



HAL
open science

Discrete Synchronization under the Hierarchical viewpoint

Camille Poignard

► **To cite this version:**

| Camille Poignard. Discrete Synchronization under the Hierarchical viewpoint. 2012. hal-00755688v1

HAL Id: hal-00755688

<https://hal.science/hal-00755688v1>

Preprint submitted on 22 Nov 2012 (v1), last revised 27 Mar 2013 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DISCRETE SYNCHRONIZATION UNDER THE HIERARCHICAL VIEW POINT

CAMILLE POIGNARD

ABSTRACT. We study the synchronization problem of dynamical systems in case of a hierarchical structure among them, of which interest comes from the growing necessity of understanding properties of complex systems, that often exhibit such an organization. Starting with a set of 2^n systems, we define a hierarchical structure inside it by a matrix representing all the steps of a matching process in groups of size 2. This leads us naturally to the synchronization of a Cantor set of systems, indexed by $\{0, 1\}^N$: we obtain a global synchronization result generalizing the finite case. In the same context, we deal with this question when some defects appear in the hierarchy, that is to say when some links between certain systems are broken. We prove a local synchronization result under the condition that these defects are present at the N smallest scales of the hierarchy (for a fixed integer N) and their number be in \sqrt{n} when n goes to infinity.

1. INTRODUCTION

Complex systems are systems presenting a very high number of components, all interacting each other, in such a way some macroscopic phenomena emerge, that cannot be deduced from the knowledge of the dynamics inherent to each unit. Understanding what are the main macroscopic properties that usually appear in such general structures is a real important challenge, for their omnipresence in a very large number of domains (see [19]): from natural sciences (with cells, neural networks, pacemaker cells of the heart), chemistry, computer sciences, social sciences (with economic networks inside a country), to the mathematics of weather and climate. With the well-known Lorenz model, this last subject takes its origin in chaotic systems (that is to say deterministic systems presenting a high sensibility in initial conditions, that make them have an unpredictable dynamics) which have motivated so many studies from the last century until now.

Among emergent dynamical phenomena, one which is widely observed in concrete life is synchronization, which is the property that all the entities tend to have the same behavior. Without this phenomenon, many actions such as running could not take place. More theoretically, synchronization has been exhibited in systems of coupled oscillators ([6], [16]), notably when each oscillator presents a chaotic behavior ([17]). Such a chaotic synchronization is remarkable for in a certain sense it's a way to control the initial erratic behavior (see [10]).

Other interesting properties to understand are more related to the nature of the complex system (rather than its behavior), more precisely to the geometry of its spatial configuration. One that constantly appears both in experiments and in theoretical frameworks is hierarchy.

The typical example of a hierarchical structure arises from the tilings of grounds (mathematically of manifolds): given a finite set of tiles filling the ground with copies of themselves, they may sometimes cluster each other into patches which are finitely distinct at fixed size, in such a way a new tiling can be made by these patches. And so on, this process repeats infinitely, each time considering the last patches obtained as the new (finite) set of tiles.

Obviously any periodic configuration presents such a structure, but mathematicians demonstrated the interest of the hierarchical ones, when was proven the existence of a finite set of tiles filling the plane aperiodically (i.e no translation let the tiling invariant) and hierarchically, but for which no periodic tiling of the plane could be obtained. R. Berger was the first to give such a set of tiles in 1966 (see [2]), then followed other ones with less tiles ([21],[18]). In 1982, the physicist D.Shechtman discovered the existence of a solid (an aluminium-manganese alloy) presenting a 5-fold symmetry in its diffraction diagram, but lacking translational symmetry, upsetting by the way the current belief that any solid was either a crystal perfectly ordered (i.e periodic) or an amorphous totally disordered one.

These advances showed the periodic configurations constituted in fact a small set in the world of ordered structures, the set of hierarchical ones containing it being definitely much bigger. Over the years, the ubiquity of hierarchy in organization of matter has been strongly reinforced, confirming what was concretely under our eyes from the beginning, but at bigger scales: the way rivers split into smaller ones, galaxies gather, but also tree shapes, lobes of compound leaves (see [20]), muscular fibers etc... Indeed, in the growing theory of quasi-crystals, most of the aperiodic tilings discovered until now were constructed by forcing them to have a hierarchy ([12], [15]). This is the case of substitution tilings, where the set of prototiles gather in homothetic copies of themselves (see [7], [8]). Recently natural quasi-crystals such as the icosahedrite were discovered in Russia. In fluid mechanics, patterns in the Faraday experiments presenting a multi-level structure were discovered in 1996 ([4]). Such an omnipresence suggests that hierarchy has a strong robustness and stability to perturbations.

The goal of this text is to deal with the Synchronization problem in case of a hierarchical structure among the dynamical systems considered. This can be understood in many ways, for this reason alone there is not yet a precise (mathematical) definition of such a structure: even in the tilings theory in which this concept is very understandable, we do not have yet a clear definition of a hierarchical tiling. However, the ubiquity of such structures in nature as well as in purely mathematical objects ensures us the interest of such a question. Some numerical investigations in this direction have been done yet in the past, with purpose the analysis of the route to synchronization. For instance in [3], A. Díaz-Guilera considers a finite set of oscillators described by a particular model issued from the Kuramoto continuous one (see [13]), and shows notably the difference of rapidity presented by the local clusters that merge together time after time, through complete synchronization (see also [1], in which the dynamics of each oscillator is described by the logistic map). Here we aim at proving rigorous results on synchronization in case of a hierarchy, and especially (as we are going to see below) to look at its influence on the dynamics of the whole structure when this one has some local defects.

As explained above, the idea is to look at the situation where each system is linked to some of them, in such a way exists in the whole set (regarded as a huge dynamical system) an imbricated structure in greater and greater ones. More precisely, we first consider 2^n systems (the general case p^n is completely similar), each of them having a dynamics represented by an expanding (so possibly chaotic) function f defined on a compact K of \mathbb{R} . We represent the hierarchy among them by a matrix B_n , defined as the product of all the structure matrices associated to its n stages, with the assumption that each stage is associated to one parameter: this comes to assuming all the couplings are the same at a given stage (see Figure 1 in which the links represent the couplings).

Then, the synchronization of our systems write in terms of the coupled map $G_n = B_n \circ F_n$, where F_n is the vector-valued function of which components are all equal to f : it happens if the successive iterations $(G_n)^2, (G_n)^3, \dots$ of G_n asymptotically approach the diagonal in K^{2^n} . Obviously, finding conditions under which the map G_n synchronizes is a very well known problem that has been fully studied for three decades (see [11], [5], [14], [9],...). Nevertheless, our approach is of interest since it leads us naturally to the limiting case of the synchronization of a very high number of systems, namely an uncountable one, which constitutes the novelty of this work. Indeed, looking at the limit (when n tends to infinity) of our cascade structure, in which our 2^n elements are ordered by a tuple of 0 and 1, leads us to a Cantor set of systems, indexed by the set $\{0, 1\}^{\mathbb{N}}$ of sequences in 0 and 1.

In this context, we thus define a new setting for the synchronization problem (in which matrix couplings are replaced by operators, initial conditions become functions, etc...) and we look at conditions under which this phenomenon happens globally (i.e every point in K^{2^n} is attracted by the diagonal) or locally (only the points in a neighborhood of the diagonal is attracted by it). Using the same approach as B. Fernandez in [5], we obtain a global synchronization theorem that generalizes its result, providing the sequence of parameters defining the stages of the hierarchy converge slowly to $1/2$.

A natural extension of this first part of our work is to consider the situation where the entire set of our systems contains some that are uncoupled, regarding by the way whether the hierarchy forces them to synchronize with the other ones or not (see Figure 2 in which the broken links represent the uncoupled systems). If it is the case, can we authorize a high number of uncoupled systems, namely an infinite one in our Cantor set of systems? How do their positions in the hierarchy influence the synchronization?

We answer these two questions by proving that providing the number of uncoupled systems increase in \sqrt{n} when n goes to infinity, and they stay at the smallest N scales of the hierarchy (for a fixed integer N), then local synchronization takes place on a neighborhood of the diagonal, that does not depend on the position of the broken links inside the last N scales.

The paper is organized in three parts. The first one (Part 2) contains the basic notations. The second (Part 3) deals with our hierarchical synchronization problem in the finite case of 2^n systems, which constitutes the preliminaries for the third part about the case of a Cantor set.

In Subsection 3.1, we begin by presenting the global synchronization result on the dynamical system $G_n = B_{n,\epsilon} \circ F_n$, where the notation $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ refers to the n parameters associated to the n stages of the hierarchy, and recall briefly the proof of this already known theorem.

Subsection 3.2 deals with our synchronization problem when some links are broken in the hierarchy, which comes to replacing some blocks inside the structure matrices that define $B_{n,\epsilon}$ by identical blocks $I_{2^{k-1}}$, for some k in $[1, n]$. The resulting new coupling matrix $\tilde{B}_{n,\epsilon}$ is no more symmetric, which prevents us from applying the same technique as previously. In fact, there is no chance to have a global synchronization in this case, for the new dynamical system $\tilde{G}_{n,\epsilon}$ admits in general some fixed points outside the diagonal (see Example 3.7). We thus look at a local result. In Paragraph 3.2.a, given an integer N , we prove that, if the broken links stay at the first N scales of the hierarchy (i.e the N smallest ones), and if their number is smaller than \sqrt{n} , then the eigenvalues of $\tilde{B}_{n,\epsilon}$ tends to 0 as n goes to infinity, except the one corresponding to the direction of the diagonal, which is obviously one (see Lemma 3.6). With this spectrum information, using a Taylor development at first order, we show in Paragraph 3.2.b (Lemma (3.8)) that for every $n \geq N$ enough great, any dynamical system $\tilde{G}_{n,\epsilon}$ presenting such a configuration synchronizes on a neighborhood of the diagonal. Besides, we prove that the size of this neighborhood does not depend on n , which will be the crucial point for the study of the same question in the infinite case.

In the last section (Part 4), we go on our investigations by considering the limiting case of synchronization of a Cantor set of systems, indexed by $\{0, 1\}^{\mathbb{N}}$. This time, initial conditions are no more vectors but functions from $\{0, 1\}^{\mathbb{N}}$ to \mathbb{R} , and the previous structure matrices naturally become operators defined on the set of these functions. In accordance with Part 3, we define the hierarchical structure among our systems as the limit \mathcal{U}_ϵ of the composition of all these structure operators, (where ϵ stands for the sequence of parameters $(\epsilon_k)_{k \geq 1}$ associated to the stages of the hierarchy), of which existence must obviously be justified.

So, we begin Subsection 4.1 by proving the existence of $\mathcal{U}_\epsilon(h)$ for any continuous function h on $\{0, 1\}^{\mathbb{N}}$: indeed, as the synchronization naturally involves the infinity norm, it is normal to first restrict our study to the set of continuous ones (in fact, our two results will only work for such functions). Following [5], we construct a scalar product on the set of functions on $\{0, 1\}^{\mathbb{N}}$, and obtain under the condition that ϵ converge slowly to 1/2, a global synchronization result (Theorem (4.6)) in this setting, which constitutes the first theorem of this paper. Like all the results of Part 4, the proof relies on a strong use of the functions constant on the closed-open subsets of the Cantor set, which are dense in it (see Proposition (4.1)).

Finally in Subsection 4.2, we deal (as in the finite case) with the synchronization in case of broken links inside the hierarchy. In order to get an infinity of broken links in \sqrt{n} staying at the smallest N scales of the structure, we need to define again our structure operators this time on the closed-open subsets of $\{0, 1\}^{\mathbb{N}}$, and then to take the limit of the composition of all of them. As above, the resulting new operator \mathcal{W}_ϵ is well-defined on the set of continuous functions (see Lemma (4.8)). By construction, it acts on the constant functions on the closed-open sets of size 2^n exactly as the matrix $\tilde{B}_{n,\epsilon}$ does on \mathbb{R}^{2^n} . This allows us to use the same estimations as in Subsection 3.2. Using Proposition (4.1) again, we obtain a local synchronization result (Theorem (4.10)), providing the sequence of parameters ϵ converge rapidly to 1/2, which constitutes our second result. We finish the paper by giving the general versions of our results (Corollaries (4.11) and (4.12)) in case of a matching in groups of size p in our construction, that is to say on a Cantor set $\{a_0, \dots, a_{p-1}\}^{\mathbb{N}}$ defined by an alphabet with $p \geq 3$ letters.

