Discrete synchronization of hierarchically organized dynamical systems.

Camille Poignard

DISCRETE SYNCHRONIZATION OF HIERARCHICALLY ORGANIZED DYNAMICAL SYSTEMS

CAMILLE POIGNARD

Abstract. We study the synchronization problem of dynamical systems in case of a hierarchical structure among them, of which interest comes from the growing necessity of understanding properties of complex systems, that often exhibit such an organization. Starting with a set of 2^n systems, we define a hierarchical structure inside it by a matrix representing all the steps of a matching process in groups of size 2. This leads us naturally to the synchronization of a Cantor set of systems, indexed by $\{0,1\}^{\mathbb{N}}$: we obtain a global synchronization result generalizing the finite case. In the same context, we deal with this question when some defects appear in the hierarchy, that is to say when some links between certain systems are broken. We prove we can allow an infinite number of broken links inside the hierarchy while keeping a local synchronization, under the condition that these defects are present at the N smallest scales of the hierarchy (for a fixed integer N) and they be enough spaced out in those scales.

1. Introduction

1.1. **Motivations.** Complex systems are systems presenting a very high number of components, all interacting each other, in such a way some macroscopic phenomena emerge, that cannot be deduced from the knowledge of the dynamics inherent to each unit. Understanding what are the main macroscopic properties that usually appear in such general structures is a real important challenge, for their omnipresence in a very large number of domains (see [14]): from natural sciences (with cells, neural networks, pacemaker cells of the heart...), chemistry, computer sciences, social sciences (with economic networks inside a country), to the mathematics of weather and climate.
Among emergent dynamical phenomena, one which is widely observed in concrete life is synchronization, which is the property that all the entities tend to have the same behavior. This phenomenon turns to be essential in many situations: for instance the synchronous transfer of digital or analog signals in communication networks. More theoretically, synchronization has been exhibited in systems of coupled oscillators ([4], [12]), notably when each oscillator presents a chaotic dynamics ([13]). Such a chaotic synchronization is remarkable for in a certain sense it is a way to control the initial erratic behavior (see [8]).

Other interesting properties to understand are more related to the nature of the complex system (rather than its behavior), more precisely to the geometry of its spatial configuration, often resulting from the evolutionary adaptive process at the origin of this system. One that constantly appears is hierarchy, characterized by an organization in cascade: the components (for instance agents in a social context, or particles in a physical one) gather themselves in groups, which gather again in groups of second-level, these last ones matching again in groups of third-level and so on.
The typical example of a hierarchical structure arises from tilings theory: given a finite set of tiles filling a plane with copies of themselves, they may sometimes cluster each other into patches which are finitely distinct at fixed size, in such a way a new tiling can be made by these patches. And so on, this process repeats infinitely, each time considering the last patches obtained as the new set of tiles (see [16], [5],[6]). This is the case of substitutive tilings, where the tiles gather into homothethic copies of themselves as it is shown in Picture 1 (see [5], [6]).
Since the 70's, the study of quasi-crystals (discovered by D. Shechtman in 1982) have unveiled the ubiquity of hierarchical structures, for quasi-crystals turn to naturally own such organizations. This only confirms what was concretely under our eyes from the beginning: the way people live in societies, but also tree shapes,

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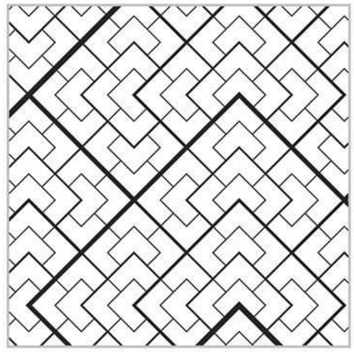


FIGURE 1. L-substitution Tiling

lobes of compound leaves (see [15]), muscular fibers etc...are further manifestations in everyday life of this omnipresence.

1.2. **Objectives.** Despite this ubiquity of hierarchy in nature as well as in purely mathematical objects, very few studies have been done on the dynamics of systems organized in such a way. Almost all the investigations have concerned lattice organizations, even though these appear much less often.
In this context, the goal of this paper is to deal with the synchronization of dynamical systems organized hierarchically.
Notice that some numerical investigations in this direction have been done yet in the past, with purpose the analysis of the route to synchronization. For instance in [2], A. Díaz-Guilera considers a finite set of oscillators described by a particular model issued from the Kuramoto continuous one (see [10]), and shows notably the difference of rapidity presented by the local clusters that merge together time after time, through complete synchronization (see also [1], in which the dynamics of each oscillator is described by the logistic map).

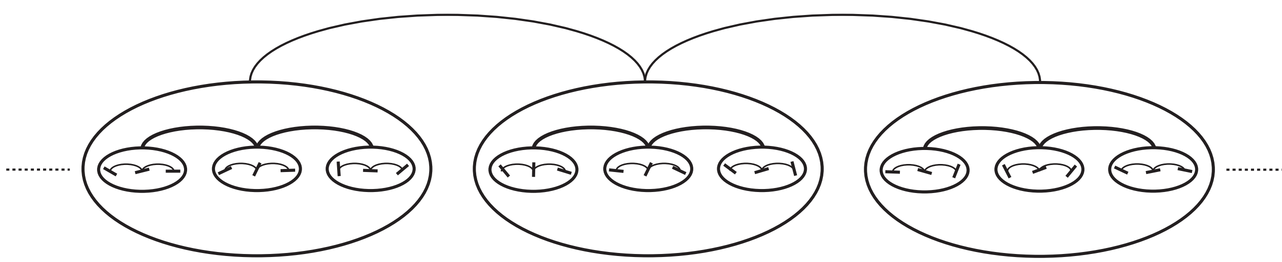


FIGURE 2. Hierarchical structure in case of a matching in groups of size $p = 3$. The strokes represent the states of the particles at a fixed time.

Here we aim at proving rigorous results, and especially (as we are going to see below) to look at the influence of the hierarchy on the dynamics of the whole structure, when this one has some local defects. We look at a finite or infinite set of dynamical systems, viewed as particles (or agents), each one interacting directly with some of them, in such a way exists in the whole set (regarded as a huge dynamical system) an imbricated structure in greater and greater ones. We make the following assumptions on their dynamics, and the interactions between them:

- All the particles evolve according to the same function $\mathrm{f} : \mathrm{K} \to \mathrm{K}$, expanding (so possibly chaotic) on a segment K of $\mathbb{R}$.
- If the particles are in the same state at a fixed time, they remain synchronized at the next times.
- In all the structure, the gathering process happens always between the same number p of elements, and this number is the same for all the stages.
- In each stage of the hierarchy, all the couplings are the same.

There are thus two cases that we deal with: the finite one of p^n particles, and the infinite one of a Cantor set of particles, indexed by the set $\{a_0, \cdots, a_{p-1}\}^{\mathbb{N}}$ of sequences in p letters $a_0, \cdots, a_{p-1}$. We have illustrated the infinite case when $p = 3$ in Figure 2: the links represent the couplings, and the strokes represent the states of the particles at a fixed time, that tend to asymptotically incline in the same direction in case of synchronization.

1.3. **Presentation of the results.** We first consider the finite case 2^n, the general situation p^n being completely similar. To each stage of the hierarchy we associate a *structure matrix* of size 2^n. In accordance with the second and fourth assumptions above, we can assume it is defined by only one parameter and its rows sum to one. We denote by $\mathrm{A}^n_{1,\epsilon_1}, \cdots, \mathrm{A}^n_{n,\epsilon_n}$ these n matrices defined by n parameters. Then we represent the hierarchical structure among the systems by the matrix $\mathrm{B}_{n,\epsilon}$ (where $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$) defined as the product of all the $\mathrm{A}^n_{k,\epsilon_k}$.
Now, the synchronization of our systems writes in terms of the coupled map $\mathrm{G}_{n,\epsilon} = \mathrm{B}_{n,\epsilon} \circ \mathrm{F}_n$, where $\mathrm{F}_n : \mathbb{R}^{2^n} \to \mathbb{R}^{2^n}$ is the vector-valued function of which components are all equal to the function f above: $\mathrm{F}_n = (\mathrm{f}, \cdots, \mathrm{f})$. It happens if we have:

$$\max_{1 \le i,j \le 2^n} \left| \left(\mathrm{G}_{n,\epsilon}\right)^m (X)_{(i)} - \left(\mathrm{G}_{n,\epsilon}\right)^m (X)_{(j)} \right| \underset{m \to \infty}{\longrightarrow} 0,$$

for any initial condition $X = \left(X_{(1)}, \cdots, X_{(2^n)}\right)$ in the entire set K^{2^n} (in which case the synchronization is called *global*) or only for the ones close to the diagonal in K^{2^n} (in which case it is said *local*).
For instance, let us consider the case $n = 2$, that is to say 2^2 particles, illustrated by Figure 3 below. We have two stages defined by two parameters ϵ_1, ϵ_2 and we associate them to the matrices:

$$\mathrm{A}^2_{1,\epsilon_1} = \begin{bmatrix} 1-\epsilon_1 & \epsilon_1 & 0 & 0 \\ \epsilon_1 & 1-\epsilon_1 & 0 & 0 \\ 0 & 0 & 1-\epsilon_1 & \epsilon_1 \\ 0 & 0 & \epsilon_1 & 1-\epsilon_1 \end{bmatrix}, \ \mathrm{A}^2_{2,\epsilon_2} = \begin{bmatrix} 1-\epsilon_2 & 0 & \epsilon_2 & 0 \\ 0 & 1-\epsilon_2 & 0 & \epsilon_2 \\ \epsilon_2 & 0 & 1-\epsilon_2 & 0 \\ 0 & \epsilon_2 & 0 & 1-\epsilon_2 \end{bmatrix}.$$

Then we represent the hierarchical organization among the particles by the matrix $\mathrm{B}_{2,\epsilon} = \mathrm{A}^2_{2,\epsilon_2} \mathrm{A}^2_{1,\epsilon_1}$.

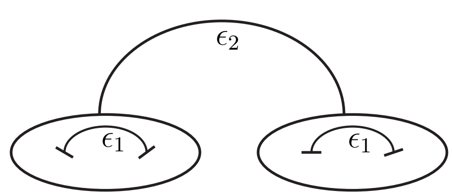


FIGURE 3. The finite case of 2^2 particles.

Finding conditions under which the map $\mathrm{G}_{n,\epsilon}$ synchronizes is a very well known problem that has been fully studied for three decades (see [9], [3], [11], [7], ...). All the interest of our approach is to lead us naturally to the limiting case of the synchronization of a very high number of systems, namely an uncountable one, which constitutes the novelty of this work. Indeed, looking at the limit (when n tends to infinity) of our cascade structure, in which our 2^n elements are ordered by a n-tuple of 0 and 1, leads us to a Cantor set of systems, indexed by the set $\{0,1\}^{\mathbb{N}}$ of sequences in 0 and 1.

We thus define a new setting for the synchronization problem of a Cantor set of systems (in which matrix couplings are replaced by operators, initial conditions become functions, etc...) and using the same approach as B. Fernandez in [3], we obtain a global synchronization result (see Theorem 4.6) that generalizes its theorem, providing the sequence of parameters defining the stages of the hierarchy converge slowly to 1/2. This constitutes the first result of the paper.

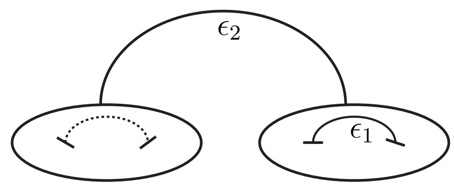


FIGURE 4. The finite case of 2^2 particles in presence of a broken link at the first stage.

A natural extension of this first part of our work is to consider the situation where the entire set of our systems contains some that are uncoupled, regarding by the way whether the hierarchy forces them to synchronize with the other ones or not. For instance in Figure 4 we have illustrated the case of 2^2 particles when two of them are uncoupled (which is represented by a broken link). This comes to replacing the matrix $\mathrm{A}^2_{1,\epsilon_1}$ above by the following one:

$$\tilde{\mathrm{A}}^2_{1,\epsilon_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-\epsilon_1 & \epsilon_1 \\ 0 & 0 & \epsilon_1 & 1-\epsilon_1 \end{bmatrix}.$$

If there is such a forcing by the hierarchy, can we authorize a high number of uncoupled systems, namely an infinite one in our Cantor set of systems? How do their positions influence the synchronization?
We give an answer to these questions by showing that providing the uncoupled systems are only present at the smallest N scales of the hierarchy (for a fixed integer N), and that they be enough spaced out in them, then local synchronization takes place on a neighborhood of the diagonal, that does not depend on the position of the broken links inside the last N scales (see Theorem 4.14). This constitutes the second result of this paper.

1.4. **Outline of the article.** The rest of the article is organized in three parts. Part 2 contains the basic notations. In Part 3, we consider the finite case of 2^n systems, which constitutes the preliminaries for the last part about the case of a Cantor set. After having presented the global synchronization result in Subsection 3.1, we deal with the situation when some links are broken in the hierarchy (Subsection 3.2). Here, there is no chance to obtain a global synchronization result, for the new dynamical system $\tilde{\mathrm{G}}_{n,\epsilon} = \tilde{\mathrm{B}}_{n,\epsilon} \circ \mathrm{F}_n$ admits in general some fixed points outside the diagonal (see Example 3.10). We define in Paragraph 3.2.b the notion of admissible nested family of submatrices for $\tilde{\mathrm{B}}_{n,\epsilon}$, for which the broken links stay at the smallest N scales of the hierarchy (for a fixed integer $N \geq 1$), while being enough spaced out in them. In the hierarchical diagrams containing such families, the number of broken links can go to infinity when n does. We prove (Lemma 3.11) that for any dynamical system $\tilde{\mathrm{G}}_{n,\epsilon}$ (with $n \geq N+1$) defined by a coupling matrix having an admissible nested family of submatrices, local synchronization takes place on a neighborhood of the diagonal. Besides, we prove that the size of this neighborhood does not depend on n, which will be the crucial point for the study of the same question in the infinite case.

In the last section (Part 4), we consider the limiting case of a Cantor set of systems, indexed by $\{0,1\}^{\mathbb{N}}$. This time, initial conditions are no more vectors but functions from $\{0,1\}^{\mathbb{N}}$ to $\mathbb{R}$. In accordance with Part 3, we define the hierarchical structure among our systems as the limit $\mathcal{U}_\epsilon$ of the composition of all the structure operators replacing the previous structure matrices (where ϵ stands for the sequence of parameters $(\epsilon_k)_{k\geq 1}$ associated to the stages of the hierarchy). After having justified the existence of $\mathcal{U}_\epsilon$ on the set of continuous functions on $\{0,1\}^{\mathbb{N}}$, we prove our first result asserting that global synchronization takes place in such a setting, under the condition that ϵ converge slowly to $1/2$ (Theorem 4.6).
Finally in Subsection 4.2, we deal with the case of an infinity of broken links inside the hierarchy. This time, for every $n \geq N+1$, we define our operators with broken links on the closed-open subsets of $\{0,1\}^{\mathbb{N}}$ of size n, so that their composition $\tilde{\mathcal{W}}_{n,\epsilon}$ act in an exactly similar way as the matrices $\tilde{\mathrm{B}}_{n,\epsilon}$ do on $\mathbb{R}^{2^n}$. This permits us to define the notion of admissible nested families of sub-operators. Then we establish the existence of the new operator $\tilde{\mathcal{W}}_\epsilon$, obtained as the limit of the operators $\tilde{\mathcal{W}}_{n,\epsilon}$ (see Lemma 4.13). From this, we prove our second result ensuring that local synchronization takes place if for every $n \geq N+1$, $\tilde{\mathcal{W}}_{n,\epsilon}$ has an admissible nested family of sub-operators and if the sequence of parameters ϵ converge rapidly to $1/2$ (Theorem 4.14). We finish the paper by giving the general versions of our results (Corollaries 4.15 and 4.17) in case of a matching in groups of size p in our construction, that is to say on a Cantor set $\{a_0, \cdots, a_{p-1}\}^{\mathbb{N}}$ defined by an alphabet with $p \geq 3$ letters.

2. Basic notations and definitions

2.1. **The system.** In all the text, we consider a segment K of $\mathbb{R}$, stable under a map $\mathrm{f}:\mathbb{R}\to\mathbb{R}$ (all the results we present are easily extendable on $\mathbb{R}^{2^n}$, as mentioned at the end of the text). The map f is assumed to be of class $\mathcal{C}^2$ and satisfies $\sup_{z\in \mathrm{K}} \left|\mathrm{f}'(z)\right| > 1$. In particular f can have a chaotic dynamics.

Example 2.1. *The typical example is obviously the logistic map defined by* $\mathrm{f}(x) = \mu x\,(1-x)$ *for* x *in the segment* $[0,1]$*, which exhibits chaos (strictly positive topological entropy) for values of parameter beyond the threshold* $\mu = 3.57$*.*

We are interested in discrete dynamical systems of the form $X_{k+1} = \mathrm{A}\circ\mathrm{F}\,(X_k)$ where A is a linear map and F is a non-linear one, the components of F being equal to the same real-valued function f. Depending on the space on which this system will be considered, the linear part A will be either an endomorphism acting on $\mathbb{R}^{2^n}$ or an operator acting on the infinite dimensional space of continuous functions on a Cantor set X (see Part 4).
The goal is to study the synchronization of such a system, that is to say its convergence to the diagonal of the underlying space, that will be the set of vectors of which coordinates are all the same (denoted by $\mathfrak{I}_{2^n}$) in Part 3, and the set of constant functions on X (denoted by $\mathfrak{I}$) in Part 4.

