Optimal observability of the multi-dimensional wave and Schrödinger equations in quantum ergodic domains

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Yannick Privat* Emmanuel Trélat† Enrique Zuazua‡§

Abstract

We consider the wave and Schrödinger equations on a bounded open connected subset Ω of a Riemannian manifold, with Dirichlet, Neumann or Robin boundary conditions whenever its boundary is nonempty. We observe the restriction of the solutions to a measurable subset ω of Ω during a time interval $[0,T]$ with $T>0$. It is well known that, if the pair (ω,T) satisfies the Geometric Control Condition (ω being an open set), then an observability inequality holds guaranteeing that the total energy of solutions can be estimated in terms of the energy localized in $\omega\times(0,T)$.

We address the problem of the optimal location of the observation subset ω among all possible subsets of a given measure or volume fraction. A priori this problem can be modeled in terms of maximizing the observability constant, but from the practical point of view it appears more relevant to model it in terms of maximizing an average either over random initial data or over large time. This leads us to define a new notion of observability constant, either randomized, or asymptotic in time. In both cases we come up with a spectral functional that can be viewed as a measure of eigenfunction concentration. Roughly speaking, the subset ω has to be chosen so to maximize the minimal trace of the squares of all eigenfunctions. Considering the convexified formulation of the problem, we prove a no-gap result between the initial problem and its convexified version, under appropriate quantum ergodicity assumptions on Ω, and compute the optimal value. Our results reveal intimate relations between shape and domain optimization, and the theory of quantum chaos (more precisely, quantum ergodicity properties of the domain Ω).

We prove that in 1D a classical optimal set exists only for exceptional values of the volume fraction, and in general one expects relaxation to occur and therefore classical optimal sets not to exist. We then provide spectral approximations and present some numerical simulations that fully confirm the theoretical results in the paper and support our conjectures.

Finally, we provide several remedies to nonexistence of an optimal domain. We prove that when the spectral criterion is modified to consider a weighted one in which the high frequency components are penalized, the problem has then a unique classical solution determined by a finite number of low frequency modes. In particular the maximizing sequence built from spectral approximations is stationary.

Keywords: wave equation, Schrödinger equation, observability inequality, optimal design, spectral decomposition, ergodic properties, quantum ergodicity.

AMS classification: 35P20, 93B07, 58J51, 49K20

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Contents

1 Introduction

1.1 Problem formulation and overview of the main results

In this article we model and solve the problem of optimal observability for wave and Schrödinger equations posed on any open bounded connected subset of a Riemannian manifold, with various possible boundary conditions.

In order to briefly highlight the main ideas and contributions of the paper, in this introduction let us focus on a particular case of our study, starting from a practical problem. Assume that Ω is a given bounded open subset of $\mathbb{R}^n$, representing for instance a cavity in which some signals are propagating according to the wave equation

$$\partial_{tt} y = \triangle y, \tag{1}$$

with Dirichlet boundary conditions. Having in mind certainly some reconstruction inverse problem, assume that one is allowed to place some sensors in the cavity, in order to make some measurements of the signals propagating in Ω over a certain horizon of time. We assume that we have the choice not only of the placement of the sensors but also of their shape. Let us address the question of knowing what is the best possible shape and location of sensors, achieving the best possible observation, in some sense to be made precise. This problem of optimal observability, inspired by control theoretical considerations, is intimately related to those of optimal controllability and stabilization (see Section 6 for a discussion of these issues).

At this step, the question is too much informal and a first challenge is to settle properly this question in the mathematical world, so that the resulting problem will be both mathematically solvable and relevant in view of practical issues.

A first obvious but important remark is that, for any problem consisting of optimizing the observation, certainly the best policy consists of observing the solutions over the whole domain Ω. This is however clearly not reasonable and in practice the domain covered by sensors is limited, due for instance to cost considerations. From the mathematical point of view, we model this basic limitation by considering as the set of unknowns, the set of all possible measurable subsets ω of Ω that are of Lebesgue measure $|\omega| = L|\Omega|$, where $L \in (0,1)$ is some fixed real number. Any such subset represents the sensors put in Ω, and we assume that we are able to measure the restrictions of the solutions of (1) to ω.

Modeling. Let us now model the notion of best observation. At this step it is useful to recall some well known facts on the observability of the wave equation. For all $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a unique solution $y \in C^0(0,T; L^2(\Omega)) \cap C^1(0,T; H^{-1}(\Omega))$ of (1) such that $y(0,\cdot) = y^0(\cdot)$ and $y_t(0,\cdot) = y^1(\cdot)$. Let $T > 0$. We say that (1) is observable on ω in time T if there exists $C > 0$ such that

$$C\|(y^0, y^1)\|^2_{L^2(\Omega,\mathbb{C})\times H^{-1}(\Omega,\mathbb{C})} \leqslant \int_0^T \int_\omega |y(t,x)|^2 \, dxdt, \tag{2}$$

for all $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. This inequality is called an observability inequality, and is of great importance in view of showing the well-posedness of some inverse problems. It is well known that within the class of $\mathcal{C}^\infty$ domains Ω, this observability property holds if the pair (ω, T) satisfies the *Geometric Control Condition* in Ω (see [3]), according to which every ray of geometrical optics that propagates in the cavity Ω and is reflected on its boundary $\partial\Omega$ intersects ω within time T. The observability constant is defined by

$$C_T^{(W)}(\chi_\omega) = \inf \left\{ \frac{\int_0^T \int_\Omega \chi_\omega(x) |y(t,x)|^2 \, dx \, dt}{\|(y^0, y^1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}} \;\Big|\; (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \setminus \{(0,0)\} \right\}. \tag{3}$$

It is the largest possible constant for which (2) holds. It depends both on the time T (the horizon time of observation) and on the subset ω on which the measurements are done.

A priori, it might appear as natural to model the problem of best observability as the problem of maximizing the functional $\chi_\omega \mapsto C_T^{(W)}(\chi_\omega)$ over the set

$$\mathcal{U}_L = \{\chi_\omega \mid \omega \text{ is a measurable subset of } \Omega \text{ of Lebesgue measure } |\omega| = L|\Omega|\}. \tag{4}$$

However, his choice of model leads to a mathematical problem that is difficult to handle from the theoretical point of view, and more importantly, this model is not relevant in view of practical issues. Let us explain these two difficulties.

First of all, making a spectral expansion of the solutions shows the emergence of crossed terms that are difficult to treat. Indeed, let $(\phi_j)_{j\in\mathbb{N}^*}$ be a Hilbertian basis of $L^2(\Omega)$ consisting of eigenfunctions of the Dirichlet-Laplacian operator on Ω, associated with the negative eigenvalues $(-\lambda_j^2)_{j\in\mathbb{N}^*}$. Then any solution y (1) can be expanded as

$$y(t,x) = \sum_{j=1}^{+\infty} \left(a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t}\right) \phi_j(x), \tag{5}$$

where the coefficients a_j and b_j account for initial data. It follows that

$$C_T^{(W)}(\chi_\omega) = \frac{1}{2} \inf_{\substack{(a_j),(b_j)\in\ell^2(\mathbb{C}) \\ \sum_{j=1}^{+\infty}(|a_j|^2+|b_j|^2)=1}} \int_0^T \int_\omega \left| \sum_{j=1}^{+\infty} \left(a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t}\right) \phi_j(x) \right|^2 dx\, dt,$$

and then maximizing this functional over $\mathcal{U}_L$ appears to be very difficult from the theoretical point of view, due to the crossed terms $\int_\omega \phi_j \phi_k dx$ measuring the interaction over ω between distinct eigenfunctions.

The second difficulty with this model is its lack of relevance in practice. Indeed, the observability constant defined by (3) is deterministic and provides an account for the worst possible case. Hence, in this sense, it is a pessimistic constant. In practice where an engineer realizes a large number of measures, it may be expected that this worst case does not occur so often, and one would like that the observation be optimal for most of experiments. This leads us to consider rather an averaged version of the observability inequality over random initial data. More details will be given in Section 2.3 on the randomization procedure, but in few words, we define what we call the *randomized observability constant* by

$$C_{T,\mathrm{rand}}^{(W)}(\chi_\omega) = \frac{1}{2} \inf_{\substack{(a_j),(b_j)\in\ell^2(\mathbb{C}) \\ \sum_{j=1}^{+\infty}(|a_j|^2+|b_j|^2)=1}} \mathbb{E}\left(\int_0^T \int_\omega \left| \sum_{j=1}^{+\infty} \left(\beta_{1,j}^\nu a_j e^{i\lambda_j t} + \beta_{2,j}^\nu b_j e^{-i\lambda_j t}\right) \phi_j(x) \right|^2 dx\, dt \right), \tag{6}$$

where $(\beta_{1,j}^\nu)_{j\in\mathbb{N}^*}$ and $(\beta_{2,j}^\nu)_{j\in\mathbb{N}^*}$ are two sequences of (for example) i.i.d. Bernoulli random laws on a probability space $(\mathcal{X}, \mathcal{A}, \mathbb{P})$, and $\mathbb{E}$ is the expectation over the $\mathcal{X}$ with respect to the probability measure $\mathbb{P}$. It corresponds to an averaged version of the observability inequality over random initial data. Actually, we have the following result.

Theorem 1 (Characterization of the randomized observability constant)**.** *For every measurable subset ω of Ω, there holds*

$$C_{T,rand}^{(W)}(\chi_\omega) = \frac{T}{2} \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\, dx. \tag{7}$$

It is interesting to note that there always holds $C_T^{(W)}(\chi_\omega) \leqslant C_{T,\mathrm{rand}}^{(W)}(\chi_\omega)$, and that the strict inequality holds for instance in each of the following cases:

- in 1D, with $\Omega = (0,\pi)$ and Dirichlet boundary conditions, whenever T is not an integer multiple of π;
- in multi-D, with Ω stadium-shaped, whenever ω contains an open neighborhood of the wings (in that case there even holds $C_T^{(W)}(\chi_\omega) = 0$, see Remark 4 for details).

Taking into account the fact that, in practice, it is expected that a large number of measurements is to be done, we finally model the problem of best observability in the following more relevant way: maximize the functional

$$J(\chi_\omega) = \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\,dx \tag{8}$$

over the set $\mathcal{U}_L$.

The functional J appears as a criterion giving an account for eigenfunctions concentration properties. It can be noted that it can be as well recovered by considering, instead of an averaged version of the observability inequality over random initial data, a time-asymptotic version of it. More precisely, we claim that, if the eigenvalues of the Dirichlet-Laplacian are simple (which is a generic property), then $J(\chi_\omega)$ is the largest possible constant such that

$$C\|(y^0,y^1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leqslant \lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_\omega |y(t,x)^2|\,dx\,dt,$$

for all $(y^0,y^1)\in L^2(\Omega)\times H^{-1}(\Omega)$ (see Section 2.5).

This model discussion and the introduction of these new notions of averaged observability inequalities (Section 2) are the first contribution of our article.

Solving. In view of solving the uniform optimal design problem

$$\sup_{\chi_\omega\in\mathcal{U}_L} J(\chi_\omega), \tag{9}$$

we first consider a convexified version of the problem, by considering the convex closure of the set $\mathcal{U}_L$ for the L^∞ weak star topology, that is $\overline{\mathcal{U}}_L = \{a\in L^\infty(\Omega,[0,1]) \mid \int_\Omega a(x)\,dx = L|\Omega|\}$. The convexified problem then consists of maximizing the functional

$$J(a) = \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2\,dx$$

over $\overline{\mathcal{U}}_L$. Clearly there exists a maximizer, but since the functional J is not lower semi-continuous it is not clear whether or not there may be a gap between the problem (9) and its convexified version. The analysis of this question happens to be very interesting and reveals deep connections with the theory of quantum chaos, more precisely with quantum ergodicity properties of Ω. We prove for instance the following result (see Section 3.2 for other related statements).

Theorem 2 (No-gap result and optimal value of J)**.** *Assume that there exists a subsequence of the sequence of probability measures $\mu_j = \phi_j^2(x)\,dx$ converging vaguely to the uniform measure $\frac{1}{|\Omega|}\,dx$ (Weak Quantum Ergodicity assumption), and that the sequence of eigenfunctions ϕ_j is uniformly bounded in $L^\infty(\Omega)$. Then*

$$\sup_{\chi_\omega\in\mathcal{U}_L} J(\chi_\omega) = \max_{a\in\overline{\mathcal{U}}_L} J(a) = L,$$

for every $L\in(0,1)$. In other words, there is no gap between the problem (9) and its convexified version.

Several remarks are in order.

- The quantum ergodicity assumptions of the above result hold true in any hypercube with Dirichlet boundary conditions (and as well with Neumann, mixed or Robin boundary conditions, or periodic conditions).
- They are sufficient but not necessary to derive such a no-gap statement: indeed we can prove that it still holds true if Ω is a two-dimensional disk, although the eigenfunctions do not equidistribute as the eigenfrequencies increase, as illustrated by the well known whispering galleries effect (see Proposition 1 in Section 3.2).
- We are not aware of any example in which there is a gap between the problem (9) and its convexified version.
- At this step, it follows from Theorems 1 and 2 that, under quantum ergodicity assumptions, the optimal possible value of $C^{(W)}_{T,\mathrm{rand}}(\chi_\omega)$ (over the set $\mathcal{U}_L$) is equal to $TL/2$.
- It is interesting to note that, since the spectral criterion J defined by (8) depends on the specific choice of the orthonormal basis $(\phi_j)_{j\in\mathbb{N}^*}$ of eigenfunctions of the Dirichlet-Laplacian, one can consider an intrinsic version of the problem, consisting of maximizing the spectral functional
$$J_{int}(\chi_\omega) = \inf_{\phi\in\mathcal{E}} \int_\omega \phi(x)^2\,dx$$
over $\mathcal{U}_L$, where $\mathcal{E}$ denotes the set of all normalized eigenfunctions of the Dirichlet-Laplacian. For this problem we have a result similar to the one above (Theorems 8 and 9 in Section 3.7), but we are moreover able to provide an explicit example where a gap occurs between the problem and its convexified formulation, by considering for instance the unit half-sphere with Dirichlet boundary conditions, and certain quantum limits of a Dirac type.

  These results show intimate connections between domain optimization and quantum ergodicity properties of Ω. Such a relation was suggested in the early work [13] concerning the exponential decay properties of dissipative wave equations.
- Under the stronger assumption that the whole sequence of probability measures $\mu_j = \phi_j^2(x)\,dx$ converges vaguely to the uniform measure $\frac{1}{|\Omega|}\,dx$ (Quantum Unique Ergodicity assumption), and assuming that the sequence of eigenfunctions ϕ_j is uniformly bounded in some $L^p(\Omega)$ with $p > 2$, we can prove that the supremum of J over the set of Jordan measurable subsets of measure $L|\Omega|$ is equal to L. Moreover, the proof of this fact (done in Section 3.5), based on a kind of homogenization procedure, is constructive and builds a maximizing sequence of subsets for the problem of maximizing J, showing that it is possible to increase the values of J by considering subsets having an increasing number of connected components.

Nonexistence of an optimal set and remedies. The maximum of J over $\overline{\mathcal{U}}_L$ is clearly reached (in general, even in an infinite number of ways). The question of the reachability of the supremum of J over $\mathcal{U}_L$, that is, the existence of an optimal classical set, is a difficult question in general. In particular cases it can however be addressed using harmonic analysis. For instance in dimension one, we can prove that the supremum is reached if and only if $L = 1/2$ (and there is an infinite number of optimal sets). In higher dimension, the question is completely open, and we conjecture that, for generic domains Ω and generic values of L, the supremum is not reached and hence there does not exist any optimal set. It can however be noted that, in the two-dimensional Euclidean square, if we restrict the search of optimal sets to Cartesian products of 1D subsets, then the supremum is reached if and only if $L \in \{1/4, 1/2, 3/4\}$ (see Section 4.1 for details).