2. BASIC NOTATIONS AND DEFINITIONS

2.1. The system. In all the text, we consider a segment K of \mathbb{R} , stable under a map $f : \mathbb{R} \rightarrow \mathbb{R}$, that is to say such that $f(K) \subset K$ (all the results we present are totally similar for functions defined on \mathbb{R}^n , as

mentioned at the end of the text). The map f is assumed to be of class \mathcal{C}^2 and satisfies $\sup_{z \in K} |f'(z)| > 1$. In particular f can have a chaotic dynamics.

Example 2.1. *The typical example is obviously the logistic map defined by $f(x) = \mu x(1-x)$ for x in the segment $[0, 1]$, which exhibits chaos for the value of parameter $\mu \approx 3.57$.*

As said in the introduction, we are interested in discrete dynamical systems of the form $X_{k+1} = A \circ F(X_k)$ where A is a linear map and F is a non-linear one, the components of F being equal to the same real-valued function f . Depending on the space on which this system will be considered, the linear part A will be either an endomorphism (in the first part the space is the finite dimensional one \mathbb{R}^{2^n}) or an operator acting on the infinite dimensional space of continuous functions on a Cantor set X (see the second part).

The goal is to study the synchronization of such a system, that is to say its convergence to the diagonal of the underlying space, that will be the set of vectors of which coordinates are all the same (denoted by \mathfrak{J}_{2^n}) in the first part, and the set of constant functions on X (denoted by \mathfrak{J}) in the second one.

In the finite dimensional case, we use the notation A either to mention the endomorphism or its matrix represented in the canonical basis $(e_j)_{1 \leq j \leq 2^n}$ of \mathbb{R}^{2^n} . The hierarchical structure will lead to the use of the Kronecker product (that we'll denote by \otimes) to define this endomorphism. Given two square matrices $M = (m_{i,j})_{1 \leq i,j \leq m}$ and $N = (n_{i,j})_{1 \leq i,j \leq n}$ respectively of size m and n , the Kronecker product $M \otimes N$ is the matrix of size mn defined by the equality:

$$M \otimes N = \begin{bmatrix} m_{1,1}N & \dots & m_{1,n}N \\ \vdots & & \vdots \\ m_{n,1}N & \dots & m_{n,n}N \end{bmatrix}.$$

Obviously the same definition works for non-square matrices but we recall it just in this case that interests us.

2.2. Vectors, matrices and norms. Each element X of \mathbb{R}^{2^n} will be denoted by a capital letter and its coordinates in the canonical basis $(e_j)_{1 \leq j \leq 2^n}$ in the following way $(X_{(1)}, \dots, X_{(2^n)})$. Given such an element X in \mathbb{R}^{2^n} , we will naturally consider the associated sum-vector X_Σ belonging to \mathfrak{J}_{2^n} , defined by the equality:

$$X_\Sigma = \left(\frac{(X_{(1)} + \dots + X_{(2^n)})}{2^n}, \dots, \frac{(X_{(1)} + \dots + X_{(2^n)})}{2^n} \right).$$

By convexity the vector X_Σ belongs to K^{2^n} whenever X is in this set.

Given a matrix A , the notations χ_A , $\mathfrak{S}(A)$, $\det(A)$, $\text{rank}(A)$ will classically stand for the characteristic polynomial, spectrum, determinant and rank of A . The algebraic multiplicity of an eigenvalue in χ_A will be denoted by the symbol \otimes .

Two matrices will be constantly used in the first part of the paper. These are:

$$\mathcal{T}_\epsilon = \begin{bmatrix} 1 - \epsilon & \epsilon \\ \epsilon & 1 - \epsilon \end{bmatrix},$$

for a real parameter ϵ , and

$$\mathbf{1}_k = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{bmatrix},$$

of size k . We define similarly $\mathbf{0}_k$.

Concerning the norms, we denote by $\|\cdot\|_{\infty, n}$ the infinity norm on \mathbb{R}^{2^n} , defined by the relation:

$$\|X\|_{\infty, n} = \max_{1 \leq i \leq 2^n} |X_{(i)}|.$$

In the second part, the same notation $\|\cdot\|_\infty$ will also stand for the infinity norm on the complete space $\mathcal{C}(X, \mathbb{R})$ of continuous functions on a Cantor set X .

3. THE FINITE CASE

In all this part we fix an integer $n \geq 1$, and deal with the synchronization problem in \mathbb{R}^{2^n} .

3.1. Global synchronization. Let us take an initial condition X_0 in K^{2^n} and consider the dynamical system $X_{k+1} = G_n^k(X_0)$, where $G_n = B_n \circ F_n$, with B_n a matrix of size 2^n representing a hierarchical structure that we define below, and $F_n = \underbrace{(f, \dots, f)}_{2^n \text{ terms}}$.

To define B_n , we can think of the 2^n components of X_0 as the initial state of some particles evolving in time. We want to synchronize their dynamics by acting on all the systems issued from the following successive two-by-two matching process: we first match the particles two-by-two, getting 2^{n-1} systems in \mathbb{R}^2 . In mathematical terms, the synchronized behavior of each of these systems is naturally associated to matrices of the form \mathcal{T}_ϵ , for some (possibly distinct) reals ϵ very close to $1/2$.

Then we gather again those 2^{n-1} systems two-by-two, obtaining 2^{n-2} systems of 4 components, of which synchronized behavior is related to the matrix $\mathcal{T}_\epsilon \otimes I_2$.

We reproduce this process n times until we reach at the end two big dynamical systems of 2^{n-1} particles, and the corresponding matrix $\mathcal{T}_\epsilon \otimes I_{2^{n-1}}$.

This hierarchical process can be represented by a diagram as in figure 1 (where the points at the bottom represent the particles to synchronize): If at each of its steps we have the separated synchronizations (which is symbolized by some links), then all of the 2^n particles will behave identically. Each step corresponds to a particular *scale* of synchronization: the first step corresponds to the smallest scale (since we act on the systems of smallest dimension, that is of dimension two), while the n^{th} corresponds to the greatest one, with an action on the systems of largest dimension, that is 2^{n-1} . For the moment the usefulness of this process does not seem to be relevant, for many other coupling matrices that do not represent this hierarchy could have been used (an extensive literature exists on this subject, see [9], [14]), but it will appear in Part 4.

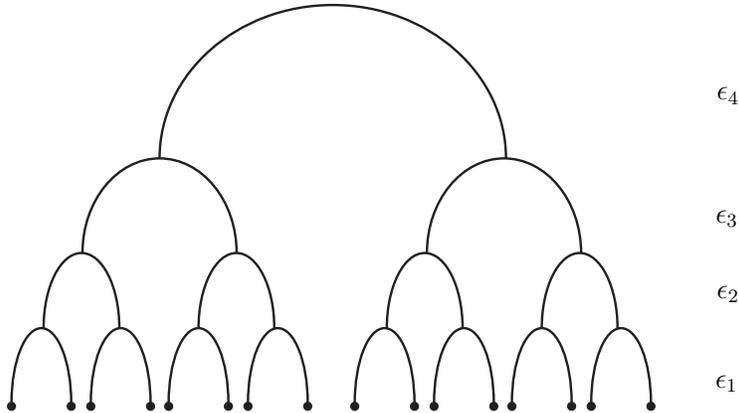


FIGURE 1. Diagram of synchronization in the case $n = 4$. Each scale is represented by one parameter.

By construction we have obtained a tuple of matrices that we'll call the *structure matrices* associated to this hierarchical process. They represent each of its n steps:

$$A_{1,\epsilon_1}^n = \begin{bmatrix} \mathcal{T}_{\epsilon_1} & & \\ & \ddots & \\ & & \mathcal{T}_{\epsilon_1} \end{bmatrix}, A_{2,\epsilon_2}^n = \begin{bmatrix} \mathcal{T}_{\epsilon_2} \otimes I_2 & & \\ & \ddots & \\ & & \mathcal{T}_{\epsilon_2} \otimes I_2 \end{bmatrix}, \dots, A_{n,\epsilon_n}^n = \mathcal{T}_{\epsilon_n} \otimes I_{2^{n-1}}.$$

We now define the matrix $B_{n,\epsilon}$ by:

$$B_{n,\epsilon} = A_{n,\epsilon_n}^n \cdots A_{1,\epsilon_1}^n,$$

where ϵ stands for the tuple $(\epsilon_1, \dots, \epsilon_n)$ defining the matrices A_{k,ϵ_k}^n . Each of them is defined by only one parameter ϵ_k to make it commute with the other ones. We have mentioned the dimension n in this notation, since our goal is to understand what happens when n goes to infinity (see Part 4).

Remark 3.1. *Obviously the matching two-by-two is arbitrary: the exactly same construction and results established below work in this general case of a p -by- p matching applied to p^n particles (see Subsection (4.3)). So for a clearer presentation (avoiding more complicated notations) we prefer to deal only with the two-by-two gathering, and then to present briefly the general case at the end of the text.*

In this context, the global synchronization result was obtained by B.Fernandez (see [5]):

Theorem 3.2. *Assuming the tuple $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ defining the structure matrices $A_{1,\epsilon_1}^n, \dots, A_{n,\epsilon_n}^n$ satisfies the relation:*

$$\forall k \in \{1, \dots, n\}, |1 - 2\epsilon_k| \sup_{z \in K} |f'(z)| < 1,$$

then the dynamical system $G_{n,\epsilon} = B_{n,\epsilon} \circ F_n$ globally synchronizes, i.e we have:

$$\forall X \in K^n, \max_{1 \leq i, j \leq 2^n} |G_{n,\epsilon}^m(X)_{(i)} - G_{n,\epsilon}^m(X)_{(j)}| \xrightarrow{m \rightarrow \infty} 0.$$

Proof. Following [5] we introduce:

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } C_1 = \begin{bmatrix} J & & \\ & \ddots & \\ & & J \end{bmatrix}, C_2 = \begin{bmatrix} J \otimes I_2 & & \\ & \ddots & \\ & & J \otimes I_2 \end{bmatrix}, \dots, C_n = J \otimes I_{2^{n-1}}.$$

For every integers k, l in \mathbb{N} , the matrices C_l and A_{k,ϵ_k}^n commute. Then it clearly comes, for every k in $\{1, \dots, n\}$ and every X in K :

$$\|G_{n,\epsilon}(X) - C_k(G_{n,\epsilon}(X))\|_{\infty} \leq |1 - 2\epsilon| \sup_{z \in K} |f'(z)| \cdot \|X - C_k(X)\|_{\infty, n},$$

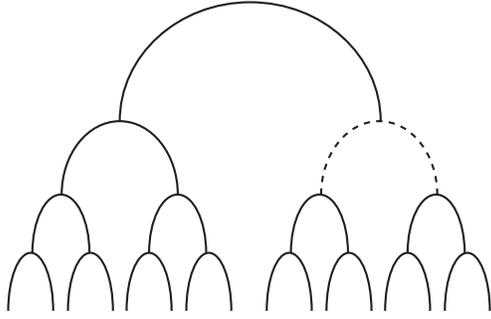
from which we get, for every $m \geq 0$:

$$\max_{1 \leq i, j \leq 2^n} |G_{n,\epsilon}^m(X)_{(i)} - G_{n,\epsilon}^m(X)_{(j)}| \leq \sum_{k=0}^m \left(|1 - 2\epsilon| \sup_{z \in K} |f'(z)| \right)^k \cdot \|X - C_k(X)\|_{\infty, n}$$

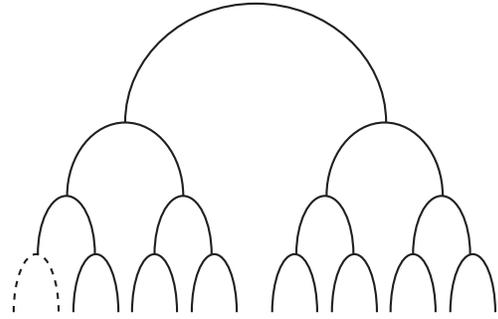
and the result. \square

Remark 3.3. *As explained in [5], this result also works for a Lipschitz function f on an interval which can be the whole set \mathbb{R} . In this case the Lipschitz constant replace the supremum of the derivative in the assumption of the theorem.*

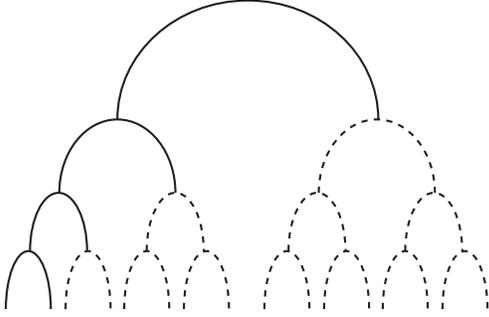
3.2. Local synchronization in presence of broken links.