In the finite dimensional case, we use the notation A either to mention the endomorphism or its matrix represented in the canonical basis $(\mathrm{e}_j)_{1\leq j\leq 2^n}$ of $\mathbb{R}^{2^n}$. The hierarchical structure will lead to the use of the Kronecker product (that we will denote by $\otimes$) to define this endomorphism. Given two square matrices $\mathrm{M} = (m_{i,j})_{1\leq i,j\leq m}$ and $\mathrm{N} = (n_{i,j})_{1\leq i,j\leq n}$ respectively of size m and n, the Kronecker product $\mathrm{M}\otimes\mathrm{N}$ is the matrix of size mn defined by the equality:

$$M\otimes N = \begin{bmatrix} m_{1,1}\mathrm{N} & \dots & m_{1,n}\mathrm{N} \\ \vdots & & \vdots \\ m_{n,1}\mathrm{N} & \dots & m_{n,n}\mathrm{N} \end{bmatrix}.$$

Obviously the same definition works for non-square matrices but we recall it just in this case that interests us.

2.2. **Vectors, matrices and norms.** Each element X of $\mathbb{R}^{2^n}$ will be denoted by a capital letter and its coordinates in the canonical basis $(\mathrm{e}_j)_{1\leq j\leq 2^n}$ in the following way $\left(X_{(1)},\cdots,X_{(2^n)}\right)$. Given such an element X in $\mathbb{R}^{2^n}$, we will naturally consider the associated sum-vector X_Σ belonging to $\mathfrak{I}_{2^n}$, defined by the equality:

$$X_\Sigma = \left(\frac{\left(X_{(1)}+\cdots+X_{(2^n)}\right)}{2^n},\cdots,\frac{\left(X_{(1)}+\cdots+X_{(2^n)}\right)}{2^n}\right).$$

By convexity the vector X_Σ belongs to K^{2^n} whenever X is in this set.

Given a matrix A, the notations χ_{A}, $\mathfrak{S}(A)$, $\det(A)$, $\mathrm{rank}(A)$ will classically stand for the characteristic polynomial, spectrum, determinant and rank of A. The algebraic multiplicity of an eigenvalue in χ_A will be denoted by the symbol $^{\otimes}$. Moreover, the notation $[A]_{i_1,\cdots,i_k}$ will stand for the square submatrix of size $i_k - i_1 + 1$, defined by the elements $a_{i,j}$ of A for i, j varying in $\{i_1, \cdots, i_k\}$.
Two matrices will be constantly used in the first part of the paper. These are:

$$\mathcal{T}_\epsilon = \begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix},$$

for a real parameter ϵ, and

$$\mathbf{1}_k = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix},$$

of size k. We define similarly $\mathbf{0}_k$.

Concerning the norms, we denote by $\big|\big|\cdot\big|\big|_{\infty,n}$ the uniform norm on $\mathbb{R}^{2^n}$, defined by the relation:

$$\big|\big|X\big|\big|_{\infty,n} = \max_{1\leq i\leq 2^n} \big|X_{(i)}\big|.$$

In the second part, the same notation $\big|\big|\cdot\big|\big|_\infty$ will also stand for the uniform norm on the complete space $\mathscr{C}(X,\mathbb{R})$ of continuous functions on a Cantor set X.

3. The finite case

In all this part we fix an integer $n \geq 1$, and deal with the synchronization problem in $\mathbb{R}^{2^n}$.

3.1. **Global synchronization.** As explained in the introduction, we consider the dynamics of particles evolving in time, and interacting hierarchically with each other according to a matching process in groups of size 2 at all the stages of the hierarchy. Namely, setting $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ the n-tuple of parameters defining each of them, this organization is represented by the matrix $\mathrm{B}_{n,\epsilon} = \mathrm{A}^n_{n,\epsilon_n} \cdots \mathrm{A}^n_{1,\epsilon_1}$, where we have set:

$$\mathrm{A}^n_{1,\epsilon_1} = \begin{bmatrix} \mathcal{T}_{\epsilon_1} & & \\ & \ddots & \\ & & \mathcal{T}_{\epsilon_1} \end{bmatrix}, \mathrm{A}^n_{2,\epsilon_2} = \begin{bmatrix} \mathcal{T}_{\epsilon_2}\otimes I_2 & & \\ & \ddots & \\ & & \mathcal{T}_{\epsilon_2}\otimes I_2 \end{bmatrix}, \cdots, \mathrm{A}^n_{n,\epsilon_n} = \mathcal{T}_{\epsilon_n}\otimes I_{2^{n-1}}.$$

Each of these structure matrices corresponds to a particular *scale* of synchronization: the first one corresponds to the smallest *scale* (since we act on the systems of smallest dimension, that is of dimension two), while the n^{th} corresponds to the greatest one, with an action on the systems of largest dimension, that is 2^{n-1}. We have mentioned the integer n in the notation $\mathrm{A}^n_{k,\epsilon_k}$, since our goal is to understand what happens when n goes to infinity.
Of course the hierarchical structure we have defined is not used here for the synchronization, as other coupling matrices than $\mathrm{B}_{n,\epsilon}$ could have been used. However, it is the hierarchy which will allow us to synchronize a Cantor set of systems (see Part 4).

With these notations, our problem writes in terms of the coupled map $\mathrm{G}_{n,\epsilon} = \mathrm{B}_{n,\epsilon}\circ \mathrm{F}_n$. The following result extends the one obtained by B.Fernandez (see [3]):

Theorem 3.1. *Assuming the n-tuple $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ defining the structure matrices $\mathrm{A}^n_{1,\epsilon_1}, \cdots, \mathrm{A}^n_{n,\epsilon_n}$ satisfies the relation:*

$$\forall k \in \{1, \cdots, n\},\ |1-2\epsilon_k| \sup_{z\in \mathrm{K}} \big|\mathrm{f}'(z)\big| < 1,$$

then the dynamical system $\mathrm{G}_{n,\epsilon} = \mathrm{B}_{n,\epsilon} \circ \mathrm{F}_n$ *globally synchronizes, i.e we have:*

$$\forall X \in \mathrm{K}^n, \ \max_{1\leq i,j\leq 2^n} \left|{\mathrm{G}_{n,\epsilon}}^m\left(X\right)_{(i)} - {\mathrm{G}_{n,\epsilon}}^m\left(X\right)_{(j)}\right| \underset{m\to\infty}{\longrightarrow} 0.$$

Proof. Following [3] we introduce:

$$\mathrm{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } \mathrm{C}_1 = \begin{bmatrix} \mathrm{J} & & \\ & \ddots & \\ & & \mathrm{J} \end{bmatrix}, \mathrm{C}_2 = \begin{bmatrix} \mathrm{J}\otimes I_2 & & \\ & \ddots & \\ & & \mathrm{J}\otimes I_2 \end{bmatrix}, \cdots, \mathrm{C}_n = \mathrm{J}\otimes I_{2^{n-1}}.$$

For every integers k, l in $\mathbb{N}$, the matrices C_l and $\mathrm{A}^n_{k,\epsilon_k}$ commute. Then it clearly comes, for every k in $\{1, \cdots, n\}$ and every X in K:

$$\left|\left|\mathrm{G}_{n,\epsilon}\left(X\right) - \mathrm{C}_k\left(\mathrm{G}_{n,\epsilon}\left(X\right)\right)\right|\right|_\infty \leq |1-2\epsilon| \sup_{z\in\mathrm{K}} \left|\mathrm{f}'(z)\right| \cdot \left|\left|X - \mathrm{C}_k\left(X\right)\right|\right|_{\infty,n},$$

from which we get, for every $m \geq 0$:

$$\max_{1\leq i,j\leq 2^n} \left|{\mathrm{G}_{n,\epsilon}}^m\left(X\right)_{(i)} - {\mathrm{G}_{n,\epsilon}}^m\left(X\right)_{(j)}\right| \leq \sum_{k=0}^{n} \left(|1-2\epsilon| \sup_{z\in\mathrm{K}}\left|\mathrm{f}'(z)\right|\right)^m \cdot \left|\left|X - \mathrm{C}_k\left(X\right)\right|\right|_{\infty,n}$$

and the result. □

Remark 3.2. *As explained in* [3]*, this result also works for a Lipschitz function* f *on an interval which can be the whole set* $\mathbb{R}$*. In this case the Lipschitz constant replace the supremum of the derivative in the assumption of the theorem.*

3.2. **Local synchronization in presence of broken links.**

3.2.a. *Position of the problem.* Now we address the following question: in the set of our 2^n particles, suppose there is a subset composed of uncoupled elements, can we still make them synchronize with the other ones? In other words, if there are some links that are broken in the hierarchical structure constructed above (see Figure 5), can we still synchronize the whole set of particles?

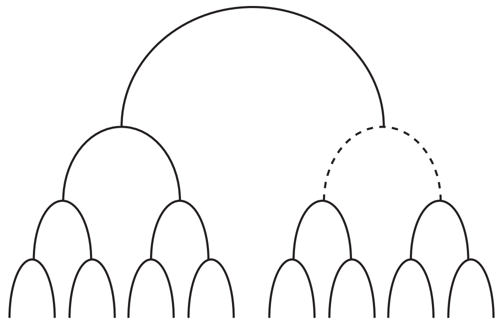
The possible synchronization depends on the new structure matrices $\tilde{\mathrm{A}}^n_{k,\epsilon_k}$ that no more commute since some blocks $\mathcal{T}_2 \otimes I_{2^{k-1}}$ (with $1 \leq k \leq n$) have been replaced by blocks I_{2^k}. For instance, the structure matrices associated to the scheme of Figure 5(a) are $\mathrm{A}^n_{1,\epsilon_1}, \cdots, \mathrm{A}^n_{n-2,\epsilon_{n-2}}$ and for the last two ones:

$$\tilde{\mathrm{A}}^n_{n-1,\epsilon_{n-1}} = \begin{bmatrix} \mathcal{T}_{\epsilon_{n-1}} \otimes I_{2^{n-2}} & \\ & \mathrm{I}_{2^{n-2}} \end{bmatrix}, \mathrm{A}^n_{n,\epsilon_n} = \mathcal{T}_{\epsilon_n} \otimes I_{2^{n-1}}.$$

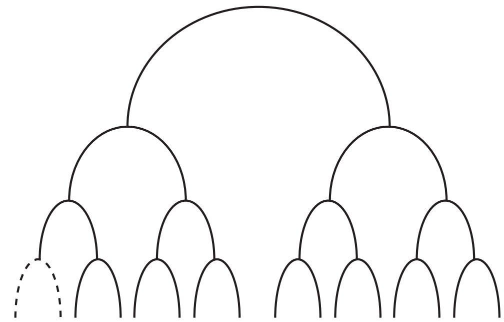
Computing the spectrum of the new matrix $\tilde{\mathrm{B}}_{n,\epsilon} = \tilde{\mathrm{A}}^n_{n,\epsilon_n} \cdots \tilde{\mathrm{A}}^n_{1,\epsilon_1}$ is complicated in general because of this loss of commutativity. But as the reals ϵ_k are always taken very close to $1/2$ in our setting, we compute it in the case where we have $\epsilon = (1/2, \cdots, 1/2)$, having in mind the continuity of the eigenvalues (in the coefficients of the matrices).
The following lemma presents the main different schemes met in this case.

Lemma 3.3. *Let us consider the different structure matrices* $\tilde{\mathrm{B}}_{n,\frac{1}{2}}$ *associated to the schemes of Figure 5 . Then all those matrices* $\tilde{\mathrm{B}}_{n,\frac{1}{2}}$ *are diagonalizable and we have:*

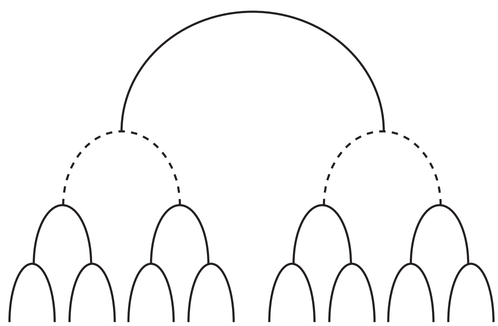
1. *The spectrum associated to the diagram of Figure 5(a) is* $\{1, \frac{1}{2}, 0^{\otimes 2^n-2}\}$.
2. *The spectrum associated to the diagram of Figure 5(b) is* $\{1, 1, 0^{\otimes 2^n-2}\}$.
3. *The spectrum associated to the diagram of Figure 5(c) is* $\{1, 1, 0^{\otimes 2^n-2}\}$.



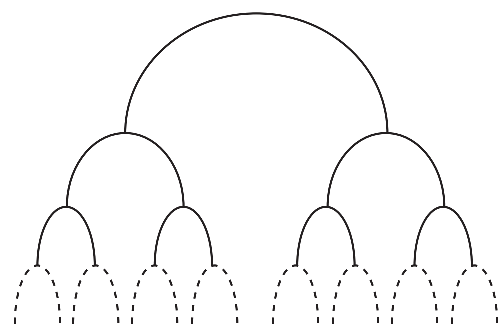
(a) One broken link at the scale $n-1$.



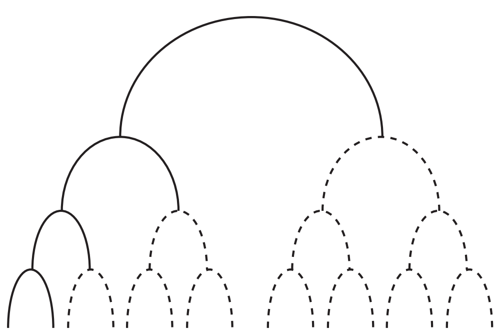
(b) One broken link at the 1^{st} scale.



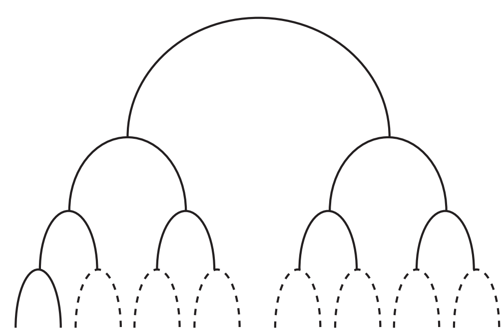
(c) 2 broken links at the scale $n-1$.



(d) 2^{n-1} broken links at the 1^{st} scale.



(e) $2^n-(n+1)$ broken links at the first $n-1$ scales.



(f) $2^{n-1}-1$ broken links at the 1^{st} scale.

FIGURE 5. Diagrams with broken links, in the case $n=4$.

(4) *The spectrum associated to the diagram of Figure 5(d) is* $\{1, \frac{1}{2^{n-1}}, 0^{\otimes 2^n-2}\}$.

(5) *The spectrum of the diagram of Figure 5(e) is* $\{1, \left(\frac{2^l-1}{2^l}\right)^{\otimes 2^{n-l-1}}, 0^{\otimes 2^n-1},\ l = 1, \cdots, n-1\}$.

(6) *The spectrum of the diagram of Figure 5(f) is* $\{1, \left(\frac{2^{n-1}-1}{2^{n-1}}\right), 0^{\otimes 2^n-2}\}$.

The proof of Lemma 3.3 makes use of the following little lemma:

Lemma 3.4. *Let* (A, B) *be two square matrices in* $\mathbb{R}^{k^2}$ *(for an integer* $k \geq 1$*) and* Z *the following matrix of size* $2k$*:*

$$\mathrm{Z} = \begin{bmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{A} & \mathrm{B} \end{bmatrix}.$$

Assume that $\mathrm{A}+\mathrm{B}$ *be diagonalizable, and that we have:* $\mathrm{rank}\,(\mathrm{Z}) = \mathrm{rank}\,(\mathrm{A}+\mathrm{B})$*. Then,* Z *is diagonalizable.*

Proof. Let us call $(\lambda_1, \cdots, \lambda_p)$ the distinct eigenvalues of $\mathrm{A}+\mathrm{B}$ and $(\alpha_1, \cdots, \alpha_p)$ the dimensions of their associated eigenspaces.
Suppose that 0 is an eigenvalue of $\mathrm{A}+\mathrm{B}$, say $\lambda_1 = 0$ (this will be the case for our structure matrices $\tilde{\mathrm{B}}_{n,\epsilon}$). We have, by the Schur theorem:

$$\chi_{\mathrm{Z}} = \det\left(-X\,\mathrm{I}_k\right)\chi_{\mathrm{A}+\mathrm{B}},$$

so the spectra are the same for both Z and $\mathrm{A}+\mathrm{B}$. By assumption, the dimension of the eigenspace associated to 0 for the matrix Z is $k+\alpha_1$, which is the multiplicity of 0 in the characteristic polynomial χ_{Z}.
The same happens to the other eigenvalues since for any $X = (X_1, X_2)$ we have:

$$\mathrm{Z}X = \lambda_i X \iff \{(\mathrm{A}+\mathrm{B})\,X_1 = \lambda_i X_1,\ X_1 = X_2\}.$$

Thus the matrix Z is diagonalizable.
The same reasoning works for the other case where 0 is not an eigenvalue of $\mathrm{A}+\mathrm{B}$. $\square$

proof of Lemma 3.3. We only do the case (1) since the demonstrations of the other ones are similar. From the definition of our hierarchical structure it clearly comes we have, for every $n \geq 1$:

$$\tilde{\mathrm{B}}_{n+1,\frac{1}{2}} = \left(\begin{array}{c|c} \frac{1}{2^{n+1}}\begin{pmatrix} \mathbf{1}_{2^{n-1}} & \mathbf{1}_{2^{n-1}} \\ \mathbf{1}_{2^{n-1}} & \mathbf{1}_{2^{n-1}} \end{pmatrix} & \frac{1}{2^n}\begin{pmatrix} \mathbf{1}_{2^{n-1}} & \mathbf{0}_{2^{n-1}} \\ \mathbf{0}_{2^{n-1}} & \mathbf{1}_{2^{n-1}} \end{pmatrix} \\ \hline \frac{1}{2^{n+1}}\begin{pmatrix} \mathbf{1}_{2^{n-1}} & \mathbf{1}_{2^{n-1}} \\ \mathbf{1}_{2^{n-1}} & \mathbf{1}_{2^{n-1}} \end{pmatrix} & \frac{1}{2^n}\begin{pmatrix} \mathbf{1}_{2^{n-1}} & \mathbf{0}_{2^{n-1}} \\ \mathbf{0}_{2^{n-1}} & \mathbf{1}_{2^{n-1}} \end{pmatrix} \end{array}\right).$$

By commutativity the sum of the two square matrices defining $\tilde{\mathrm{B}}_{n+1,\frac{1}{2}}$ is diagonalizable with eigenvalues $1, 1/2$ and 0. The result follows from Lemma 3.4. $\square$

This lemma gives some important informations for the second part of our work.
As we could have expected, looking at the cases (3) and (4), we see that synchronization is not possible if there is only broken links at a given stage. According to the cases (5), (6) the more the broken links are numerous, the more the eigenvalues associated to the transverse directions of the diagonal are close to the value 1. In (5), the diagram contains $2^n - (n+1)$ broken links meanwhile the one associated to the case (6) contains $2^{n-1} - 1$ broken ones (recall that the total number of links in any diagram is $2^n - 1$). As a consequence the structures (5) and (6) will not be chosen when we deal with the infinite case in Part 4.
More interesting is the case (1) which reveals the emphasis of the scale at which stand the broken links: to make the eigenvalues associated to the transverse direction tend to 0 as n goes to infinity, they must not be placed at the greatest scales of the diagram, i.e they must not link the biggest sub-systems of our set of particles. Lastly, the case (2) tells us the rate of convergence to zero of the transverse eigenvalues associated to the matrix (corresponding to this case) cannot be better than $1/2^n$.