In view of that, it is then natural to study a finite-dimensional spectral approximation of the problem, namely the problem of maximizing the functional

$$J_N(\chi_\omega) = \min_{1 \leqslant j \leqslant N} \int_\omega \phi_j(x)^2 \, dx$$

over $\mathcal{U}_L$, for $N \in \mathbb{N}^*$. The existence and uniqueness of an optimal set ω^N is then not difficult to prove, as well as a Γ-convergence property of J_N towards J for the weak star topology of L^∞. Moreover, the sets ω^N have a finite number of connected components, expected to increase in function of N. Several numerical simulations (provided in Section 4.2) will show the shapes of these sets; their increasing complexity which can be observed as N increases is in accordance with the conjecture of the nonexistence of an optimal set maximizing J. It can be noted that, in the one-dimensional case, for L sufficiently small, loosely speaking the optimal domain ω^N for N modes is the worst possible one when considering the truncated problem with $N+1$ modes (spillover phenomenon; see [24, 47]).

This intrinsic instability is in some sense due to the fact that in the definition of the spectral criterion (8) all modes have the same weight, and the same criticism can be made on the observability inequality (2). Due to the increasing complexity of the geometry of highfrequency eigenfunctions, it could indeed be expected that the optimal shape and placement problem would be complicated. This leads to the intuition that lower frequencies should be more weighted than the higher ones, and then it seems relevant to introduce a weighted version of the observability inequality (2), by considering the (equivalent) inequality

$$C^{(W)}_{T,\sigma}(\chi_\omega) \left(\|(y^0, y^1)\|^2_{L^2 \times H^{-1}} + \sigma \|y^0\|^2_{H^{-1}} \right) \leqslant \int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt, \tag{10}$$

where $\sigma \geqslant 0$ is some weight. There holds $C^{(W)}_{T,\sigma}(\chi_\omega) \leqslant C^{(W)}_T(\chi_\omega)$, and considering as before an averaged version of this weighted observability inequality over random initial data leads to $2\, C^{(W)}_{T,\sigma,\mathrm{rand}}(\chi_\omega) = T J_\sigma(\chi_\omega)$, where the weighted spectral criterion J_σ is defined by

$$J_\sigma(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \sigma_j \int_\omega \phi_j(x)^2 \, dx,$$

with $\sigma_j = \lambda_j^2/(\sigma + \lambda_j^2)$ (increasing sequence of positive real numbers converging to 1; see Section 4.4 for details). The truncated criterion $J_{\sigma,N}$ is then defined accordingly, by keeping only the N first modes. We then have the following result.

Theorem 3 (Weighted spectral criterion)**.** *Assume that the whole sequence of probability measures $\mu_j = \phi_j^2(x)\, dx$ converges vaguely to the uniform measure $\frac{1}{|\Omega|}\, dx$ (Quantum Unique Ergodicity assumption), and that the sequence of eigenfunctions ϕ_j is uniformly bounded in $L^\infty(\Omega)$. Then, for every $L \in (\sigma_1, 1)$, there exists $N_0 \in \mathbb{N}^*$ such that*

$$\max_{\chi_\omega \in \mathcal{U}_L} J_\sigma(\chi_\omega) = \max_{\chi_\omega \in \mathcal{U}_L} J_{\sigma,N}(\chi_\omega) \leqslant \sigma_1 < L,$$

for every $N \geqslant N_0$. In particular, the problem of maximizing J_σ over $\mathcal{U}_L$ has a unique solution $\chi_{\omega^{N_0}}$, and moreover the set ω^{N_0} has a finite number of connected components.

It has to be noted that the assumptions of the above theorem (referred to as L^∞-QUE as discussed further) are strong ones. Up to now, except in the one-dimensional case where these assumptions obviously hold, in the multi-dimensional case no domain is known where they are satisfied, and it is one of the deepest open problems in mathematical physics to exhibit such

a domain (as discussed in Section 3.3). We are however able to prove that the conclusion of Theorem 3 holds true in a hypercube with Dirichlet boundary conditions, although QUE is not satisfied in such a domain (see Proposition 5 in Section 4.4).

The theorem says that, for the problem of maximizing $J_{\sigma,N}$ over $\mathcal{U}_L$, the sequence of optimal sets ω^N is stationary whenever L is large enough, and ω^{N_0} is then the (unique) optimal set, solution of the problem of maximizing J_σ. It can be noted that the lower threshold in L depends on the chosen weights, and the numerical simulations that we will provide indicate that this threshold is sharp in the sense that, if $L < \sigma_1$ then the sequence of maximizing sets loses its stationarity feature.

As a conclusion, this weighted version of our spectral criterion can be viewed as a remedy for the spillover phenomenon. Note that, of course, other more evident remedies can be discussed, such as the search of an optimal domain among a set of subdomains sharing nice compactness properties (such as having a uniform perimeter or BV norm; see Section 4.3), however our aim is here to investigate domains as general as possible (only measurable) and rather to discuss the mathematical, physical and practical relevance of the criterion encoding the notion of optimal observability.

Let us finally note that all our results hold for wave and Schrödinger equations on any open bounded connected subset of a Riemannian manifold (then replacing $\triangle$ with the Laplace-Beltrami operator), with various possible boundary conditions (Dirichlet, Neumann, mixed, Robin) or no boundary conditions in case the manifold is compact without boundary. The abstract framework and generalizations are described in Section 5.

1.2 Brief state of the art

The literature on optimal observation or sensor location problems is abundant in engineering applications (see, e.g., [34, 44, 54, 57, 60] and references therein), but very few mathematical theoretical contributions do exist. In engineering applications, the aim is to optimize the number, the place and the type of sensors in order to improve the estimation of the state of the system. Fields of applications are very numerous and concern for example active structural acoustics, piezoelectric actuators, vibration control in mechanical structures, damage detection and chemical reactions, just to name a few of them. In most of these applications however the method consists in approximating appropriately the problem by selecting a finite number of possible optimal candidates and of recasting the problem as a finite dimensional combinatorial optimization problem. Among these approaches, the closest one to ours consists of considering truncations of Fourier expansion representations. Adopting such a Fourier point of view, the authors of [23, 24] studied optimal stabilization issues of the one-dimensional wave equation and, up to our knowledge, these are the first articles in which one can find rigorous mathematical arguments and proofs to characterize the optimal set whenever it exists, for the problem of determining the best possible shape and position of the damping subdomain of a given measure. In [5] the authors investigate the problem modeled in [54] of finding the best possible distributions of two materials (with different elastic Young modulus and different density) in a rod in order to minimize the vibration energy in the structure. For this optimal design problem in wave propagation, the authors of [5] prove existence results and provide convexification and optimality conditions. The authors of [1] also propose a convexification formulation of eigenfrequency optimization problems applied to optimal design. In [17] the authors discuss several possible criteria for optimizing the damping of abstract wave equations in Hilbert spaces, and derive optimality conditions for a certain criterion related to a Lyapunov equation. In [47] we investigated the problem presented previously in the one-dimensional case. We also quote the article [48] where we study the related problem of finding the optimal location of the support of the control for the one-dimensional wave equation.

In this article we provide a complete model and mathematical analysis of the optimal observability problem overviewed in Section 1.1. The article is structured as follows.

Section 2 is devoted to discuss and define a relevant mathematical criterion, modeling the optimal observability problem. We first introduce the context and recall the classical observability inequality, and then using spectral considerations we introduce randomized or time asymptotic observability inequalities, and we finally come up with a spectral criterion which is at the heart of our study.

The resulting optimal design problem is solved in Section 3, where we derive, under appropriate quantum ergodicity assumptions, a no-gap result between our problem and a convexified version. We put in evidence some deep relations between shape optimization and concentration properties of eigenfunctions.

The existence of an optimal set is investigated in Section 4. We study a spectral approximation of our problem, providing a maximizing sequence of optimal sets which does not converge in general. We then provide some remedies, in particular by defining a weighted spectral criterion and showing the existence and uniqueness of an optimal set.

Section 5 is devoted to generalize all results to the wave and Schrödinger equations, on any open bounded connected subset of a Riemannian manifold, with various possible boundary conditions.

Further comments are provided in Section 6, concerning the problem of optimal shape and location of internal controllers, as well as several open problems and issues.

2 Modeling the optimal observability problem

This section is devoted to discuss and model mathematically the problem of maximizing the observability of wave equations. A first natural model is to settle the problem of maximizing the observability constant, but it appears that this problem is both difficult to treat from a theoretical point of view, and actually not relevant with respect to practice. Using spectral considerations, we will then define a spectral criterion based on averaged versions of the observability inequalities, which is better suited to model what is expected in practice.

2.1 The framework

Let $n \geqslant 1$, T be a positive real number and Ω be an open bounded connected subset of $\mathbb{R}^n$. We consider the wave equation

$$\partial_{tt} y = \triangle y, \tag{11}$$

in $(0,T)\times\Omega$, with Dirichlet boundary conditions. Let ω be an arbitrary measurable subset of Ω of positive measure. Throughout the paper, the notation χ_ω stands for the characteristic function of ω. The equation (11) is said to be *observable* on ω in time T if there exists $C_T^{(W)}(\chi_\omega) > 0$ such that

$$C_T^{(W)}(\chi_\omega)\|(y^0,y^1)\|^2_{L^2(\Omega,\mathbb{C})\times H^{-1}(\Omega,\mathbb{C})} \leqslant \int_0^T \int_\omega |y(t,x)|^2\,dx\,dt, \tag{12}$$

for all $(y^0,y^1) \in L^2(\Omega,\mathbb{C})\times H^{-1}(\Omega,\mathbb{C})$. This is the so-called *observability inequality*, relevant in inverse problems or in control theory because of its dual equivalence with the property of controllability (see [41]). It is well known that within the class of $\mathcal{C}^\infty$ domains Ω, this observability property holds, roughly, if the pair (ω,T) satisfies the *Geometric Control Condition (GCC)* in Ω (see [3, 9]), according to which every geodesic ray in Ω and reflected on its boundary according to the laws of geometrical optics intersects the observation set ω within time T. In particular, if at least one ray does not reach $\overline{\omega}$ within time T then the observability inequality fails because of

the existence of gaussian beam solutions concentrated along the ray and, therefore, away from the observation set.

In the sequel, the *observability constant* $C_T^{(W)}(\chi_\omega)$ denotes the largest possible nonnegative constant for which the inequality (12) holds, that is,

$$C_T^{(W)}(\chi_\omega) = \inf \left\{ \frac{\int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt}{\|(y^0,y^1)\|^2_{L^2(\Omega,\mathbb{C})\times H^{-1}(\Omega,\mathbb{C})}} \;\Big|\; (y^0,y^1) \in L^2(\Omega,\mathbb{C}) \times H^{-1}(\Omega,\mathbb{C}) \setminus \{(0,0)\} \right\}, \quad (13)$$

We next discuss the question of modeling mathematically the notion of maximizing the observability of wave equations. It is a priori natural to consider the problem of maximizing the observability constant $C_T^{(W)}(\chi_\omega)$ over all possible subsets ω of Ω of Lebesgue measure $|\omega| = L|\Omega|$ for a given time $T > 0$. In the next two subsections, using spectral expansions, we discuss the difficulty and the relevance of this problem, leading us to consider a more adapted spectral criterion.

2.2 Spectral expansion of the solutions

From now on, we fix an orthonormal Hilbertian basis $(\phi_j)_{j\in\mathbb{N}^*}$ of $L^2(\Omega,\mathbb{C})$ consisting of eigenfunctions of the Dirichlet-Laplacian on Ω, associated with the positive eigenvalues $(\lambda_j^2)_{j\in\mathbb{N}^*}$.

Let $(y^0,y^1) \in L^2(\Omega,\mathbb{C})\times H^{-1}(\Omega,\mathbb{C})$ be some arbitrary initial data. The solution $y \in C^0(0,T;L^2(\Omega,\mathbb{C}))\cap C^1(0,T;H^{-1}(\Omega,\mathbb{C}))$ of (11) such that $y(0,\cdot) = y^0(\cdot)$ and $\partial_t y(0,\cdot) = y^1(\cdot)$ can be expanded as

$$y(t,x) = \sum_{j=1}^{+\infty} \left(a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t}\right) \phi_j(x), \quad (14)$$

where the sequences $(a_j)_{j\in\mathbb{N}^*}$ and $(b_j)_{j\in\mathbb{N}^*}$ belong to $\ell^2(\mathbb{C})$ and are determined in terms of the initial data (y^0,y^1) by

$$\begin{aligned} a_j &= \frac{1}{2}\left(\int_\Omega y^0(x)\phi_j(x)\,dx - \frac{i}{\lambda_j}\int_\Omega y^1(x)\phi_j(x)\,dx\right), \\ b_j &= \frac{1}{2}\left(\int_\Omega y^0(x)\phi_j(x)\,dx + \frac{i}{\lambda_j}\int_\Omega y^1(x)\phi_j(x)\,dx\right). \end{aligned} \quad (15)$$

for every $j \in \mathbb{N}^*$. Moreover,

$$\|(y^0,y^1)\|^2_{L^2(\Omega,\mathbb{C})\times H^{-1}(\Omega,\mathbb{C})} = 2\sum_{j=1}^{+\infty}(|a_j|^2 + |b_j|^2). \quad (16)$$

It follows from (14) that

$$\int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt \quad = \quad \sum_{j,k=1}^{+\infty} \alpha_{jk} \int_\omega \phi_i(x)\phi_j(x)\,dx, \quad (17)$$

where

$$\alpha_{jk} = \int_0^T (a_j e^{i\lambda_j t} - b_j e^{-i\lambda_j t})(\bar{a}_k e^{-i\lambda_k t} - \bar{b}_k e^{i\lambda_k t})\,dt. \quad (18)$$

The coefficients α_{jk}, $(j,k) \in (\mathbb{N}^*)^2$, depend only on the initial data (y^0,y^1), and their precise expression is given by

$$\begin{aligned} \alpha_{jk} = {} & \frac{2a_j\bar{a}_k}{\lambda_j-\lambda_k}\sin\left((\lambda_j-\lambda_k)\frac{T}{2}\right)e^{i(\lambda_j-\lambda_k)\frac{T}{2}} - \frac{2a_j\bar{b}_k}{\lambda_j+\lambda_k}\sin\left((\lambda_j+\lambda_k)\frac{T}{2}\right)e^{i(\lambda_j+\lambda_k)\frac{T}{2}} \\ & - \frac{2b_j\bar{a}_k}{\lambda_j+\lambda_k}\sin\left((\lambda_j+\lambda_k)\frac{T}{2}\right)e^{-i(\lambda_j+\lambda_k)\frac{T}{2}} + \frac{2b_j\bar{b}_k}{\lambda_j-\lambda_k}\sin\left((\lambda_j-\lambda_k)\frac{T}{2}\right)e^{-i(\lambda_j-\lambda_k)\frac{T}{2}} \end{aligned} \quad (19)$$

whenever $\lambda_j \neq \lambda_k$, and

$$\alpha_{jk} = T(a_j\bar{a}_k + b_j\bar{b}_k) - \frac{\sin(\lambda_j T)}{\lambda_j}(a_j\bar{b}_k e^{i\lambda_j T} + b_j\bar{a}_k e^{-i\lambda_j T}) \tag{20}$$

whenever $\lambda_j = \lambda_k$.

Remark 1. In dimension one, set $\Omega = (0,\pi)$. Then $\phi_j(x) = \sqrt{\frac{2}{\pi}}\sin(jx)$ and $\lambda_j = j$ for every $j \in \mathbb{N}^*$. In this one-dimensional case, it can be noticed that when the time T is a multiple of 2π all nondiagonal terms vanish. Indeed, if $T = 2p\pi$ with $p \in \mathbb{N}^*$, then $\alpha_{ij} = 0$ whenever $i \neq j$, and

$$\alpha_{jj} = p\pi(|a_j|^2 + |b_j|^2), \tag{21}$$

for all $(i,j) \in (\mathbb{N}^*)^2$, and therefore

$$\int_0^{2p\pi}\int_\omega |y(t,x)|^2\,dx\,dt = \sum_{j=1}^{+\infty}\alpha_{jj}\int_\omega \sin^2(jx)\,dx. \tag{22}$$

Hence in that case there are no crossed terms. The optimal observability problem for this one-dimensional case was studied in detail in [47].

Using the above spectral expansions, the observability constant is given by

$$C_T^{(W)}(\chi_\omega) = \frac{1}{2}\inf_{\substack{(a_j),(b_j)\in\ell^2(\mathbb{C})\\ \sum_{j=1}^{+\infty}(|a_j|^2+|b_j|^2)=1}}\int_0^T\int_\omega\left|\sum_{j=1}^{+\infty}\left(a_j e^{i\lambda_j t} - b_j e^{-i\lambda_j t}\right)\phi_j(x)\right|^2 dx\,dt, \tag{23}$$

and where the coefficients a_j and b_j in the expressions above are the Fourier coefficients of the initial data, defined by (15).