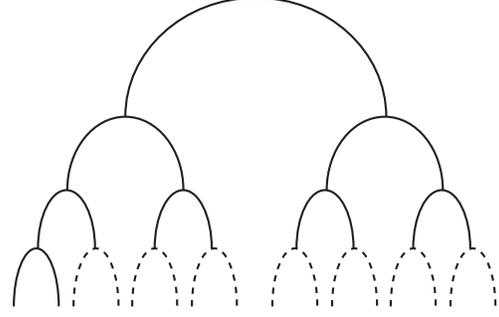
(a) One broken link at the 3^{rd} scale.



(b) One broken link at the 1^{st} scale.



(c) $2^n - (n + 1)$ broken links at the first three scales.



(d) $2^{n-1} - 1$ broken links at the 1^{st} scale.

FIGURE 2. Diagrams with broken links, in the case $n = 4$.

3.2.a. *Position of the problem.* Now we address the following question: in the set of our 2^n particles, suppose there is a subset composed of uncoupled elements, can we still make them synchronize with the other ones? In other words, if there are some links that are broken in the hierarchical structure constructed above (see Figure 2), can we still synchronize the whole set of particles?

The possible synchronization depends on the new structure matrices $\tilde{A}_{k,\epsilon_k}^n$ that no more commute since some blocks $\mathcal{T}_2 \otimes I_{2^{k-1}}$ (with $1 \leq k \leq n$) have been replaced by blocks I_{2^k} . For instance, the structure matrices associated to the scheme of Figure 2(a) are $A_{1,\epsilon_1}^n, \dots, A_{n-2,\epsilon_{n-2}}^n$ and for the last two ones:

$$\tilde{A}_{n-1,\epsilon_{n-1}}^n = \begin{bmatrix} \mathcal{T}_{\epsilon_{n-1}} \otimes I_{2^{n-2}} & \\ & I_{2^{n-2}} \end{bmatrix}, A_{n,\epsilon_n}^n = \mathcal{T}_{\epsilon_n} \otimes I_{2^{n-1}}.$$

Computing the spectrum of the new matrix $\tilde{B}_{n,\epsilon} = \tilde{A}_{n,\epsilon_n}^n \dots \tilde{A}_{1,\epsilon_1}^n$ is complicated in general because of this loss of commutativity. But as the reals ϵ_k are always taken very close to $1/2$ in our setting, we compute it in the case where we have $\epsilon = (1/2, \dots, 1/2)$, having in mind the continuity of the eigenvalues (in the coefficients of the matrices).

The following lemma presents the main different schemes met in this case.

Lemma 3.4. *Let us consider the different structure matrices $\tilde{B}_{n,\frac{1}{2}}$ associated to the schemes of Figure 2. Then all those matrices $\tilde{B}_{n,\frac{1}{2}}$ are diagonalizable and we have:*

- (1) *The spectrum associated to the diagram of Figure 2(a) is $\{1, \frac{1}{2}, 0^{\otimes 2^n - 2}\}$.*
- (2) *The spectrum associated to the diagram of Figure 2(b) is $\{1, \frac{1}{2^{n-1}}, 0^{\otimes 2^n - 2}\}$.*
- (3) *The spectrum of the diagram of Figure 2(c) is $\{1, \left(\frac{2^l - 1}{2^l}\right)^{\otimes 2^{n-l-1}}, 0^{\otimes 2^n - 1}, l = 1, \dots, n - 1\}$.*
- (4) *The spectrum of the diagram of Figure 2(d) is $\{1, \left(\frac{2^{n-1} - 1}{2^{n-1}}\right), 0^{\otimes 2^n - 2}\}$.*

The proof of Lemma (3.4) makes use of the following little lemma:

Lemma 3.5. *Let (A, B) two square matrices in \mathbb{R}^{k^2} , for $k \geq 1$, such that $A + B$ be diagonalizable. Assume the rank of $A + B$ is greater or equal to the rank of B . Then the following matrix Z :*

$$Z = \begin{bmatrix} A & B \\ A & B \end{bmatrix}$$

is diagonalizable.

Proof. Let us call $(\lambda_1, \dots, \lambda_p)$ the distinct eigenvalues of $A + B$ and $(\alpha_1, \dots, \alpha_p)$ the dimensions of their associated eigenspaces.

Suppose that 0 is an eigenvalue of $A + B$, say $\lambda_1 = 0$ (this will be the case for our structure matrices \tilde{B}_n). We have, by the Schur theorem:

$$\chi_Z = \det(-X I_k) \chi_{A+B},$$

so the spectra are the same for both Z and $A + B$. Then the assumption tells us the following:

$$\text{rank}(Z) = \text{rank}(A + B),$$

so the dimension of the eigenspace associated to 0 for the matrix Z is $k + \alpha_1$, which is the multiplicity of 0 in the characteristic polynomial χ_Z .

The same happens to the other eigenvalues since for any $X = (X_1, X_2)$ we have:

$$ZX = \lambda_i X \iff \{(A + B)X_1 = \lambda_i X_1, X_1 = X_2\}.$$

Thus the matrix Z is diagonalizable.

The same reasoning works for the other case where 0 is not an eigenvalue of $A + B$. □

Proof. We only do the case (1) since the demonstrations of the other ones are similar. From the definition of our hierarchical structure it clearly comes we have, for every $n \geq 1$:

$$\tilde{B}_{n+1,\frac{1}{2}} = \left(\begin{array}{c|c} \frac{1}{2^{n+1}} \begin{pmatrix} \mathbf{1}_{2^{n-1}} & \mathbf{1}_{2^{n-1}} \\ \mathbf{1}_{2^{n-1}} & \mathbf{1}_{2^{n-1}} \end{pmatrix} & \frac{1}{2^n} \begin{pmatrix} \mathbf{1}_{2^{n-1}} & \mathbf{0}_{2^{n-1}} \\ \mathbf{0}_{2^{n-1}} & \mathbf{1}_{2^{n-1}} \end{pmatrix} \\ \hline \frac{1}{2^{n+1}} \begin{pmatrix} \mathbf{1}_{2^{n-1}} & \mathbf{1}_{2^{n-1}} \\ \mathbf{1}_{2^{n-1}} & \mathbf{1}_{2^{n-1}} \end{pmatrix} & \frac{1}{2^n} \begin{pmatrix} \mathbf{1}_{2^{n-1}} & \mathbf{0}_{2^{n-1}} \\ \mathbf{0}_{2^{n-1}} & \mathbf{1}_{2^{n-1}} \end{pmatrix} \end{array} \right).$$

By commutativity the sum of the two square matrices defining $\tilde{B}_{n+1,\frac{1}{2}}$ is diagonalizable with eigenvalues 1, 1/2 and 0. The result follows from Lemma (3.5). □

This lemma gives some important informations for the second part of our work. As we could have expected, according to the cases (3) and (4), the more the broken links are numerous, the more the eigenvalues associated to the transverse directions of the diagonal are close to the value 1. In (3), the diagram contain $2^n - (n + 1)$ broken links meanwhile the one associated to the case (4) contains $2^{n-1} - 1$ broken ones (recall that the total number of links in any diagram is $2^n - 1$). As a consequence the structures

(3) and (4) will not be chosen when we deal with the infinite case in Part 4. More interesting is the case (1) which reveals the emphasis of the scale at which stand the broken links: to make the eigenvalues associated to the transverse direction tend to 0 as n goes to infinity, they must not be placed at the greatest scales of the diagram, i.e they must not link the biggest sub-systems of our set of particles. Lastly, the case (2) tells us the rate of convergence to zero of the transverse eigenvalues associated to the matrix (corresponding to this case) cannot be better than $1/2^n$.

Those ideas are confirmed by the next lemma, which proves that imposing there are at most $E(\sqrt{n})$ broken links (where $E(x)$ stands for the integer part of x) suffices to ensure the eigenvalues associated to the transverse directions of the diagonal tend to zero as n goes to infinity.

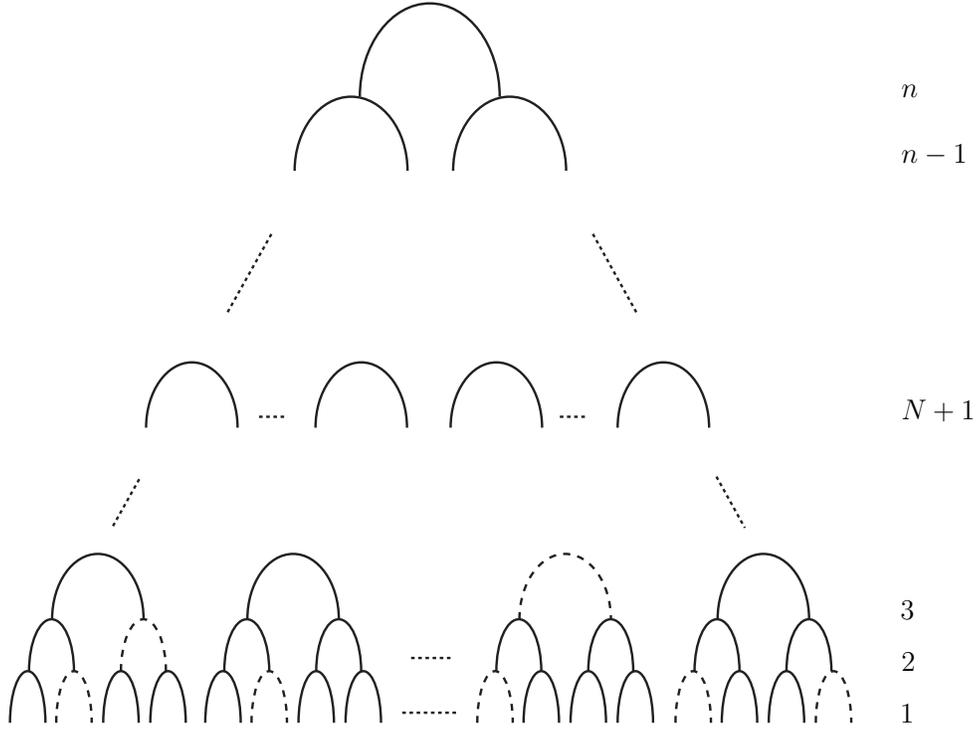


FIGURE 3. Diagram of synchronization with $E(\sqrt{n})$ broken links at the first N scales (i.e the N smallest ones).

Lemma 3.6. *Let us fix an integer $N \geq 1$ and assume there are exactly $E(\sqrt{n})$ broken links spread at the scales $1, \dots, N$ of our diagram (see Figure 3). That is to say assume we have:*

$$\tilde{B}_{n, \frac{1}{2}} = A_{n, 1/2}^n \cdots A_{N+1, 1/2}^n \tilde{A}_{N, 1/2}^n \cdots \tilde{A}_{1, 1/2}^n,$$

where $\tilde{A}_{N, 1/2}^n, \dots, \tilde{A}_{1, 1/2}^n$ contain $E(\sqrt{n})$ identity blocks I_{2^k} for $k \leq N-1$.

Then $\tilde{B}_{n, \frac{1}{2}}$ is diagonalizable with following spectrum:

$$\mathfrak{S}(\tilde{B}_{n, \frac{1}{2}}) = \{1, \lambda_{n, 1}, \dots, \lambda_{n, N}, 0^{\otimes 2^n - N - 1}\},$$

the $\lambda_{n, i}$ being some (possibly equal) positive numbers satisfying:

$$\forall i \in \{1, \dots, N\}, \lambda_{n, i} \leq \frac{1}{2^{n-N-1}}.$$

Proof. The demonstration is made by induction on n , using Lemma (3.5).