With these informations in mind, the question is thus to know how to put a number of broken links inside the hierarchy that tends to infinity when n tends to infinity, while keeping the coupling matrix diagonalizable with eigenvalues associated to the directions transverse to the diagonal tending to zero.
(Note that it is easy to construct some examples of diagrams with broken links for which the coupling matrix is not diagonalizable: such an example is given by the one associated to the following Figure 6.)
Lemma 3.9 above is an answer to this question, under the following condition: the broken links must stand at the first N stages of the hierarchy (for a fixed integer N), and they must be enough spaced out inside them. In order to establish this lemma, we need the following one.

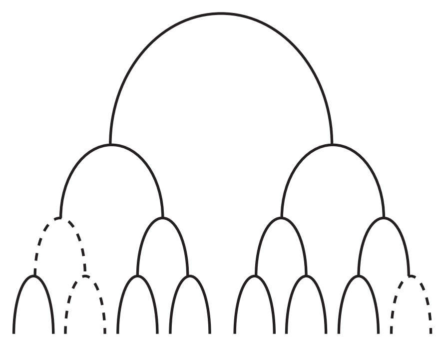


FIGURE 6. A diagram with broken links for which the associated coupling matrix is not diagonalizable.

Lemma 3.5. *Let $N \geq 1$ an integer, and consider two coupling matrices of the same form $\tilde{\mathrm{B}}_{N,\frac{1}{2}} = \tilde{\mathrm{A}}^N_{N,1/2} \cdots \tilde{\mathrm{A}}^N_{1,1/2}$ and $\tilde{\mathrm{B}}'_{N,\frac{1}{2}} = \tilde{\mathrm{A}}'^N_{N,1/2} \cdots \tilde{\mathrm{A}}'^N_{1,1/2}$, where $\tilde{\mathrm{B}}_{N,\frac{1}{2}}, \tilde{\mathrm{B}}'_{N,\frac{1}{2}}$ both contain one broken link, of which position is arbitrary (it can even be at the last Nth stage, that is in $\tilde{\mathrm{A}}^N_{N,1/2}$ or in $\tilde{\mathrm{A}}'^N_{N,1/2}$). Then the matrix $\frac{1}{2}\tilde{\mathrm{B}}_{N,\frac{1}{2}} + \frac{1}{2}\tilde{\mathrm{B}}'_{N,\frac{1}{2}}$ is diagonalizable with following spectrum:*

$$\mathfrak{S}\left(\frac{1}{2}\tilde{\mathrm{B}}_{N,\frac{1}{2}} + \frac{1}{2}\tilde{\mathrm{B}}'_{N,\frac{1}{2}}\right) = \{1, \lambda_1, \cdots, \lambda_N, 0^{\otimes 2^N - N - 1}\},$$

where $\lambda_i \leq 1$, for every $i = 1, \cdots N$.

Proof. We make an induction on N.
For $N = 1$, there is nothing to do, since we have $\frac{1}{2}\tilde{\mathrm{B}}_{1,\frac{1}{2}} + \frac{1}{2}\tilde{\mathrm{B}}'_{1,\frac{1}{2}} = I_2$.
Assume the result is true for an integer $N \geq 2$, and consider two matrices $\tilde{\mathrm{B}}_{N+1,\frac{1}{2}}$, $\tilde{\mathrm{B}}'_{N+1,\frac{1}{2}}$ both containing one broken link at an arbitrary position.

- Assume in a first case that the broken link of $\tilde{\mathrm{B}}_{N+1,\frac{1}{2}}$ is in the last scale of the hierarchy, that is: $\tilde{\mathrm{A}}^{N+1}_{N+1,1/2} = I_{2^{N+1}}$. Then we have:

$$\tilde{\mathrm{B}}_{N+1,\frac{1}{2}} = \left(\begin{array}{c|c} \frac{1}{2^N}\mathbf{1}_{2^N} & \mathbf{0}_{2^N} \\ \hline \mathbf{0}_{2^N} & \frac{1}{2^N}\mathbf{1}_{2^N} \end{array}\right).$$

If $\tilde{\mathrm{B}}'_{N+1,\frac{1}{2}} = \tilde{\mathrm{B}}_{N+1,\frac{1}{2}}$, the result is clear. Otherwise, by symmetry we can assume (without loss of generality) that $\tilde{\mathrm{B}}'_{N+1,\frac{1}{2}}$ has the following form:

$$\tilde{B'}_{N+1,\frac{1}{2}} = \left(\begin{array}{c|c} \frac{1}{2}\tilde{B'}_{N,\frac{1}{2}} & \frac{1}{2^{N+1}}\mathbf{1}_{2^N} \\ \hline \frac{1}{2}\tilde{B'}_{N,\frac{1}{2}} & \frac{1}{2^{N+1}}\mathbf{1}_{2^N} \end{array}\right),$$

where $\tilde{B'}_{N,\frac{1}{2}}$ contains one broken link. Clearly $\tilde{B'}_{N,\frac{1}{2}}$ is diagonalizable (it is easy to see this as in Lemma 3.3), and as it is one row-sum and one column-sum, it commutes with $\mathbf{1}_{2^N}$. Thus by Lemma 3.4, $\tilde{B'}_{N+1,\frac{1}{2}}$ is diagonalizable, and its eigenvalues are those of $\frac{1}{2}\tilde{B'}_{N,\frac{1}{2}}$. Once again, we remark that $\tilde{B'}_{N+1,\frac{1}{2}}$ and $\tilde{B}_{N+1,\frac{1}{2}}$ commute, thus the term $\dfrac{1}{2}\tilde{B}_{N+1,\frac{1}{2}} + \dfrac{1}{2}\tilde{B'}_{N+1,\frac{1}{2}}$ is diagonalizable. The spectrum of this matrix has the desired form, since $\tilde{B'}_{N,\frac{1}{2}}$ have only two non zero eigenvalues.

- The second case to consider is when both $\tilde{B}_{N+1,\frac{1}{2}}$ and $\tilde{B'}_{N+1,\frac{1}{2}}$ have their broken link which are not at the last $(N+1)$th scale. By symmetry, we can write, without loss of generality:

$$\tilde{B}_{N+1,\frac{1}{2}} = \left(\begin{array}{c|c} \frac{1}{2}\tilde{B}_{N,\frac{1}{2}} & \frac{1}{2^{N+1}}\mathbf{1}_{2^N} \\ \hline \frac{1}{2}\tilde{B}_{N,\frac{1}{2}} & \frac{1}{2^{N+1}}\mathbf{1}_{2^N} \end{array}\right),$$

and

$$\tilde{B'}_{N+1,\frac{1}{2}} = \left(\begin{array}{c|c} \frac{1}{2^{N+1}}\mathbf{1}_{2^N} & \frac{1}{2}\tilde{B'}_{N,\frac{1}{2}} \\ \hline \frac{1}{2^{N+1}}\mathbf{1}_{2^N} & \frac{1}{2}\tilde{B'}_{N,\frac{1}{2}} \end{array}\right).$$

Then we have to verify the assumption on the rank required by Lemma 3.4, i.e that the following holds:

$$\operatorname{rank}\left(\frac{1}{2}\tilde{B}_{N+1,\frac{1}{2}} + \frac{1}{2}\tilde{B'}_{N+1,\frac{1}{2}}\right) = \operatorname{rank}\left(\frac{1}{4}\tilde{B}_{N,\frac{1}{2}} + \frac{1}{4}\tilde{B'}_{N,\frac{1}{2}} + \frac{1}{2^{N+1}}\mathbf{1}_{2^N}\right)$$

or in other words that we have:

(1) $$\operatorname{rank}\left(\frac{1}{2}\tilde{B}_{N+1,\frac{1}{2}} + \frac{1}{2}\tilde{B'}_{N+1,\frac{1}{2}}\right) = \operatorname{rank}\left(\frac{1}{4}\tilde{B}_{N,\frac{1}{2}} + \frac{1}{4}\tilde{B'}_{N,\frac{1}{2}}\right),$$

the last equality coming from the fact the matrices $\tilde{B}_{N,\frac{1}{2}}$ and $\tilde{B'}_{N,\frac{1}{2}}$ are one-column sum.
We first notice that whatever the positions of the broken links inside $\tilde{B}_{N,\frac{1}{2}}$ and $\tilde{B'}_{N,\frac{1}{2}}$, we have:

$$\operatorname{rank}\left(\frac{1}{2}\tilde{B}_{N+1,\frac{1}{2}} + \frac{1}{2}\tilde{B'}_{N+1,\frac{1}{2}}\right) = \operatorname{rank}\left(\begin{bmatrix}\tilde{B}_{N,\frac{1}{2}} & \tilde{B'}_{N,\frac{1}{2}}\end{bmatrix}\right).$$

Now, assume that the broken link of $\tilde{B}_{N,\frac{1}{2}}$ is at the (last) Nth scale. If this is also the case for $\tilde{B'}_{N,\frac{1}{2}}$, then we can conclude the relation (1) holds. Otherwise, the broken link of $\tilde{B'}_{N,\frac{1}{2}}$ stands in the first $N-1$ scales, from which we get:

$$\operatorname{rank}\left(\begin{bmatrix}\tilde{B}_{N,\frac{1}{2}} & \tilde{B'}_{N,\frac{1}{2}}\end{bmatrix}\right) = \operatorname{rank}\left(\begin{array}{c|c} \frac{1}{2^{N-1}}\mathbf{1}_{2^{N-1}} & \frac{1}{2}\tilde{B'}_{N-1,\frac{1}{2}} \\ \hline \mathbf{0}_{2^{N-1}} & \frac{1}{2}\tilde{B'}_{N-1,\frac{1}{2}} \end{array}\right),$$

and

$$\operatorname{rank}\left(\tilde{\mathrm{B}}_{N,\frac{1}{2}} + \tilde{\mathrm{B}}'_{N,\frac{1}{2}}\right) = \operatorname{rank}\left(\begin{array}{c|c} \dfrac{1}{2^{N-1}}\mathbf{1}_{2^{N-1}} & \dfrac{1}{2}\tilde{\mathrm{B}}'_{N-1,\frac{1}{2}} \\ \hline \dfrac{1}{2^{N}}\mathbf{1}_{2^{N-1}} & \dfrac{1}{2}\tilde{\mathrm{B}}'_{N-1,\frac{1}{2}} \end{array}\right),$$

which implies the relation (1) again.
Then, by Lemma 3.4 and by assumption of induction, we conclude again that $\frac{1}{2}\tilde{\mathrm{B}}_{N+1,\frac{1}{2}} + \frac{1}{2}\tilde{\mathrm{B}}'_{N+1,\frac{1}{2}}$ is diagonalizable. Its spectrum is the same as $\frac{1}{4}\tilde{\mathrm{B}}_{N,\frac{1}{2}} + \frac{1}{4}\tilde{\mathrm{B}}'_{N,\frac{1}{2}}$, so has the desired form.

□

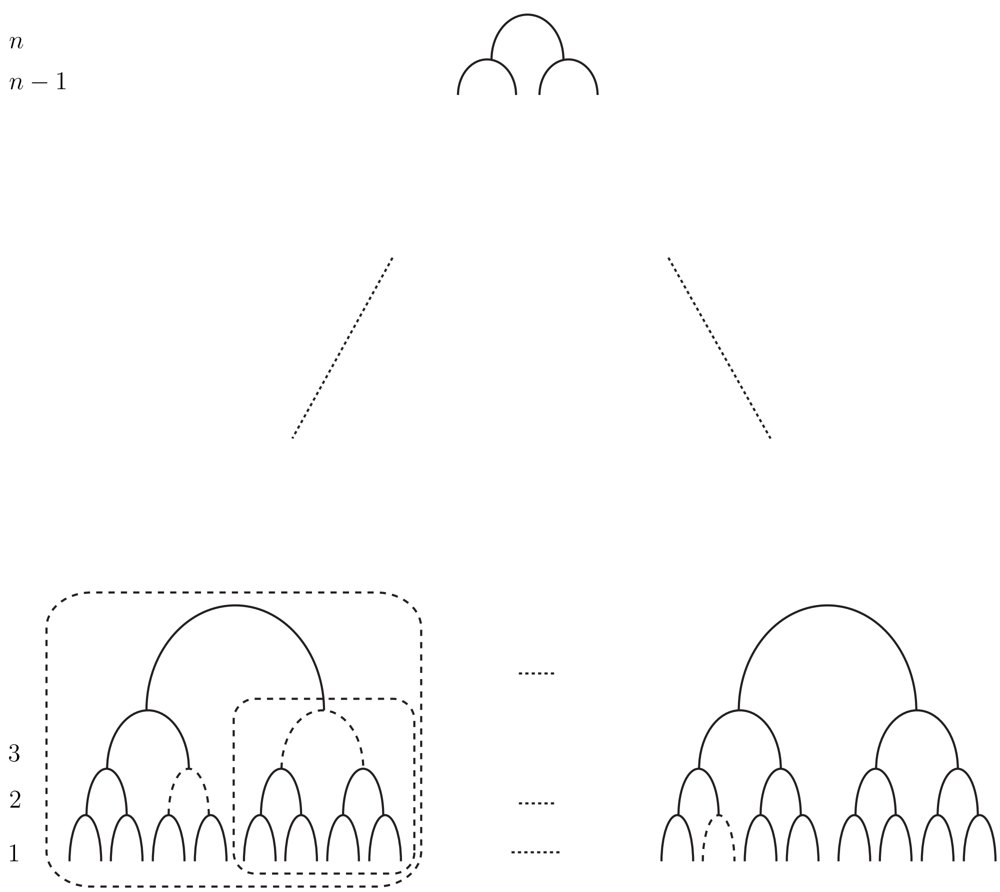


FIGURE 7. A nested family of blocks in the diagram of synchronization with broken links at the first 3 stages.

Definition 3.6. *Let us fix an integer $N \geq 1$ and take a tuple $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ in $[0,1]^n$, for an integer $n \geq N+1$. Consider the coupling matrix*

$$\tilde{\mathrm{B}}_{n,\epsilon} = \mathrm{A}^n_{n,\epsilon_n} \cdots \mathrm{A}^n_{N+1,\epsilon_{N+1}} \tilde{\mathrm{A}}^n_{N,\epsilon_N} \cdots \tilde{\mathrm{A}}^n_{1,\epsilon_1},$$

containing some broken links (i.e identity blocks I_{2^l} with $l \leq N-1$) at the first N stages of its associated diagram, that is to say in $\tilde{\mathrm{A}}^n_{N,\epsilon_N}, \cdots, \tilde{\mathrm{A}}^n_{1,\epsilon_1}$. For each $k = N, \cdots, n-1$ let us denote by $\mathscr{D}_k$ the following family of submatrices of size 2^k:

$$\mathscr{D}_k = \Big\{ \frac{1}{(1-\epsilon_n)\cdots(1-\epsilon_{k+1})} \cdot [\tilde{\mathrm{B}}_{n,\epsilon}]_{1,\cdots,2^k}, \frac{1}{(1-\epsilon_n)\cdots(1-\epsilon_{k+2})\,\epsilon_{k+1}} \cdot [\tilde{\mathrm{B}}_{n,\epsilon}]_{2^k+1,\cdots,2^{k+1}}, \ \cdots, \\ \frac{1}{\epsilon_n\cdots\epsilon_{k+2}\,(1-\epsilon_{k+1})} \cdot [\tilde{\mathrm{B}}_{n,\epsilon}]_{2^n-2^{k+1}+1,\cdots,2^n-2^k}, \frac{1}{\epsilon_n\cdots\epsilon_{k+1}} \cdot [\tilde{\mathrm{B}}_{n,\epsilon}]_{2^n-2^k+1,\cdots,2^n} \Big\},$$

and let us set $\mathscr{D}_n = \{\tilde{\mathrm{B}}_{n,\epsilon}\}$. Then, a tuple $D_N, \cdots, D_n$ is said to be a nested family of submatrices for $\tilde{\mathrm{B}}_{n,\epsilon}$ if for each $k = N, \cdots, n$, D_k belongs to $\mathscr{D}_k$, and D_k is a submatrix of D_{k+1}.