Due to the crossed terms appearing in (17), the problem of maximizing $C_T^{(W)}(\chi_\omega)$ over all possible subsets ω of Ω of measure $|\omega| = L|\Omega|$, is very difficult to handle, at least from a theoretical point of view. The difficulty related with the cross terms already appears in one-dimensional problems (see [47]). Actually, this question is very much related with classical problems in non harmonic Fourier analysis, such as the one of determining the best constants in Ingham's inequalities (see [28, 29]).

This problem is then let open, but as we will see next, although it is very interesting, it is not so relevant from the practical point of view.

2.3 Randomized observability inequality

As mentioned above, the problem of maximizing the *deterministic* (classical) observability constant $C_T^{(W)}(\chi_\omega)$ defined by (13) over all possible measurable subsets ω of Ω of measure $|\omega| = L|\Omega|$, is open and is probably very difficult. However, when considering the practical problem of locating sensors in an optimal way, the optimality should rather be thought in terms of an *average* with respect to a large number of experiments. From this point of view, the observability constant $C_T^{(W)}(\chi_\omega)$, which is by definition deterministic, is expected to be pessimistic in the sense that they give an account for the worst possible case. In practice, when carrying out a large number of experiments, it can however be expected that the worst possible case does not occur very often. Having this remark in mind, we next define a new notion of observability inequality by considering an average over random initial data.

The observability constant defined by (13) is defined as an infimum over *all possible* (deterministic) initial data. We are going to modify slightly this definition by randomizing the initial data in some precise sense, and considering an averaged version of the observability inequality with a new (randomized) observability constant.

Consider the expression of $C_T^{(W)}(\chi_\omega)$ given by (23) in terms of spectral expansions. Following the works of N. Burq and N. Tzvetkov on nonlinear partial differential equations with random initial data (see [7, 10, 11]) using early ideas of Paley and Zygmund (see [45]), we randomize the coefficients a_j, b_j, c_j, accounting for the initial conditions, by multiplying each of them by some well chosen random law. This random selection of all possible initial data for the wave equation (74) consists of replacing $C_T^{(W)}(\chi_\omega)$ by the randomized version

$$C_{T,\mathrm{rand}}^{(W)}(\chi_\omega) = \frac{1}{2} \inf_{\substack{(a_j),(b_j)\in\ell^2(\mathbb{C}) \\ \sum_{j=1}^{+\infty}(|a_j|^2+|b_j|^2)=1}} \mathbb{E}\left(\int_0^T \int_\omega \left| \sum_{j=1}^{+\infty} \left(\beta_{1,j}^\nu a_j e^{i\lambda_j t} - \beta_{2,j}^\nu b_j e^{-i\lambda_j t}\right) \phi_j(x) \right|^2 dx\, dt \right), \tag{24}$$

where $(\beta_{1,j}^\nu)_{j\in\mathbb{N}^*}$ and $(\beta_{2,j}^\nu)_{j\in\mathbb{N}^*}$ are two sequences of independent Bernoulli random variables on a probability space $(\mathcal{X}, \mathcal{A}, \mathbb{P})$, satisfying

$$\mathbb{P}(\beta_{1,j}^\nu = \pm 1) = \mathbb{P}(\beta_{2,j}^\nu = \pm 1) = \frac{1}{2} \qquad \text{and} \qquad \mathbb{E}(\beta_{1,j}^\nu \beta_{2,k}^\nu) = 0,$$

for every j and k in $\mathbb{N}^*$ and every $\nu \in \mathcal{X}$. Here, the notation $\mathbb{E}$ stands for the expectation over the space $\mathcal{X}$ with respect to the probability measure $\mathbb{P}$. In other words, instead of considering the deterministic observability inequality (12) for the wave equation (74), we consider the *randomized observability inequality*

$$C_{T,\mathrm{rand}}^{(W)}(\chi_\omega) \|(y^0, y^1)\|^2_{L^2(\Omega,\mathbb{C})\times H^{-1}(\Omega,\mathbb{C})} \leqslant \mathbb{E}\left(\int_0^T \int_\omega |y_\nu(t,x)|^2 \, dx\, dt \right), \tag{25}$$

for all $y^0(\cdot) \in L^2(\Omega,\mathbb{C})$ and $y^1(\cdot) \in H^{-1}(\Omega,\mathbb{C})$, where y_ν denotes the solution of the wave equation with the random initial data $y_\nu^0(\cdot)$ and $y_\nu^1(\cdot)$ determined by their Fourier coefficients $a_j^\nu = \beta_{1,j}^\nu a_j$ and $b_j^\nu = \beta_{2,j}^\nu b_j$ (see (15) for the explicit relation between the Fourier coefficients and the initial data), that is,

$$y_\nu(t,x) = \sum_{j=1}^{+\infty} \left(\beta_{1,j}^\nu a_j e^{i\lambda_j t} + \beta_{2,j}^\nu b_j e^{-i\lambda_j t}\right) \phi_j(x). \tag{26}$$

This new constant $C_{T,\mathrm{rand}}^{(W)}(\chi_\omega)$ is called *randomized observability constant*.

Theorem 4. *There holds*

$$2\, C_{T,rand}^{(W)}(\chi_\omega) = T \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx$$

for every measurable subset ω of Ω.

Proof. The proof is immediate by expanding the square in (24), using Fubini's theorem and the fact that the random laws are independent, of zero mean and of variance 1. □

Remark 2. It can be easily checked that Theorem 4 still holds true when considering, in the above randomization procedure, more general real random variables that are independent, have mean equal to 0, variance 1, and have a super exponential decay. We refer to [7, 10] for more details on these randomization issues. Bernoulli and Gaussian random variables satisfy such appropriate

assumptions. As proved in [11], for all initial data $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$, the Bernoulli randomization keeps constant the $L^2 \times H^{-1}$ norm, whereas the Gaussian randomization generates a dense subset of $L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$ through the mapping $R_{(y^0,y^1)} : \nu \in \mathcal{X} \mapsto (y^0_\nu, y^1_\nu)$ provided that all Fourier coefficients of (y^0, y^1) are nonzero and that the measure θ charges all open sets of $\mathbb{R}$. The measure $\mu_{(y^0,y^1)}$ defined as the image of $\mathcal{P}$ by $R_{(y^0,y^1)}$ strongly depends both on the choice of the random variables and on the choice of the initial data (y^0, y^1). Properties of these measures are established in [11].

Remark 3. It is easy to see that $C^{(W)}_{T,\mathrm{rand}}(\chi_\omega) \geqslant C^{(W)}_T(\chi_\omega)$, for every measurable subset ω of Ω, and every $T > 0$.

Remark 4. As mentioned previously, the problem of maximizing the *deterministic* (classical) observability constant $C^{(W)}_T(\chi_\omega)$ defined by (13) over all possible measurable subsets ω of Ω of measure $|\omega| = L|\Omega|$, is open and is probably very difficult. For practical issues it is actually more natural to consider the problem of maximizing the *randomized* observability constant defined by (24). Indeed, when considering for instance the practical problem of locating sensors in an optimal way, the optimality should be thought in terms of an average with respect to a large number of experiments. From this point of view, the deterministic observability constant is expected to be pessimistic with respect to their randomized version. Indeed, in general it is expected that $C^{(W)}_{T,\mathrm{rand}}(\chi_\omega) > C^{(W)}_T(\chi_\omega)$.

In dimension one, with $\Omega = (0, \pi)$ and Dirichlet boundary conditions, it follows from [47, Proposition 2] (where this one-dimensional case is studied in detail) that these strict inequalities hold if and only if T is not an integer multiple of π (note that if T is a multiple of 2π then the equalities follow immediately from Parseval's Theorem). Note that, in the one-dimensional case, the GCC is satisfied for every $T \geqslant 2\pi$, and the fact that the deterministic and the randomized observability constants do not coincide is due to crossed Fourier modes in the deterministic case.

In dimension greater than one, there is a class of examples where the strict inequality holds: this is indeed the case when one is able to assert that $C^{(W)}_T(\chi_\omega) = 0$ whereas $C^{(W)}_{T,\mathrm{rand}}(\chi_\omega) > 0$. Let us provide several examples.

An example of such a situation for the wave equation is provided by considering $\Omega = (0, \pi)^2$ with Dirichlet boundary conditions and $L = 1/2$. It is indeed proved further (see Lemma 4 and Remark 20) that the domain $\omega = \{(x, y) \in \Omega \mid x < \pi/2\}$ maximizes J over $\mathcal{U}_L$, and that $J(\chi_\omega) = 1/2$. Clearly, such a domain does not satisfy the Geometric Control Condition, and one has $C^{(W)}_T(\chi_\omega) = 0$, whereas $C^{(W)}_\infty(\chi_\omega) = 1/4$.

Another class of examples for the wave equation is provided by the well known Bunimovich stadium with Dirichlet boundary conditions. Setting $\Omega = R \cup W$, where R is the rectangular part and W the circular wings, it is proved in [12] that, for any open neighborhood ω of the closure of W (or even, any neighborhood ω of the vertical intervals between R and W) in Ω, there exists $c > 0$ such that $\int_\omega \phi_j(x)^2\, dx \geqslant c$ for every $j \in \mathbb{N}^*$. It follows that $J(\chi_\omega) > 0$, whereas $C^{(W)}_T(\chi_\omega) = 0$ since ω does not satisfy the Geometric Control Condition. It can be noted that the result still holds if one replaces the wings W by any other manifold glued along R, so that Ω is a partially rectangular domain.

2.4 Conclusion: a relevant criterion

In the previous section we have shown that it is more relevant in practice to model the problem of maximizing the observability as the problem of maximizing the randomized observability constant.

Using Theorem 4, this leads us to consider the following spectral problem.

Let $L \in (0,1)$ be fixed. We consider the problem of maximizing the spectral functional

$$J(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx, \tag{27}$$

over all possible measurable subsets ω of Ω of measure $|\omega| = L|\Omega|$.

Note that this spectral criterion is independent of T and is of diagonal nature, not involving any crossed term. However it depends on the choice of the specific Hilbertian basis $(\phi_j)_{j \in \mathbb{N}^*}$ of eigenfunctions of A, at least, whenever the spectrum of A is not simple. We will come back on this issue in Section 3.7 by considering an intrinsic spectral criterion, where the infimum runs over all possible normalized eigenfunctions of A.

The study of the maximization of J will be done in Section 3, and will lead to an unexpectedly rich field of investigations, related to quantum ergodicity properties of Ω.

Before going on with that study, let us provide another way of coming out with this spectral functional (27). In the previous section we have seen that $TJ(\chi_\omega)$ can be interpreted as a randomized observability constant, corresponding to a randomized observability inequality. We will see next that $J(\chi_\omega)$ can be obtained as well by performing a time averaging procedure on the classical observability inequality.

2.5 Time asymptotic observability inequality

First of all, we claim that, for all $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$, the quantity

$$\frac{1}{T} \int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt,$$

where $y \in C^0(0,T; L^2(\Omega, \mathbb{C})) \cap C^1(0,T; H^{-1}(\Omega, \mathbb{C}))$ is the solution of the wave equation (74) such that $y(0,\cdot) = y^0(\cdot)$ and $\partial_t y(0,\cdot) = y^1(\cdot)$, has a limit as T tends to $+\infty$ (this fact is proved in lemmas 6 and 7 further). This leads to define the concept of *time asymptotic observability constant*

$$C_\infty^{(W)}(\chi_\omega) = \inf \left\{ \lim_{T \to +\infty} \frac{1}{T} \frac{\int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt}{\|(y^0, y^1)\|^2_{L^2 \times H^{-1}}} \;\middle|\; (y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C}) \setminus \{(0,0)\} \right\}. \tag{28}$$

This constant appears as the largest possible nonnegative constant for which the time asymptotic observability inequality

$$C_\infty^{(W)}(\chi_\omega) \|(y^0, y^1)\|^2_{L^2(\Omega,\mathbb{C}) \times H^{-1}(\Omega,\mathbb{C})} \leqslant \lim_{T \to +\infty} \frac{1}{T} \int_0^T \int_\omega |y(t,x)^2| \, dx \, dt, \tag{29}$$

holds for all $y^0(\cdot) \in L^2(\Omega, \mathbb{C})$ and $y^1(\cdot) \in H^{-1}(\Omega, \mathbb{C})$.

We have the following results.

Theorem 5. *For every measurable subset ω of Ω, there holds*

$$2\, C_\infty^{(W)}(\chi_\omega) = \inf \left\{ \frac{\int_\omega \sum_{\lambda \in U} \left| \sum_{k \in I(\lambda)} c_k \phi_k(x) \right|^2 dx}{\sum_{k=1}^{+\infty} |c_k|^2} \;\middle|\; (c_j)_{j \in \mathbb{N}^*} \in \ell^2(\mathbb{C}) \setminus \{0\} \right\},$$

where U is the set of all distinct eigenvalues λ_k and $I(\lambda) = \{j \in \mathbb{N}^* \mid \lambda_j = \lambda\}$.

Corollary 1. *There holds* $2C_\infty^{(W)}(\chi_\omega) \leqslant J(\chi_\omega)$, *for every measurable subset* ω *of* Ω. *If the domain* Ω *is such that every eigenvalue of the Dirichlet-Laplacian is simple, then*

$$2\,C_\infty^{(W)}(\chi_\omega) = \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\,dx = J(\chi_\omega),$$

for every measurable subset ω *of* Ω.

The proof of these results is done in Appendix A. Note that, as is well known, the assumption of the simplicity of the spectrum of the Dirichlet-Laplacian is generic with respect to the domain Ω (see e.g. [43, 58, 26]).

Remark 5. It follows obviously from the definitions of the observability constants that

$$\limsup_{T\to+\infty} \frac{C_T^{(W)}(\chi_\omega)}{T} \leqslant C_\infty^{(W)}(\chi_\omega)$$

for every measurable subset ω of Ω. However, the equalities do not hold in general. Indeed, consider a set Ω with a smooth boundary, and a pair (ω, T) not satisfying the Geometric Control Condition. Then there must hold $C_T^{(W)}(\chi_\omega) = 0$. Besides, $J(\chi_\omega)$ may be positive, as already discussed in Remark 4 where we gave several classes of examples having this property.

3 Solving of the optimal observability problem under quantum ergodicity assumptions

We define the set

$$\mathcal{U}_L = \{\chi_\omega \mid \omega \text{ is a measurable subset of } \Omega \text{ of measure } |\omega| = L|(\Omega|\}. \tag{30}$$

In Section 2, our discussions have led us to model the problem of optimal observability as the problem

$$\sup_{\chi_\omega\in\mathcal{U}_L} J(\chi_\omega), \tag{31}$$

with

$$J(\chi_\omega) = \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\,dx,$$

where $(\phi_j)_{j\in\mathbb{N}^*}$ is a Hilbertian basis of the Hilbert space $L^2(\Omega,\mathbb{C})$ (defined in Section 2.1), consisting of eigenfunctions of $\triangle$.

The criterion $J(\chi_\omega)$ can be seen as a spectral energy (de)concentration criterion. For every $j \in \mathbb{N}^*$, the integral $\int_\omega \phi_j(x)^2\,dx$ is the energy of the j^{th} eigenfunction restricted to ω, and the problem is to maximize the infimum over j of these energies, over all subsets ω of measure $|\omega| = L|\Omega|$.

This section is organized as follows. Section 3.1 is devoted to some preliminary remarks and in particular to the introduction of a convexified version of the problem (31). Our main results are stated in Section 3.2. They provide the optimal value of (31) under quantum ergodicity assumptions on Ω, by proving moreover that there is no gap between the problem (31) and its convexified version. These assumptions are commented in Section 3.3. Sections 3.4, 3.5 and 3.6 are devoted to prove our main results. Finally, in Section 3.7 we consider an intrinsic spectral variant of (31) where, as announced in Section 2.4, the infimum runs over all possible normalized eigenfunctions of $\triangle$. Moreover for this intrinsic problem we provide an explicit example where there is a gap with the convexified version.