For every $n \geq N + 1$, let us set $\tilde{\mathcal{A}}_N^n = \tilde{A}_{N,1/2}^n \cdots \tilde{A}_{1,1/2}^n$, this matrix containing all the broken links, that is to say all the blocks I_{2^k} , for some integers k smaller than $N - 1$. At the step $n + 1$, $\tilde{\mathcal{A}}_N^{n+1}$ admits $E(\sqrt{n+1})$ ones and so contains at most one more such link than $\tilde{\mathcal{A}}_N^n$. Thus we have:

$$\tilde{\mathcal{A}}_N^{n+1} = \left(\begin{array}{c|c} \tilde{\mathcal{A}}_N^n & \mathbf{0}_{2^n} \\ \hline \mathbf{0}_{2^n} & \Gamma_n \end{array} \right),$$

where the matrix (of size 2^n) Γ_n is still a product of the same form as $\tilde{\mathcal{A}}_N^n$, but in which there is only one or zero matrix I_{2^k} (for $k \leq N - 1$). Remark there are $1 + 2^n - 2^{n-N}$ different such matrices Γ_n , depending on the position at which we choose to place the possible broken link. We will denote all these possible matrices by $\Gamma_{n,k}$ and finally define the matrix $\mathcal{C}_{n,k}$ by:

$$\forall k \in \{1, \dots, 1 + 2^n - 2^{n-N}\}, \mathcal{C}_{n,k} = A_{n,1/2}^n \cdots A_{N+1,1/2}^n \Gamma_{n,k}.$$

Consequently, we have for every $n \geq N + 1$, the following equality:

$$\tilde{\mathcal{B}}_{n+1, \frac{1}{2}} = \left(\begin{array}{c|c} \frac{1}{2} \tilde{\mathcal{B}}_{n, \frac{1}{2}} & \frac{1}{2} \mathcal{C}_n \\ \hline \frac{1}{2} \tilde{\mathcal{B}}_{n, \frac{1}{2}} & \frac{1}{2} \mathcal{C}_n \end{array} \right)$$

where $\mathcal{C}_n = \mathcal{C}_{n,k}$ for a certain integer k smaller than $1 + 2^n - 2^{n-N}$.

Now let's prove by induction that for every $n \geq N + 1$, the following property is true:

H_n: "For every integer k in $[1, 1 + 2^n - 2^{n-N}]$, the associated matrix $\mathcal{C}_{n,k}$ satisfies the following:

$$\mathfrak{S} \left(\frac{1}{2} \tilde{\mathcal{B}}_{n, \frac{1}{2}} + \frac{1}{2} \mathcal{C}_{n,k} \right) = \{1, \lambda_{n,k,1}, \dots, \lambda_{n,k,N}, 0^{\otimes 2^n - N - 1}\},$$

the $\lambda_{n,k,i}$ being some (possibly equal) positive numbers verifying:

$$\forall i \in \{1, \dots, N\}, \lambda_{n,k,i} \leq \frac{1}{2^{n-N-1}}.$$

Moreover this matrix $(1/2) \tilde{\mathcal{B}}_{n, \frac{1}{2}} + (1/2) \mathcal{C}_{n,k}$ is diagonalizable."

For $n = N + 1$, the result is clear, as in Lemma (3.4). Let $n \geq 1$ and assume **H_n** is true. Let's also fix an integer k smaller than $1 + 2^{n+1} - 2^{n+1-N}$ and look at the term $(1/2) \tilde{\mathcal{B}}_{n+1, \frac{1}{2}} + (1/2) \mathcal{C}_{n+1,k}$.

- Assume in a first case, the matrix \mathcal{C}_n contains one broken link, which imposes $\mathcal{C}_{n+1,k}$ does not. In this case we have:

$$\mathcal{C}_{n+1,k} = \left(\begin{array}{c|c} \frac{1}{2^{n+1}} \mathbf{1}_{2^n} & \frac{1}{2^{n+1}} \mathbf{1}_{2^n} \\ \hline \frac{1}{2^{n+1}} \mathbf{1}_{2^n} & \frac{1}{2^{n+1}} \mathbf{1}_{2^n} \end{array} \right),$$

from which follows the relation:

$$\chi_{(1/2) \tilde{\mathcal{B}}_{n+1, \frac{1}{2}} + (1/2) \mathcal{C}_{n+1,k}} = \det(-X I_{2^n}) \chi_{(1/4) \tilde{\mathcal{B}}_{n, \frac{1}{2}} + (1/4) \mathcal{C}_n + (1/2^{n+1}) \mathbf{1}_{2^n}}.$$

The matrices $\tilde{B}_{n,\frac{1}{2}}$ and C_n are clearly one-row sum (i.e each row of them sums to one) and one-row column, as products of the structure matrices that verify this property. So the term $(1/4)\tilde{B}_{n,\frac{1}{2}} + (1/4)C_n$ which is diagonalizable according to \mathbf{H}_n , commutes with the matrix $\mathbf{1}_{2^n}$ and thus the sum of these two terms is diagonalizable.

And since the matrix $(1/4)C_n + (1/2^{n+1})\mathbf{1}_{2^n}$ contains only one broken link we have:

$$\text{rank} \left(\frac{1}{4}\tilde{B}_{n,\frac{1}{2}} + \frac{1}{4}C_n + \frac{1}{2^{n+1}}\mathbf{1}_{2^n} \right) \geq \text{rank} \left(\frac{1}{4}C_n + \frac{1}{2^{n+1}}\mathbf{1}_{2^n} \right),$$

from which we get by Lemma (3.5) that the desired term $(1/2)\tilde{B}_{n+1,\frac{1}{2}} + (1/2)C_{n+1,k}$ is diagonalizable. To finish this case it suffices to remark we have:

$$\mathfrak{S} \left(\frac{1}{4}\tilde{B}_{n,\frac{1}{2}} + \frac{1}{4}C_n + \frac{1}{2^{n+1}}\mathbf{1}_{2^n} \right) = \left\{ 1, \frac{\lambda_{n,k,1}}{2}, \dots, \frac{\lambda_{n,k,N}}{2}, 0^{\otimes 2^n - N - 1} \right\},$$

as required.

- The second possibility is no broken link in Γ_n (or C_n) and one in $\Gamma_{n+1,k}$. Without loss of generality, we can assume this broken link is in the first 2^n rows of this matrix i.e we have:

$$C_{n+1,k} = \left(\begin{array}{c|c} \frac{1}{2}C_{n,i} & \frac{1}{2^{n+1}}\mathbf{1}_{2^n} \\ \hline \frac{1}{2}C_{n,i} & \frac{1}{2^{n+1}}\mathbf{1}_{2^n} \end{array} \right),$$

for an integer i smaller than $1 + 2^n - 2^{n-N}$ depending on k . Indeed the other case (corresponding to other choices $\Gamma_{n+1,k'}$) is completely symmetric.

Here we observe the following equality:

$$\frac{1}{2}\tilde{B}_{n+1,\frac{1}{2}} + \frac{1}{2}C_{n+1,k} = \frac{1}{4} \left(\begin{array}{c|c} \tilde{B}_{n,\frac{1}{2}} + C_{n,i} - \frac{1}{2^n}\mathbf{1}_{2^n} & \frac{1}{2^n}\mathbf{1}_{2^n} \\ \hline \tilde{B}_{n,\frac{1}{2}} + C_{n,i} - \frac{1}{2^n}\mathbf{1}_{2^n} & \frac{1}{2^n}\mathbf{1}_{2^n} \end{array} \right) + \frac{1}{4} \left(\begin{array}{c|c} \frac{1}{2^n}\mathbf{1}_{2^n} & \frac{1}{2^n}\mathbf{1}_{2^n} \\ \hline \frac{1}{2^n}\mathbf{1}_{2^n} & \frac{1}{2^n}\mathbf{1}_{2^n} \end{array} \right),$$

which permits to apply the exactly same reasoning as above.

- The last case in which there are no broken links in both Γ_n and $\Gamma_{n+1,k}$ is similar and easier.

These three cases dealt with all the possible choices of Γ_{n+1} , thus \mathbf{H}_{n+1} is true.

Finally, applying again Lemma (3.5) to $\tilde{B}_{n,\frac{1}{2}}$ for any $n \geq N + 1$, permits to conclude the proof. \square

3.2.b. *A local synchronization lemma.* Now let us establish the synchronization result in this context. As the structure matrices $\tilde{A}_{k,\epsilon_k}^n$ no more commute, the method of B. Fernandez cannot be applied again. More, there is no hope to synchronize globally our system since in general it admits some fixed points outside the diagonal.

Example 3.7. Consider the case $n = 2$ (i.e dimension 4), with f as the logistic map (taken at the value of parameter $\mu = 3.57$, as in Example (2.1)) and the following structure matrices:

$$\tilde{A}_{1,\epsilon_1}^2 = \begin{bmatrix} \mathcal{T}_{\epsilon_1} & \\ & I_2 \end{bmatrix}, \quad A_{2,\epsilon_2}^2 = \mathcal{T}_{\epsilon_2} \otimes I_2.$$

This is the only interesting structure with broken link in this dimension. Then for $(\epsilon_1, \epsilon_2) = (0.45, 0.476)$, we have $\tilde{G}_{2,\epsilon}(X) = \tilde{B}_{2,\epsilon} \circ F_2(X) = X$ for $X \approx (0.3394508235, 0.2491080749, 0.7404987705, -0.2406491883)$.

For this reason we adopt another approach based on a Taylor development at first order: although the approximations deduced are coarse, it permits us to recover a commutativity on the differential of G_n , leading to the existence of a contracting neighborhood of the diagonal \mathfrak{I}_{2^n} . This result has been yet established notably by W. Lu in [14] (see also [9]) in a general setting (that does not take into account the hierarchical structure and the broken links we are dealing with) but obviously without exhibiting how the contracting neighborhood depends or not on the dimension of the space (here 2^n). And as we need this information for the second part, we demonstrate this result again:

Lemma 3.8. *Let us fix an integer $N \geq 1$ and take a tuple $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ with $n \geq N + 1$. We consider the system $\tilde{G}_{n,\epsilon} = \tilde{B}_{n,\epsilon} \circ F_n$ defining the same structure as in Lemma (3.6). Assume the tuple ϵ is enough close to $1/2$ so that we have:*

$$6 \sup_{z \in K} |f'(z)|^{2N} \sum_{k=1}^n |1 - 2\epsilon_k| < 1.$$

Then, there is a constant Λ_n such that, for every $\epsilon' > 0$ exists a real $\eta_N > 0$ (that does not depend on n) defining the following neighborhood Ω_n of the diagonal:

$$\Omega_n = \{X \in K^{2^n}, \forall i \neq j |X_{(i)} - X_{(j)}| \leq \eta_N\},$$

on which the map $(\tilde{G}_{n,\epsilon})^{2N}$ satisfies the inequality:

$$\max_{1 \leq i, j \leq 2^n} |(\tilde{G}_{n,\epsilon})^{2N}(X)_{(i)} - (\tilde{G}_{n,\epsilon})^{2N}(X)_{(j)}| \leq 2 \left(\epsilon' + \Lambda_n + 3 \sup_{z \in K} |f'(z)|^{2N} \sum_{k=1}^n |1 - 2\epsilon_k| \right) |X_{(i)} - X_{(j)}|.$$

The constant Λ_n tends to zero as n tends to infinity. In particular choosing ϵ' enough small, the dynamical system $\tilde{G}_{n,\epsilon}$ synchronizes on the set Ω_n for a sufficiently large n .

Proof. We first prove the result for the map $\tilde{G}_{n,\frac{1}{2}}^{2N}$.