The fact that the broken links are only present at the first N scales of the hierarchy implies that the matrices of the family $\mathscr{D}_N, \cdots, \mathscr{D}_n$ correspond to the blocks of size $2^N, \cdots, 2^n$ at the bottom of the diagram associated to $\tilde{\mathrm{B}}_{n,\epsilon}$ (see Figure 7). Indeed we have:

$$\tilde{\mathrm{B}}_{n,\epsilon} = \left(\begin{array}{c|c} (1-\epsilon_n)\,\tilde{\mathrm{B}}_{n-1,\epsilon} & \epsilon_n\,\tilde{\mathrm{B}'}_{n-1,\epsilon} \\ \hline \epsilon_n\,\tilde{\mathrm{B}}_{n-1,\epsilon} & (1-\epsilon_n)\,\tilde{\mathrm{B}'}_{n-1,\epsilon} \end{array} \right),$$

where the coupling matrices $\tilde{\mathrm{B}}_{n-1,\epsilon}, \tilde{\mathrm{B}'}_{n-1,\epsilon}$ (associated to the tuple $(\epsilon_1, \cdots, \epsilon_{n-1})$) also contain the broken links at the first N stages. Those two matrices form the set $\mathscr{D}_{n-1}$. That is why we set:

$$\mathscr{D}_{n-1} = \{\frac{1}{1-\epsilon_n} \cdot [\tilde{\mathrm{B}}_{n,\epsilon}]_{1,\cdots,2^{n-1}}, \frac{1}{\epsilon_n} \cdot [\tilde{\mathrm{B}}_{n,\epsilon}]_{2^{n-1}+1,\cdots,2^n}\}.$$

Similarly for $\mathscr{D}_{n-2}, \cdots, \mathscr{D}_N$.
Obviously this definition could also work for a matrix $\mathrm{B}_{n,\epsilon}$ with no broken links, but it is usefulness in this case. In fact, the nested families that interest us are the following ones:

Definition 3.7. *Let us fix an integer $N \geq 1$ and consider (for $n \geq N+1$), the coupling matrix*

$$\tilde{\mathrm{B}}_{n,\epsilon} = \mathrm{A}^n_{n,\epsilon_n} \cdots \mathrm{A}^n_{N+1,\epsilon_{N+1}} \tilde{\mathrm{A}}^n_{N,\epsilon_N} \cdots \tilde{\mathrm{A}}^n_{1,\epsilon_1},$$

where $\tilde{\mathrm{A}}^n_{N,\epsilon_N}, \cdots, \tilde{\mathrm{A}}^n_{1,\epsilon_1}$ are supposed to contain some broken links.
Given a nested family $D_N, \cdots, D_n$ of submatrices for $\tilde{\mathrm{B}}_{n,\epsilon}$, let's denote by a_k the number of broken links of the element D_k. We say that $D_N, \cdots, D_n$ is an admissible nested family if the following conditions are satisfied:

- $a_N \leq 1$
- $(a_{N+1} - a_N, \cdots, a_n - a_{n-1})$ *is only composed of* 0 *and* 1, *and does not contain two successive* 1.

We remark that if the coupling matrix $\tilde{\mathrm{B}}_{n,\epsilon}$ has an admissible nested family of submatrices, it may not be unique. Moreover, if $D_N, \cdots, D_n$ is such a family, then $D_N, \cdots, D_{n-1}$ is also an admissible nested family for D_n. As a consequence, among the admissible nested families for a coupling matrix $\tilde{\mathrm{B}}_{n,\epsilon}$ derived from $\tilde{\mathrm{B}}_{n-1,\epsilon}$, there are some that extend the ones of $\tilde{\mathrm{B}}_{n-1,\epsilon}$. This will permit us to require in a certain sense the existence of an infinite admissible nested family of sub-operators in the infinite case (see Theorem 4.14).

Example 3.8. *A nested family associated to the tuple $(a_{N+1} - a_N, \cdots, a_n - a_{n-1}) = (1,0,1,0,\cdots,1,0)$ is admissible. In this case, the coupling matrix admits a number a_n of broken links greater than $\mathrm{E}\,(n/2)$. Of course, this case is interesting since $\mathrm{E}\,(n/2)$ goes to infinity when n does.*

Let us go back to the situation where all the parameters are equal to 1/2, that is $(\epsilon_1, \cdots, \epsilon_n) = (1/2, \cdots, 1/2)$. In this case, the admissible nested families permit us to put a high number of broken links inside the hierarchy while keeping the associated coupling matrix diagonalizable with eigenvalues (corresponding to directions transverse to the diagonal) decreasing rapidly.

Lemma 3.9. *Let us fix an integer $N \geq 1$ and consider (for $n \geq N+1$), the coupling matrix*

$$\tilde{\mathrm{B}}_{n,\frac{1}{2}} = \mathrm{A}^n_{n,1/2} \cdots \mathrm{A}^n_{N+1,1/2} \tilde{\mathrm{A}}^n_{N,1/2} \cdots \tilde{\mathrm{A}}^n_{1,1/2},$$

where $\tilde{\mathrm{A}}^n_{N,1/2}, \cdots, \tilde{\mathrm{A}}^n_{1,1/2}$ contain some broken links (that is to say identity blocks I_{2^l} with $l \leq N-1$), in such a way that there exists an admissible nested family of submatrices for $\tilde{\mathrm{B}}_{n,\frac{1}{2}}$.
Then $\tilde{\mathrm{B}}_{n,\frac{1}{2}}$ is diagonalizable with following spectrum:

$$\mathfrak{S}\left(\tilde{\mathrm{B}}_{n,\frac{1}{2}}\right) = \{1, \lambda_{n,1}, \cdots, \lambda_{n,N}, 0^{\otimes 2^n - N - 1}\},$$

the $\lambda_{n,i}$ being some (possibly equal) positive numbers satisfying:

$$\forall i \in \{1, \cdots, N\}, \ \lambda_{n,i} \leq \frac{n-N+1}{2^{n-N}}.$$

proof of Lemma 3.9. (1) Let $n \geq N+1$, and consider a matrix $\tilde{\mathrm{B}}_{n,\frac{1}{2}}$ satisfying the required assumptions. Let us denote by $D_N, \cdots, D_n$ its admissible nested family of submatrices. Then without loss of generality we have:

$$\tilde{\mathrm{B}}_{n,\frac{1}{2}} \,(= D_n) = \left(\begin{array}{c|c} \frac{1}{2} D_{n-1} & \frac{1}{2}\mathcal{C}_{n-1} \\ \hline \frac{1}{2} D_{n-1} & \frac{1}{2}\mathcal{C}_{n-1} \end{array}\right), \tag{2}$$

where $\mathcal{C}_{n-1}$ is an element of $\mathscr{D}_{n-1}$ but in which there is zero or one broken link (the other symmetric case for $\tilde{\mathrm{B}}_{n,\frac{1}{2}}$ is obtained by exchanging D_{n-1} and $\mathcal{C}_{n-1}$). As in Lemma 3.5 we naturally consider the term $\frac{1}{2}D_{n-1} + \frac{1}{2}\mathcal{C}_{n-1}$: if we prove this term is diagonalizable and its spectrum has the desired form, then the same will hold for $\tilde{\mathrm{B}}_{n,\frac{1}{2}}$. And as the integer n is taken arbitrary, the result will follow.

(2) So let us prove by induction that for every $n \geq N+1$, the following property is true:
$\mathbf{H}_{\boldsymbol{n}}$: " *Let $\tilde{\mathrm{B}}_{n,\frac{1}{2}}$ be a coupling matrix satisfying the required assumptions of this lemma 3.9, and let D_{n-1}, $\mathcal{C}_{n-1}$ be the matrices defining $\tilde{\mathrm{B}}_{n,\frac{1}{2}}$ by the expression* (2) *above. Then we have:*

$$\mathfrak{S}\left(\frac{1}{2}D_{n-1} + \frac{1}{2}\mathcal{C}_{n-1}\right) = \{1, \lambda_{n,1}, \cdots, \lambda_{n,N}, 0^{\otimes 2^{n-1} - N - 1}\},$$

the $\lambda_{n,k}$, being some (possibly equal) positive numbers verifying:

$$\forall k \in \{1, \cdots, N\}, \ \lambda_{n,k} \leq \frac{n-N+1}{2^{n-N}}.$$

Moreover this matrix $\dfrac{1}{2}D_{n-1} + \dfrac{1}{2}\mathcal{C}_{n-1}$ is diagonalizable, and its rank is the one of $\tilde{\mathrm{B}}_{n,\frac{1}{2}}$."

For $n = N+1$, the result is clear if D_N or $\mathcal{C}_N$ contains no broken link. If D_N and $\mathcal{C}_N$ both contain one broken link, the result is given by Lemma 3.5.
Let $n \geq N+2$ and assume $\mathbf{H}_{\boldsymbol{n}}$ is true. Let $\tilde{\mathrm{B}}_{n+1,\frac{1}{2}}$ admitting a nested admissible family $D_N, \cdots, D_{n+1}$, and $a_N, \cdots, a_{n+1}$ the successive numbers of broken links associated to this family. We look at the expression of $\tilde{\mathrm{B}}_{n+1,\frac{1}{2}}$ in terms of the matrices D_n and $\mathcal{C}_n$ (in a similar way as in the expression (2)).

- Assume in a first case $\mathcal{C}_n$ contains one broken link, i.e $a_{n+1} - a_n = 1$ (or in other words D_{n+1} admits one more broken link than D_n). As before without loss of generality we have:

$$\mathcal{C}_n = \left(\begin{array}{c|c} \frac{1}{2}\mathcal{C}_{n-1} & \frac{1}{2^n}\mathbf{1}_{2^{n-1}} \\ \hline \frac{1}{2}\mathcal{C}_{n-1} & \frac{1}{2^n}\mathbf{1}_{2^{n-1}} \end{array}\right),$$

where $\mathcal{C}_{n-1}$ contains one broken link. Now, as $D_N, \cdots, D_{n+1}$ is an admissible sequence thus we have $a_n - a_{n-1} = 0$, (D_n and D_{n-1} have the same number of broken links). So we can write:

$$D_n = \left(\begin{array}{c|c} \frac{1}{2^n}\mathbf{1}_{2^{n-1}} & \frac{1}{2}D_{n-1} \\ \hline \frac{1}{2^n}\mathbf{1}_{2^{n-1}} & \frac{1}{2}D_{n-1} \end{array}\right).$$

Since $\mathbf{H}_{\boldsymbol{n}}$ is true, the term $\frac{1}{4}D_{n-1} + \frac{1}{4}\mathcal{C}_{n-1}$ is diagonalizable and its spectrum has the required form. As in Lemma 3.5 we conclude that the sum $\frac{1}{4}D_{n-1} + \frac{1}{4}\mathcal{C}_{n-1} + \frac{1}{2^n}\mathbf{1}_{2^{n-1}}$ is diagonalizable. As the terms D_{n-1} and $\mathcal{C}_{n-1}$ are one-column sum, we clearly have:

$$\operatorname{rank}\left(\frac{1}{2}D_n + \frac{1}{2}\mathcal{C}_n\right) = \operatorname{rank}\left(\begin{bmatrix}\frac{1}{4}\mathcal{C}_{n-1} & \frac{1}{4}D_{n-1}\end{bmatrix}\right)$$

from which we obtain (by $\mathbf{H}_{\boldsymbol{n}}$ and by Lemma 3.4) that the desired term $\frac{1}{2}D_n + \frac{1}{2}\mathcal{C}_n$ is diagonalizable. This equality tells us also that $\operatorname{rank}\left(\tilde{\mathrm{B}}_{n+1,\frac{1}{2}}\right) = \operatorname{rank}\left(\frac{1}{2}D_n + \frac{1}{2}\mathcal{C}_n\right)$. To conclude it suffices to remark we have:

$$\mathfrak{S}\left(\frac{1}{4}D_{n-1} + \frac{1}{4}\mathcal{C}_{n-1} + \frac{1}{2^n}\mathbf{1}_{2^n}\right) = \{1, \frac{\lambda_{n,1}}{2}, \cdots, \frac{\lambda_{n,N}}{2}, 0^{\otimes 2^n - N - 1}\},$$

as required.
- In the second case, $a_{n+1} - a_n = 0$ (i.e $\mathcal{C}_n$ contains no broken link), and $a_n - a_{n-1} = 1$ (that is to say D_n has one more broken link than D_{n-1}). Then D_n is of the form:

$$D_n = \left(\begin{array}{c|c} \frac{1}{2}D_{n-1} & \frac{1}{2}\mathcal{C}_{n-1} \\ \hline \frac{1}{2}D_{n-1} & \frac{1}{2}\mathcal{C}_{n-1} \end{array}\right).$$

Since $\mathcal{C}_n = \frac{1}{2^n}\mathbf{1}_{2^n}$, the assumption $\mathbf{H}_{\boldsymbol{n}}$ and the Lemma 3.4 permit to conclude as in the first case.
- The last case is $a_{n+1} - a_n = 0$ and $a_n - a_{n-1} = 0$, and is dealt with the exactly same reasoning.

$\square$

3.2.b. *A local synchronization lemma.* Now let us prove the synchronization result in this context. As the structure matrices $\tilde{\mathrm{A}}^n_{k,\epsilon_k}$ no more commute, the classical method exposed in [3] cannot be applied again. More, there is no hope to synchronize globally our system since in general it admits some fixed points outside the diagonal.

Example 3.10. *Consider the case $n = 2$ (i.e dimension 4), with* f *as the logistic map (taken at the value of parameter $\mu = 3.57$, as in Example* (2.1)*) and the following structure matrices:*

$$\tilde{\mathrm{A}}^2_{1,\epsilon_1} = \begin{bmatrix} \mathcal{T}_{\epsilon_1} & \\ & I_2 \end{bmatrix}, \ \mathrm{A}^2_{2,\epsilon_2} = \mathcal{T}_{\epsilon_2} \otimes I_2.$$

This is the only interesting structure with broken link in this dimension. Then for $(\epsilon_1, \epsilon_2) = (0.45, 0.476)$*, we have* $\tilde{\mathrm{G}}_{2,\epsilon}(X) = \tilde{B}_{2,\epsilon} \circ \mathrm{F}_2(X) = X$ *for* $X \approx (0.3394508235, 0.2491080749, 0.7404987705, -0.2406491883)$.

For this reason we adopt another approach based on a Taylor development at first order: although the approximations deduced are coarse, it permits us to recover a commutativity on the differential of G_n, leading to the existence of a contracting neighborhood of the diagonal $\mathfrak{I}_{2^n}$. This result has been already established notably by W. Lu in [11] (see also [7]) in a general setting (that does not take into account the hierarchical structure and the broken links we are dealing with) but obviously without exhibiting how the contracting neighborhood depends or not on the dimension of the space (here 2^n). And as we need this information for the second part, we demonstrate this result again:

Lemma 3.11. *Let us fix an integer $N \geq 1$ and take a n-tuple $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ with $n \geq N + 1$. We consider the system $\tilde{\mathrm{G}}_{n,\epsilon} = \tilde{\mathrm{B}}_{n,\epsilon} \circ \mathrm{F}_n$, where the coupling matrix $\tilde{\mathrm{B}}_{n,\epsilon}$ admits an admissible nested family of submatrices (as in Lemma 3.9). Assume the n-tuple ϵ is enough close to $1/2$ so that we have:*

$$6 \sup_{z \in \mathrm{K}} |\mathrm{f}'(z)|^{2N} \sum_{k=1}^{n} |1 - 2\epsilon_k| < 1.$$

Then, there is a constant Λ_n such that, for every $\epsilon' > 0$ exists a real $\eta_N > 0$ (that does not depend on n) defining the following neighborhood Ω_n of the diagonal:

$$\Omega_n = \{X \in \mathrm{K}^{2^n}, \forall i \neq j \ |X_{(i)} - X_{(j)}| \leq \eta_N\},$$

on which the map $\left(\tilde{\mathrm{G}}_{n,\epsilon}\right)^{2N}$ satisfies the inequality:

$$\max_{1 \leq i,j \leq 2^n} | \left(\tilde{\mathrm{G}}_{n,\epsilon}\right)^{2N}(X)_{(i)} - \left(\tilde{\mathrm{G}}_{n,\epsilon}\right)^{2N}(X)_{(j)} | \leq 2\left(\epsilon' + \Lambda_n + 3 \sup_{z \in \mathrm{K}} |\mathrm{f}'(z)|^{2N} \sum_{k=1}^{n} |1 - 2\epsilon_k|\right) \left|X_{(i)} - X_{(j)}\right|.$$

The constant Λ_n tends to zero as n tends to infinity. In particular choosing ϵ' enough small, the dynamical system $\tilde{\mathrm{G}}_{n,\epsilon}$ synchronizes on the set Ω_n for a sufficiently large n.