3.1 Preliminary remarks

Since the set $\mathcal{U}_L$ does not have compactness properties ensuring the existence of a solution of (31), we consider the convex closure of $\mathcal{U}_L$ for the weak star topology of L^∞,

$$\overline{\mathcal{U}}_L = \left\{ a \in L^\infty(\Omega, [0,1]) \;\middle|\; \int_\Omega a(x)\,dx = L|\Omega| \right\}. \tag{32}$$

This convexification procedure is standard in shape optimization problems where an optimum may fail to exist because of hard constraints (see e.g. [6]). Replacing $\chi_\omega \in \mathcal{U}_L$ with $a \in \overline{\mathcal{U}}_L$, we define a convexified formulation of the second problem (31) by

$$\sup_{a \in \overline{\mathcal{U}}_L} J(a), \tag{33}$$

where

$$J(a) = \inf_{j \in \mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2\,dx. \tag{34}$$

Since $J(a)$ is defined as the infimum of linear continuous functionals for the weak star topology of L^∞, it is upper semi continuous for this topology. This yields to the following result.

Lemma 1. *The problem (33) has at least one solution.*

Obviously, there holds

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_\Omega \chi_\omega(x)\phi_j(x)^2\,dx \;\leqslant\; \sup_{a \in \overline{\mathcal{U}}_L} \inf_{j \in \mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2\,dx. \tag{35}$$

Note that, since the constant function $a(\cdot) = L$ belongs to $\overline{\mathcal{U}}_L$, it follows that $\sup_{a \in \overline{\mathcal{U}}_L} J(a) \geqslant L$. In the next section, under an additional ergodicity assumption, we compute the optimal value (33) of this convexified problem and investigate the question of knowing whether the above inequality is strict or not. In other words we investigate whether there is a gap or not between the problem (31) and its convexified version (33).

Remark 6. *Comments on the choice of the topology.*
In our study we consider measurable subsets ω of Ω, and we endow the set $L^\infty(\Omega, \{0,1\})$ of all characteristic functions of measurable subsets with the weak-star topology. Other topologies are used in shape optimization problems, such as the Hausdorff topology. Note however that, although the Hausdorff topology shares nice compactness properties, it cannot be used in our study because of the measure constraint on ω. Indeed, the Hausdorff convergence does not preserve measure, and the class of admissible domains is not closed for this topology. Topologies associated with convergence in the sense of characteristic functions or in the sense of compact sets (see for instance [25, Chapter 2]) do not guarantee easily the compactness of minimizing sequences of domains, unless one restricts the class of admissible domains, imposing for example some kind of uniform regularity.

Remark 7. We stress that the question of the possible existence of a gap between the original problem and its convexified version is not obvious and cannot be handled with usual Γ-convergence tools, in particular because the function J defined by (34) is is not lower semi-continuous for the weak star topology of L^∞ (it is however upper semi-continuous for that topology, as an infimum of linear functions). To illustrate this fact, consider the one-dimensional case of Remark 1. In this specific situation, since $\phi_j(x) = \sqrt{\frac{2}{\pi}}\sin(jx)$ for every $j \in \mathbb{N}^*$, one has

$$J(a) = \frac{2}{\pi} \inf_{j \in \mathbb{N}^*} \int_0^\pi a(x)\sin^2(jx)\,dx,$$

for every $a \in \overline{\mathcal{U}}_L$. Since the functions $x \mapsto \sin^2(jx)$ converge weakly to $1/2$, it clearly follows that $J(a) \leqslant L$ for every $a \in \overline{\mathcal{U}}_L$. Therefore,

$$\sup_{a \in \overline{\mathcal{U}}_L} J(a) = L,$$

and the supremum is reached with the constant function $a(\cdot) = L$. Consider the sequence of subsets ω_N of $(0, \pi)$ of measure $L\pi$ defined by

$$\omega_N = \bigcup_{k=1}^{N} \left(\frac{k\pi}{N+1} - \frac{L\pi}{2N}, \frac{k\pi}{N+1} + \frac{L\pi}{2N} \right),$$

for every $N \in \mathbb{N}^*$. Clearly, the sequence of functions χ_{ω_N} converges to the constant function $a(\cdot) = L$ for the weak star topology of L^∞, but nevertheless, an easy computation shows that

$$\int_{\omega_N} \sin^2(jx)\, dx = \begin{cases} \frac{L\pi}{2} - \frac{N}{2j} \sin\left(\frac{jL\pi}{N}\right) & \text{if } \ (N+1) \mid j, \\ \frac{L\pi}{2} + \frac{1}{2j} \sin\left(\frac{jL\pi}{N}\right) & \text{otherwise,} \end{cases}$$

and hence,

$$\limsup_{N \to +\infty} \frac{2}{\pi} \inf_{j \in \mathbb{N}^*} \int_{\omega_N} \sin^2(jx)\, dx < L.$$

This simple example illustrates the difficulty in understanding the limiting behavior of the functional because of the lack of the lower semicontinuity, what makes possible the occurrence of a gap in the convexification procedure. In Section 3.2, we will prove that there is no such a gap under an additional geometric spectral assumption.

3.2 Optimal value of the problem

In what follows, we make the following assumptions on the basis $(\phi_j)_{j \in \mathbb{N}^*}$ of eigenfunctions under consideration.

> ***Weak Quantum Ergodicity on the base (WQE) property.*** *There exists a subsequence of the sequence of probability measures $\mu_j = \phi_j^2\, dx$ converging vaguely to the uniform measure $\frac{1}{|\Omega|}\, dx$.*

> ***Uniform L^∞-boundedness property.*** *There exists $A > 0$ such that $\|\phi_j\|_{L^\infty(\Omega)} \leqslant A$, for every $j \in \mathbb{N}^*$.*

Note that the two assumptions above imply what we call the *L^∞-Weak Quantum Ergodicity on the base (L^∞-WQE) property*[1], that is, there exists a subsequence of $(\phi_j^2)_{j \in \mathbb{N}^*}$ converging to $\frac{1}{|\Omega|}$ for the weak star topology of $L^\infty(\Omega)$.

Obviously, this property implies that

$$\sup_{a \in \overline{\mathcal{U}}_L} \inf_{j \in \mathbb{N}^*} \int_\Omega a(x) \phi_j(x)^2\, dx = L, \tag{36}$$

and moreover the supremum is reached with the constant function $a = L$ on Ω.

[1] The wording used here is motivated and explained further in a series of remarks.

Remark 8. In general the convexified problem (33) does not admit a unique solution. Indeed, under symmetry assumptions on Ω there exists an infinite number of solutions. For example, in dimension one, with $\Omega = (0, \pi)$, all solutions of (33) are given by all functions of $\overline{\mathcal{U}}_L$ whose Fourier expansion series is of the form $a(x) = L + \sum_{j=1}^{+\infty}(a_j \cos(2jx) + b_j \sin(2jx))$ with coefficients $a_j \leqslant 0$.

It follows from (35) and (36) that

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx \ \leqslant \ L.$$

The next result states that this inequality is actually an equality.

Theorem 6. *If the WQE and uniform L^∞-boundedness properties hold, then*

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx = L, \tag{37}$$

for every $L \in (0, 1)$. In other words, under these assumptions there is no gap between the original problem (31) and the convexified one.

Remark 9. In fact, we can weaken the uniform L^∞-boundedness property into the following property: for almost every $x \in \Omega$, there exists $M_x > 0$ such that $\sup_{j \in \mathbb{N}^*} \phi_j(x)^2 \leqslant M_x$.

It follows from this result, from Corollary 1 and Theorem 4, that the maximal value of the randomized observability constant $C^{(W)}_{T,\mathrm{rand}}(\chi_\omega)$ over the set $\mathcal{U}_L$ is equal to $TL/2$, and that, if the spectrum of $\triangle$ is simple, the maximal value of the time asymptotic observability constant $C^{(W)}_{\infty}(\chi_\omega)$ over the set $\mathcal{U}_L$ is equal to $L/2$.

The question of knowing whether the supremum in (37) is reached (existence of an optimal set) is investigated in Section 4.1.

Theorem 6 is established within the class of measurable subsets. We next state a similar (but distinct) result within the class of measurable subsets whose boundary is of measure zero. We define the set

$$\mathcal{U}_L^b = \{\chi_\omega \in \mathcal{U}_L \mid |\partial\omega| = 0\}. \tag{38}$$

This is the set of all characteristic functions of Jordan measurable subsets of Ω of measure $L|\Omega|$. We make the following assumptions.

> ***Quantum Unique Ergodicity on the base (QUE) property.*** *The whole sequence of probability measures $\mu_j = \phi_j^2 \, dx$ converges vaguely to the uniform measure $\frac{1}{|\Omega|} \, dx$.*
>
> ***Uniform L^p-boundedness property.*** *There exist $p \in (1, +\infty]$ and $A > 0$ such that $\|\phi_j\|_{L^{2p}(\Omega)} \leqslant A$, for every $j \in \mathbb{N}^*$.*

Theorem 7. *Assume that $\partial\Omega$ is Lipschitz whenever it is nonempty. If the QUE and uniform L^p-boundedness properties hold, then*

$$\sup_{\chi_\omega \in \mathcal{U}_L^b} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx = L, \tag{39}$$

for every $L \in (0, 1)$.

Theorems 6 and 7 are proved in Sections 3.4 and 3.5 respectively.

Remark 10. It follows from the proof of Theorem 7 that this statement holds true as well whenever the set $\mathcal{U}_L^b$ is replaced with the set of all measurable subsets ω of Ω, of measure $|\omega| = L|\Omega|$, that are moreover either open with a Lipschitz boundary, or open with a bounded perimeter.

Remark 11. The assumptions made in Theorems 6 or 7 are sufficient conditions implying (37) or (39), but they are however not sharp, as proved in the next proposition.

Proposition 1. *Assume that Ω is the unit disk of the Euclidean two-dimensional space. Then, for every $p \in (1, +\infty]$ and for any basis of eigenfunctions of $\triangle$, the uniform L^p-boundedness property is not satisfied, and QUE does not hold as well. However, the equalities (37) and (39) hold true.*

To establish this result, in the proof of this proposition (done in Section 3.6) we use the explicit expression of certain semi-classical measures in the disk (weak limits of the probability measures $\phi_j^2\, dx$). Among these quantum limits, one can find the Dirac measure along the boundary which causes the well known phenomenon of whispering galleries. Having in mind this phenomenon, it could be expected that there exists an optimal set, concentrating around the boundary. The calculations show that it is however not the case, and (37) and (39) are proved to hold.

The next section is devoted to gather some comments on the quantum ergodicity assumptions made in these theorems.

3.3 Comments on quantum ergodicity assumptions

This section is organized as a series of remarks.

Remark 12. The assumptions of Theorems 6 and 7 hold true in dimension one. Indeed, it has already been mentioned that the eigenfunctions of the Dirichlet-Laplacian operator on $\Omega = (0, \pi)$ are given by $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$, for every $j \in \mathbb{N}^*$. Therefore clearly the whole sequence (not only a subsequence) $(\phi_j^2)_{j\in\mathbb{N}^*}$ converges weakly to $\frac{1}{\pi}$ for the weak star topology of $L^\infty(0, \pi)$. The same property clearly holds for all other boundary conditions considered in this article.

Remark 13. In dimension greater than one the situation is more intricate, but we have the following facts.

In any hypercube (tensorised version of the previous one-dimensional case) or flat torus, any orthonormal basis of eigenfunctions is uniformly bounded and satisfies WQE on the base.

Generally speaking, these assumptions are related to ergodicity properties of Ω. Before providing precise results, we recall the following well known definition.

> ***Quantum Ergodicity on the base (QE) property.*** *There exists a subsequence of the sequence of probability measures $\mu_j = \phi_j^2\, dx$ of density one converging vaguely to the uniform measure $\frac{1}{|\Omega|} dx$.*

Here, *density one* means that there exists $\mathcal{I} \subset \mathbb{N}^*$ such that $\#\{j \in \mathcal{I} \mid j \leqslant N\}/N$ converges to 1 as N tends to $+\infty$. Obviously, QE implies WQE[2]. It is well known that, if the domain Ω (seen as a billiard where the geodesic flow moves at unit speed and bounces at the boundary according to the Geometric Optics laws) is ergodic, then the property QE is satisfied. This is the contents of Shnirelman Theorem, proved in [14, 19, 53, 63] in various contexts (manifolds with or without boundary, with a certain regularity). Actually the results proved in these references are stronger, for two reasons. Firstly, they are valid for any Hilbertian basis of eigenfunctions of $\triangle$, whereas

[2]Note that, up to our knowledge, the notion of WQE is new, whereas the notions of QE and QUE are classical in mathematical physics.

here we make this kind of assumption only for the specific basis $(\phi_j)_{j\in\mathbb{N}^*}$ that has been fixed at the beginning of the study. Secondly, they establish that a stronger microlocal version of the QE property holds for pseudodifferential operators, in the unit cotangent bundle $S^*\Omega$ of Ω, and not just only on the configuration space Ω. Here however we do not need (de)concentration results in the full phase space, but only in the configuration space. This is why, following [62], we use the wording "on the base".

Note that the vague convergence of the measures μ_j is weaker than the convergence of the functions ϕ_j^2 for the weak star topology of $L^\infty(\Omega)$. Since Ω is bounded, the property of vague convergence is equivalent to saying that, for a subsequence of density one, $\int_\omega \phi_j(x)^2\,dx$ converges to $|\omega|/|\Omega|$ for every measurable subset ω of Ω such that $|\partial\omega| = 0$ (Portmanteau theorem). In contrast, the property of convergence for the weak star topology of $L^\infty(\Omega)$ is equivalent to saying that, for a subsequence of density one, $\int_\omega \phi_j(x)^2\,dx$ converges to $|\omega|/|\Omega|$ for every measurable subset ω of Ω. Under the assumption that all eigenfunctions are uniformly bounded in $L^\infty(\Omega)$, both notions are equivalent. This is the case for instance in flat tori. But, for instance, if Ω is a ball or a sphere of any dimension, then the eigenfunctions of the Laplacian are not uniformly bounded. This is well known to be a delicate issue (see [62]). It is conjectured that flat tori are the sole compact manifolds without boundary where the whole family of eigenfunctions is uniformly bounded in L^∞.

Note that the notion of L^∞-QE property, meaning that the above QE property holds for the weak star topology of L^∞, is defined and mentioned in [62] as a delicate open problem. As said above we stress that, under the assumption that all eigenfunctions are uniformly bounded in $L^\infty(\Omega)$, QE and L^∞-QE are equivalent.

To the best of our knowledge, nothing seems to be known on the uniform L^p-boundedness property. This property holds for flat tori but does not hold for balls or spheres.

Remark 14. Shnirelman Theorem lets however open the possibility of having an exceptional subsequence of measures μ_j converging vaguely to some other measure. The QUE assumption consists of assuming that the whole sequence converges vaguely to the uniform measure. It is an important issue in quantum and mathematical physics. Note indeed that the quantity $\int_\omega \phi_j^2(x)\,dx$ is interpreted as the probability of finding the quantum state of energy λ_j^2 in ω. We stress again on the fact that, here, we consider a version of QUE in the configuration space only, not in the full phase space. Moreover, we consider the QUE property for the basis $(\phi_j)_{j\in\mathbb{N}^*}$ under consideration, but not necessarily for any such basis of eigenfunctions.

QUE obviously holds true in the one-dimensional case of Remark 1 (see also Remark 7) but it does however not hold true for multi-dimensional hypercubes.

More generally, only partial results do exist. The question of determining what are the possible weak limits of the μ_j's (semi-classical measures, or quantum limits) is widely open in general. It could happen that, even in the framwork of Shnirelman Theorem, a subsequence of density zero converge to an invariant measure like for instance a measure carried by closed geodesics (these are the so-called strong *scars*, see, e.g., [18]). Note however that, as already mentioned, here we are concerned with concentration results in the configuration space only.

The QUE property on the base, stating that the whole sequence of measures $\mu_j = \phi_j^2\,dx$ converges vaguely to the uniform measure, postulates that there is no such concentration phenomenon. Note that, although rational polygonal billiards are not ergodic in the phase space, while polygonal billiards are generically ergodic (see [32]), the property QE on the base holds in any rational polygon[3] (see [42]), and L^∞-QE on the base holds in any flat torus (see [51]). Apart from these recent results, and in spite of impressive recent results around QUE (see, e.g., the survey [52]), up to now no example of multi-dimensional domain is known where QUE on the base holds true.