For every $r \geq 1$, we have, by the Taylor formula with bounded remainder applied at first order:

$$\begin{aligned} \left\| \left(\tilde{G}_{n,\frac{1}{2}} \right)^r(X) - \left(\tilde{G}_{n,\frac{1}{2}} \right)^r(X_\Sigma) \right\|_{\infty,n} &\leq \sup_{t \in [0,1]} \left\| D_{(1-t)X_\Sigma + tX} \left(\tilde{G}_{n,\frac{1}{2}} \right)^r(X - X_\Sigma) - D_{X_\Sigma} \left(\tilde{G}_{n,\frac{1}{2}} \right)^r(X - X_\Sigma) \right\|_{\infty,n} \\ &\quad + \left\| D_{X_\Sigma} \left(\tilde{G}_{n,\frac{1}{2}} \right)^r(X - X_\Sigma) \right\|_{\infty,n}. \end{aligned}$$

Now we fix a small number $\epsilon' > 0$. The (uniform) continuity of the map $X \mapsto D_X \tilde{G}_{n,\frac{1}{2}}$ on the compact K^{2^n} , gives us the existence of a number $\eta_r > 0$ such that:

$$\forall X, Y \in K^{2^n} \times K^{2^n}, \|X - Y\|_{\infty,n} < \eta_r \Rightarrow \|D_X \left(\tilde{G}_{n,\frac{1}{2}} \right)^r - D_Y \left(\tilde{G}_{n,\frac{1}{2}} \right)^r\|_{\infty,n} < \epsilon'.$$

Indeed, for every $X, Y \in K^{2^n}$, applying successively the mean value inequality leads to the estimation:

$$\begin{aligned} \left\| D_X \left(\tilde{G}_{n,\frac{1}{2}} \right)^r - D_Y \left(\tilde{G}_{n,\frac{1}{2}} \right)^r \right\|_{\infty,n} &\leq \sup_{z \in K} |f'(z)|^{r-1} \cdot \sup_{z \in K} |f''(z)| \left(\sum_{i=1}^r \left\| \left(\tilde{G}_{n,\frac{1}{2}} \right)^{r-i}(X) - \left(\tilde{G}_{n,\frac{1}{2}} \right)^{r-i}(Y) \right\|_{\infty,n} \right) \\ &\leq r \left(\sup_{z \in K} |f'(z)|^{2r-2} \cdot \sup_{z \in K} |f''(z)| \right) \|X - Y\|_{\infty,n}, \end{aligned}$$

which permits us to choose a convenient real η_r depending only on the integer r and the function f .

We consider the set $\mathcal{C}_{\eta_r} = \{Z = (z_{(1)}, \dots, z_{(2^n)}) \in K^{2^n} : \forall i \neq j |z_{(i)} - z_{(j)}| < \eta_r\}$. If X is in \mathcal{C}_{η_r} , the upper bound in the Taylor inequality above is smaller than $\epsilon' \|X - X_\Sigma\|_{\infty,n}$. We now estimate the second term of the sum for X in this set.

3.3. Remarks concerning the definition of $G_{n,\epsilon}$. To finish this first part, let us underly there are mainly two possibilities for the definition of our dynamical system $G_{n,\epsilon}$, that represent the hierarchical structure we have constructed. Besides the one we have chosen $G_{n,\epsilon} = A_{n,\epsilon_n}^n \circ \dots \circ A_{1,\epsilon_1}^n \circ F_n$ in Section 3.1, we could have set:

$$G_{n,\epsilon} = A_{n,\epsilon_n}^n \circ F_n \circ \dots \circ A_{1,\epsilon_1}^n \circ F_n,$$

and develop the same approach concerning the local synchronization in presence of broken links. The result of lemma (3.8) is also true with this definition, but this time the size of the synchronization neighborhood depends on n , for is added a term F_n at each increasing of the dimension.

4. GENERALIZATION TO A CANTOR SET.

Consider again a set of 2^n particles, coupled together according to the hierarchical structure established in Section 3.1. As we have gathered them two-by-two at each step, it is natural to number them by a code with only two letters, say 0 and 1, representing their path to the top of the graph associated to this process: we use 0 if the subsystem is at the left of the one to which it is linked, and 1 if it is at its right. This numbering is one-to-one since every particle admits one and only one such path. As mentioned above, in this finite case we could have synchronized our particles with a different matrix coupling.

This is no more the case if we let n tends to infinity. In this limiting case, our set of particles forms a Cantor set, thus (infinite) uncountable, indexed by the set $X = \{0, 1\}^{\mathbb{N}}$ of sequences in 0 and 1, and the previous numbering describe all the possible sequences of this set. Instead of having a vector as initial condition (representing the values at each position k in $1, \dots, 2^n$), we now have a set $h(X)$ for a function h from X to \mathbb{R} . But because of this uncountability, there is no straight way to synchronize all our particles as in [5], [14], [9], for there is no way to write explicitly a series in all the images $h(c)$ for c in X . It is here that the hierarchical structure permits to overpass naturally this difficulty.

The matrices $\left(A_{k,\epsilon_k}^n \right)_{1 \leq k \leq n}$ of Part 3 become the following operators $(\mathcal{L}_{k,\epsilon_k})_{k \in \mathbb{N}^*}$ acting on the space of real-valued functions on X :

$$\mathcal{L}_{k,\epsilon_k}(h) = (1 - \epsilon_k)h + \epsilon_k J_k(h),$$

where the function $J_k(h)$ is defined by:

$$\forall c \in X, J_k(h)(c) = h(c^{*,k}),$$

the sequence $c^{*,k}$ being obtained from c by only replacing the letter c_k with $1 - c_k$. Obviously, these operators J_k play the same role as the matrices C_k in the finite case (see the proof of Theorem (3.2)).

We look at the infinite composition of our new operators $\mathcal{L}_{k,\epsilon_k}$, i.e at the limit:

$$\lim_{n \rightarrow \infty} \mathcal{L}_{n,\epsilon_n} \circ \dots \circ \mathcal{L}_{1,\epsilon_1}(h),$$

for a given function h on X . Assuming for the moment such a limit exists (see below), it defines an operator \mathcal{U}_ϵ (where ϵ is the sequence $(\epsilon_k)_{k \geq 1}$) from which we construct, as in Part 3 with G_n , a dynamical system leading to the synchronization. Applied to a function h on X , this operator acts on all the terms $h(c)$, as desired.

In this context the diagonal writes $\mathfrak{J} = \{h : X \rightarrow \mathbb{R}, h \text{ is constant}\}$. Geometrically, given an initial condition h , the new dynamical system will tend to flatten its associated graph $\{(c, h(c)) : c \in X\}$. We are now able to apply the same reasoning as in Part 3, the main issue being the existence of the operator \mathcal{U}_ϵ .

4.1. Global synchronization.

4.1.a. *Existence of the limit operator \mathcal{U}_ϵ .* We equip X with the metric:

$$d(c, c') = \sum_{n=0}^{\infty} \frac{|c_n - c'_n|}{2^n}.$$

The metric space (X, d) is compact, totally disconnected (i.e its greatest connected component is a point), and without isolated point. We recall that the metric spaces presenting these three characteristics are all homeomorphic. They are called Cantor spaces. It is important to tell that all our results below are true for any other Cantor space than (X, d) (which is not always the case), as they do not depend on the metric chosen (see the point (2) of Remark 4.3 below).

Given a tuple (a_1, \dots, a_n) in $\{0, 1\}^n$, consider the set $\mathcal{C}_{a_1, \dots, a_n} = \{c \in X : (c_1, \dots, c_n) = (a_1, \dots, a_n)\}$. It is easy to verify this set is closed and open for the topology defined by d . For every $n \geq 1$, we will denote by $(\mathcal{C}_{n,k})_{1 \leq k \leq 2^n}$ all the 2^n such sets, no matter the way they are ordered. Each $\mathcal{C}_{n,k}$ has diameter $1/2^{n-1}$ and we have:

$$\forall n \geq 1, X = \bigsqcup_{k=1}^{2^n} \mathcal{C}_{n,k}.$$

The fundamental property is that the set \mathcal{C} of all those sets:

$$\mathcal{C} = \{\mathcal{C}_{1,1}, \mathcal{C}_{1,2}, \dots, \mathcal{C}_{n,k}, \dots\},$$

forms a countable basis of open sets of the topological space X .

As explained above we are lead to consider the functions from X to \mathbb{R} , that will be our new points in the phase space. In fact our results will only concern the space $\mathcal{C}(X, \mathbb{R})$ of continuous functions, which we recall is complete for the infinity norm $\|\cdot\|_\infty$. The reason is the synchronization is expressed with this norm. The sum of the series $\sum_{n \geq 0} c_n/2^n$ is an example of such a function. We have:

Proposition 4.1. *The set of constant functions on the closed-open sets \mathcal{C} is dense in the set $(\mathcal{C}(X, \mathbb{R}), \|\cdot\|_\infty)$.*

Proof. Any function constant on the closed-open sets of order n for some $n \geq 1$, is clearly continuous on X . Then given an element h of $\mathcal{C}(X, \mathbb{R})$, define the sequence of functions $(h_n)_{n \geq 1}$ by:

$$\forall c \in X, h_n(c) = h(c_1, \dots, c_n, \mathbf{1}),$$

where $\mathbf{1}$ denotes the constant sequence equals to one. For every $n \geq 1$, the function h_n is constant on the $(\mathcal{C}_{n,k})_{1 \leq k \leq 2^n}$, and by the uniform continuity of h on the compact X , converges to this function for the infinity norm. \square

Lemma 4.2. *For every sequence $\epsilon = (\epsilon_n)_{n \geq 1}$ of real numbers in $[0, 1]$, and every continuous function h on X , the following limit:*

$$\mathcal{U}_\epsilon(h) := \lim_{n \rightarrow \infty} \mathcal{L}_{n, \epsilon_n} \circ \dots \circ \mathcal{L}_{1, \epsilon_1}(h),$$

exists as a function of $(\mathcal{C}(X, \mathbb{R}), \|\cdot\|_\infty)$.

Proof. For h in $(\mathcal{C}(X, \mathbb{R}), \|\cdot\|_\infty)$, let's consider the sequence $(h_n)_{n \geq 1}$ defined as in the proof of Proposition 4.1, that converges to h . Introducing the notation $K_n(h) = \mathcal{L}_{n, \epsilon_n} \circ \dots \circ \mathcal{L}_{1, \epsilon_1}(h)$ we then have, for every integers p, q :

$$\|K_{p+q}(h) - K_p(h)\|_\infty \leq \|K_{p+q}(h) - K_{p+q}(h_p)\|_\infty + \|K_{p+q}(h_p) - K_p(h_p)\|_\infty + \|K_p(h_p) - K_p(h)\|_\infty$$

Clearly, the operator norm of each operator $\mathcal{L}_{n, \epsilon_n}$ is one. It then suffices to remark we have,

$$\forall n > p, \mathcal{L}_{n, \epsilon_n}(h_p) = h_p,$$

to conclude the first member of the inequality is smaller than $2\|h - h_p\|_\infty$. The considered sequence is thus a Cauchy one. \square

Remark 4.3. (1) *Nothing tells us the sequence $\mathcal{L}_{n, \epsilon_n} \circ \dots \circ \mathcal{L}_{1, \epsilon_1}$ converges for the operator norm, i.e in the set of linear maps on $\mathcal{C}(X, \mathbb{R})$, for the speed of convergence of the term $\|h - h_p\|_\infty$ depends on the function h .*

- (2) *The existence of \mathcal{U}_ϵ does not require the convergence of the sequence ϵ . Nor does it involve the metric d taken on X , and especially the diameter of the closed-open sets $\mathcal{C}_{n,k}$, of which size could decrease much more slowly on other Cantor spaces.*

Now as in Part 3, we set $G_\epsilon = \mathcal{U}_\epsilon \circ F$ where the map F is defined by $F(h) = f \circ h$ for every real-valued function h defined on X .

4.1.b. *Global synchronization result on $\mathcal{C}(X, \mathbb{K})$.* In this context, the global synchronization result is proved by the same technique as in [5]. We equip X with the Borel σ -algebra and consider the probability measure μ verifying:

$$\forall n \geq 1, \forall k \in \{1, \dots, 2^n\}, \mu(\mathcal{C}_{n,k}) = \frac{1}{2^n}.$$

Since \mathcal{C} generates the topology of X , and for every $n \geq 1$ the $(\mathcal{C}_{n,k})_{1 \leq k \leq 2^n}$ form a partition of X , it is very easy to construct such a probability measure.