Proof. We first prove the result for the map $\tilde{\mathrm{G}}^{2N}_{n,\frac{1}{2}}$.
For every $r \geq 1$, we have, by the Taylor formula with bounded remainder applied at first order:

$$\begin{aligned} || \left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^r (X) - \left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^r (X_\Sigma) ||_{\infty,n} &\leq \sup_{t \in [0,1]} ||\mathrm{D}_{(1-t)X_\Sigma + tX} \left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^r (X - X_\Sigma) - \mathrm{D}_{X_\Sigma} \left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^r (X - X_\Sigma) ||_{\infty,n} \\ &+ ||\mathrm{D}_{X_\Sigma} \left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^r (X - X_\Sigma) ||_{\infty,n}. \end{aligned}$$

Now we fix a small number $\epsilon' > 0$. The (uniform) continuity of the map $X \mapsto \mathrm{D}_X \tilde{\mathrm{G}}_{n,\frac{1}{2}}$ on the compact K^{2^n}, gives us the existence of a number $\eta_r > 0$ such that:

$$\forall X, Y \in \mathrm{K}^{2^n} \times \mathrm{K}^{2^n}, \ ||X - Y||_{\infty,n} < \eta_r \Rightarrow ||\mathrm{D}_X \left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^r - \mathrm{D}_Y \left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^r ||_{\infty,n} < \epsilon'.$$

Indeed, for every $X, Y \in \mathrm{K}^{2^n}$, applying successively the mean value inequality leads to the estimation:

$$\begin{aligned} ||\mathrm{D}_X \left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^r - \mathrm{D}_Y \left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^r ||_{\infty,n} &\leq \sup_{z \in \mathrm{K}} \left|\mathrm{f}'(z)\right|^{r-1} \cdot \sup_{z \in \mathrm{K}} \left|\mathrm{f}''(z)\right| \left(\sum_{i=1}^{r} || \left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^{r-i} (X) - \left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^{r-i} (Y) ||_{\infty,n}\right) \\ &\leq r \left(\sup_{z \in \mathrm{K}} \left|\mathrm{f}'(z)\right|^{2r-2} \cdot \sup_{z \in \mathrm{K}} \left|\mathrm{f}''(z)\right|\right) ||X - Y||_{\infty,n}, \end{aligned}$$

which permits us to choose a convenient real η_r depending only on the integer r and the function f.
We consider the set $\mathcal{C}_{\eta_r} = \{Z = (z_{(1)}, \cdots, z_{(2^n)}) \in \mathrm{K}^{2^n} : \ \forall i \neq j \ \left|z_{(i)} - z_{(j)}\right| < \eta_r\}$. If X is in $\mathcal{C}_{\eta_r}$, the upper bound in the Taylor inequality above is smaller than $\epsilon' \left|\left|X - X_\Sigma\right|\right|_{\infty,n}$. We now estimate the second term of the sum for X in this set.

Let $\mathcal{P}_n$ the matrix of $GL_{2^n}(\mathbb{R})$ conjugating $\tilde{\mathrm{B}}_{n,\frac{1}{2}}$ to its diagonal form (see Lemma 3.9):

$$\begin{bmatrix} 1 & & & & & & \\ & \lambda_{n,1} & & & & & \\ & & \ddots & & & & \\ & & & \lambda_{n,N} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}.$$

Clearly ${\mathcal{P}_n}^{-1}(X - X_\Sigma)$ belongs to the space vect $(e_2, \cdots, e_{2^n})$, as $X - X_\Sigma$ is orthogonal to the diagonal. From this we get:

$$\left|\left|\mathrm{D}_{X_\Sigma}\left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^r (X - X_\Sigma)\right|\right|_{\infty,n} \leq \left|\left|\mathcal{P}_n\right|\right|_{\infty,n} \left|\left|{\mathcal{P}_n}^{-1}\right|\right|_{\infty,n} \left((n - N + 1) \sup_{z \in \mathrm{K}} \left|\mathrm{f}'(z)\right| \cdot \frac{1}{2^{n-N}}\right)^r \left|\left|X - X_\Sigma\right|\right|_{\infty,n}.$$

The embarrassing condition number $\left|\left|\mathcal{P}_n\right|\right|_{\infty,n} \left|\left|{\mathcal{P}_n}^{-1}\right|\right|_{\infty,n}$ may goes to infinity as n does (this is the case numerically) but it's easy to see by induction this number is smaller than 2^{nN}. Thus if we choose the integer r enough great, for instance $r = 2N$ we get:

$$\left|\left|\mathrm{D}_{X_\Sigma}\left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^{2N} (X - X_\Sigma)\right|\right|_{\infty,n} \leq \Lambda_n \left|\left|X - X_\Sigma\right|\right|_{\infty,n},$$

where the term Λ_n defined by:

$$\Lambda_n = \frac{(n - N + 1)^{2N} \cdot 2^{2N^2}}{2^{nN}} \sup_{z \in \mathrm{K}} \left|\mathrm{f}'(z)\right|^{2N},$$

goes to zero as n tends to infinity. The result is proved in the case $\epsilon = (1/2, \cdots, 1/2)$.

Then, to prove the general case it suffices to compare our map $\tilde{\mathrm{G}}_{n,\epsilon}$ with the map $\tilde{\mathrm{G}}_{n,1/2}$ we have just studied. Here again the mean value inequality gives us:

$$\begin{aligned}\left|\left|\mathrm{D}_{X_\Sigma}\left(\tilde{\mathrm{G}}_{n,\epsilon}\right)^{2N} (X - X_\Sigma)\right|\right|_{\infty,n} &\leq 3 \sup_{z \in \mathrm{K}} \left|\mathrm{f}'(z)\right|^{2N} \left|\left|\tilde{\mathrm{B}}_{n,1/2} - \tilde{\mathrm{B}}_{n,\epsilon}\right|\right|_{\infty,n} \left|\left|X - X_\Sigma\right|\right|_{\infty,n} \\ &\quad + \left|\left|\mathrm{D}_{X_\Sigma}\left(\tilde{\mathrm{G}}_{n,\frac{1}{2}}\right)^{2N} (X - X_\Sigma)\right|\right|_{\infty,n} \\ &\leq \left(\Lambda_n + 3 \sup_{z \in \mathrm{K}} \left|\mathrm{f}'(z)\right|^{2N} \sum_{k=1}^{n} \left|1 - 2\epsilon_k\right|\right) \left|\left|X - X_\Sigma\right|\right|_{\infty,n}.\end{aligned}$$

Taking ϵ' smaller if necessary we get the desired inequality.

Finally, for n enough large the map $\left(\tilde{\mathrm{G}}_{n,\epsilon}\right)^{2N}$ is transversally contracting on $\Omega_n = \mathcal{C}_{\eta_{2N}}$ and thus synchronizes. There exists an integer M such that for every X in Ω_n, all the iterated $\left(\tilde{\mathrm{G}}_{n,\epsilon}\right)^{2NM+s}(X)$ with s in $\{0, \cdots, 2N - 1\}$, belong to this set. And by euclidean division, for every $m \geq 1$ exists an integer q such that $m = 2N(q - M) + 2NM + s$, with s in $\{0, \cdots, 2N - 1\}$. So we conclude:

$$\forall X \in \Omega_n, \ \max_{1 \leq i,j \leq 2^n} \left| \left(\tilde{\mathrm{G}}_{n,\epsilon}\right)^m (X)_{(i)} - \left(\tilde{\mathrm{G}}_{n,\epsilon}\right)^m (X)_{(j)} \right| \underset{m \to \infty}{\longrightarrow} 0.$$

□

Remark 3.12. *The crucial point of this lemma lies in the non dependance of the size* η_{2N} *(on* n*) of the neighborhood* Ω_n *on which our system synchronizes: this size stays constant as the dimension* n *goes to infinity, which will ensure us the existence of a non trivial neighborhood of synchronization in the infinite dimensional case.*

3.3. **Remarks concerning the definition of** $\mathrm{G}_{n,\epsilon}$**.** To finish this first part, let us underly there are mainly two possibilities for the definition of our dynamical system $\mathrm{G}_{n,\epsilon}$, that represent the hierarchical structure we have constructed. Besides the one we have chosen $\mathrm{G}_{n,\epsilon} = \mathrm{A}^n_{n,\epsilon_n} \circ \cdots \circ \mathrm{A}^n_{1,\epsilon_1} \circ \mathrm{F}_n$ in Section 3.1, we could have set:

$$\mathrm{G}_{n,\epsilon} = \mathrm{A}^n_{n,\epsilon_n} \circ \mathrm{F}_n \circ \cdots \circ \mathrm{A}^n_{1,\epsilon_1} \circ \mathrm{F}_n,$$

and develop the same approach concerning the local synchronization in presence of broken links. The result of lemma (3.11) is also true with this definition, but this time the size of the synchronization neighborhood depends on n, for is added a term F_n at each increasing of the dimension.

4. Generalization to a Cantor set.

Consider again a set of 2^n particles, coupled together according to the hierarchical structure established in Section 3.1. As we have gathered them two-by-two at each step, it is natural to number them by a code with only two letters, say 0 and 1, representing their path to the top of the graph associated to this process: we use 0 if the subsystem is at the left of the one to which it is linked, and 1 if it is at its right. This numbering is one-to-one since every particle admits one and only one such path. As mentioned above, in this finite case we could have synchronized our particles with a different matrix coupling.

This is no more the case if we let n tends to infinity. In this limiting case, our set of particles forms a Cantor set, thus (infinite) uncountable, indexed by the set $\mathrm{X} = \{0,1\}^{\mathbb{N}}$ of sequences in 0 and 1, and the previous numbering describe all the possible sequences of this set. Instead of having a vector as initial condition (representing the values at each position k in $1, \cdots, 2^n$), we now have a set $\mathrm{h}(\mathrm{X})$ for a function h from X to $\mathbb{R}$. But because of this uncountability, there is no straight way to synchronize all our particles as in [3], [11], [7], for there is no way to write explicitly a series in all the images $\mathrm{h}(c)$ for c in X.
It is here that the hierarchical structure permits to overpass naturally this difficulty.
The matrices $\left(\mathrm{A}^n_{k,\epsilon_k}\right)_{1\le k\le n}$ of Part 3 become the following operators $\left(\mathcal{L}_{k,\epsilon_k}\right)_{k\in\mathbb{N}^*}$ acting on the space of real-valued functions on X:

$$\mathcal{L}_{k,\epsilon_k}(\mathrm{h}) = (1-\epsilon_k)\,\mathrm{h} + \epsilon_k \mathrm{J}_k(\mathrm{h}),$$

where the function $\mathrm{J}_k(\mathrm{h})$ is defined by:

$$\forall c \in \mathrm{X},\ \mathrm{J}_k(\mathrm{h})(c) = \mathrm{h}\left(c^{\star,k}\right),$$

the sequence $c^{\star,k}$ being obtained from c by only replacing the letter c_k with $1-c_k$. Obviously, these operators J_k play the same role as the matrices C_k in the finite case (see the proof of Theorem 3.1).
We look at the infinite composition of our new operators $\mathcal{L}_{k,\epsilon_k}$, i.e at the limit:

$$\lim_{n\to\infty} \mathcal{L}_{n,\epsilon_n} \circ \cdots \circ \mathcal{L}_{1,\epsilon_1}(\mathrm{h}),$$

for a given function h on X. Assuming for the moment such a limit exists (see below), it defines an operator $\mathcal{U}_\epsilon$ (where ϵ is the sequence $(\epsilon_k)_{k\ge1}$) from which we construct, as in Part 3 with G_n, a dynamical system leading to the synchronization. Applied to a function h on X, this operator acts on all the terms $\mathrm{h}(c)$, as desired.

In this context the diagonal writes $\mathfrak{I} = \{\mathrm{h} : \mathrm{X} \to \mathbb{R}, \mathrm{h} \text{ is constant}\}$. Geometrically, given an initial condition h, the new dynamical system will tend to flatten its associated graph $\{(c, \mathrm{h}(c)) : c \in \mathrm{X}\}$. We are now able to apply the same reasoning as in Part 3, the main issue being the existence of the operator $\mathcal{U}_\epsilon$.

4.1. **Global synchronization.**

4.1.a. *Existence of the limit operator* $\mathcal{U}_\epsilon$. We equip X with the metric:

$$d(c,c') = \sum_{n=0}^{\infty} \frac{|c_n - c'_n|}{2^n}.$$

The metric space (X, d) is compact, totally disconnected (i.e its greatest connected component is a point), and without isolated point. We recall that the metric spaces presenting these three characteristics are all homeomorphic. They are called Cantor spaces. It is important to tell that all our results below are true for any other Cantor space than (X, d) (which is not always the case), as they do not depend on the metric chosen (see the point (2) of Remark 4.3 below).

Given a n-tuple $(a_1, \cdots, a_n)$ in $\{0,1\}^n$, consider the set $\mathscr{C}_{a_1,\cdots,a_n} = \{c \in \mathrm{X} : (c_1, \cdots, c_n) = (a_1, \cdots, a_n)\}$. It is easy to verify this set is closed and open for the topology defined by d. For every $n \geq 1$, we will denote by $(\mathscr{C}_{n,k})_{1\leq k\leq 2^n}$ all the 2^n such sets, no matter the way they are ordered. Each $\mathscr{C}_{n,k}$ has diameter $1/2^{n-1}$ and we have:

$$\forall n \geq 1, \ \mathrm{X} = \sqcup_{k=1}^{2^n} \mathscr{C}_{n,k}.$$

The fundamental property is that the set $\mathscr{C}$ of all those sets:

$$\mathscr{C} = \{\mathscr{C}_{1,1}, \mathscr{C}_{1,2}, \cdots, \mathscr{C}_{n,k}, \cdots\},$$

forms a countable basis of open sets of the topological space X.
As explained above we are lead to consider the functions from X to $\mathbb{R}$, that will be our new points in the phase space. In fact our results will only concern the space $\mathscr{C}(\mathrm{X}, \mathbb{R})$ of continuous functions, which we recall is complete for the infinity norm $||\cdot||_\infty$. The reason is the synchronization is expressed with this norm. The sum of the series $\sum_{n\geq 0} c_n/2^n$ is an example of such a function. We have:

Proposition 4.1. *The set of constant functions on the closed-open sets* $\mathscr{C}$ *is dense in the set* $(\mathscr{C}(\mathrm{X}, \mathbb{R}), ||\cdot||_\infty)$.

Proof. Any function constant on the closed-open sets of order n for some $n \geq 1$, is clearly continuous on X. Then given an element h of $\mathscr{C}(\mathrm{X}, \mathbb{R})$, define the sequence of functions $(\mathrm{h}_n)_{n\geq 1}$ by:

$$\forall c \in \mathrm{X}, \ \mathrm{h}_n(c) = \mathrm{h}(c_1, \cdots, c_n, \mathbf{1}),$$

where $\mathbf{1}$ denotes the constant sequence equals to one. For every $n \geq 1$, the function h_n is constant on the $(\mathscr{C}_{n,k})_{1\leq k\leq 2^n}$, and by the uniform continuity of h on the compact X, converges to this function for the infinity norm. ☐

Lemma 4.2. *For every sequence* $\epsilon = (\epsilon_n)_{n\geq 1}$ *of real numbers in* $[0,1]$*, and every continuous function* h *on* X*, the following limit:*

$$\mathcal{U}_\epsilon(\mathrm{h}) := \lim_{n\to\infty} \mathcal{L}_{n,\epsilon_n} \circ \cdots \circ \mathcal{L}_{1,\epsilon_1}(\mathrm{h}),$$

exists as a function of $(\mathscr{C}(\mathrm{X}, \mathbb{R}), ||\cdot||_\infty)$.

Proof. For h in $(\mathscr{C}(\mathrm{X}, \mathbb{R}), ||\cdot||_\infty)$, let's consider the sequence $(\mathrm{h}_n)_{n\geq 1}$ defined as in the proof of Proposition 4.1, that converges to h. Introducing the notation $\mathrm{K}_n(\mathrm{h}) = \mathcal{L}_{n,\epsilon_n} \circ \cdots \circ \mathcal{L}_{1,\epsilon_1}(\mathrm{h})$ we then have, for every integers p, q:

$$||\mathrm{K}_{p+q}(\mathrm{h}) - \mathrm{K}_p(\mathrm{h})||_\infty \leq ||\mathrm{K}_{p+q}(\mathrm{h}) - \mathrm{K}_{p+q}(\mathrm{h}_p)||_\infty + ||\mathrm{K}_{p+q}(\mathrm{h}_p) - \mathrm{K}_p(\mathrm{h}_p)||_\infty + ||\mathrm{K}_p(\mathrm{h}_p) - \mathrm{K}_p(\mathrm{h})||_\infty$$

Clearly, the operator norm of each operator $\mathcal{L}_{n,\epsilon_n}$ is one. It then suffices to remark we have,

$$\forall n > p, \ \mathcal{L}_{n,\epsilon_n}(\mathrm{h}_p) = \mathrm{h}_p,$$

to conclude the first member of the inequality is smaller than $2||\mathrm{h} - \mathrm{h}_p||_\infty$. The considered sequence is thus a Cauchy one. ☐

Remark 4.3. (1) *Nothing tells us the sequence* $\mathcal{L}_{n,\epsilon_n} \circ \cdots \circ \mathcal{L}_{1,\epsilon_1}$ *converges for the operator norm, i.e in the set of linear maps on* $\mathscr{C}(\mathrm{X}, \mathbb{R})$*, for the speed of convergence of the term* $||\mathrm{h} - \mathrm{h}_p||_\infty$ *depends on the function* h.

(2) *The existence of* $\mathcal{U}_\epsilon$ *does not require the convergence of the sequence* ϵ. *Nor does it involve the metric* d *taken on* X*, and especially the diameter of the closed-open sets* $\mathscr{C}_{n,k}$*, of which size could decrease much more slowly on other Cantor spaces.*

Now as in Part 3, we set $\mathrm{G}_\epsilon = \mathcal{U}_\epsilon \circ \mathrm{F}$ where the map F is defined by $\mathrm{F}(\mathrm{h}) = \mathrm{f} \circ \mathrm{h}$ for every real-valued function h defined on X.