[3] A rational polygon is a planar polygon whose interior is connected and simply connected and whose vertex angles are rational multiples of π.

Remark 15. The results of Theorems 6 and 7 are similar but distinct. The QUE assumption is a very strong one as said above. The proofs of these results, provided in Sections 3.4 and 3.5, are of a completely different nature. In particular, our proof of Theorem 6 is short but does not permit to get an insight on the possible theoretical construction of a maximizing sequence of subsets. In contrast, our proof of Theorem 7 is constructive and provides a theoretical way of building a maximizing sequence of subsets, by implementing a kind of homogenization procedure. Moreover, this proof highlights the following interesting feature:

> *It is possible to increase the values of J by considering subsets having an increasing number of connected components.*

Remark 16. The question of knowing whether there exists an example where there is a gap between the convexified problem (33) and the original one (31), is an open problem. We think that, if such an example exists, then the underlying geodesic flow ought to be completely integrable and have strong concentration properties. As already mentioned in our framework we have fixed a given basis $(\phi_j)_{j\in\mathbb{N}^*}$ of eigenvectors, and we consider only the weak limits of the measures $\phi_j^2\,dx$. With a fixed given basis, we are not aware of any example having concentration properties strong enough to derive a gap statement. We refer to Section 3.7 and in particular to Proposition 2 for an example of a gap for an intrinsic variant of the second problem where the infimum runs over all possible eigenfunctions (and not only over a basis).

Remark 17. Our results here show that shape optimization problems are intimately related with the ergodicity properties of Ω. Notice that, in the early article [13], the authors suggested such connections. They analyzed the exponential decay of solutions of damped wave equations. Their results reflected that the quantum effects of bouncing balls or whispering galleries play an important role in the success of failure of the exponential decay property. At the end of the article, the authors conjectured that such considerations could be useful in the placement and design of actuators or sensors. Our results of this section provide precise results showing these connections and new perspectives on those intuitions. In our view they are the main contribution of our article, in the sense that they have pointed out the close relations existing between shape optimization and ergodicity, and provide new open problems and directions to domain optimization analysis.

3.4 Proof of Theorem 6

Since we already have the inequality $\sup_{\chi_\omega\in\mathcal{U}_L}\inf_{j\in\mathbb{N}^*}\int_\omega\phi_j(x)^2\,dx\ \leqslant\ L$, it suffices to prove that, for every $\varepsilon>0$, there exists $\chi_\omega\in\mathcal{U}_L$ such that

$$\left|\int_\omega\phi_j(x)^2\,dx-L\right|\leqslant\varepsilon,$$

for every $j\in\mathbb{N}^*$. To prove this fact, we consider the function f defined by $f(x)=(\phi_j(x)^2)_{j\in\mathbb{N}^*}$, for every $x\in\Omega$. Using the fact that the eigenfunctions are uniformly bounded in $L^\infty(\Omega)$, it is clear that $f(x)\in\ell^\infty$, for every $x\in\Omega$. Then, clearly, $f\in L^1(\Omega,\ell^\infty)$ (using the Bochner integral), and $\int_\Omega f\,dx$ is the constant sequence of ℓ^∞ equal to 1. For every $\varepsilon>0$, there exists a partition of $\Omega=\cup_{k=1}^n\Omega_k$, with Ω_k measurable, such that $\int_\Omega\|(f(x)-f_n(x))\|_{\ell^\infty}dx\leqslant\varepsilon/(L+1)$, with $f_n=\sum_{k=1}^n\alpha_k\chi_{\Omega_k}$. For every $k\in\{1,\dots,n\}$, let ω_k be a measurable subset of Ω_k such that $|\omega_k|=L|\Omega_k|$. We set $\omega=\cup_{k=1}^n\omega_k$. Note that, by construction, one has $\chi_\omega\in\mathcal{U}_L$, and

$$\int_\Omega(\chi_\omega(x)-L)f_n(x)\,dx=\sum_{k=1}^n\alpha_k\int_\Omega(\chi_\omega(x)-L)\chi_{\Omega_k}(x)\,dx=\sum_{k=1}^n\alpha_k(|\omega_k|-L|\Omega_k|)=0.$$

Therefore, there holds

$$\begin{aligned}\left\|\int_\omega f(x)\,dx - L\int_\Omega f(x)\,dx\right\|_{\ell^\infty} &= \left\|\int_\Omega (\chi_\omega(x)-L)f(x)\,dx\right\|_{\ell^\infty}\\ &\leqslant \left\|\int_\Omega (\chi_\omega(x)-L)f_n(x)\,dx\right\|_{\ell^\infty}\\ &\quad + \left\|\int_\Omega (\chi_\omega(x)-L)(f(x)-f_n(x))\,dx\right\|_{\ell^\infty} \leqslant \varepsilon\end{aligned}$$

and the conclusion follows.

3.5 Proof of Theorem 7

In what follows, for every measurable subset ω of Ω, we set $I_j(\omega) = \int_\omega \phi_j(x)^2\,dx$, for every $j \in \mathbb{N}^*$. By definition, there holds $J(\omega) = \inf_{j\in\mathbb{N}^*} I_j(\omega)$. Note that it follows from QUE and from the Portmanteau theorem (see Remark 13) that, for every measurable subset ω of Ω such that $|\omega| = L|\Omega|$ and $|\partial\omega| = 0$, one has $I_j(\omega) \to L$ as $j \to +\infty$, and hence $J(\omega) \leqslant L$.

Let ω_0 be an open connected subset of Ω of measure $L|\Omega|$ having a Lipschitz boundary. In the sequel we assume that $J(\omega_0) < L$, otherwise there is nothing to prove. Using QUE, there exists an integer j_0 such that

$$I_j(\omega_0) \geqslant L - \frac{1}{4}(L - J(\omega_0)), \tag{40}$$

for every $j > j_0$. Our proof below consists of implementing a kind of homogenization procedure by constructing a sequence of open subsets ω_k (starting from ω_0) such that $|\omega_k| = L|\omega_k|$ and $\lim_{k\to+\infty} J(\omega_k) = L$. Denote by $\overline{\omega_0}$ the closure of ω_0, and by ω_0^c the complement of ω_0 in Ω. Since Ω and ω_0 have a Lipschitz boundary, it follows that ω_0 and $\Omega\backslash\omega_0$ satisfy a δ-cone property[4], for some $\delta > 0$ (see [25, Theorem 2.4.7]). Consider partitions of $\overline{\omega_0}$ and ω_0^c,

$$\overline{\omega_0} = \bigcup_{i=1}^{K} F_i \quad \text{and} \quad \omega_0^c = \bigcup_{i=1}^{\tilde{K}} \widetilde{F}_i, \tag{41}$$

to be chosen later. As a consequence of the δ-cone property, there exists $c_\delta > 0$ and a choice of partition $(F_i)_{1\leqslant i\leqslant K}$ (resp. $(\widetilde{F}_i)_{1\leqslant i\leqslant \tilde{K}}$) such that, for $|F_i|$ small enough,

$$\forall i \in \{1,\cdots,K\}\ \left(\text{resp. } \forall i \in \{1,\cdots,\tilde{K}\}\right),\ \frac{\eta_i}{\operatorname{diam} F_i} \geqslant c_\delta\ \left(\text{resp. } \frac{\widetilde{\eta}_i}{\operatorname{diam}\widetilde{F}_i} \geqslant c_\delta\right), \tag{42}$$

where η_i (resp., $\widetilde{\eta}_i$) is the inradius[5] of F_i (resp., $\widetilde{F}_i$), and $\operatorname{diam} F_i$ (resp., $\operatorname{diam}\widetilde{F}_i$) the diameter of F_i (resp., of $\widetilde{F}_i$).

It is then clear that, for every $i \in \{1,\dots,K\}$ (resp., for every $i \in \{1,\dots,\tilde{K}\}$), there exists $\xi_i \in F_i$ (resp., $\tilde{\xi}_i \in \widetilde{F}_i$) such that $B(\xi_i,\eta_i/2) \subset F_i \subset B(\xi_i,\eta_i/c_\delta)$ (resp., $B(\tilde{\xi}_i,\widetilde{\eta}_i/2) \subset \widetilde{F}_i \subset B(\tilde{\xi}_i,\widetilde{\eta}_i/c_\delta)$), where the notation $B(\xi,\eta)$ stands for the open ball centered at ξ with radius η. These features characterize a substantial family of sets (also called nicely shrinking sets), as is well known in

[4] We recall that an open subset Ω of $\mathbb{R}^n$ verifies a δ-cone property if, for every $x \in \partial\Omega$, there exists a normalized vector ξ_x such that $C(y,\xi_x,\delta) \subset \Omega$ for every $y \in \overline{\Omega} \cap B(x,\delta)$, where $C(y,\xi_x,\delta) = \{z \in \mathbb{R}^n \mid \langle z-y,\xi\rangle \geqslant \cos\delta\|z-y\|$ and $0 < \|z-y\| < \delta\}$. For manifolds, the definition is done accordingly in some charts, for $\delta > 0$ small enough.

[5] In other words, the largest radius of balls contained in F_i.

measure theory. By continuity, the points ξ_i and $\tilde{\xi}_i$ are Lebesgue points of the functions ϕ_j^2, for every $j \leqslant j_0$. This implies that, for every $j \leqslant j_0$, there holds

$$\int_{F_i} \phi_j(x)^2\,dx = |F_i|\phi_j(\xi_i)^2 + \mathrm{o}(|F_i|) \quad \text{as } \eta_i \to 0,$$

for every $i \in \{1,\dots,K\}$, and

$$\int_{\widetilde{F}_i} \phi_j(x)^2\,dx = |\widetilde{F}_i|\phi_j(\xi_i)^2 + \mathrm{o}(|\widetilde{F}_i|) \quad \text{as } \widetilde{\eta}_i \to 0,$$

for every $i \in \{1,\dots,\tilde{K}\}$. Setting $\eta = \max\left(\max_{1\leqslant i\leqslant K} \operatorname{diam} F_i,\ \max_{1\leqslant i\leqslant \tilde{K}} \operatorname{diam} \widetilde{F}_i\right)$, it follows that

$$\begin{aligned} I_j(\omega_0) &= \int_{\omega_0} \phi_j(x)^2\,dx = \sum_{i=1}^{K} |F_i|\phi_j(\xi_i)^2 + \mathrm{o}(\eta^d) \quad \text{as } \eta \to 0,\\ I_j(\omega_0^c) &= \int_{\omega_0^c} \phi_j(x)^2\,dx = \sum_{i=1}^{\tilde{K}} |\widetilde{F}_i|\phi_j(\tilde{\xi}_i)^2 + \mathrm{o}(\eta^d) \quad \text{as } \eta \to 0, \end{aligned} \tag{43}$$

for every $j \leqslant j_0$. Note that, since ω_0^c is the complement of ω_0 in Ω, there holds

$$I_j(\omega_0) + I_j(\omega_0^c) = \int_{\omega_0} \phi_j(x)^2\,dx + \int_{\omega_0^c} \phi_j(x)^2\,dx = 1, \tag{44}$$

for every j. Note also that $\sum_{i=1}^{K} V_g(F_i) = L|\Omega|$ and $\sum_{i=1}^{\tilde{K}} V_g(\widetilde{F}_i) = (1-L)|\Omega|$. Set $h_i = (1-L)V_g(F_i)$ and $\ell_i = LV_g(\widetilde{F}_i)$. Then, we infer from (43) and (44) that

$$\begin{aligned} (1-L)\,I_j(\omega_0) &= \sum_{i=1}^{K} h_i\phi_j(\xi_i)^2 + \mathrm{o}(\eta^d) \quad \text{as } \eta \to 0,\\ L\,I_j(\omega_0) &= L - \sum_{i=1}^{\tilde{K}} \ell_i\phi_j(\tilde{\xi}_i)^2 + \mathrm{o}(\eta^d) \quad \text{as } \eta \to 0, \end{aligned} \tag{45}$$

for every $j \leqslant j_0$. For $\varepsilon > 0$ to be chosen later, define the perturbation ω^ε of ω_0 by

$$\omega^\varepsilon = \left(\omega_0 \setminus \bigcup_{i=1}^{K} \overline{B(\xi_i,\varepsilon_i)}\right) \quad \bigcup \quad \bigcup_{i=1}^{\tilde{K}} B(\tilde{\xi}_i,\widetilde{\varepsilon}_i),$$

where $\varepsilon_i = \varepsilon h_i^{1/n}/V_g(B(\xi_i,1))^{1/n}$ and $\widetilde{\varepsilon}_i = \varepsilon \ell_i^{1/n}/V_g(B(\tilde{\xi}_i,1))^{1/n}$. Note that it is possible to define such a perturbation, provided that

$$0 < \varepsilon < \min\left(\min_{1\leqslant i\leqslant K} \frac{\eta_i V_g(B(\xi_i,1))^{1/n}}{h_i^{1/n}},\ \min_{1\leqslant i\leqslant \tilde{K}} \frac{\widetilde{\eta}_i V_g(B(\tilde{\xi}_i,1))^{1/n}}{\ell_i^{1/n}}\right).$$

It follows from the well known isodiametric inequality[6] and from a compactness argument that there exists a constant $V_n > 0$ (only depending on Ω) such that $|F_i| \leqslant V_n(\operatorname{diam} F_i)^n$ for every

[6]The isodiametric inequality states that, for every compact K of the Euclidean space $\mathbb{R}^n$, there holds $|K| \leqslant |B(0,\operatorname{diam}(K)/2)|$.

$i \in \{1, \cdots, K\}$, and $|\tilde{F}_i| \leqslant V_n(\operatorname{diam} \tilde{F}_i)^n$ for every $i \in \{1, \cdots, \tilde{K}\}$, independently on the partitions considered. Again, by compactness of Ω, there exists $v_n > 0$ (only depending on Ω) such that $|B(x,1)| \geqslant v_n$ for every $x \in \Omega$. Set $\varepsilon_0 = \min(1, c_\delta v_n / V_n^{1/n})$. Using (42), we get

$$\frac{\eta_i |B(\xi_i,1)|^{1/n}}{h_i^{1/n}} \geqslant \frac{v_n}{(1-L)^{1/n} V_n^{1/n}} \frac{\eta_i}{\operatorname{diam} F_i} \geqslant \varepsilon_0,$$

for every $i \in \{1, \cdots, K\}$, and similarly,

$$\frac{\widetilde{\eta}_i |B(\tilde{\xi}_i,1)|^{1/n}}{\ell_i^{1/n}} \geqslant \varepsilon_0,$$

for every $i \in \{1, \cdots, \tilde{K}\}$. It follows that the previous perturbation is well defined for every $\varepsilon \in (0, \varepsilon_0)$. Note that, by construction,

$$\begin{aligned}
|\omega^\varepsilon| &= |\omega_0| - \sum_{i=1}^K \varepsilon_i^n |B(\xi_i,1)| + \sum_{i=1}^{\tilde{K}} \widetilde{\varepsilon}_i^n |B(\tilde{\xi}_i,1)| \\
&= |\omega_0| - \varepsilon^n \sum_{i=1}^K h_i + \varepsilon^n \sum_{i=1}^{\tilde{K}} \ell_i \\
&= |\omega_0| - \varepsilon^n (1-L) \sum_{i=1}^K |F_i| + \varepsilon^n L \sum_{i=1}^{\tilde{K}} |\tilde{F}_i| \\
&= |\omega_0|) - \varepsilon^n (1-L) L |\Omega| + \varepsilon^n L (1-L) |\Omega| \\
&= |\omega_0| = L|\Omega|.
\end{aligned}$$

Moreover, one has

$$I_j(\omega^\varepsilon) = \int_{\omega^\varepsilon} \phi_j(x)^2 \, dx = I_j(\omega_0) - \sum_{i=1}^K \int_{B(\xi_i, \varepsilon_i)} \phi_j(x)^2 \, dx + \sum_{i=1}^{\tilde{K}} \int_{B(\tilde{\xi}_i, \widetilde{\varepsilon}_i)} \phi_j(x)^2 \, dx,$$

and using again the fact that the ξ_i and $\tilde{\xi}_i$ are Lebesgue points of the functions ϕ_j^2, for every $j \leqslant j_0$, we infer that

$$\begin{aligned}
I_j(\omega^\varepsilon) &= I_j(\omega_0) - \sum_{i=1}^K \varepsilon_i^n |B(\xi_i,1)| \phi_j(\xi_i)^2 + \sum_{i=1}^{\tilde{K}} \widetilde{\varepsilon}_i^n |B(\tilde{\xi}_i,1)| \phi_j(\tilde{\xi}_i)^2 + \mathrm{o}(\eta^d) \quad \text{as } \eta \to 0 \\
&= I_j(\omega_0) - \varepsilon^n \left(\sum_{i=1}^K h_i \phi_j(\xi_i)^2 - \sum_{i=1}^{\tilde{K}} \ell_i \phi_j(\tilde{\xi}_i)^2 \right) + \mathrm{o}(\eta^d) \quad \text{as } \eta \to 0,
\end{aligned}$$

and hence, using (45),

$$I_j(\omega^\varepsilon) = I_j(\omega_0) + \varepsilon^n \left(L - I_j(\omega_0) \right) + \varepsilon^n \mathrm{o}(\eta^d) \quad \text{as } \eta \to 0,$$

for every $j \leqslant j_0$ and every $\varepsilon \in (0, \varepsilon_0)$. Since $\varepsilon_0^n \leqslant 1$, it then follows that

$$I_j(\omega^\varepsilon) \geqslant J(\omega_0) + \varepsilon^n (L - J(\omega_0)) + \varepsilon^n \mathrm{o}(\eta^d) \quad \text{as } \eta \to 0, \tag{46}$$

for every $j \leqslant j_0$ and every $\varepsilon \in (0, \varepsilon_0)$, where the functional J is defined by (27).