Proposition 4.4. *For every $n \geq 1$, the map $c \mapsto c^{*,n}$ preserves the measure μ . Consequently for every measurable function h from X to \mathbb{R} , we have:*

$$\int_X h(c) d\mu(c) = \int_X h(c^{*,n}) d\mu(c).$$

Proof. The map $c \mapsto c^{*,n}$ is an involution. Let's take an element $\mathcal{C}_{p,k}$ in \mathcal{C} . If we have $p < n$ then $(\mathcal{C}_{p,k})^{*,n} = \mathcal{C}_{p,k}$, otherwise the measure of these two sets is the same by definition of μ . This yields the result by σ -additivity. \square

From this measure μ , we recover a scalar product $\langle \cdot, \cdot \rangle$ on the space of measurable real-valued functions on X , defined by:

$$\langle h, g \rangle = \int_X hg d\mu.$$

We denote by $^\perp$ the orthogonality for this scalar product. Proposition 4.4 gives us the following:

Lemma 4.5. *For every $k \geq 1$, let us define the subset \mathfrak{J}_k of $\mathcal{C}(X, \mathbb{R})$ by:*

$$\mathfrak{J}_k = \{h \in \mathcal{C}(X, \mathbb{R}) : J_k(h) = h\}.$$

Then the diagonal \mathfrak{J} is equal to the intersection of all the \mathfrak{J}_k , and we have the following relation:

$$\mathfrak{J}_k^\perp = \{h \in \mathcal{C}(X, \mathbb{R}) : J_k(h) = -h\}.$$

Proof. The first assertion is clear. For the second one, if a function h is in \mathfrak{J}_k^\perp , then we have:

$$\int_X h(c) (h(c) + h(c^{*,k})) d\mu(c) = 0,$$

and thus by Proposition 4.4,

$$\int_X (h(c) + h(c^{*,k}))^2 d\mu(c) = 0,$$

which gives the equality $h(c^{*,k}) = -h(c)$ for almost every sequence c . But as the measure μ is an exterior one this holds for every c in X . \square

Theorem 4.6. *Let $\epsilon = (\epsilon_k)_{k \geq 1}$ be a sequence of real numbers in $[0, 1]$, defining the operator \mathcal{U}_ϵ as in Lemma 4.2. Assume the following condition holds:*

$$\exists a > 1, \exists \alpha > 0, \forall k \geq 1, |1 - 2\epsilon_k| \sup_{z \in K} |f'(z)| \leq \frac{1}{(ak)^\alpha}.$$

Then the dynamical system $G_\epsilon = \mathcal{U}_\epsilon \circ F$ globally synchronizes on the set $\mathcal{C}(X, \mathbb{K})$ of continuous functions having values in \mathbb{K} .

Proof. Let us fix an integer $k \geq 1$ and take a continuous function h on X , having values in K . As in the finite dimensional case, the operators $(J_k)_{k \geq 1}$ commute with the operators $(\mathcal{L}_{n, \epsilon_n})_{n \geq 1}$ and with F . So we have:

$$\begin{aligned} \|\mathcal{G}_\epsilon(h) - J_k(\mathcal{G}_\epsilon(h))\|_\infty &= \|\mathcal{U}_\epsilon(f \circ h - J_k(f \circ h))\|_\infty \\ &\leq \sup_{z \in K} |f'(z)| \cdot \|\mathcal{U}_\epsilon|_{\mathfrak{J}_k^\perp}\|_\infty \|h - J_k(h)\|_\infty. \end{aligned}$$

Then we observe the equality $\|\mathcal{U}_\epsilon|_{\mathfrak{J}_k^\perp}\|_\infty = |1 - 2\epsilon_k|$ (the inequality would have sufficed for our proof). Indeed, for a function h in \mathfrak{J}_k^\perp , we have:

$$\mathcal{L}_{k, \epsilon_k} \circ \cdots \circ \mathcal{L}_{1, \epsilon_1}(h) = (1 - 2\epsilon_k) \mathcal{L}_{k-1, \epsilon_{k-1}} \circ \cdots \circ \mathcal{L}_{1, \epsilon_1}(h),$$

which imposes, as the infinity norm of each $\mathcal{L}_{n, \epsilon_n}$ is one:

$$\|\mathcal{U}_\epsilon|_{\mathfrak{J}_k^\perp}\| \leq |1 - 2\epsilon_k|.$$

The equality is reached for the function $c \mapsto c_k - (1 - c_k)$. This leads us to the following, for every elements c, c' of X :

$$\begin{aligned} \forall m \geq 1 : |\mathcal{G}_\epsilon^m(h)(c) - \mathcal{G}_\epsilon^m(h)(c')| &\leq \sum_{k=1}^{+\infty} \left(|1 - 2\epsilon_k| \sup_{z \in K} |f'(z)| \right)^m \|h - J_k(h)\|_\infty, \\ &\leq \frac{1}{a^{m\alpha}} \sum_{k=1}^{+\infty} \frac{1}{k^{m\alpha}} \|h - J_k(h)\|_\infty, \end{aligned}$$

which gives the result. \square

Remark 4.7. (1) *The sequence ϵ is asked to converge slowly to 1/2, since we do not know how fast is the convergence to zero of the term $\|h - J_k(h)\|_\infty$. This convergence depends in a certain sense, on the metric taken on the Cantor space. For instance, if we take a Lipschitz function h on our set (X, d) as initial condition, it just suffices to ask the terms $|1 - 2\epsilon_k| \sup_{z \in K} |f'(z)|$ are all strictly smaller than one, because the diameters of the closed-open sets decrease enough strongly to 0.*
(2) *There again, the result also works for a Lipschitz function f on an interval, in which case it is true on $\mathcal{C}(X, \mathbb{R})$.*

4.2. Local synchronization in presence of broken links. Now, in this last subsection we ask the same question as in Subsection 3.2: assume in our Cantor set of particles, there are infinitely many ones that are not coupled. Can we still synchronize the entire set of particles?

As said previously, the work has been yet prepared by Lemma 3.6 and 3.8: it just suffices to let the dimension n goes to infinity in our estimations, using the fact that the set of constant functions on the closed-open sets $\mathcal{C}_{n, k}$ form a 2^n -dimensional space, that we identify with \mathbb{R}^{2^n} .

In order to make this passing to the limit, we have to define again our sequence of structure operators, acting at each step n on this space of constant functions on the $\mathcal{C}_{n, k}$. We begin by the operators having only strong links, setting for every $n \geq 1$ and every function h in $\mathcal{C}(X, \mathbb{R})$:

$$\forall k \in \{1, \dots, n\} : \mathcal{I}_{k, \epsilon_k}^n(h)(c) = (1 - \epsilon_k) h(c_1, \dots, c_n, \mathbf{1}) + \epsilon_k h\left((c_1, \dots, c_n, \mathbf{1})^{*, k}\right).$$

As in Part 3, we decide that the number of broken links for each diagram of size 2^n be at most $E(\sqrt{n})$, and that they stay at the smallest N scales of the hierarchical structure, for a fixed $N \geq 1$. In this purpose, for every $n \geq N + 1$ and every function h in $\mathcal{C}(X, \mathbb{R})$, we consider the composition:

$$\mathcal{I}_{1, \epsilon_1}^n \circ \cdots \circ \mathcal{I}_{n-N, \epsilon_{n-N}}^n \circ \tilde{\mathcal{I}}_{n-N+1, \epsilon_{n-N+1}}^n \circ \cdots \circ \tilde{\mathcal{I}}_{n, \epsilon_n}^n(h),$$

where the operators $\left(\tilde{\mathcal{I}}_{k, \epsilon_k}^n\right)_{n-N+1 \leq k \leq n}$ contain the broken links (note the order of composition has been reversed compared to the definition of Part 3, see Remark 4.9). It is always possible to define them. For

instance, let us fix an integer n enough great so that we have $E(\sqrt{n}) = N$, and set:

$$\begin{aligned} \tilde{\mathcal{I}}_{n-N+1, \epsilon_{n-N+1}}^n(h)(c) &= \begin{cases} h(c_1, \dots, c_n, \mathbf{1}) & \text{if } (c_1, \dots, c_{n-N}) = (a_1^{(1)}, \dots, a_{n-N}^{(1)}) \\ \mathcal{I}_{n-N+1, \epsilon_{n-N+1}}^n(h)(c) & \text{otherwise} \end{cases} \\ \vdots & \\ \tilde{\mathcal{I}}_{n-1, \epsilon_{n-1}}^n(h)(c) &= \begin{cases} h(c_1, \dots, c_n, \mathbf{1}) & \text{if } (c_1, \dots, c_{n-2}) = (a_1^{(N-1)}, \dots, a_{n-2}^{(N-1)}) \\ \mathcal{I}_{n-1, \epsilon_{n-1}}^n(h)(c) & \text{otherwise} \end{cases} \\ \tilde{\mathcal{I}}_{n, \epsilon_n}^n(h)(c) &= \begin{cases} h(c_1, \dots, c_n, \mathbf{1}) & \text{if } (c_1, \dots, c_{n-1}) = (a_1^{(N)}, \dots, a_{n-1}^{(N)}) \\ \mathcal{I}_{n, \epsilon_n}^n(h)(c) & \text{otherwise,} \end{cases} \end{aligned}$$

where the elements $(a_1^{(N-i)}, \dots, a_{n-i-1}^{(N-i)})$ designate some distinct tuples of 0's and 1's. In this configuration, each of the N scales admits one broken link. Then two cases appear. If we have $E(\sqrt{n}) = N$, i.e if no broken link is added, we set:

$$\begin{aligned} \tilde{\mathcal{I}}_{n-N+2, \epsilon_{n-N+2}}^{n+1}(h)(c) &= \begin{cases} h(c_1, \dots, c_{n+1}, \mathbf{1}) & \text{if } (c_1, \dots, c_{n-N+1}) = (0, a_1^{(1)}, \dots, a_{n-N}^{(1)}) \\ \mathcal{I}_{n-N+2, \epsilon_{n-N+2}}^{n+1}(h)(c) & \text{otherwise,} \end{cases} \\ \vdots & \\ \tilde{\mathcal{I}}_{n+1, \epsilon_{n+1}}^{n+1}(h)(c) &= \begin{cases} h(c_1, \dots, c_{n+1}, \mathbf{1}) & \text{if } (c_1, \dots, c_n) = (0, a_1^{(N)}, \dots, a_{n-1}^{(N)}) \\ \mathcal{I}_{n+1, \epsilon_{n+1}}^{n+1}(h)(c) & \text{otherwise.} \end{cases} \end{aligned}$$

If on the contrary we have $E(\sqrt{n+1}) = N+1$, then we keep the same definition, and we add a condition holding on the closed-open set of sequences beginning by 1, for instance the following:

$$\tilde{\mathcal{I}}_{n+1, \epsilon_{n+1}}^{n+1}(h)(c) = h(c_1, \dots, c_{n+1}, \mathbf{1}) \text{ if } (c_1, \dots, c_n) = (1, \dots, 1).$$

We define similarly all the operators $(\tilde{\mathcal{I}}_{k, \epsilon_k}^p)_{p-N+1 \leq k \leq p}$ for any p greater than n .

Now, as in the global synchronization case, the following limit

$$\lim_{n \rightarrow \infty} \mathcal{I}_{1, \epsilon_1}^n \circ \dots \circ \mathcal{I}_{n-N, \epsilon_{n-N}}^n \circ \tilde{\mathcal{I}}_{n-N+1, \epsilon_{n-N+1}}^n \circ \dots \circ \tilde{\mathcal{I}}_{n, \epsilon_n}^n(h)$$

exists. This limit is the desired limit operator since it acts on (all) the sequences of 0 and 1.

Lemma 4.8. *Let $\epsilon = (\epsilon_k)_{k \geq 1}$ a sequence in $[0, 1]^{\mathbb{N}^*}$. For every function h in $\mathcal{C}(X, \mathbb{R})$, we set:*

$$\mathcal{W}_\epsilon(h) = \lim_{n \rightarrow \infty} \mathcal{I}_{1, \epsilon_1}^n \circ \dots \circ \mathcal{I}_{n-N, \epsilon_{n-N}}^n \circ \tilde{\mathcal{I}}_{n-N+1, \epsilon_{n-N+1}}^n \circ \dots \circ \tilde{\mathcal{I}}_{n, \epsilon_n}^n(h),$$

where the operators $(\tilde{\mathcal{I}}_{k, \epsilon_k}^n)_{n-N+1 \leq k \leq n}$ contain $E(\sqrt{n})$ broken links of which positions are arbitrary. Then, \mathcal{W}_ϵ is a well-defined operator on $\mathcal{C}(X, \mathbb{R})$.

