \\ h_{P \setminus I}^I = 0}} (P \setminus I) \otimes I.$$

Let  $I$  be a biideal of  $P$  such that  $h_{P \setminus I}^I = 0$ . Let  $x \in P \setminus I$  and  $y \in I$ . As  $I$  is a biideal,  $x > y$  is not possible, so  $x < y$ . As  $h_{P \setminus I}^I = 0$ ,  $x <_h y$  is not possible, so  $x <_r y$ : finally,  $P = (P \setminus I)I$ . Conversely, if  $P = P_1 P_2$ , then  $P_2$  is a biideal of  $P$ , and  $h_{P \setminus P_2}^{P_2} = h_{P_1}^{P_2} = 0$ .

2.  $\Leftarrow$ . We obviously have  $\iota(\cdot^n) \leq \cdot^n$  and  $\phi(\cdot^n, \cdot^n) = 0$ , so  $\langle \cdot^n, \cdot^n \rangle_0 = 1$ .

$\implies$ . Let us assume that  $\langle P, Q \rangle_q \neq 0$ . Then  $P$  and  $Q$  have the same degree, which we denote by  $n$ . Moreover,  $\iota(P) \leq Q$  and  $\phi(P, Q) = 0$ . Let  $x', y' \in Q$ . We put  $x' = \theta_{P, Q}(x)$  and  $y' = \theta_{P, Q}(y)$ . If  $x' <_h y'$  in  $Q$ , as  $\iota(P) \leq Q$  necessarily  $x <_r y$  in  $P$ , so:

$$\phi(P, Q) \geq \#\{(x, y) \in P^2 \mid x <_r y \text{ and } \theta_{P, Q}(x) <_h \theta_{P, Q}(y)\} > 0.$$

This is a contradiction. Hence, if  $x', y'$  are two different vertices of  $Q$ , they are not comparable for  $\leq_h$ :  $Q = \cdot^n$ . Symmetrically,  $P = \cdot^n$ .  $\square$

### 3.3 Level of a plane poset

**Definition 26** Let  $P$  be a plane poset. Its level is the integer:

$$\ell(P) = \#\{(x, y) \in P^2 \mid x <_r y\}.$$

**Examples.** Here are the level of plane posets of cardinality  $\leq 3$ :

$P$	$\cdot$	$\mathbf{1}$	$\dots$	$\mathbf{1}$	$\vee$	$\wedge$	$\mathbf{1} \cdot$	$\cdot \mathbf{1}$	$\dots$
$\ell(P)$	0	0	1	0	1	1	2	2	3

**Proposition 27** Let  $P$  and  $Q$  be two plane posets. If  $P \leq Q$ , then any path in the Hasse (oriented) graph of the poset  $(\mathcal{PP}, \leq)$  from  $P$  to  $Q$  has length  $\ell(Q) - \ell(P)$ .

**Proof.** *First step.* Let  $R$  be a plane poset of cardinality  $n$ . Then:

$$\ell(R) = \#\{(x, y) \in R^2 \mid x < y\} - \#\{(x, y) \in R^2 \mid x <_h y\} = \frac{n(n-1)}{2} - \#E(R). \quad (1)$$

Consequently, if  $R < S$  in  $\mathcal{PP}(n)$ , then  $E(S) \subsetneq E(R)$ , so  $\ell(R) < \ell(S)$ .

*Second step.* Let  $R$  and  $S$  be two plane posets of the same cardinality  $n$ , such that there is an edge from  $R$  to  $S$  in the Hasse graph of  $(\mathcal{PP}(n), \leq)$ . Then  $R < S$ , so  $E(S) \subsetneq E(R)$ . Let us put  $k = \#E(R) - \#E(S)$ . Note that  $k \geq 1$ . By the first step,  $\ell(S) - \ell(R) = k$ . If  $k \geq 2$ , by lemma 13, there exists  $P_1, \dots, P_{k-1} \in \mathcal{PP}(n)$ , such that  $R \preceq P_1 \preceq \dots \preceq P_{k-1} \preceq S$ . Consequently,  $R < P_1 < \dots < P_{k-1} < S$ , so there is no edge from  $R$  to  $S$  in the Hasse graph: contradiction. So  $k = 1$ .

*Conclusion.* Let  $P = P_0 < P_1 < \dots < P_{k-1} < P_k = Q$  be a path from  $P$  to  $Q$  in the Hasse graph. For all  $0 \leq i \leq k-1$ , there is an edge from  $P_i$  to  $P_{i+1}$ , so  $\ell(P_{i+1}) = \ell(P_i) + 1$  from the second step. Finally,  $\ell(S) = \ell(R) + k$ .  $\square$

**Proposition 28** Let  $P, Q \in \mathcal{PP}(n)$ , such that  $\iota(P) \leq Q$ . Then:

$$\langle P, Q \rangle_q = q^{n(n-1) - \ell(P) - \ell(Q)}.$$

**Proof.** As  $\iota(P) \leq Q$ , for all  $x, y \in P$ ,  $\theta_{P, Q}(x) <_h \theta_{P, Q}(y)$  in  $Q$  implies that  $x <_r y$  in  $P$ . So, with the help of (1):

$$\begin{aligned} \#\{(x, y) \in P^2 \mid x <_r y \text{ and } \theta_{P, Q}(x) <_h \theta_{P, Q}(y)\} &= \#\{(x, y) \in P^2 \mid \theta_{P, Q}(x) <_h \theta_{P, Q}(y)\} \\ &= \#\{(x', y') \in Q^2 \mid x' <_h y'\} \\ &= \frac{n(n-1)}{2} - \ell(Q). \end{aligned}$$

Moreover,  $\iota(Q) \leq \iota^2(P) = P$ , so, for all  $x, y \in P$ ,  $x \leq_h y$  in  $P$  implies that  $\theta_{P, Q}(x) <_r \theta_{P, Q}(y)$  in  $Q$ . So, with the help of (1):

$$\begin{aligned} \#\{(x, y) \in P^2 \mid x <_h y \text{ and } \theta_{P, Q}(x) <_r \theta_{P, Q}(y)\} &= \#\{(x, y) \in P^2 \mid x <_h y\} \\ &= \frac{n(n-1)}{2} - \ell(P). \end{aligned}$$

Summing, we obtain  $\phi(P, Q) = n(n-1) - \ell(P) - \ell(Q)$ .  $\square$

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