4.1.b. *Global synchronization result on* $\mathscr{C}(\mathrm{X}, \mathrm{K})$. In this context, the global synchronization result is proved by the same technique as in [3]. We equip X with the Borel σ-algebra and consider the probability measure μ verifying:

$$\forall n \geq 1,\ \forall k \in \{1, \cdots, 2^n\},\ \mu(\mathscr{C}_{n,k}) = \frac{1}{2^n}.$$

Since $\mathscr{C}$ generates the topology of X, and for every $n \geq 1$ the $(\mathscr{C}_{n,k})_{1 \leq k \leq 2^n}$ form a partition of X, it is very easy to construct such a probability measure.

Proposition 4.4. *For every* $n \geq 1$*, the map* $c \mapsto c^{\star,n}$ *preserves the measure* μ*. Consequently for every measurable function* h *from* X *to* $\mathbb{R}$*, we have:*

$$\int_{\mathrm{X}} \mathrm{h}(c)\, \mathrm{d}\mu(c) = \int_{\mathrm{X}} \mathrm{h}(c^{\star,n})\, \mathrm{d}\mu(c).$$

Proof. The map $c \mapsto c^{\star,n}$ is an involution. Let's take an element $\mathscr{C}_{p,k}$ in $\mathscr{C}$. If we have $p < n$ then $(\mathscr{C}_{p,k})^{\star,n} = \mathscr{C}_{p,k}$, otherwise the measure of these two sets is the same by definition of μ. This yields the result by σ-additivity. □

From this measure μ, we recover a scalar product $\langle\, ,\, \rangle$ on the space of measurable real-valued functions on X, defined by:

$$\langle \mathrm{h}\, , \mathrm{g} \rangle = \int_{\mathrm{X}} \mathrm{hg}\, \mathrm{d}\mu.$$

We denote by $^\perp$ the orthogonality for this scalar product. Proposition 4.4 gives us the following:

Lemma 4.5. *For every* $k \geq 1$*, let us define the subset* $\mathfrak{J}_k$ *of* $\mathscr{C}(\mathrm{X}, \mathbb{R})$ *by:*

$$\mathfrak{J}_k = \{\mathrm{h} \in \mathscr{C}(\mathrm{X}, \mathbb{R}) :\ \mathrm{J}_k(\mathrm{h}) = \mathrm{h}\}.$$

Then the diagonal $\mathfrak{J}$ *is equal to the intersection of all the* $\mathfrak{J}_k$*, and we have the following relation:*

$${\mathfrak{J}_k}^\perp = \{\mathrm{h} \in \mathscr{C}(\mathrm{X}, \mathbb{R}) :\ \mathrm{J}_k(\mathrm{h}) = -\mathrm{h}\}.$$

Proof. The first assertion is clear. For the second one, if a function h is in ${\mathfrak{J}_k}^\perp$, then we have:

$$\int_X \mathrm{h}(c)\left(\mathrm{h}(c) + \mathrm{h}\left(c^{\star,k}\right)\right) \mathrm{d}\mu(c) = 0,$$

and thus by Proposition 4.4,

$$\int_X \left(\mathrm{h}(c) + \mathrm{h}\left(c^{\star,k}\right)\right)^2 \mathrm{d}\mu(c) = 0,$$

which gives the equality $\mathrm{h}\left(c^{\star,k}\right) = -\mathrm{h}(c)$ for almost every sequence c. But as the measure μ is an exterior one this holds for every c in X. □

Theorem 4.6. *Let* $\epsilon = (\epsilon_k)_{k \geq 1}$ *be a sequence of real numbers in* $[0,1]$*, defining the operator* $\mathcal{U}_\epsilon$ *as in Lemma 4.2. Assume the following condition holds:*

$$\exists a > 1,\ \exists \alpha > 0,\ \forall k \geq 1,\ |1 - 2\epsilon_k| \sup_{z \in \mathrm{K}} \left|\mathrm{f}'(z)\right| \leq \frac{1}{(ak)^\alpha}.$$

Then the dynamical system $\mathrm{G}_\epsilon = \mathcal{U}_\epsilon \circ \mathrm{F}$ *globally synchronizes on the set* $\mathscr{C}(\mathrm{X}, \mathrm{K})$ *of continuous functions having values in* K.

Proof. Let us fix an integer $k \geq 1$ and take a continuous function h on X, having values in K. As in the finite dimensional case, the operators $(\mathrm{J}_k)_{k\geq 1}$ commute with the operators $(\mathcal{L}_{n,\epsilon_n})_{n\geq 1}$ and with F. So we have:

$$\begin{aligned}\left|\left|\mathrm{G}_\epsilon(\mathrm{h}) - \mathrm{J}_k(\mathrm{G}_\epsilon(\mathrm{h}))\right|\right|_\infty &= \left|\left|\mathcal{U}_\epsilon(\mathrm{f}\circ\mathrm{h} - \mathrm{J}_k(\mathrm{f}\circ\mathrm{h}))\right|\right|_\infty \\ &\leq \sup_{z\in\mathrm{K}}\left|\mathrm{f}'(z)\right| \cdot ||\mathcal{U}_\epsilon{}_{\big|_{\mathfrak{J}_k{}^\perp}}||_\infty ||\mathrm{h} - \mathrm{J}_k(\mathrm{h})||_\infty.\end{aligned}$$

Then we observe the equality $\left|\left|\mathcal{U}_\epsilon{}_{\big|_{\mathfrak{J}_k{}^\perp}}\right|\right|_\infty = |1-2\epsilon_k|$ (the inequality would have sufficed for our proof). Indeed, for a function h in $\mathfrak{J}_k{}^\perp$, we have:

$$\mathcal{L}_{k,\epsilon_k}\circ\cdots\circ\mathcal{L}_{1,\epsilon_1}(\mathrm{h}) = (1-2\epsilon_k)\,\mathcal{L}_{k-1,\epsilon_{k-1}}\circ\cdots\circ\mathcal{L}_{1,\epsilon_1}(\mathrm{h}),$$

which imposes, as the infinity norm of each $\mathcal{L}_{n,\epsilon_n}$ is one:

$$||\mathcal{U}_\epsilon{}_{\big|_{\mathfrak{J}_k{}^\perp}}|| \leq |1-2\epsilon_k|.$$

The equality is reached for the function $c \mapsto c_k - (1-c_k)$. This leads us to the following, for every elements c, c' of X:

$$\begin{aligned}\forall m \geq 1:\ |\mathrm{G}_\epsilon^m(\mathrm{h})(c) - \mathrm{G}_\epsilon^m(\mathrm{h})(c')| &\leq \sum_{k=1}^{+\infty}\left(|1-2\epsilon_k|\sup_{z\in\mathrm{K}}|\mathrm{f}'(z)|\right)^m ||\mathrm{h}-\mathrm{J}_k(\mathrm{h})||_\infty, \\ &\leq \frac{1}{a^{m\alpha}}\sum_{k=1}^{+\infty}\frac{1}{k^{m\alpha}}||\mathrm{h}-\mathrm{J}_k(\mathrm{h})||_\infty,\end{aligned}$$

which gives the result. □

Remark 4.7. (1) *The sequence ϵ is asked to converge slowly to $1/2$, since we do not know how fast is the convergence to zero of the term $||\mathrm{h} - \mathrm{J}_k(\mathrm{h})||_\infty$. This convergence depends in a certain sense, on the metric taken on the Cantor space. For instance, if we take a Lipschitz function* h *on our set* (X, d) *as initial condition, it just suffices to ask the terms* $|1-2\epsilon_k|\sup_{z\in\mathrm{K}}|\mathrm{f}'(z)|$ *are all strictly smaller than one, because the diameters of the closed-open sets decrease enough strongly to* 0.

(2) *There again, the result also works for a Lipschitz function* f *on an interval, in which case it is true on* $\mathscr{C}(\mathrm{X},\mathbb{R})$.

4.2. **Local synchronization in presence of broken links.** Finally, as announced before we ask the same question as in Subsection 3.2: assume in our Cantor set of particles, there are infinitely many ones that are not coupled. Can we still synchronize the entire set of particles?
As said previously, the work has been already prepared by Lemma 3.9 and 3.11: it just suffices to let the dimension n goes to infinity in our estimations, using the fact that the set of constant functions on the closed-open sets $\mathscr{C}_{n,k}$ form a 2^n-dimensional space, that we identify with $\mathbb{R}^{2^n}$.

In order to make this passing to the limit, we have to define again our sequence of structure operators, acting at each step n on this space of constant functions on the $\mathscr{C}_{n,k}$. We begin by the operators having only strong links, setting for every $n \geq 1$ and every function h in $\mathscr{C}(\mathrm{X},\mathbb{R})$:

$$\forall k \in \{1,\cdots,n\}:\ \mathcal{I}^n_{k,\epsilon_k}(\mathrm{h})(c) = (1-\epsilon_k)\,\mathrm{h}(c_1,\cdots,c_n,\mathbf{1}) + \epsilon_k\mathrm{h}\left((c_1,\cdots,c_n,\mathbf{1})^{\star,k}\right).$$

As in Part 3, we ask that for each diagram of size 2^n, the broken links stay at the smallest N stages of the hierarchical structure (for a fixed $N \geq 1$), while being enough spread in those stages (see Lemma 3.9). In this purpose, for every $n \geq N+1$ and every function h in $\mathscr{C}(\mathrm{X},\mathbb{R})$, we consider the composition:

$$\tilde{\mathcal{W}}_{n,\epsilon}(\mathrm{h}) = \mathcal{I}^n_{1,\epsilon_1}\circ\cdots\circ\mathcal{I}^n_{n-N,\epsilon_{n-N}}\circ\tilde{\mathcal{I}}^n_{n-N+1,\epsilon_{n-N+1}}\circ\cdots\circ\tilde{\mathcal{I}}^n_{n,\epsilon_n}(\mathrm{h}),$$

where $\epsilon = (\epsilon_1,\cdots,\epsilon_n)$ and the operators $\left(\tilde{\mathcal{I}}^n_{k,\epsilon_k}\right)_{n-N+1\leq k\leq n}$ contain the broken links (note the order of composition has been reversed compared to the definition of Part 3, see Remark 4.9). To define the operators $\tilde{\mathcal{I}}^n_{k,\epsilon_k}$, we view the 2^n closed-open sets as points of $\mathbb{R}^{2^n}$, ordered according to the same rule as the one described

at the beginning of Part 4. That is to say, at each stage of the hierarchical diagram of size 2^n, we place on the left the tuples of which letter corresponding to this stage is 0, and on the right those for which this letter is 1 (see Figure 8). Obviously, taking the reverse order (where the letter 1 corresponds to the left) changes nothing.
Once this order is chosen (for the diagrams of any size), we denote for any function h in $\mathscr{C}(X,\mathbb{R})$, and any $n \geq 1$, the vector $\mathrm{H}_n = (\mathrm{h}_{n,1}, \cdots, \mathrm{h}_{n,2^n})$ in $\mathbb{R}^{2^n}$, of which coordinates (numbered according to this order) are the 2^n distinct values taken by the function h_n defined in the proof of Proposition 4.1. That is to say, we set:

$$
\begin{cases}
\mathrm{h}_{n,1} & = \mathrm{h}(0, \cdots, 0, \mathbf{1}) \\
\mathrm{h}_{n,2} & = \mathrm{h}(0, \cdots, 0, 1, \mathbf{1}) \\
& \vdots \\
\mathrm{h}_{n,2^n} & = \mathrm{h}(1, \cdots, 1, \mathbf{1}).
\end{cases}
\tag{3}
$$

It is now possible to define the operators $\left(\tilde{\mathcal{I}}^n_{k,\epsilon_k}\right)_{n-N+1 \leq k \leq n}$ so that they act in the same way as the matrices $\left(\tilde{\mathrm{A}}^n_{k,\epsilon_k}\right)_{1 \leq k \leq N}$ do on $\mathbb{R}^{2^n}$, by requiring they satisfy the relations:

$$
\forall p \in \{1, \cdots, 2^n\} : \left(\tilde{\mathcal{I}}^n_{n-k+1,\epsilon_{n-k+1}}(\mathrm{h})\right)_p = \left(\tilde{\mathrm{A}}^n_{k,\epsilon_k}\mathrm{H}_n\right)_p,
$$

for k varying in $\{1, \cdots, N\}$.
Let us give an example when n is small.

Example 4.8. *We illustrate the transition from the step $n = 3$ to $n = 4$, in case of two broken links placed as in Figure 8. The operators $\tilde{\mathcal{I}}^3_{2,\epsilon_2}, \tilde{\mathcal{I}}^3_{3,\epsilon_3}$ representing the structure of Figure 8 are the ones defined by the relations:*

$$
\tilde{\mathcal{I}}^3_{2,\epsilon_2}(\mathrm{h})(c) = \begin{cases} \mathrm{h}(c_1, c_2, c_3, \mathbf{1}) & \text{if } c_1 = 0 \\ \mathcal{I}^3_{2,\epsilon_2}(\mathrm{h})(c) & \text{otherwise} \end{cases}
$$

and

$$
\tilde{\mathcal{I}}^3_{3,\epsilon_3}(\mathrm{h}) = \mathcal{I}^3_{3,\epsilon_3}(\mathrm{h}).
$$

As one more broken link is added at the step $n = 4$, the operators $\tilde{\mathcal{I}}^4_{3,\epsilon_3}, \tilde{\mathcal{I}}^4_{4,\epsilon_4}$ are defined by:

$$
\tilde{\mathcal{I}}^4_{3,\epsilon_3}(\mathrm{h})(c) = \begin{cases} \mathrm{h}(c_1, c_2, c_3, c_4, \mathbf{1}) & \text{if } (c_1, c_2) = (0, 0) \\ \mathcal{I}^4_{3,\epsilon_3}(\mathrm{h})(c) & \text{otherwise} \end{cases}
$$

and

$$
\tilde{\mathcal{I}}^4_{4,\epsilon_4}(\mathrm{h})(c) = \begin{cases} \mathrm{h}(c_1, c_2, c_3, c_4, \mathbf{1}) & \text{if } (c_1, c_2, c_3) = (1, 0, 0) \\ \mathcal{I}^4_{4,\epsilon_4}(\mathrm{h})(c) & \text{otherwise.} \end{cases}
$$

Remark 4.9. *By considering (for every $n \geq N + 1$) the operator $\tilde{\mathcal{W}}_{n,\epsilon}$, we have reversed the direction of iteration compared with the finite-dimensional case (i.e with the definition of $\tilde{\mathrm{B}}_{n,\epsilon}$). This comes from the fact in the hierarchical structure constructed in Part 3, the smallest scale (which is the one linking two successive elements $X_{(i)}$, $X_{(i+1)}$) corresponds to the matrix $\mathcal{T}_{\epsilon_1}$, and thus remains fixed as the iteration goes on, whereas the smallest scale on the closed-open sets of size n is given by the operator J_n. The lector could be easily convinced by himself this reversing is needed for the existence of the limit of $\tilde{\mathcal{W}}_{n,\epsilon}$, when n goes to infinity. This existence will be established in Lemma 4.13.*

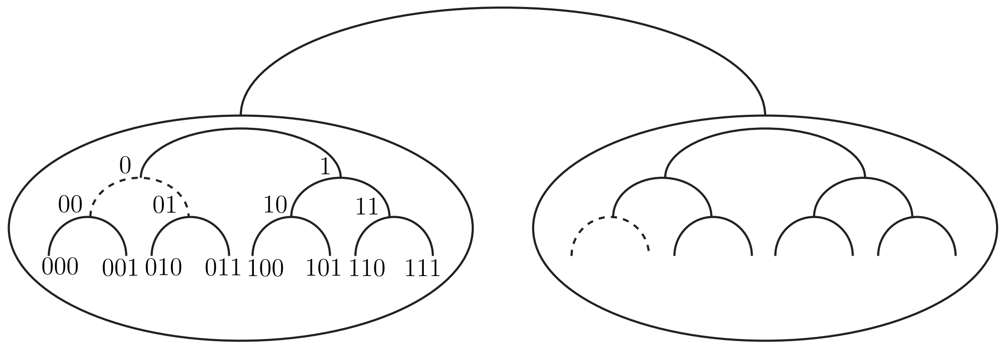


FIGURE 8. Transition from the step $n = 3$ to $n = 4$. The numbering represents the closed-open sets $(\mathscr{C}_{3,k})_{k=1,\cdots,2^3}$.

With this construction of the operators $\left(\tilde{\mathcal{I}}^n_{k,\epsilon_k}\right)_{n-N+1\leq k\leq n}$, we can now view the operator $\tilde{\mathcal{W}}_{n,\epsilon}$ as a coupling matrix of the form $\tilde{\mathrm{B}}_{n,\epsilon}$, except that here the order of the parameters has been reversed:

Definition 4.10. *Let us fix an integer $N \geq 1$, and consider (for $n \geq N+1$) the composition:*

$$\tilde{\mathcal{W}}_{n,\epsilon} = \mathcal{I}^n_{1,\epsilon_1} \circ \cdots \circ \mathcal{I}^n_{n-N,\epsilon_{n-N}} \circ \tilde{\mathcal{I}}^n_{n-N+1,\epsilon_{n-N+1}} \circ \cdots \circ \tilde{\mathcal{I}}^n_{n,\epsilon_n},$$

where $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ and the operators $\left(\tilde{\mathcal{I}}^n_{k,\epsilon_k}\right)_{n-N+1\leq k\leq n}$ (containing the broken links) are associated to the matrices $\left(\tilde{\mathrm{A}}^n_{k,\epsilon_k}\right)_{1\leq k\leq N}$. Then the following matrix:

$$\hat{\mathrm{B}}_{n,\epsilon} = \mathrm{A}^n_{n,\epsilon_1} \circ \cdots \circ \mathrm{A}^n_{N+1,\epsilon_{n-N}} \circ \tilde{\mathrm{A}}^n_{N,\epsilon_{n-N+1}} \circ \cdots \circ \tilde{\mathrm{A}}^n_{1,\epsilon_n},$$

is called the coupling matrix of the operator $\tilde{\mathcal{W}}_{n,\epsilon}$.