Proof. The demonstration is the same as the one of Lemma 4.2. Let's take a continuous function h , and the associated sequence $(h_n)_{n \geq 1}$ defined as in the proof of Proposition 4.1, that converges to h . Denoting again $\mathcal{K}_n(h) = \mathcal{I}_{1, \epsilon_1}^n \circ \dots \circ \mathcal{I}_{n-N, \epsilon_{n-N}}^n \circ \tilde{\mathcal{I}}_{n-N+1, \epsilon_{n-N+1}}^n \circ \dots \circ \tilde{\mathcal{I}}_{n, \epsilon_n}^n(h)$, we have for any integers $p, q \geq 1$:

$$\|\mathcal{K}_{p+q}(h) - \mathcal{K}_p(h)\|_\infty \leq 2\|h - h_p\|_\infty + \|\mathcal{K}_{p+q}(h_p) - \mathcal{K}_p(h_p)\|_\infty.$$

Taking q greater if necessary, we can assume the inequality: $p + q - N \geq p$. Then, as h_p is constant on the closed-open sets $\mathcal{C}_{p,k}$ of size p , we have:

$$\begin{aligned} \|\mathbf{K}_{p+q}(h_p) - \mathbf{K}_p(h_p)\|_\infty &\leq \|\mathcal{I}_{1,\epsilon_1}^p \circ \cdots \circ \mathcal{I}_{p,\epsilon_p}^p(h_p) - \mathcal{I}_{1,\epsilon_1}^p \circ \cdots \circ \mathcal{I}_{p-N,\epsilon_{p-N}}^p \circ \tilde{\mathcal{I}}_{p-N+1,\epsilon_{p-N+1}}^p \circ \cdots \circ \tilde{\mathcal{I}}_{p,\epsilon_p}^p(h_p)\|_\infty \\ &\leq \|\mathcal{I}_{p-N+1,\epsilon_{p-N+1}}^p \circ \cdots \circ \mathcal{I}_{p,\epsilon_p}^p(h_p) - \tilde{\mathcal{I}}_{p-N+1,\epsilon_{p-N+1}}^p \circ \cdots \circ \tilde{\mathcal{I}}_{p,\epsilon_p}^p(h_p)\|_\infty \\ &\leq \sum_{k=1}^N \epsilon_{p-N+k} \|h_p - \mathbf{J}_{p-N+k}(h_p)\|_\infty, \end{aligned}$$

the last inequality coming from the commutativity of the operators \mathbf{J}_l with the operators $\mathcal{I}_{k,\epsilon_k}^p$, for any integers k, l . The last term tends to zero as p goes to infinity, and this independently on q . Thus we still have a Cauchy sequence. \square

Remark 4.9. *We have reversed the direction of iteration compared with the finite-dimensional case (see the definition of $\tilde{\mathbf{B}}_{n,\epsilon}$). This comes from the fact in the hierarchical structure constructed in Part 3, the smallest scale (which is the one linking two successive elements $X_{(i)}, X_{(i+1)}$) corresponds to the matrix \mathcal{T}_{ϵ_1} , and thus remains fixed as the iteration goes on, whereas the smallest scale on the closed-open sets of size n is given by the operator \mathbf{J}_n . The lector could be easily convinced by himself this reversing is compulsory for the existence of our limit operator \mathcal{W}_ϵ .*

From this lemma we define the new dynamical system $\tilde{\mathbf{G}}_\epsilon = \mathcal{W}_\epsilon \circ \mathbf{F}$ and finally get the second result of our paper:

Theorem 4.10. *Let us fix $N \geq 1$ and $\epsilon = (\epsilon_k)_{k \geq 1}$ a sequence of real numbers in $[0, 1]$ defining the operator \mathcal{W}_ϵ on $\mathcal{C}(X, \mathbb{R})$ by the relation: $\mathcal{W}_\epsilon(h) = \lim_{n \rightarrow \infty} \mathcal{I}_{1,\epsilon_1}^n \circ \cdots \circ \mathcal{I}_{n-N,\epsilon_{n-N}}^n \circ \tilde{\mathcal{I}}_{n-N+1,\epsilon_{n-N+1}}^n \circ \cdots \circ \tilde{\mathcal{I}}_{n,\epsilon_n}^n(h)$, where the operators $(\tilde{\mathcal{I}}_{k,\epsilon_k}^n)_{n-N+1 \leq k \leq n}$ contain $E(\sqrt{n})$ broken links of which positions are arbitrary. Assume the sequence ϵ satisfies the following:*

$$6 \sup_{z \in \mathbf{K}} |f'(z)|^{2N} \sum_{k=1}^{+\infty} |1 - 2\epsilon_k| < 1.$$

Then the dynamical system $\tilde{\mathbf{G}}_\epsilon = \mathcal{W}_\epsilon \circ \mathbf{F}$ synchronizes on a non trivial neighborhood Ω_N of the diagonal $\mathcal{J} \cap \mathcal{C}(X, \mathbf{K})$ in $\mathcal{C}(X, \mathbb{R})$.

Proof. (1) We first make the same Taylor approximations as in the proof of Lemma (3.8).

As above for the convenience of the notations, we set:

$$\mathbf{K}_n = \mathcal{I}_{1,\epsilon_1}^n \circ \cdots \circ \mathcal{I}_{n-N,\epsilon_{n-N}}^n \circ \tilde{\mathcal{I}}_{n-N+1,\epsilon_{n-N+1}}^n \circ \cdots \circ \tilde{\mathcal{I}}_{n,\epsilon_n}^n.$$

Let us fix a real $\epsilon' > 0$. By Lemma 3.8, there exists a real $\eta_{2N} > 0$ such that for every $n \geq N + 1$, and every X, Y in \mathbf{K}^{2^n} we have:

$$\|X - Y\|_{\infty, n} \leq \eta_{2N} \Rightarrow \|D_X(\hat{\mathbf{G}}_{n,\epsilon})^{2N} - D_Y(\hat{\mathbf{G}}_{n,\epsilon})^{2N}\|_{\infty, n} \leq \epsilon'$$

where the function $\hat{\mathbf{G}}_{n,\epsilon} = \hat{\mathbf{B}}_{n,\epsilon} \circ \mathbf{F}_n$ is defined by:

$$\hat{\mathbf{B}}_{n,\epsilon} = \mathbf{A}_{n,\epsilon_1}^n \circ \cdots \circ \mathbf{A}_{N+1,\epsilon_{n-N}}^n \circ \tilde{\mathbf{A}}_{N,\epsilon_{n-N+1}}^n \circ \cdots \circ \tilde{\mathbf{A}}_{1,\epsilon_n}^n,$$

that is to say $\hat{\mathbf{B}}_{n,\epsilon}$ is obtained from the matrix $\tilde{\mathbf{B}}_{n,\epsilon}$ by reversing the order of the ϵ_k in accordance with Remark 4.9. The positions of the identical blocks in $\hat{\mathbf{B}}_{n,\epsilon}$ are taken similarly as in the operator \mathbf{K}_n .

Now let h a function belonging to $\mathcal{C}(X, \mathbf{K})$ and $(h_n)_{n \geq 1}$ its associated sequence of Proposition 4.1. For every $n \geq 1$, denote by $(h_{n,k})_{1 \leq k \leq 2^n}$ the 2^n distinct values taken by the function h_n on X . As the closed-open sets form a partition of X , we have:

$$\int_X h_n(c) d\mu(c) = \frac{1}{2^n} \sum_{k=1}^{2^n} h_{n,k},$$

and since X is compact, we remark the following:

$$\int_X h_n(c) d\mu(c) \xrightarrow{n \rightarrow \infty} \int_X h(c) d\mu(c).$$

We define the vector $H_n = (h_{n,1}, \dots, h_{n,2^n})$ in \mathbb{R}^{2^n} and its associated sum-vector in \mathfrak{J}_{2^n} :

$$H_{n,\Sigma} = \left(\int_X h_n d\mu, \dots, \int_X h_n d\mu \right).$$

Now take an integer $n \geq N + 1$. For any function g_n constant on the closed-open sets of size n , we have $\mathcal{W}_\epsilon(g_n) = K_n(g_n)$. Renumbering the coordinates of the vector H_n if necessary, it comes:

$$\begin{aligned} \left\| \left(\tilde{G}_\epsilon \right)^{2N} (h_n) - \left(\tilde{G}_\epsilon \right)^{2N} \left(\int_X h_n d\mu \right) \right\|_\infty &= \left\| (K_n \circ F)^{2N} (h_n) - (K_n \circ F)^{2N} \left(\int_X h_n d\mu \right) \right\|_\infty \\ &= \left\| \left(\hat{G}_{n,\epsilon} \right)^{2N} (H_n) - \left(\hat{G}_{n,\epsilon} \right)^{2N} (H_{n,\Sigma}) \right\|_{\infty,n}, \end{aligned}$$

and so by Lemma 3.8 the term on the left is smaller than:

$$(\epsilon' + \Lambda_n) \|H_n - H_{n,\Sigma}\|_{\infty,n} + \|D_{H_{n,\Sigma}} \left(\hat{G}_{n,\epsilon} \right)^{2N} - D_{H_{n,\Sigma}} \left(\hat{G}_{n,1/2} \right)^{2N}\|_{\infty,n} \|H_n - H_{n,\Sigma}\|_{\infty,n},$$

providing that our function h satisfies: $|h(c) - h(c')| \leq \eta_{2N}$ for every c, c' in X . Now as in the proof of Lemma 4.9 we notice that:

$$\begin{aligned} \|D_{H_{n,\Sigma}} \left(\hat{G}_{n,\epsilon} \right)^{2N} - D_{H_{n,\Sigma}} \left(\hat{G}_{n,1/2} \right)^{2N}\|_{\infty,n} &\leq 3 \sup_{z \in K} |f'(z)|^{2N} \|\hat{B}_{n,1/2} - \hat{B}_{n,\epsilon}\|_{\infty,n} \\ &\leq 3 \sup_{z \in K} |f'(z)|^{2N} \sum_{k=1}^n |1 - 2\epsilon_k|. \end{aligned}$$

Passing to the limit at infinity, we finally get:

$$\left\| \left(\tilde{G}_\epsilon \right)^{2N} (h) - \left(\tilde{G}_\epsilon \right)^{2N} \left(\int_X h d\mu \right) \right\|_\infty \leq \left(\epsilon' + 3 \sup_{z \in K} |f'(z)|^{2N} \sum_{k=1}^{+\infty} |1 - 2\epsilon_k| \right) \|h - \int_X h d\mu\|_\infty.$$

(2) From this it comes for every sequences c, c' in X :

$$\left| \left(\tilde{G}_\epsilon \right)^{2N} (h)(c) - \left(\tilde{G}_\epsilon \right)^{2N} (h)(c') \right| \leq 2 \left(\epsilon' + 3 \sup_{z \in K} |f'(z)|^{2N} \sum_{k=1}^{+\infty} |1 - 2\epsilon_k| \right) \sup_{x, x' \in X} |h(x) - h(x')|,$$

which gives the synchronization of the dynamical system $\left(\tilde{G}_\epsilon \right)^{2N}$ on the neighborhood Ω_N of the diagonal $\mathfrak{J} \cap \mathcal{C}(X, K)$ in $\mathcal{C}(X, \mathbb{R})$, defined by:

$$\Omega_N = \{h \in \mathcal{C}(X, K) : \sup_{x, x' \in X} |h(x) - h(x')| \leq \eta_{2N}\}.$$

Doing again the same trick as in the end of the proof of Lemma 3.8, we finally obtain the synchronization of \tilde{G}_ϵ on Ω_N . QED. □

4.3. Corollaries of the results in the case p^n . In this last subsection, we present the corollaries of our two theorems on a Cantor set $X = \{a_0, \dots, a_{p-1}\}^{\mathbb{N}}$ associated to an alphabet with $p \geq 3$ letters, which is the limiting case corresponding to a finite matching process in groups of size p : given a tuple $\epsilon^{(k)} = (\epsilon_1^{(k)}, \dots, \epsilon_{p-1}^{(k)})$ defining the k^{th} stage of the hierarchical structure, the coupling between the gathered elements is now defined by a circulant matrix $\mathcal{R}_{\epsilon^{(k)}}$, of which entries are $1 - \sum_{i=0}^{p-1} \epsilon_i^{(k)}, \epsilon_1^{(k)}, \dots, \epsilon_{p-1}^{(k)}$.