We now choose the subdivisions (41) fine enough (that is, $\eta > 0$ small enough) so that, for every $j \leqslant j_0$, the remainder term $\underset{\eta\to 0}{\mathrm{o}}(\eta^d)$ in (46) is bounded by $\frac{1}{2}(L - J(\omega_0))$. It follows from (46) that

$$I_j(\omega^\varepsilon) \geqslant J(\omega_0) + \frac{\varepsilon^n}{2}(L - J(\omega_0)), \tag{47}$$

for every $j \leqslant j_0$ and every $\varepsilon \in (0, \varepsilon_0)$.

Let us first show that the set ω^ε still satisfies an inequality of the type (40) for ε small enough. Using the uniform L^p-boundedness property and Hölder's inequality, we have

$$|I_j(\omega^\varepsilon) - I_j(\omega_0)| = \left| \int_\Omega (\chi_{\omega^\varepsilon}(x) - \chi_{\omega_0}(x))\, \phi_j(x)^2\, dx \right| \leqslant A^2 \left(\int_\Omega |\chi_{\omega^\varepsilon}(x) - \chi_{\omega_0}(x)|^q\, dx \right)^{1/q},$$

for every integer j and every $\varepsilon \in (0, \varepsilon_0)$, where q is defined by $\frac{1}{p} + \frac{1}{q} = 1$. Moreover,

$$\int_\Omega |\chi_{\omega^\varepsilon}(x) - \chi_{\omega_0}(x)|^q\, dx = \int_\Omega |\chi_{\omega^\varepsilon}(x) - \chi_{\omega_0}(x)|\, dx = \varepsilon^n \left(\sum_{i=1}^{K} h_i + \sum_{i=1}^{\tilde{K}} \ell_i \right) = 2\varepsilon^n L(1-L)|\Omega|,$$

and hence $|I_j(\omega^\varepsilon) - I_j(\omega_0)| \leqslant \left(2A^{2q}\varepsilon^n L(1-L)|\Omega|\right)^{1/q}$. Therefore, setting

$$\varepsilon_1 = \min\left(\varepsilon_0, \left(\frac{(L - J(\omega_0))^q}{2^{2q+1} A^{2q} L(1-L)|\Omega|} \right)^{\frac{1}{n}} \right),$$

it follows from (40) that

$$I_j(\omega^\varepsilon) \geqslant L - \frac{1}{2}(L - J(\omega_0)) \tag{48}$$

for every $j \geqslant j_0$ and every $\varepsilon \in (0, \varepsilon_1)$.

Now, using the fact that $J(\omega_0) + \frac{\varepsilon^n}{2}(L - J(\omega_0)) \leqslant L - \frac{1}{2}(L - J(\omega_0))$ for every $\varepsilon \in (0, \varepsilon_0)$, we infer from (47) and (48) that

$$J(\omega^\varepsilon) \geqslant J(\omega_0) + \frac{\varepsilon^n}{2}(L - J(\omega_0)), \tag{49}$$

for every $\varepsilon \in (0, \varepsilon_1)$. In particular, this inequality holds for ε such that $\varepsilon^n = C_1 \min(C_2, L - J(\omega_0))$, where the positive constants $C_1 = 1/8AL(1-L)|\Omega|$ and $C_2 = 1/2^n C_1$. For this specific value of ε, we set $\omega_1 = \omega^\varepsilon$, and hence we have obtained

$$J(\omega_1) \geqslant J(\omega_0) + \frac{C_1}{2} \min(C_2, L - J(\omega_0))\, (L - J(\omega_0)). \tag{50}$$

Note that the constants involved in this inequality depend only on L, A and Ω. Note also that by construction ω_1 satisfies a δ-cone property.

If $J(\omega_1) \geqslant L$ then we are done. Otherwise, we apply all the previous arguments to this new set ω_1: using QUE, there exists an integer still denoted j_0 such that (40) holds with ω_0 replaced with ω_1. This provides a lower bound for highfrequencies. The lower frequencies $j \leqslant j_0$ are then handled as previously, and we end up with (47) with ω_0 replaced with ω_1. Finally, this leads to the existence of ω_2 such that (50) holds with ω_1 replaced with ω_2 and ω_0 replaced with ω_1.

By iteration, we construct a sequence of subsets ω_k of Ω (satisfying a δ-cone property) of measure $V_g(\omega_k) = L|\Omega|$, as long as $J(\omega_k) < L$, satisfying

$$J(\omega_{k+1}) \geqslant J(\omega_k) + \frac{C_1}{2} \min(C_2, L - J(\omega_k))\, (L - J(\omega_k)).$$

If $J(\omega_k) < L$ for every integer k, then clearly the sequence $J(\omega_k)$ is increasing, bounded above by L, and converges to L. This finishes the proof.

Remark 18. It can be noted that, in the above construction, the subsets ω_k are open, Lipschitz and of bounded perimeter. Hence, if the second problem is considered on the class of measurable subsets ω of Ω, of measure $|\omega| = L|\Omega|$, that are moreover either open with a Lipschitz boundary, or open with a bounded perimeter, then the conclusion holds as well that the supremum is equal to L. This proves the contents of Remark 10.

3.6 Proof of Proposition 1

Assume that Ω is the unit (Euclidean) disk of $\mathbb{R}^2$, $\Omega = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$. It is well known that the normalized eigenfunctions of the Dirichlet-Laplacian are a triply indexed sequence given by

$$\phi_{jkm}(r,\theta) = \begin{cases} R_{0k}(r)/\sqrt{2\pi} & \text{if } j = 0, \\ R_{jk}(r)Y_{jm}(\theta) & \text{if } j \geqslant 1, \end{cases}$$

for $j \in \mathbb{N}$, $k \in \mathbb{N}^*$ and $m = 1, 2$, where (r, θ) are the usual polar coordinates. The functions $Y_{jm}(\theta)$ are defined by $Y_{j1}(\theta) = \frac{1}{\sqrt{\pi}}\cos(j\theta)$ and $Y_{j2}(\theta) = \frac{1}{\sqrt{\pi}}\sin(j\theta)$, and the functions R_{jk} are defined by

$$R_{jk}(r) = \sqrt{2}\,\frac{J_j(z_{jk}r)}{|J_j'(z_{jk})|},$$

where J_j is the Bessel function of the first kind of order j, and $z_{jk} > 0$ is the k^{th}-zero of J_j. The eigenvalues of the Dirichlet-Laplacian are given by the double sequence of $-z_{jk}^2$ and are of multiplicity 1 if $j = 0$, and 2 if $j \geqslant 1$. Many properties are known on these functions and, in particular (see [35]):

- for every $j \in \mathbb{N}$, the sequence of probability measures $r \mapsto R_{jk}(r)^2 r dr$ converges vaguely to 1 as k tends to $+\infty$,
- for every $k \in \mathbb{N}^*$, the sequence of probability measures $r \mapsto R_{jk}(r)^2 r dr$ converges vaguely to the Dirac at $r = 1$ as j tends to $+\infty$.

These convergence properties permit to identify certain quantum limits, the second property accounting for the well known phenomenon of whispering galleries. Less known is the convergence of the above sequence of measures when the ratio j/k is kept constant. Simple computations (due to [8]) show that, when taking the limit of $R_{jk}(r)^2 r dr$ with a fixed ratio j/k, and making this ratio vary, we obtain the family of probability measures

$$\mu_s = f_s(r)\,dr = \frac{1}{\sqrt{1-s^2}}\,\frac{r}{\sqrt{r^2-s^2}}\chi_{(s,1)}(r)\,dr,$$

parametrized by $s \in [0, 1)$. We can even extend to $s = 1$ by defining μ_1 as the Dirac at $r = 1$. It easily follows that

$$\sup_{a \in \overline{\mathcal{U}}_L} J(a) = \sup_{a \in \overline{\mathcal{U}}_L} \inf_{\substack{j \in \mathbb{N}, k \in \mathbb{N}^* \\ m \in \{1,2\}}} \int_0^{2\pi}\int_0^1 a(r,\theta)\phi_{jkm}(r,\theta)^2\, r dr d\theta \leqslant \sup_{a \in \overline{\mathcal{U}}_L} K(a),$$

where

$$K(a) = \inf_{s \in [0,1]} \int_0^1 \int_0^{2\pi} a(r,\theta)\,d\theta\, f_s(r)\,dr.$$

Lemma 2. *There holds* $\sup_{a \in \overline{\mathcal{U}}_L} K(a) = L$, *and the supremum is reached with the constant function* $a = L$ *on* Ω.

Proof of Lemma 2. First, note that $K(a = L) = L$ and that the infimum in the definition of K is then reached for every $s \in [0, 1]$. Since K is concave (as infimum of linear functions), it suffices to prove that $\langle DK(a = L), h\rangle \leqslant 0$ (directional derivative), for every function h defined on Ω such that $\int_\Omega h(x)\,dx = 0$. Using Danskin's Theorem (see [16, 4]), we have

$$\langle DK(a = L), h\rangle = \inf_{s\in[0,1]} \int_0^1 \int_0^{2\pi} h(r, \theta)\,d\theta\, f_s(r)\,dr.$$

By contradiction, let us assume that there exists a function h on Ω such that $\int_\Omega h(x)\,dx = 0$ and such that $\int_0^1 \int_0^{2\pi} h(r, \theta)\,d\theta\, f_s(r)\,dr > 0$ for every $s \in [0, 1]$. Then, it follows that

$$\int_s^1 \int_0^{2\pi} h(r, \theta)\,d\theta\, \frac{r}{\sqrt{r^2 - s^2}}\,dr > 0$$

for every $s \in [0, 1]$, and integrating in s over $[0, 1]$, we get

$$\begin{aligned} 0 < \int_0^1 \int_s^1 \int_0^{2\pi} h(r, \theta)\,d\theta\, \frac{r}{\sqrt{r^2 - s^2}}\,dr ds &= \int_0^1 \int_0^r \frac{r}{\sqrt{r^2 - s^2}}\,ds \int_0^{2\pi} h(r, \theta)\,d\theta\,dr \\ &= \frac{\pi}{2} \int_0^1 r \int_0^{2\pi} h(r, \theta)\,d\theta\,dr \\ &= \frac{\pi}{2} \int_\Omega h(x)\,dx = 0, \end{aligned}$$

which is a contradiction. The lemma is proved. □

It follows from this lemma that $\sup_{a\in\overline{\mathcal{U}}_L} J(a) = L$ (note that $a = L$ realizes the maximum), and hence, $\sup_{\chi_\omega\in\mathcal{U}_L} J(\chi_\omega) \leqslant L$. To prove the no-gap statement, we use particular (radial) subsets ω, of the form $\omega = \{(r, \theta) \in [0, 1] \times [0, 2\pi] \mid \theta \in \omega_\theta\}$, where $|\omega_\theta| = 2L\pi$, as drawn on Figure 1. For

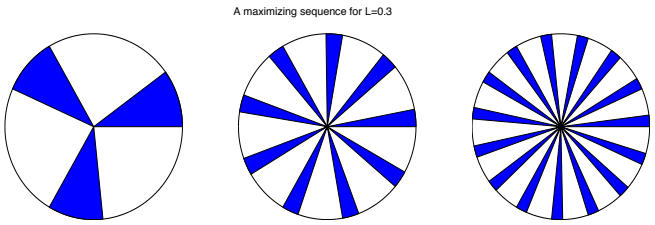

Figure 1: Particular radial subsets

such a subset ω, one has

$$\int_\omega \phi_{jkm}(x)^2\,dx = \int_0^1 R_{jk}(r)^2 r\,dr \int_{\omega_\theta} Y_{jm}(\theta)^2\,d\theta = \int_{\omega_\theta} Y_{jm}(\theta)^2\,d\theta,$$

for all $j \in \mathbb{N}^*$, $k \in \mathbb{N}^*$ and $m = 1, 2$. For $j = 0$, there holds

$$\int_\omega \phi_{0km}(x)^2\,dx = \int_0^1 R_{jk}(r)^2 r\,dr \int_{\omega_\theta} d\theta = |\omega_\theta|.$$

Besides, since $L\pi = |\Omega| = \int_0^1 r\,dr \int_{\omega_\theta} d\theta = \frac{1}{2}|\omega_\theta|$, it follows that $|\omega_\theta| = 2L\pi$. By applying the no-gap result in dimension one (clearly, it can be applied as well with the cosine functions), one has

$$\sup_{\substack{\omega_\theta \subset [0,2\pi] \\ |\omega_\theta| = 2L\pi}} \inf_{j\in\mathbb{N}^*} \int_{\omega_\theta} \sin^2(j\theta)\,d\theta = \sup_{\substack{\omega_\theta \subset [0,2\pi] \\ |\omega_\theta| = 2L\pi}} \inf_{j\in\mathbb{N}^*} \int_{\omega_\theta} \cos^2(j\theta)\,d\theta = L\pi.$$

Therefore, we deduce that

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{\substack{j\in\mathbb{N}, k\in\mathbb{N}^* \\ m\in\{1,2\}}} \int_\omega \phi_{jkm}(x)^2\,dx = L,$$

and the conclusion follows.

3.7 An intrinsic spectral variant of the problem

The problem (27), defined in Section 2.4, depends a priori on the Hilbertian basis $(\phi_j)_{j\in\mathbb{N}^*}$ of $L^2(\Omega,\mathbb{C})$ under consideration, at least whenever the spectrum of $\triangle$ is not simple. In this section we assume that the eigenvalues $(\lambda_j^2)_{j\in\mathbb{N}^*}$ of $\triangle$ are multiple, so that the choice of the basis $(\phi_j)_{j\in\mathbb{N}^*}$ enters into play.

We have already seen in Theorem 5 (see Section 2.3) that, in the case of multiple eigenvalues, the spectral expression for the time-asymptotic observability constant is more intricate and it does not seem that our analysis can be adapted in an easy way to that case.

Besides, recall that the criterion J defined by (27) has been motivated in Section 2.3 by means of randomizing initial data, and has been interpreted as a randomized observability constant (see Theorem 4), but then this criterion depends a priori on the preliminary choice of the basis $(\phi_j)_{j\in\mathbb{N}^*}$ of eigenfunctions.

In order to get rid of this dependence, and to deal with a more intrinsic criterion, it makes sense to consider the infimum of the criteria J defined by (27) over all possible choices of orthonormal bases of eigenfunctions. This leads us to consider the following intrinsic variant of our second problem.

> **Intrinsic uniform optimal design problem.** *We investigate the problem of maximizing the functional*
>
> $$J_{int}(\chi_\omega) = \inf_{\phi\in\mathcal{E}} \int_\omega \phi(x)^2\,dx, \tag{51}$$
>
> *over all possible subsets ω of Ω of measure $|\omega| = L|\Omega|$, where $\mathcal{E}$ denotes the set of all normalized eigenfunctions of A.*

Here, the word intrinsic means that this problem does not depend on the choice of the basis of eigenfunctions of $\triangle$.