Note that we have $\hat{\mathrm{B}}_{n,\epsilon} = \tilde{\mathrm{B}}_{n,\epsilon^{-1}}$, where ϵ^{-1} denotes the reverse tuple of parameters $(\epsilon_n, \cdots, \epsilon_1)$.

Definition 4.11. *Let us consider (for $n \geq N+1$) the operator $\tilde{\mathcal{W}}_{n,\epsilon}$ above. We define, for $k = N, \cdots, n-1$, for h in $\mathscr{C}(X, \mathbb{R})$ and c in X, the following family of 2^{n-k} real numbers:*

$$\mathscr{D}_k(\mathrm{h})(c) = \bigcup_{(a_1,\cdots,a_{n-k})\in\{0,1\}^{n-k}} \alpha_k(\mathrm{h})(c)$$

where we have set:

$$\alpha_k(\mathrm{h})(c) = \mathcal{I}^n_{n-k+1,\epsilon_{n-k+1}} \circ \cdots \circ \mathcal{I}^n_{n-N,\epsilon_{n-N}} \circ \tilde{\mathcal{I}}^n_{n-N+1,\epsilon_{n-N+1}} \circ \cdots \circ \tilde{\mathcal{I}}^n_{n,\epsilon_n}(\mathrm{h})(a_1, \cdots, a_{n-k}, c),$$

and we set $\mathscr{D}_n(\mathrm{h})(c) = \tilde{\mathcal{W}}_{n,\epsilon}(\mathrm{h})(c)$. This set $\mathscr{D}_k(\mathrm{h})(c)$ defines a family $\mathscr{D}_k$ of operators W_k associated to $\tilde{\mathcal{W}}_{n,\epsilon}$, defined by an expression $\mathrm{W}_k(\mathrm{h})(c)$ belonging to it. Such an operator W_k in $\mathscr{D}_k$ is called a sub-operator for $\tilde{\mathcal{W}}_{n,\epsilon}$.

It is important to notice that each sub-operator W_k is of the form $\tilde{\mathcal{W}}_{k,(\epsilon_{n-k+1},\cdots,\epsilon_n)}$. More precisely, we have:

$$\mathrm{W}_k(\mathrm{h})(c) = \mathcal{I}^k_{1,\epsilon_{n-k+1}} \circ \cdots \circ \mathcal{I}^k_{k-N,\epsilon_{n-N}} \circ \tilde{\mathcal{I}}^k_{k-N+1,\epsilon_{n-N+1}} \circ \cdots \circ \tilde{\mathcal{I}}^k_{k,\epsilon_n}(\mathrm{g})(c),$$

where $\mathrm{g}(c) = \mathrm{h}(a_1, \cdots, a_{n-k}, c)$.
This permits us to inherit the notion of nested sub-operators for $\tilde{\mathcal{W}}_{n,\epsilon}$. For instance, in the example 4.8 above, we have $\tilde{\mathcal{W}}_{4,\epsilon} = \mathcal{I}^4_{1,\epsilon_1} \circ \mathcal{I}^4_{2,\epsilon_2} \circ \tilde{\mathcal{I}}^4_{3,\epsilon_3} \circ \tilde{\mathcal{I}}^4_{4,\epsilon_4}$ and the operator W_3 defined by the relation $\mathrm{W}_3(\mathrm{h})(c) =$

$\mathcal{I}^4_{2,\epsilon_2} \circ \tilde{\mathcal{I}}^4_{3,\epsilon_3} \circ \tilde{\mathcal{I}}^4_{4,\epsilon_4} (\mathrm{h}) (1, c)$ is the sub-operator of $\tilde{\mathcal{W}}_{4,\epsilon}$ associated to the sub-diagram on the right of picture 8. We have the relation $\mathrm{W}_3 (\mathrm{h}) (c) = \tilde{\mathcal{W}}_{3,(\epsilon_2,\epsilon_3,\epsilon_4)} (\mathrm{g}) (c)$, where $\mathrm{g} (c) = \mathrm{h} (1, c)$.

Definition 4.12. *Let us consider (for $n \geq N + 1$) the operator $\tilde{\mathcal{W}}_{n,\epsilon}$ above. Then, a family $\mathrm{W}_N, \cdots, \mathrm{W}_n$ is said to be a nested family of sub-operators for $\tilde{\mathcal{W}}_{n,\epsilon}$ if for each $k = N, \cdots, n$, W_k belongs to $\mathscr{D}_k$, and W_k is a sub-operator of W_{k+1}.*
A nested family of sub-operators $\mathrm{W}_N, \cdots, \mathrm{W}_n$ is called admissible if its associated family of coupling matrices $\mathrm{D}_\mathrm{N}, \cdots, \mathrm{D}_n$ is an admissible nested one for $\hat{\mathrm{B}}_{n,\epsilon}$.

Now, as in the global synchronization case, for any continuous function h on X, the term $\lim_{n\to\infty} \tilde{\mathcal{W}}_{n,\epsilon} (\mathrm{h})$ exists (no matter the positions and the numbers of broken links, inside the N smallest stages). This limit is the desired limit operator since it acts on (all) the sequences of 0 and 1.

Lemma 4.13. *Let $\epsilon = (\epsilon_k)_{k\geq 1}$ a sequence in $[0, 1]^{\mathbb{N}^*}$ and consider for every $n \geq N + 1$ the operator $\tilde{\mathcal{W}}_{n,\epsilon} (\mathrm{h}) = \mathcal{I}^n_{1,\epsilon_1} \circ \cdots \circ \mathcal{I}^n_{n-N,\epsilon_{n-N}} \circ \tilde{\mathcal{I}}^n_{n-N+1,\epsilon_{n-N+1}} \circ \cdots \circ \tilde{\mathcal{I}}^n_{n,\epsilon_n} (\mathrm{h})$ containing some broken links at the first N stages of its hierarchical diagram. For every function h in $\mathscr{C} (\mathrm{X}, \mathbb{R})$, we set:*

$$\tilde{\mathcal{W}}_\epsilon (\mathrm{h}) = \lim_{n\to\infty} \tilde{\mathcal{W}}_{n,\epsilon} (\mathrm{h}) .$$

Then, $\tilde{\mathcal{W}}_\epsilon$ is a well-defined operator on $\mathscr{C} (\mathrm{X}, \mathbb{R})$.

Proof. The demonstration is the same as the one of Lemma 4.2. Let's take a continuous function h, and the associated sequence $(\mathrm{h}_n)_{n\geq 1}$ defined as in the proof of Proposition 4.1, that converges to h. We have for any integers $p, q \geq 1$:

$$||\tilde{\mathcal{W}}_{p+q,\epsilon} (\mathrm{h}) - \tilde{\mathcal{W}}_{p,\epsilon} (\mathrm{h}) ||_\infty \leq 2||\mathrm{h} - \mathrm{h}_p||_\infty + ||\tilde{\mathcal{W}}_{p+q,\epsilon} (\mathrm{h}_p) - \tilde{\mathcal{W}}_{p,\epsilon} (\mathrm{h}_p) ||_\infty.$$

Without loss of generality we can assume $q \geq N$. Then, as h_p is constant on the closed-open sets $\mathscr{C}_{p,k}$ of size p, we have:

$$\begin{aligned} ||\tilde{\mathcal{W}}_{p+q,\epsilon} (\mathrm{h}_p) - \tilde{\mathcal{W}}_{p,\epsilon} (\mathrm{h}_p) ||_\infty &\leq ||\mathcal{I}^p_{1,\epsilon_1} \circ \cdots \circ \mathcal{I}^p_{p,\epsilon_p} (\mathrm{h}_p) - \mathcal{I}^p_{1,\epsilon_1} \circ \cdots \circ \mathcal{I}^p_{p-N,\epsilon_{p-N}} \circ \tilde{\mathcal{I}}^p_{p-N+1,\epsilon_{p-N+1}} \circ \\ &\qquad \cdots \circ \tilde{\mathcal{I}}^p_{p,\epsilon_p} (\mathrm{h}_p) ||_\infty \\ &\leq ||\mathcal{I}^p_{p-N+1,\epsilon_{p-N+1}} \circ \cdots \circ \mathcal{I}^p_{p,\epsilon_p} (\mathrm{h}_p) - \tilde{\mathcal{I}}^p_{p-N+1,\epsilon_{p-N+1}} \circ \cdots \circ \tilde{\mathcal{I}}^p_{p,\epsilon_p} (\mathrm{h}_p) ||_\infty \\ &\leq \sum_{k=1}^{N} \epsilon_{p-N+k} ||\mathrm{h}_p - \mathrm{J}_{p-N+k} (\mathrm{h}_p) ||_\infty, \end{aligned}$$

the last inequality coming from the commutativity of the operators J_l with the operators $\mathcal{I}^p_{k,\epsilon_k}$, for any integers k, l. The last term tends to zero as p goes to infinity, and this independently on q. Thus we still have a Cauchy sequence. ☐

From this lemma we define the new dynamical system $\tilde{\mathrm{G}}_\epsilon = \tilde{\mathcal{W}}_\epsilon \circ \mathrm{F}$ and finally get the second result of our paper:

Theorem 4.14. *Let us fix $N \geq 1$ and $\epsilon = (\epsilon_k)_{k\geq 1}$ a sequence of real numbers in $[0, 1]$ defining the operator $\tilde{\mathcal{W}}_\epsilon$ on $\mathscr{C} (\mathrm{X}, \mathbb{R})$ by the relation: $\tilde{\mathcal{W}}_\epsilon (\mathrm{h}) = \lim_{n\to\infty} \tilde{\mathcal{W}}_{n,\epsilon} (\mathrm{h})$, where for every $n \geq N + 1$, the operator $\tilde{\mathcal{W}}_{n,\epsilon} = \mathcal{I}^n_{1,\epsilon_1} \circ \cdots \circ \mathcal{I}^n_{n-N,\epsilon_{n-N}} \circ \tilde{\mathcal{I}}^n_{n-N+1,\epsilon_{n-N+1}} \circ \cdots \circ \tilde{\mathcal{I}}^n_{n,\epsilon_n}$ has an admissible nested family of sub-operators. Assume the sequence ϵ satisfies the following:*

$$6 \sup_{z\in\mathrm{K}} |\mathrm{f}' (z) |^{2N} \sum_{k=1}^{+\infty} |1 - 2\epsilon_k| < 1.$$

Then the dynamical system $\tilde{\mathrm{G}}_\epsilon = \tilde{\mathcal{W}}_\epsilon \circ \mathrm{F}$ synchronizes on a non trivial neighborhood Ω_N of the diagonal $\mathfrak{I} \cap \mathscr{C} (\mathrm{X}, \mathrm{K})$ in $\mathscr{C} (\mathrm{X}, \mathbb{R})$.

In fact, the assumption comes to requiring the existence of an infinite admissible nested family of sub-operators for $\tilde{\mathcal{W}}_\epsilon$, constructed by extending at each step n the (finite) such family of $\tilde{\mathcal{W}}_{n,\epsilon}$.
As mentioned several times before, the proof of this theorem is just a passage to the limit of the estimations

done in Lemma 3.11. Obviously, the fact that this lemma is applied to the matrix $\hat{\mathrm{B}}_{n,\epsilon}$ instead of $\tilde{\mathrm{B}}_{n,\epsilon}$ changes nothing, for the required assumption remains the same.

Proof of Theorem 4.14. (1) We first make the same Taylor approximations as in the proof of Lemma 3.11. Let us fix a real $\epsilon' > 0$. According to the proof of this lemma, there exists a real $\eta_{2N} > 0$ such that for every $n \geq N+1$, and every X, Y in K^{2^n} we have:

$$||X-Y||_{\infty,n} \leq \eta_{2N} \Rightarrow ||D_X\left(\hat{\mathrm{G}}_{n,\epsilon}\right)^{2N} - D_Y\left(\hat{\mathrm{G}}_{n,\epsilon}\right)^{2N}||_{\infty,n} \leq \epsilon',$$

where $\hat{\mathrm{G}}_{n,\epsilon} = \hat{\mathrm{B}}_{n,\epsilon} \circ \mathrm{F}_n$. Let h a function belonging to $\mathscr{C}(\mathrm{X},\mathrm{K})$ and $\mathrm{H}_n = (\mathrm{h}_{n,1}, \cdots, \mathrm{h}_{n,2^n})$ the vector defined previously by the relations (3). As the closed-open sets form a partition of X, we have:

$$\int_X \mathrm{h}_n(c)\,\mathrm{d}\mu(c) = \frac{1}{2^n}\sum_{k=1}^{2^n}\mathrm{h}_{n,k},$$

and since X is compact, we remark the following:

$$\int_X \mathrm{h}_n(c)\,\mathrm{d}\mu(c) \underset{n\to\infty}{\longrightarrow} \int_{\mathrm{X}} \mathrm{h}(c)\,\mathrm{d}\mu(c).$$

We define the sum-vector in $\mathfrak{I}_{2^n}$:

$$\mathrm{H}_{n,\Sigma} = \left(\int_{\mathrm{X}}\mathrm{h}_n\mathrm{d}\mu, \cdots, \int_{\mathrm{X}}\mathrm{h}_n\mathrm{d}\mu\right).$$

Now, for any $n \geq N+1$ and any function g_n constant on the closed-open sets of size n, we have $\tilde{\mathscr{W}}_\epsilon(\mathrm{g}_n) = \tilde{\mathscr{W}}_{n,\epsilon}(\mathrm{g}_n)$. It comes:

$$\begin{aligned}||\left(\tilde{\mathrm{G}}_\epsilon\right)^{2N}(\mathrm{h}_n) - \left(\tilde{\mathrm{G}}_\epsilon\right)^{2N}\left(\int_{\mathrm{X}}\mathrm{h}_n\mathrm{d}\mu\right)||_\infty &= ||\left(\tilde{\mathscr{W}}_{n,\epsilon}\circ\mathrm{F}\right)^{2N}(\mathrm{h}_n) - \left(\tilde{\mathscr{W}}_{n,\epsilon}\circ\mathrm{F}\right)^{2N}\left(\int_{\mathrm{X}}\mathrm{h}_n\mathrm{d}\mu\right)||_\infty \\ &= ||\left(\hat{\mathrm{G}}_{n,\epsilon}\right)^{2N}(\mathrm{H}_n) - \left(\hat{\mathrm{G}}_{n,\epsilon}\right)^{2N}(\mathrm{H}_{n,\Sigma})||_{\infty,n},\end{aligned}$$

and so by Lemma 3.11 the term on the left is smaller than:

$$(\epsilon' + \Lambda_n)\,||\mathrm{H}_n - \mathrm{H}_{n,\Sigma}||_{\infty,n} + ||D_{\mathrm{H}_{n,\Sigma}}\left(\hat{\mathrm{G}}_{n,\epsilon}\right)^{2N} - D_{\mathrm{H}_{n,\Sigma}}\left(\hat{\mathrm{G}}_{n,1/2}\right)^{2N}||_{\infty,n}||\mathrm{H}_n - \mathrm{H}_{n,\Sigma}||_{\infty,n},$$

providing that our function h satisfies: $|\mathrm{h}(c) - \mathrm{h}(c')| \leq \eta_{2N}$ for every c, c' in X. Now as in the proof of Lemma 4.9 we notice that:

$$\begin{aligned}||D_{\mathrm{H}_{n,\Sigma}}\left(\hat{\mathrm{G}}_{n,\epsilon}\right)^{2N} - D_{\mathrm{H}_{n,\Sigma}}\left(\hat{\mathrm{G}}_{n,1/2}\right)^{2N}||_{\infty,n} &\leq 3\sup_{z\in\mathrm{K}}|\mathrm{f}'(z)|^{2N}||\hat{\mathrm{B}}_{n,1/2} - \hat{\mathrm{B}}_{n,\epsilon}||_{\infty,n} \\ &\leq 3\sup_{z\in\mathrm{K}}|\mathrm{f}'(z)|^{2N}\sum_{k=1}^{n}|1-2\epsilon_k|.\end{aligned}$$

Passing to the limit at infinity, we get:

$$||\left(\tilde{\mathrm{G}}_\epsilon\right)^{2N}(\mathrm{h}) - \left(\tilde{\mathrm{G}}_\epsilon\right)^{2N}\left(\int_{\mathrm{X}}\mathrm{h}\mathrm{d}\mu\right)||_\infty \leq \left(\epsilon' + 3\sup_{z\in\mathrm{K}}|\mathrm{f}'(z)|^{2N}\sum_{k=1}^{+\infty}|1-2\epsilon_k|\right)||\mathrm{h} - \int_{\mathrm{X}}\mathrm{h}\,\mathrm{d}\mu||_\infty.$$

(2) From this it comes for every sequences c, c' in X:

$$|\left(\tilde{\mathrm{G}}_\epsilon\right)^{2N}(\mathrm{h})(c) - \left(\tilde{\mathrm{G}}_\epsilon\right)^{2N}(\mathrm{h})(c')| \leq 2\left(\epsilon' + 3\sup_{z\in\mathrm{K}}|\mathrm{f}'(z)|^{2N}\sum_{k=1}^{+\infty}|1-2\epsilon_k|\right)\sup_{x,x'\in\mathrm{X}}|\mathrm{h}(x) - \mathrm{h}(x')|,$$

which gives the synchronization of the dynamical system $\left(\tilde{\mathrm{G}}_\epsilon\right)^{2N}$ on the neighborhood Ω_N of the diagonal $\mathfrak{J} \cap \mathscr{C}(\mathrm{X}, \mathrm{K})$ in $\mathscr{C}(\mathrm{X}, \mathbb{R})$, defined by:

$$\Omega_N = \{\mathrm{h} \in \mathscr{C}(\mathrm{X}, \mathrm{K}) : \sup_{x, x' \in \mathrm{X}} |\mathrm{h}(x) - \mathrm{h}(x')| \leq \eta_{2N}\}.$$

Doing again the same trick as in the end of the proof of Lemma 3.11, we finally obtain the synchronization of $\tilde{\mathrm{G}}_\epsilon$ on Ω_N. QED. □

4.3. **Corollaries of the results in the case p^n.** In this last subsection, we present the corollaries of our two theorems on a Cantor set $\mathrm{X} = \{a_0, \cdots, a_{p-1}\}^{\mathbb{N}}$ associated to an alphabet with $p \geq 3$ letters, which is the limiting case corresponding to a finite matching process in groups of size p: given a $(p-1)$-tuple $\epsilon^{(k)} = \left({\epsilon_1}^{(k)}, \cdots, {\epsilon_{p-1}}^{(k)}\right)$ defining the k^{th} stage of the hierarchical structure, the coupling between the gathered elements is now defined by a circulant matrix $\mathcal{R}_{\epsilon^{(k)}}$, of which entries are $1 - \sum_{i=1}^{p-1} \epsilon_i^{(k)}, {\epsilon_1}^{(k)}, \cdots, {\epsilon_{p-1}}^{(k)}$.