4.3.a. *Corollary of theorem (4.6).* For every tuple $\epsilon^{(k)} = (\epsilon_1^{(k)}, \dots, \epsilon_{p-1}^{(k)})$ in $[0, 1]^{p-1}$ the new operator structures $\mathcal{L}_{k, \epsilon^{(k)}}$ write:

$$\mathcal{L}_{k, \epsilon^{(k)}}(h) = \left(1 - \sum_{i=0}^{p-1} \epsilon_i^{(k)} \right) h + \sum_{i=1}^{p-1} J_k^i(h),$$

where this time the map $J_k(h)$ is defined by the relation:

$$\forall c \in X, \text{ with } c_k = a_i, J_k(h)(c) = h(c_1, \dots, c_{k-1}, a_{i+1 \bmod p}, c_{k+1}, \dots).$$

With this definition, nothing changes in the approach we have done previously, except the fact the orthogonal of the diagonal \mathfrak{J}_k in $\mathcal{C}(X, \mathbb{R})$ becomes:

$$\mathfrak{J}_k^\perp = \{h \in \mathcal{C}(X, \mathbb{R}) : \sum_{i=0}^{p-1} J_k^i(h) = 0\}.$$

We get:

Corollary 4.11. *Let $\epsilon = (\epsilon^{(k)})_{k \geq 1}$ be a sequence of elements in $[0, 1]^{p-1}$, defining the operator \mathcal{U}_ϵ by:*

$$\mathcal{U}_\epsilon(h) = \lim_{n \rightarrow \infty} \mathcal{L}_{n, \epsilon^{(n)}} \circ \dots \circ \mathcal{L}_{1, \epsilon^{(1)}}(h),$$

for every continuous function h on X . Assume the following condition holds:

$$\exists a > 1, \exists \alpha > 0, \forall k \geq 1, |\gamma^{(k)}| \sup_{z \in K} |f'(z)| \leq \frac{1}{(ak)^\alpha},$$

where $\gamma^{(k)}$ is the eigenvalue (distinct from one) of $\mathcal{R}_{\epsilon^{(k)}}$ having the greatest modulus.

Then the dynamical system $G_\epsilon = \mathcal{U}_\epsilon \circ F$ globally synchronizes on the set $\mathcal{C}(X, K)$ of continuous functions having values in K .

4.3.b. *Corollary of theorem (4.10).* Concerning the local synchronization in presence of broken links, this requires to consider again the spectrum of the structure matrix $\tilde{B}_{n, \frac{1}{p}} = A_{n, 1/p}^n \cdots A_{N+1, 1/p}^n \tilde{A}_{N, 1/p}^n \cdots \tilde{A}_{1, 1/p}^n$: as previously, we impose that the identical blocks be only present in the matrices $\tilde{A}_{N, 1/p}^n, \dots, \tilde{A}_{1, 1/p}^n$, but this time we allow them to contain $(p-1)E(\sqrt{n})$ such identical blocks.

In this case, setting again $\tilde{\mathcal{A}}_N^n = \tilde{A}_{N, 1/p}^n \cdots \tilde{A}_{1, 1/p}^n$ we have:

$$\tilde{\mathcal{A}}_N^{n+1} = \begin{bmatrix} \tilde{\mathcal{A}}_N^n & & & \\ & \Gamma_{n,1} & & \\ & & \ddots & \\ & & & \Gamma_{n,p-1} \end{bmatrix},$$

where the matrices (of size p^n) $\Gamma_{n,1}, \dots, \Gamma_{n,p-1}$ are still products of the same form as $\tilde{\mathcal{A}}_N^n$, but in which there are at most $p-1$ blocks I_{p^k} in the set of all of them (all these blocks can possibly be present in only one $\Gamma_{n,i}$). Denoting by $\mathcal{C}_{n,k}$ the matrix $A_{n, 1/2}^n \cdots A_{N+1, 1/2}^n \Gamma_{n,k}$, it comes:

$$\tilde{B}_{n+1, \frac{1}{p}} = \begin{bmatrix} \frac{1}{p} \tilde{B}_{n, \frac{1}{p}} & \frac{1}{p} \mathcal{C}_{n,1} & \cdots & \frac{1}{p} \mathcal{C}_{n,p-1} \\ \vdots & \vdots & & \vdots \\ \frac{1}{p} \tilde{B}_{n, \frac{1}{p}} & \frac{1}{p} \mathcal{C}_{n,1} & \cdots & \frac{1}{p} \mathcal{C}_{n,p-1} \end{bmatrix}.$$

As in Part 3, we get by induction that $\tilde{B}_{n,\frac{1}{p}}$ is diagonalizable and satisfies:

$$\mathfrak{S}\left(\tilde{B}_{n,\frac{1}{p}}\right) = \{1, \lambda_{n,1}, \dots, \lambda_{n,(p-1)N}, 0^{\otimes p^n - (p-1)N - 1}\},$$

the $\lambda_{n,i}$ being some (possibly equal) positive numbers satisfying:

$$\forall i \in \{1, \dots, (p-1)N\}, \lambda_{n,i} \leq \frac{1}{p^{n-N-1}}.$$

This leads us to the same estimation as in Lemma (3.8), from which we obtain:

Corollary 4.12. *Let us fix $N \geq 1$ and $\epsilon = (\epsilon^{(k)})_{k \geq 1}$ a sequence of elements in $[0, 1]^p$ defining the operator \mathcal{W}_ϵ on $\mathcal{C}(X, \mathbb{R})$ by the relation: $\mathcal{W}_\epsilon(h) = \lim_{n \rightarrow \infty} \mathcal{I}_{1, \epsilon^{(1)}}^n \circ \dots \circ \mathcal{I}_{n-N, \epsilon^{(n-N)}}^n \circ \tilde{\mathcal{I}}_{n-N+1, \epsilon^{(n-N+1)}}^n \circ \dots \circ \tilde{\mathcal{I}}_{n, \epsilon^{(n)}}^n(h)$, where the operators $(\tilde{\mathcal{I}}_{k, \epsilon^{(k)}}^n)_{n-N+1 \leq k \leq n}$ contain $(p-1)E(\sqrt{n})$ broken links of which positions are arbitrary. Assume the sequence ϵ satisfies the following:*

$$6\sqrt{p} \sup_{z \in K} |f'(z)|^{2N} \sum_{k=1}^{+\infty} |\gamma^{(k)}| < 1,$$

where $\gamma^{(k)}$ is the eigenvalue (distinct from one) of $\mathcal{R}_{\epsilon^{(k)}}$ having the greatest modulus.

Then the dynamical system $\tilde{G}_\epsilon = \mathcal{W}_\epsilon \circ F$ synchronizes on a non trivial neighborhood Ω_N of the diagonal $\mathcal{J} \cap \mathcal{C}(X, K)$ in $\mathcal{C}(X, \mathbb{R})$.

Finally, these corollaries directly extend to functions f defined on a convex compact set K of \mathbb{R}^n and verifying $\sup_{z \in K} \|D_z f\| > 1$ for some norm on \mathbb{R}^n , since this case just comes to replacing the coupling matrix $\tilde{B}_{n,\epsilon}$ by $\tilde{B}_{n,\epsilon} \otimes I_d$.

5. CONCLUSION

From this work we conclude that using the hierarchical structure we have exhibited in this text is the natural way to synchronize an uncountable set (namely a Cantor set) of dynamical systems. Once again, such a phenomenon echoes to tiling theory, where hierarchy is often imposed to produce aperiodicity.

Our investigation could be pursued with the same framework as the one we have used in this text.

A first question would be to look at the highest number of broken links we can afford: indeed the boundary in \sqrt{n} , could be surely improved. Another more interesting one, would be to study what happens when the structure is not repeated at each step, while keeping a hierarchy: for instance, starting at the first scale with a matching with two elements, we could gather them three by three at the second one, and then change again the type of matching at the third one, and so on. If we impose that the size of the matching process varies in a finite set $\{2, \dots, p\}$, then some results must be found. In tilings this problem appears strongly, and corresponds to the case where the shape of the patches changes at each stage, contrary to substitution tilings. A lot of work still remains to be done in this direction to understand better those general hierarchical structures.

REFERENCES

- [1] Neil J. Balmforth, Antonello Provenzale, and Roberto Sassi. A hierarchy of coupled maps. *Chaos*, 12(3):719–731, 2002.
- [2] Robert Berger. The undecidability of the domino problem. *Mem. Amer. Math. Soc. No.*, 66:72, 1966.
- [3] A. Díaz-Guilera. Dynamics towards synchronization in hierarchical networks. *J. Phys. A*, 41(22):224007, 7, 2008.
- [4] W. S. Edwards and S. Fauve. Patterns and quasi-patterns in the faraday experiment. *Journal of Fluid Mechanics*, 278:123–148, 1994.
- [5] Bastien Fernandez. Global synchronization in translation invariant coupled map lattices. *International Journal of Bifurcation and Chaos*, 18(11):3455–3459, 2008.
- [6] Hirokazu Fujisaka and Tomoji Yamada. Stability theory of synchronized motion in coupled-oscillator systems. *Progr. Theoret. Phys.*, 69(1):32–47, 1983.
- [7] Chaim Goodman-Strauss. Matching rules and substitution tilings. *Ann. of Math. (2)*, 147(1):181–223, 1998.
- [8] Chaim Goodman-Strauss. Aperiodic hierarchical tilings. In *Foams and emulsions (Cargèse, 1997)*, volume 354 of *NATO Adv. Sci. Inst. Ser. E Appl. Sci.*, pages 481–496. Kluwer Acad. Publ., Dordrecht, 1999.
- [9] J. Jost and M. P. Joy. Spectral properties and synchronization in coupled map lattices. *Phys. Rev. E*, 65:016201, Dec 2001.

- [10] Kunihiko Kaneko. Clustering, coding, switching, hierarchical ordering, and control in a network of chaotic elements. *Physica D: Nonlinear Phenomena*, 41(2):137 – 172, 1990.
- [11] Kunihiko Kaneko. *Theory and applications of coupled map lattices*. Nonlinear Science: Theory and Applications. John Wiley & Sons, Ltd., Chichester, 1993.
- [12] P. Kramer and R. Neri. Erratum: “On periodic and nonperiodic space fillings of \mathbf{E}^m obtained by projection” [Acta Cryst. Sect. A **40** (1984), no. 5, 580–587; MR0768042 (86c:52018)]. *Acta Cryst. Sect. A*, 41(6):619, 1985.
- [13] Y. Kuramoto. *Chemical oscillations, waves, and turbulence*, volume 19 of *Springer Series in Synergetics*. Springer-Verlag, Berlin, 1984.
- [14] Wenlian Lu and Tianping Chen. Synchronization analysis of linearly coupled networks of discrete time systems. *Physica D: Nonlinear Phenomena*, 198(1-2):148–168, 2004.
- [15] Yves Meyer. Quasicrystals, Diophantine approximation and algebraic numbers. In *Beyond quasicrystals (Les Houches, 1994)*, pages 3–16. Springer, Berlin, 1995.
- [16] R. Mirollo and S. Strogatz. Synchronization of pulse-coupled biological oscillators. *SIAM Journal on Applied Mathematics*, 50(6):1645–1662, 1990.
- [17] Louis M. Pecora and Thomas L. Carroll. Synchronization in chaotic systems. *Phys. Rev. Lett.*, 64:821–824, Feb 1990.
- [18] Roger Penrose. The rôle of aesthetics in pure and applied mathematical research. *The Institute of Mathematics and its Applications Bulletin*, 10:266–271, 1974.
- [19] Arkady Pikovsky, Michael Rosenblum, and Jürgen Kurths. *Synchronization: a universal concept in nonlinear sciences*, volume 12 of *Cambridge Nonlinear Science Series*. Cambridge University Press, Cambridge, 2001.
- [20] Przemyslaw Prusinkiewicz and Aristid Lindenmayer. *The algorithmic beauty of plants*. Springer-Verlag New York, Inc., New York, NY, USA, 1996.
- [21] Raphael M. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Invent. Math.*, 12:177–209, 1971.

LABORATOIRE J-A DIEUDONNÉ, UNIVERSITÉ DE NICE-SOPHIA ANTIPOLIS 06108 NICE CEDEX 02, FRANCE
E-mail address: uncam@unice.fr