As in Theorem 4, the quantity $\frac{T}{2}J_{\rm int}(\chi_\omega)$ (resp., $TJ_{\rm int}(\chi_\omega)$) can be interpreted as a constant for which the randomized observability inequality (25) for the wave equation holds, but this constant is less than or equal to $C^{(W)}_{T,\rm rand}(\chi_\omega)$. Besides, there obviously holds $C^{(W)}_T(\chi_\omega) \leqslant \frac{T}{2}J_{\rm int}(\chi_\omega)$. Indeed this inequality follows form the deterministic observability inequality applied to the particular solution $y(t,x) = \mathrm{e}^{i\lambda t}\phi(x)$, for every eigenfunction ϕ. In brief, there holds

$$C^{(W)}_T(\chi_\omega) \leqslant \frac{T}{2}J_{\rm int}(\chi_\omega) \leqslant C^{(W)}_{T,\rm rand}(\chi_\omega).$$

As in Section 3.1, the convexified version of the above problem consists of maximizing the functional

$$J_{\rm int}(a) = \inf_{\phi\in\mathcal{E}} \int_\Omega a(x)\phi(x)^2\,dx,$$

over the set $\overline{\mathcal{U}}_L$. This problem obviously has at least one solution, and

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{\phi \in \mathcal{E}} \int_\Omega \chi_\omega(x)\phi(x)^2\,dx \;\leqslant\; \sup_{a \in \overline{\mathcal{U}}_L} \inf_{\phi \in \mathcal{E}} \int_\Omega a(x)\phi(x)^2\,dx.$$

Theorem 8. *Assume that the uniform measure $\frac{1}{|\Omega|}\,dx$ is a closure point of the family of probability measures $\mu_\phi = \phi^2\,dx$, $\phi \in \mathcal{E}$, for the vague topology, and that the whole family of eigenfunctions in $\mathcal{E}$ is uniformly bounded in $L^\infty(\Omega)$. Then*

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{\phi \in \mathcal{E}} \int_\omega \phi(x)^2\,dx = \sup_{a \in \overline{\mathcal{U}}_L} \inf_{\phi \in \mathcal{E}} \int_\Omega a(x)\phi(x)^2\,dx = L, \tag{52}$$

for every $L \in (0,1)$. In other words, there is no gap between the intrinsic uniform optimal design problem and its convexified version.

Proof. The proof follows the same lines as in Section 3.4, by considering the function f defined by $f(x) = (\phi(x)^2)_{\phi \in \mathcal{E}}$. Then $f \in L^1(\Omega, X)$ with $X = L^\infty(\mathcal{E}, \mathbb{R})$ which is a Banach manifold that can be seen as an infinite product of spheres of dimension equal to the respective multiplicities of the eigenvalues. □

Similarly, the intrinsic counterpart of Theorem 7 is the following.

Theorem 9. *Assume that the uniform measure $\frac{1}{|\Omega|}\,dx$ is the unique closure point of the family of probability measures $\mu_\phi = \phi^2\,dx$, $\phi \in \mathcal{E}$, for the vague topology, and that the whole family of eigenfunctions in $\mathcal{E}$ is uniformly bounded in $L^{2p}(\Omega)$, for some $p \in (1, +\infty]$. Then*

$$\sup_{\chi_\omega \in \mathcal{U}_L^b} \inf_{\phi \in \mathcal{E}} \int_\omega \phi(x)^2\,dx = L, \tag{53}$$

for every $L \in (0,1)$.

Proof. The proof follows the same lines as in Section 3.4, replacing the integer index j with the continuous index λ (standing for the eigenvalues of A). The only thing that has to be noticed is the derivation of the estimate corresponding to (47). In Section 3.4, to obtain (47) from (46), we used the fact that only a finite number of terms have to be considered. Now the number of terms is infinite, but however one has to consider all possible normalized eigenfunctions associated with an eigenvalue $|\lambda| \leqslant |\lambda_0|$. Since this set is compact for every λ_0, there is no difficulty to extend our previous proof. □

With respect to Remark 16, it is interesting to note that, here, we are able to provide examples where there is a gap between the intrinsic second problem (51) and its convexified version.

Proposition 2. *In any of the two following examples:*

- *$\Omega = \mathcal{S}^2$, the unit Euclidean two-dimensional sphere, endowed with the usual flat metric;*
- *Ω is the unit half-sphere in $\mathbb{R}^3$, endowed with the usual flat metric, and Dirichlet conditions are imposed on the great circle which is the boundary of Ω;*

if L is close enough to 1 then $\sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega) < L$, and hence there is a gap between the problem (51) and its convexified version.

Proof. Assume first that $\Omega = \mathcal{S}^2$. In [30] it is proved that the set of semi-classical measures on $\mathcal{S}^2$ coincides with the convex set of invariant probability measures for the geodesic flow that are time-reversal invariant. In particular, the Dirac measure μ_γ of any great circle γ on $\mathcal{S}^2$ (defined as an equator, up to a rotation) is the projection of a semi-classical measure. The measure μ_γ is the arc-length measure defined by

$$\mu_\gamma(\omega) = \frac{1}{2\pi}\int_{\gamma\cap\omega} ds = \frac{1}{2\pi}|\gamma\cap\omega|,$$

for every measurable subset ω of $\mathcal{S}^2$. Besides, since the uniform measure is a quantum limit as well, $\mathcal{S}^2$ satisfies WQE and hence $\sup_{a\in\overline{\mathcal{U}}_L} J(a) = L$ (and the supremum is reached with the constant function $a = L$). Denoting by σ the Lebesgue measure of $\mathcal{S}^2$, $\mathcal{U}_L$ is the set of all measurable subsets ω of $\mathcal{S}^2$ of measure $\sigma(L) = 4\pi L$. For every $\omega \in \mathcal{U}_L$, one has

$$4\pi L = \int_0^{2\pi}\int_0^{\pi} \chi_\omega(\varphi,\theta)\sin\varphi\, d\varphi d\theta \geqslant \sin\varepsilon \int_0^{2\pi}\int_\varepsilon^{\pi-\varepsilon} \chi_\omega(\varphi,\theta)\, d\varphi d\theta \geqslant \sin\varepsilon \int_0^{2\pi} (|\gamma_\theta\cap\omega| - 2\varepsilon) d\theta,$$

for every $\varepsilon \in [0,\pi/2]$, where γ_θ denotes the great circle joining the north pole to the south pole at longitude θ (where a north pole is fixed arbitrarily). By contradiction, assume that $\mu_{\gamma_\theta}(\omega) > 3L/4$ for every $\theta \in [0,2\pi]$. Then we infer that $4\pi L > 2\pi\sin\varepsilon(3\pi L/2 - 2\varepsilon)$, which raises a contradiction when choosing e.g. $\varepsilon = \pi/4$ and L close to 1. It then follows that

$$J(\chi_\omega) = \inf_{j\in\mathbb{N}^*}\int_\omega \phi_j^2 \leqslant \inf_{\theta\in[0,2\pi]} \mu_{\gamma_\theta}(\omega) \leqslant \frac{3L}{4},$$

for every $\omega \in \mathcal{U}_L$, whence the gap.

Assume now that Ω is the unit half-sphere of $\mathbb{R}^3$. As recalled above, for every great circle C of $\mathcal{S}^2$ there exists a sequence of squares of eigenfunctions ϕ_j whose support concentrates along C. Let $\mathcal{S}$ denote the orthogonal symmetry with respect to the hyperplane passing through the origin, cutting $\mathcal{S}^2$ into two half-spheres, one of which being Ω. Then, $\psi_j = (\phi_j - \phi_j\circ\mathcal{S})/\sqrt{2}$ is an eigenfunction of the Dirichlet-Laplacian on Ω. Let us prove[7] that the support of ψ_j^2 concentrates on the union of two symmetric half-circles of Ω, as drawn on Figure 2.

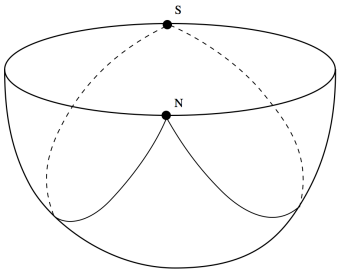

Figure 2: The half-sphere.

Indeed, since $\psi_j^2 = \frac{1}{2}(\phi_j^2 + |\phi_j\circ\mathcal{S}|^2 + \phi_j\cdot\phi_j\circ\mathcal{S})$, it suffices to prove that for every $a \in L^\infty(\Omega)$, $\int_\Omega a(x)\phi_j(x)\phi_j\circ\mathcal{S}(x)\, d\sigma(x)$ tends to 0 as j tends to $+\infty$. But this fact is obvious since the measure of the intersection of the corresponding supports tends to 0. The following (interesting in itself) fact follows: the Dirac measure along every union of symmetric half-circles on Ω is the projection of

[7] This idea emerged from discussions with Luc Hillairet.

a semi-classical measure. Note however that, in this construction, the half-circles passing through the lowest point of the half-sphere cannot be considered.

Then, the same calculation as before can be led. Indeed, let us fix a point N of the boundary of Ω, and let S be the diametrically symmetric point, as on Figure 2. If we think of N and S as a north pole and south pole, then any curve consisting of the union of two symmetric half-circles emerging from N and S can be viewed, with evident symmetries, as a great circle γ_θ of $\mathcal{S}^2$ as considered previously. Then, the same argument can be applied and leads to the desired conclusion. □

4 Nonexistence of an optimal set and remedies

In Section 4.1 we investigate the question of the existence of an optimal set, reaching the supremum in (31). Apart from simple geometries, this question remains essentially open and we conjecture that in general there does not exist any optimal set. In Section 4.2 we study a spectral approximation of (31), by keeping only the N first modes. We establish existence and uniqueness results, and provide numerical simulations showing the increasing complexity of the optimal sets. We then investigate possible remedies to the nonexistence of an optimal set of (31). As a first remark, we consider in Section 4.3 classes of subsets sharing compactness properties, in view of ensuring existence results for (31). Since our aim is however to investigate domains as general as possible (only measurable), in Section 4.4, we introduce a weighted variant of the observability inequality, where the weight is stronger on lower frequencies. We then come up with a weighted spectral variant of (31), for which we prove, in contrast with the previous results, that there exists a unique optimal set whenever L is large enough, and that the maximizing sequence built from a spectral truncation is stationary.

4.1 On the existence of an optimal set

In this section we comment on the problem of knowing whether the supremum in (37) is reached or not, in the framework of Theorem 6. This problem remains essentially open except in several particular cases.

For the one-dimensional case already mentioned in Remarks 1, 7 and 12, we have the following result.

Lemma 3. *Assume that $\Omega = (0,\pi)$. Let $L \in (0,1)$. The supremum of J over $\mathcal{U}_L$ (which is equal to L) is reached if and only if $L = 1/2$. In that case, it is reached for all measurable subsets $\omega \subset (0,\pi)$ of measure $\pi/2$ such that ω and its symmetric image $\omega' = \pi - \omega$ are disjoint and complementary in $(0,\pi)$.*

Proof. Although the proof of that result can be found in [23] and in [47], we recall it here shortly since similar arguments will be used in the proof of the forthcoming Lemma 4.

A subset $\omega \subset (0,\pi)$ of Lebesgue measure $L\pi$ is solution of (37) if and only if $\int_\omega \sin^2(jx)\,dx \geqslant L\pi/2$ for every $j \in \mathbb{N}^*$, that is, $\int_\omega \cos(2jx)\,dx \leqslant 0$. Therefore the Fourier series expansion of χ_ω on $(0,\pi)$ must be of the form $L + \sum_{j=1}^{+\infty}(a_j\cos(2jx) + b_j\sin(2jx))$, with coefficients $a_j \leqslant 0$. Let $\omega' = \pi - \omega$ be the symmetric set of ω with respect to $\pi/2$. The Fourier series expansion of $\chi_{\omega'}$ is $L + \sum_{j=1}^{+\infty}(a_j\cos(2jx) - b_j\sin(2jx))$. Set $g(x) = L - \frac{1}{2}(\chi_\omega(x) + \chi_{\omega'}(x))$, for almost every $x \in (0,\pi)$. The Fourier series expansion of g is $-\sum_{j=1}^{+\infty} a_j\cos(2jx)$, with $a_j \leqslant 0$ for every $j \in \mathbb{N}^*$. Assume that $L \neq 1/2$. Then the sets ω and ω' are not disjoint and complementary, and hence g is discontinuous. It then follows that $\sum_{j=1}^{\infty} a_j = -\infty$. Besides, the sum $\sum_{j=1}^{\infty} a_j$ is also the limit of $\sum_{k=1}^{+\infty} a_k\widehat{\Delta}_n(k)$ as $n \to +\infty$, where $\widehat{\Delta}_n$ is the Fourier transform of the positive function Δ_n whose graph is the triangle joining the points $(-\frac{1}{n},0)$, $(0,2n)$ and $(\frac{1}{n},0)$ (note that Δ_n is an

approximation of the Dirac measure, with integral equal to 1). This raises a contradiction with the fact that

$$\int_0^\pi g(t)\Delta_n(t)dt = \sum_{k=1}^{+\infty} a_k \widehat{\Delta}_n(k),$$

derived from Plancherel's Theorem. □

For the two-dimensional square $\Omega = (0,\pi)^2$ studied in Proposition 1 we are not able to provide a complete answer to the question of the existence. We are however able to characterize the existence of optimal sets that are a Cartesian product.

Lemma 4. *Assume that $\Omega = (0,\pi)^2$. Let $L \in (0,1)$. The supremum of J over the class of all possible subsets $\omega = \omega_1 \times \omega_2$ of Lebesgue measure $L\pi^2$, where ω_1 and ω_2 are measurable subsets of $(0,\pi)$, is reached if and only if $L \in \{1/4, 1/2, 3/4\}$. In that case, it is reached for all such sets ω satisfying*

$$\frac{1}{4}(\chi_\omega(x,y) + \chi_\omega(\pi-x,y) + \chi_\omega(x,\pi-y) + \chi_\omega(\pi-x,\pi-y)) = L,$$

for almost all $(x,y) \in [0,\pi^2]$.

Proof. A subset $\omega \subset (0,\pi)^2$ of Lebesgue measure $L\pi^2$ is solution of (37) if and only if the inequality $\frac{4}{\pi^2}\int_\omega \sin^2(jx)\sin^2(ky)\,dx\,dy \geqslant L$ holds for all $(j,k) \in (\mathbb{N}^*)^2$, that is,

$$\int_\omega \cos(2jx)\cos(2ky)\,dx\,dy \geqslant \int_\omega \cos(2jx)\,dx\,dy + \int_\omega \cos(2ky)\,dx\,dy. \tag{54}$$

Set $\ell_x = \int_0^\pi \chi_\omega(x,y)\,dy$ for almost every $x \in (0,\pi)$, and $\ell_y = \int_0^\pi \chi_\omega(x,y)\,dx$ for almost every $y \in (0,\pi)$. Letting either j or k tend to $+\infty$ and using Fubini's theorem in (54) leads to

$$\int_0^\pi \ell_x \cos(2jx)\,dx \leqslant 0 \text{ and } \int_0^\pi \ell_y \cos(2ky)\,dy \leqslant 0,$$

for every $j \in \mathbb{N}^*$ and every $k \in \mathbb{N}^*$.