4.3.a. *Corollary of theorem* (4.6). For every $(p-1)$-tuple $\epsilon^{(k)} = \left({\epsilon_1}^{(k)}, \cdots, {\epsilon_{p-1}}^{(k)}\right)$ in $[0,1]^{p-1}$ the new operator structures $\mathcal{L}_{k,\epsilon^{(k)}}$ write:

$$\mathcal{L}_{k,\epsilon^{(k)}}(\mathrm{h}) = \left(1 - \sum_{i=1}^{p-1} \epsilon_i^{(k)}\right) \mathrm{h} + \sum_{i=1}^{p-1} \epsilon_i^{(k)} {\mathrm{J}_k}^i(\mathrm{h}),$$

where this time the map $\mathrm{J}_k(\mathrm{h})$ is defined by the relation:

$$\forall c \in \mathrm{X}, \text{ with } c_k = a_i,\ \mathrm{J}_k(\mathrm{h})(c) = \mathrm{h}(c_1, \cdots, c_{k-1}, a_{i+1 \bmod p}, c_{k+1}, \cdots).$$

With this definition, nothing changes in the approach we have done previously, except the fact the orthogonal of the diagonal $\mathfrak{J}_k$ in $\mathscr{C}(\mathrm{X}, \mathbb{R})$ becomes:

$${\mathfrak{J}_k}^{\perp} = \{\mathrm{h} \in \mathscr{C}(\mathrm{X}, \mathbb{R}) : \sum_{i=0}^{p-1} {\mathrm{J}_k}^i(\mathrm{h}) = 0\}.$$

We get:

Corollary 4.15. *Let $\epsilon = \left(\epsilon^{(k)}\right)_{k \geq 1}$ be a sequence of elements in $[0,1]^{p-1}$, defining the operator $\mathcal{U}_\epsilon$ by:*

$$\mathcal{U}_\epsilon(\mathrm{h}) = \lim_{n \to \infty} \mathcal{L}_{n,\epsilon^{(n)}} \circ \cdots \circ \mathcal{L}_{1,\epsilon^{(1)}}(\mathrm{h}),$$

for every continuous function h *on* X*. Assume the following condition holds:*

$$\exists a > 1,\ \exists \alpha > 0,\ \forall k \geq 1,\ |\gamma^{(k)}| \sup_{z \in \mathrm{K}} \left|\mathrm{f}'(z)\right| \leq \frac{1}{(ak)^\alpha},$$

where $\gamma^{(k)}$ is the eigenvalue (distinct from one) of $\mathcal{R}_{\epsilon^{(k)}}$ having the greatest modulus.
Then the dynamical system $\mathrm{G}_\epsilon = \mathcal{U}_\epsilon \circ \mathrm{F}$ globally synchronizes on the set $\mathscr{C}(\mathrm{X}, \mathrm{K})$ of continuous functions having values in K.

4.3.b. *Corollary of theorem* (4.14). Concerning the local synchronization in presence of broken links, this requires to consider again the spectrum of the structure matrix $\tilde{\mathrm{B}}_{n,\epsilon} = \mathrm{A}^n_{n,\epsilon^{(n)}} \cdots \mathrm{A}^n_{N+1,\epsilon^{(N+1)}} \tilde{\mathrm{A}}^n_{N,\epsilon^{(N)}} \cdots \tilde{\mathrm{A}}^n_{1,\epsilon^{(1)}}$, for a n-tuple ϵ in $[0,1]^{p-1}$, close to the constant one $\epsilon^{(n)} = \cdots = \epsilon^{(1)} = \left(\frac{1}{p}, \cdots, \frac{1}{p}\right)$. As previously, we impose that the broken links be only present in the smallest N stages of the hierarchy, while keeping the same condition that they be enough placed out.
In this case, for every $n \geq N+1$, we have:

$$\tilde{\mathrm{B}}_{n,\epsilon}=\begin{bmatrix}\left(1-\sum_{i=1}^{p-1}\epsilon_i{}^{(n)}\right)\tilde{\mathrm{B}}_{n-1,\epsilon} & \epsilon_1{}^{(n)}\,\tilde{\mathrm{B}}'_{n-1,\epsilon} & \cdots & \epsilon_{p-1}{}^{(n)}\,\tilde{\mathrm{B}}^{(p-1)}_{n-1,\epsilon}\\ \vdots & \vdots & & \vdots\\ \left(1-\sum_{i=1}^{p-1}\epsilon_i{}^{(n)}\right)\tilde{\mathrm{B}}_{n-1,\epsilon} & \epsilon_1{}^{(n)}\tilde{\mathrm{B}}'_{n-1,\epsilon} & \cdots & \epsilon_{p-1}{}^{(n)}\tilde{\mathrm{B}}^{(p-1)}_{n-1,\epsilon}\end{bmatrix},$$

where the matrices $\tilde{\mathrm{B}}_{n-1,\epsilon}$, $\tilde{\mathrm{B}}'_{n-1,\epsilon}$, $,\cdots,$ $\tilde{\mathrm{B}}^{(p-1)}_{n-1,\epsilon}$ of size p^{n-1} are associated to the $(n-1)$-tuple $\left(\epsilon^{(1)},\cdots,\epsilon^{(n-1)}\right)$, and also contain their broken links at the N smallest stages of the hierarchy.
From this relation, we clearly define, as in Definition 3.6, the families $\mathscr{D}_N,\cdots,\mathscr{D}_n$ associated to $\tilde{\mathrm{B}}_{n,\epsilon}$ and recover the notion of nested family of submatrices for $\tilde{\mathrm{B}}_{n,\epsilon}$, which permits us to define again the notion of admissible nested ones:

Definition 4.16. *Let us fix an integer $N\geq 1$ and consider (for $n\geq N+1$), the coupling matrix*

$$\tilde{\mathrm{B}}_{n,\epsilon}=\mathrm{A}^n_{n,\epsilon^{(n)}}\cdots\mathrm{A}^n_{N+1,\epsilon^{(N+1)}}\tilde{\mathrm{A}}^n_{N,\epsilon^{(N)}}\cdots\tilde{\mathrm{A}}^n_{1,\epsilon^{(1)}},$$

where $\tilde{\mathrm{A}}^n_{N,\epsilon^{(N)}},\cdots,\tilde{\mathrm{A}}^n_{1,\epsilon_{(1)}}$ are supposed to contain some broken links.
Given a nested family $D_N,\cdots,D_n$ of submatrices for $\tilde{\mathrm{B}}_{n,\epsilon}$, let's denote by a_k the number of broken links of the element D_k. We say that $D_N,\cdots,D_n$ is an admissible nested family if the following conditions are satisfied:

- $a_N\leq 1$
- $(a_{N+1}-a_N,\cdots,a_n-a_{n-1})$ *does not contain two successive numbers of which sum is greater or equal to p.*
- *Each submatrix (of size p^N) belonging to the family $\mathscr{D}_N$ of $\tilde{\mathrm{B}}_{n,\epsilon}$ does not contain more than one broken link.*

Indeed, as in Lemma 3.5, it is easy to prove that if the matrices $\tilde{\mathrm{B}}_{N,\frac{1}{p}}$, $\tilde{\mathrm{B}}'_{N,\frac{1}{p}}$, $,\cdots,$ $\tilde{\mathrm{B}}^{(p-1)}_{N,\frac{1}{p}}$ all contain one broken link, then the term:

$$\frac{1}{p}\tilde{\mathrm{B}}_{N,\frac{1}{p}}+\frac{1}{p}\tilde{\mathrm{B}}'_{N,\frac{1}{p}}+\cdots+\frac{1}{p}\tilde{\mathrm{B}}^{(p-1)}_{N,\frac{1}{p}},$$

is diagonalizable with spectrum of the form $\{1,\lambda_1,\cdots,\lambda_{(p-1)N},0^{\otimes p^N-(p-1)N-1}\}$, where the eigenvalues $\lambda_1,\cdots,\lambda_{(p-1)N}$ are smaller than 1.
Remark that the third condition of Definition 4.16 was automatically satisfied in the case $p=2$.

An interesting example of admissible nested family is the one associated to the tuple

$$(a_{N+1}-a_N,\cdots,a_n-a_{n-1})=(p-1,0,p-1,0,\cdots,p-1,0).$$

In this case, the coupling matrix admits a number a_n of broken links greater than $(p-1)\,\mathrm{E}\,(n/2)$.
By induction, we prove the result equivalent to Lemma 3.9: that is, for any $n\geq N+1$, if $\tilde{\mathrm{B}}_{n,\frac{1}{p}}$ has an admissible nested family of submatrices, then this matrix is diagonalizable and satisfies:

$$\mathfrak{S}\left(\tilde{\mathrm{B}}_{n,\frac{1}{p}}\right)=\{1,\lambda_{n,1},\cdots,\lambda_{n,(p-1)N},0^{\otimes p^n-(p-1)N-1}\},$$

the $\lambda_{n,i}$ being some (possibly equal) positive numbers satisfying:

$$\forall\, i\in\{1,\cdots,(p-1)\,N\},\ \lambda_{n,i}\leq\frac{n-N+1}{p^{n-N}}.$$

This leads us to the same estimation as in Lemma 3.11, in which appears the term $||\tilde{\mathrm{B}}_{n,\epsilon} - \tilde{\mathrm{B}}_{n,\frac{1}{p}}||_{\infty,n}$. Then, denoting by Q_k the orthogonal matrix of $GL_{p^n}(\mathbb{R})$ conjugating $\mathcal{R}_{\epsilon^{(k)}}$ to its diagonal form, we have:

$$\begin{aligned}
||\tilde{\mathrm{B}}_{n,\epsilon} - \tilde{\mathrm{B}}_{n,\frac{1}{p}}||_{\infty,n} &\leq \sum_{k=1}^{n} ||\tilde{\mathrm{A}}_{k,\epsilon^{(k)}} - \tilde{\mathrm{A}}_{k,\frac{1}{p}}||_{\infty,n} \\
&\leq \sum_{k=1}^{n} ||Q_k||_{\infty,n} ||{Q_k}^{-1}||_{\infty,n} |\gamma^{(k)}| \\
&\leq \sqrt{p} \sum_{k=1}^{n} |\gamma^{(k)}|,
\end{aligned}$$

the last inequality coming from the fact that each condition number $||Q_k||_{2,n}||{Q_k}^{-1}||_{2,n}$ is equal to 1, since the matrix Q_k is orthogonal.
From this comes the final result:

Corollary 4.17. *Let us fix $N \geq 1$ and $\epsilon = \left(\epsilon^{(k)}\right)_{k\geq 1}$ a sequence of elements in $[0,1]^p$ defining the operator $\tilde{\mathcal{W}}_\epsilon$ on $\mathscr{C}(\mathrm{X},\mathbb{R})$ by the relation: $\tilde{\mathcal{W}}_\epsilon(\mathrm{h}) = \lim_{n\to\infty} \tilde{\mathcal{W}}_{n,\epsilon}(\mathrm{h})$, where for every $n \geq N+1$, the operator $\tilde{\mathcal{W}}_{n,\epsilon} = \mathcal{I}^n_{1,\epsilon^{(1)}} \circ \cdots \circ \mathcal{I}^n_{n-N,\epsilon^{(n-N)}} \circ \tilde{\mathcal{I}}^n_{n-N+1,\epsilon^{(n-N+1)}} \circ \cdots \circ \tilde{\mathcal{I}}^n_{n,\epsilon^{(n)}}(\mathrm{h})$, has an admissible nested family of sub-operators.*
Assume the sequence ϵ satisfies the following:

$$6\sqrt{p} \sup_{z\in\mathrm{K}} |\mathrm{f}'(z)|^{2N} \sum_{k=1}^{+\infty} |\gamma^{(k)}| < 1,$$

where $\gamma^{(k)}$ is the eigenvalue (distinct from one) of $\mathcal{R}_{\epsilon^{(k)}}$ having the greatest modulus.
Then the dynamical system $\tilde{\mathrm{G}}_\epsilon = \tilde{\mathcal{W}}_\epsilon \circ \mathrm{F}$ synchronizes on a non trivial neighborhood Ω_N of the diagonal $\mathfrak{I} \cap \mathscr{C}(\mathrm{X},\mathrm{K})$ in $\mathscr{C}(\mathrm{X},\mathbb{R})$.

Finally, these corollaries directly extend to functions f defined on a convex compact set K of $\mathbb{R}^n$ and verifying $\sup_{z\in\mathrm{K}} \left|\left|D_z\mathrm{f}\right|\right| > 1$ for some norm on $\mathbb{R}^n$, since this case just comes to replacing the coupling matrix $\tilde{\mathrm{B}}_{n,\epsilon}$ by $\tilde{\mathrm{B}}_{n,\epsilon} \otimes I_d$.

5. Conclusion

From this work we conclude that using the hierarchical structure we have exhibited in this article is the natural way to synchronize an uncountable set (namely a Cantor set) of dynamical systems. Once again, such a phenomenon echoes to tiling theory, where hierarchy is often imposed to produce aperiodicity.
Our investigation could be pursued with the same framework as the one we have used here.
An interesting question would be to study what happens when the structure is not repeated at each step, while keeping a hierarchy: for instance, starting at the first scale with a matching with two elements, we could gather them three by three at the second one, and then change again the type of matching at the third one, and so on. If we impose that the size of the matching process varies in a finite set $\{2, \cdots, p\}$, then some results must be found. In tilings this problem appears strongly, and corresponds to the case where the shape of the patches changes at each stage, contrary to substitution tilings. A lot of work still remains to be done in this direction to understand better those general hierarchical structures.

References

[1] Neil J. Balmforth, Antonello Provenzale, and Roberto Sassi. A hierarchy of coupled maps. *Chaos*, 12(3):719–731, 2002.
[2] A. Díaz-Guilera. Dynamics towards synchronization in hierarchical networks. *J. Phys. A*, 41(22):224007, 7, 2008.
[3] Bastien Fernandez. Global synchronization in translation invariant coupled map lattices. *International Journal of Bifurcation and Chaos*, 18(11):3455–3459, 2008.
[4] Hirokazu Fujisaka and Tomoji Yamada. Stability theory of synchronized motion in coupled-oscillator systems. *Progr. Theoret. Phys.*, 69(1):32–47, 1983.
[5] Chaim Goodman-Strauss. Matching rules and substitution tilings. *Ann. of Math. (2)*, 147(1):181–223, 1998.
[6] Chaim Goodman-Strauss. Aperiodic hierarchical tilings. In *Foams and emulsions (Cargèse, 1997)*, volume 354 of *NATO Adv. Sci. Inst. Ser. E Appl. Sci.*, pages 481–496. Kluwer Acad. Publ., Dordrecht, 1999.

[7] J. Jost and M. P. Joy. Spectral properties and synchronization in coupled map lattices. *Phys. Rev. E*, 65:016201, Dec 2001.

[8] Kunihiko Kaneko. Clustering, coding, switching, hierarchical ordering, and control in a network of chaotic elements. *Physica D: Nonlinear Phenomena*, 41(2):137 – 172, 1990.

[9] Kunihiko Kaneko. *Theory and applications of coupled map lattices.* Nonlinear Science: Theory and Applications. John Wiley & Sons, Ltd., Chichester, 1993.

[10] Y. Kuramoto. *Chemical oscillations, waves, and turbulence*, volume 19 of *Springer Series in Synergetics*. Springer-Verlag, Berlin, 1984.

[11] Wenlian Lu and Tianping Chen. Synchronization analysis of linearly coupled networks of discrete time systems. *Physica D: Nonlinear Phenomena*, 198(1-2):148–168, 2004.

[12] R. Mirollo and S. Strogatz. Synchronization of pulse-coupled biological oscillators. *SIAM Journal on Applied Mathematics*, 50(6):1645–1662, 1990.

[13] Louis M. Pecora and Thomas L. Carroll. Synchronization in chaotic systems. *Phys. Rev. Lett.*, 64:821–824, Feb 1990.

[14] Arkady Pikovsky, Michael Rosenblum, and Jürgen Kurths. *Synchronization: a universal concept in nonlinear sciences*, volume 12 of *Cambridge Nonlinear Science Series*. Cambridge University Press, Cambridge, 2001.

[15] Przemyslaw Prusinkiewicz and Aristid Lindenmayer. *The algorithmic beauty of plants.* Springer-Verlag New York, Inc., New York, NY, USA, 1996.

[16] Raphael M. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Invent. Math.*, 12:177–209, 1971.

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