Now, if $\omega = \omega_1 \times \omega_2$, where ω_1 and ω_2 are measurable subsets of $(0,\pi)$, then the functions $x \mapsto \ell_x$ and $y \mapsto \ell_y$ must be discontinuous. Using similar arguments as in the proof of Lemma 3, it follows that the functions $x \mapsto \ell_x + \ell_{\pi-x}$ and $y \mapsto \ell_y + \ell_{\pi-y}$ must be constant on $(0,\pi)$, and hence,

$$\int_0^\pi \ell_x \cos(2jx)\,dx = 0 \text{ and } \int_0^\pi \ell_y \cos(2ky)\,dy = 0,$$

for every $j \in \mathbb{N}^*$ and every $k \in \mathbb{N}^*$. Using (54), it follows that $\int_\omega \cos(2jx)\cos(2ky)\,dx\,dy \geqslant 0$, for all $(j,k) \in (\mathbb{N}^*)^2$. The function F defined by

$$F(x,y) = \frac{1}{4}(\chi_\omega(x,y) + \chi_\omega(\pi-x,y) + \chi_\omega(x,\pi-y) + \chi_\omega(\pi-x,\pi-y)),$$

for almost all $(x,y) \in (0,\pi)^2$, can only take the values 0, 1/4, 1/2, 3/4 and 1, and its Fourier series is of the form

$$L + \frac{4}{\pi^2}\sum_{j,k=1}^{+\infty}\left(\int_\omega \cos(2ju)\cos(2kv)\,du\,dv\right)\cos(2jx)\cos(2ky),$$

and all Fourier coefficients are nonnegative. Using once again similar arguments as in the proof of Lemma 3 (Fourier transform and Plancherel's Theorem), it follows that F must necessarily be continuous on $(0,\pi)^2$ and thus constant. The conclusion follows. □

Remark 19. All results of this section can obviously be generalized to multi-dimensional domains Ω written as N cartesian products of one-dimensional sets.

Remark 20. According to Lemma 4, if $L = 1/2$ then there exists an infinite number of optimal sets. Four of them are drawn on Figure 3. It is interesting to note that the optimal sets drawn on the left-side of the figure do not satisfy the Geometric Control Condition mentioned in Section 2.1, and that in this configuration the (classical, deterministic) observability constants $C_T^{(W)}(\chi_\omega)$ and $C_T^{(S)}(\chi_\omega)$ are equal to 0, whereas, according to the previous results, there holds $2\,C_{T,\mathrm{rand}}^{(W)}(\chi_\omega) = C_{T,\mathrm{rand}}^{(S)}(\chi_\omega) = TL$. This fact is in accordance with Remarks 4 and 5.

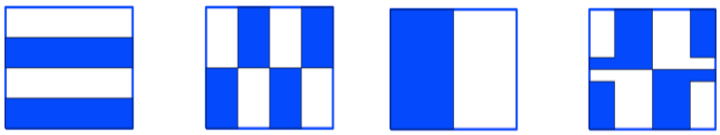

Figure 3: $\Omega = (0,\pi)^2$, $L = 1/2$.

Remark 21. Similar considerations hold for the two-dimensional unit disk. Actually it easily follows from Lemma 3 and from the proof of Proposition 1 that, for $L = 1/2$, the supremum of J over $\mathcal{U}_L$ is reached for every subset ω of the form $\omega = \{(r,\theta) \in [0,1] \times [0,2\pi] \mid \theta \in \omega_\theta\}$, where ω_θ is any subset of $[0,2\pi]$ such that ω_θ and its symmetric image $\omega'_\theta = 2\pi - \omega_\theta$ are disjoint and complementary in $[0,2\pi]$. But we do not know whether or not there are other maximizing subsets.

Remark 22. In view of the results above one could expect that when Ω is the unit N-dimensional hypercube, there exists a finite number of values of $L \in (0,1)$ such that the supremum in (37) is reached. The same result can probably be expected for generic domains Ω. But these issues are open.

4.2 Spectral approximation

In this section, we consider a spectral truncation of the functional J defined by (27), and we define

$$J_N(\chi_\omega) = \min_{1 \leqslant j \leqslant N} \int_\omega \phi_j(x)^2 \, dx, \tag{55}$$

for every $N \in \mathbb{N}^*$ and every measurable subset ω of Ω, and we consider the spectral approximation of the second problem (uniform optimal design problem)

$$\sup_{\chi_\omega \in \mathcal{U}_L} J_N(\chi_\omega). \tag{56}$$

As before, the functional J_N is naturally extended to $\overline{\mathcal{U}}_L$ by

$$J_N(a) = \min_{1 \leqslant j \leqslant N} \int_\Omega a(x)\phi_j(x)^2 \, dx,$$

for every $a \in \overline{\mathcal{U}}_L$. We have the following result, establishing existence, uniqueness and Γ-convergence properties.

Theorem 10. *1. For every measurable subset ω of Ω, the sequence $(J_N(\chi_\omega))_{N\in\mathbb{N}^*}$ is nonincreasing and converges to $J(\chi_\omega)$.*

2. There holds

$$\lim_{N\to+\infty}\ \max_{a\in\overline{\mathcal{U}}_L} J_N(a) = \max_{a\in\overline{\mathcal{U}}_L} J(a).$$

Moreover, if $(a^N)_{n\in\mathbb{N}^*}$ is a sequence of maximizers of J_N in $\overline{\mathcal{U}}_L$, then up to a subsequence, it converges to a maximizer of J in $\overline{\mathcal{U}}_L$ for the weak star topology of L^∞.

3. For every $N\in\mathbb{N}^*$, the problem (56) has a unique solution χ_{ω^N}, where $\omega^N\in\mathcal{U}_L$. Moreover, ω^N is semi-analytic[8] and has a finite number of connected components.

Proof. For every measurable subset ω of Ω, the sequence $(J_N(\chi_\omega))_{N\in\mathbb{N}^*}$ is clearly nonincreasing and thus is convergent. Note that

$$J_N(\chi_\omega) = \inf\left\{\sum_{j=1}^N \alpha_j\int_\omega \phi_j(x)^2\,dx\ \middle|\ \alpha_j\geqslant 0,\ \sum_{j=1}^N\alpha_j=1\right\},$$

$$J(\chi_\omega) = \inf\left\{\sum_{j\in\mathbb{N}^*} \alpha_j\int_\omega \phi_j(x)^2\,dx\ \middle|\ \alpha_j\geqslant 0,\ \sum_{j\in\mathbb{N}^*}\alpha_j=1\right\}.$$

Hence, for every $(\alpha_j)_{j\in\mathbb{N}^*}\in\ell^1(\mathbb{R}^+)$, one has

$$\sum_{j=1}^N \alpha_j\int_\omega \phi_j(x)^2\,dx \geqslant J_N(\chi_\omega)\sum_{j=1}^N\alpha_j,$$

for every $N\in\mathbb{N}^*$, and letting N tend to $+\infty$ yields

$$\sum_{j\in\mathbb{N}^*} \alpha_j\int_\omega \phi_j(x)^2\,dx \geqslant \lim_{N\to+\infty} J_N(\omega)\sum_{j\in\mathbb{N}^*}\alpha_j,$$

and thus $\lim_{N\to+\infty} J_N(\chi_\omega)\leqslant J(\chi_\omega)$. This proves the first item since there always holds $J_N(\chi_\omega)\geqslant J(\chi_\omega)$.

Since J_N is upper semi-continuous (and even continuous) for the L^∞ weak star topology and since $\overline{\mathcal{U}}_L$ is compact for this topology, it follows that J_N has at least one maximizer $a^N\in\overline{\mathcal{U}}_L$. Let $\bar a\in\overline{\mathcal{U}}_L$ be a closure point of the sequence $(a^N)_{n\in\mathbb{N}^*}$ in the L^∞ weak star topology. One has, for every $p\leqslant N$,

$$\sup_{a\in\overline{\mathcal{U}}_L} J(a)\leqslant \sup_{a\in\overline{U}_L} J_N(a) = J_N(a^N)\leqslant J_p(a^N),$$

and letting N tend to $+\infty$ yields

$$\sup_{a\in\overline{\mathcal{U}}_L} J(a)\leqslant \lim_{N\to+\infty} J_N(a^N)\leqslant \lim_{N\to+\infty} J_p(a^N) = J_p(\bar a),$$

[8] A subset ω of a real analytic finite dimensional manifold M is said to be semi-analytic if it can be written in terms of equalities and inequalities of analytic functions, that is, for every $x\in\omega$, there exists a neighborhood U of x in M and $2pq$ analytic functions g_{ij}, h_{ij} (with $1\leqslant i\leqslant p$ and $1\leqslant j\leqslant q$) such that

$$\omega\cap U = \bigcup_{i=1}^p\{y\in U\mid g_{ij}(y)=0 \text{ and } h_{ij}(y)>0,\ j=1,\dots,q\}.$$

We recall that such semi-analytic (and more generally, subanalytic) subsets enjoy nice properties, for instance they are stratifiable in the sense of Whitney (see [21, 27]).

for every $p \in \mathbb{N}^*$. Since $J_p(\bar{a})$ tends to $J(\bar{a}) \leqslant \sup_{a \in \overline{\mathcal{U}_L}} J(a)$ as p tends to $+\infty$, it follows that $\bar{a}$ is a maximizer of J in $\overline{\mathcal{U}_L}$. The second item is proved.

To prove the third item, let us now prove that J_N has a unique maximizer $a^N \in \overline{\mathcal{U}}_L$ of J_N, which is moreover a characteristic function. We define the simplex set

$$\mathcal{A}_N = \{\alpha = (\alpha_j)_{1 \leqslant j \leqslant N} \mid \alpha_j \geqslant 0, \ \sum_{j=1}^N \alpha_j = 1\}.$$

Note that

$$\min_{1 \leqslant j \leqslant N} \int_\Omega a(x) \phi_j(x)^2 \, dx = \min_{\alpha \in \mathcal{A}_N} \int_\Omega a(x) \sum_{j=1}^N \alpha_j \phi_j(x)^2 \, dx,$$

for every $a \in \overline{\mathcal{U}}_L$. It follows from Sion's minimax theorem (see [55]) that there exists $\alpha^N \in \mathcal{A}_N$ such that (a^N, α^N) is a saddle point of the bilinear functional

$$(a, \alpha) \mapsto \int_\Omega a(x) \sum_{j=1}^N \alpha_j \phi_j(x)^2 \, dx$$

defined on $\overline{\mathcal{U}}_L \times \mathcal{A}_N$, and

$$\begin{aligned} &\max_{a \in \overline{\mathcal{U}}_L} \min_{\alpha \in \mathcal{A}_N} \int_\Omega a(x) \sum_{j=1}^N \alpha_j \phi_j(x)^2 \, dx = \min_{\alpha \in \mathcal{A}_N} \max_{a \in \overline{\mathcal{U}}_L} \int_\Omega a(x) \sum_{j=1}^N \alpha_j \phi_j(x)^2 \, dx \\ &= \max_{a \in \overline{\mathcal{U}}_L} \int_\Omega a(x) \sum_{j=1}^N \alpha_j^N \phi_j(x)^2 \, dx = \int_\Omega a^N(x) \sum_{j=1}^N \alpha_j^N \phi_j(x)^2 \, dx. \end{aligned} \tag{57}$$

We claim that the function $x \mapsto \sum_{j=1}^N \alpha_j^N \phi_j(x)^2$ is never constant on any subset of positive measure. This fact is proved by contradiction. Indeed otherwise this function would be constant on Ω (by analyticity). We infer from the Dirichlet boundary conditions that the function $x \mapsto \sum_{j=1}^N \alpha_j^N \phi_j(x)^2$ vanishes on $\bar{\Omega}$, which is a contradiction.

It follows from this fact and from (57) that there exists $\lambda^N > 0$ such that $a^N(x) = 1$ if $\sum_{j=1}^N \alpha_j^N \phi_j(x)^2 \geqslant \lambda^N$, and $a^N(x) = 0$ otherwise, for almost every $x \in \Omega$. Hence there exists $\omega^N \in \mathcal{U}_L$ such that $a^N = \chi_{\omega^N}$. Since the eigenfunctions ϕ_j are analytic in Ω (by analytic hypoellipticity), it follows that ω^N is semi-analytic (see Footnote 8) and has a finite number of connected components. □

Remark 23. Note that the third item of Theorem 10 can be seen as a generalization of [24, Theorem 3.1] and [46, Theorem 3.1]. We have also provided a shorter proof.

Remark 24. In the one-dimensional case $\Omega = (0, \pi)$ with Dirichlet boundary conditions, it can be proved that the optimal set ω_N maximizing J_N is the union of N intervals concentrating around equidistant points and that ω_N is actually the worst possible subset for the problem of maximizing J_{N+1}. This is the *spillover phenomenon*, observed in [24] and rigorously proved in [47].

We provide hereafter several numerical simulations based on the modal approximation described previously, which permit to put in evidence some maximizing sequences of sets.

Assume first that $\Omega = (0, \pi)^2$, the Euclidean two-dimensional square. The normalized eigenfunctions of the Dirichlet-Laplacian are

$$\phi_{j,k}(x_1, x_2) = \frac{2}{\pi} \sin(jx_1) \sin(kx_2),$$

for every $(x_1, x_2) \in (0, \pi)^2$. Let $N \in \mathbb{N}^*$. We use an interior point line search filter method to solve the spectral approximation of the second problem $\sup_{\chi_\omega \in \mathcal{U}_L} J_N(\chi_\omega)$, where

$$J_N(\chi_\omega) = \min_{1 \leqslant j,k \leqslant N} \int_0^\pi \int_0^\pi \chi_\omega(x_1, x_2) \phi_{j,k}(x_1, x_2)^2 \, dx_1 \, dx_2.$$

Some results are provided on Figure 4.

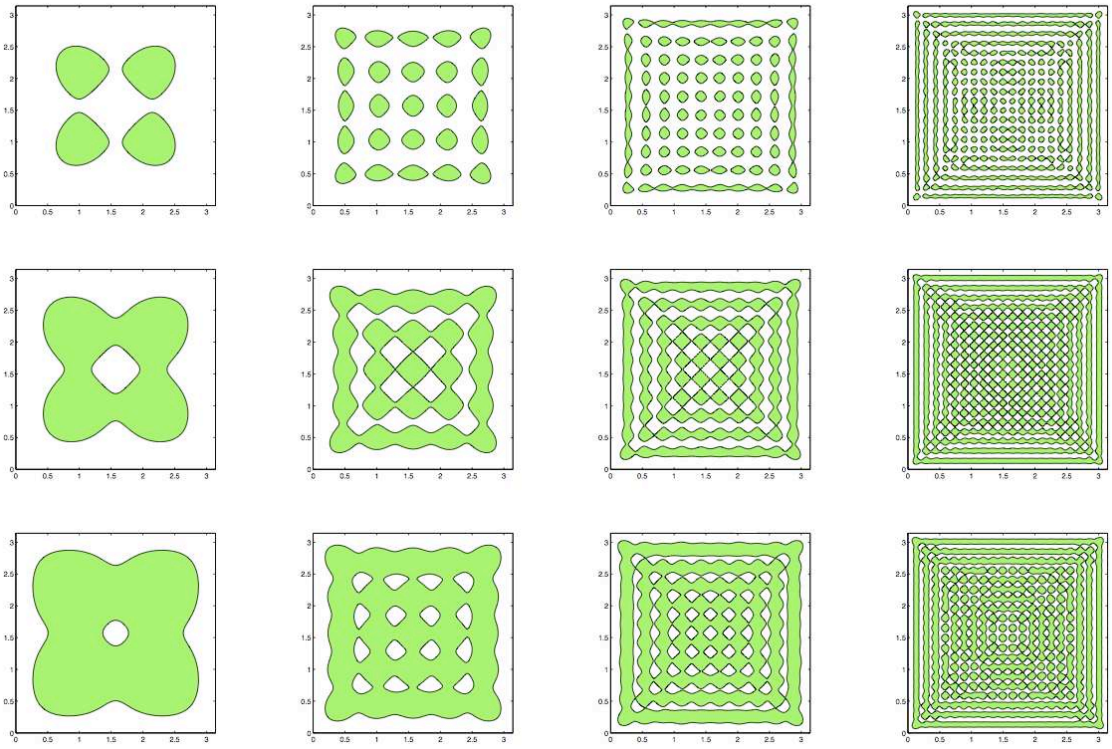

Figure 4: $\Omega = (0, \pi)^2$, with Dirichlet boundary conditions. Row 1: $L = 0.2$; row 2: $L = 0.4$; row 3: $L = 0.6$. From left to right: $N = 2$ (4 eigenmodes), $N = 5$ (25 eigenmodes), $N = 10$ (100 eigenmodes), $N = 20$ (400 eigenmodes). The optimal domain is in green.

Assume now that $\Omega = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$, the unit Euclidean disk of $\mathbb{R}^2$. The normalized eigenfunctions of the Dirichlet-Laplacian are a triply indexed sequence given by

$$\phi_{jkm}(r, \theta) = \begin{cases} R_{0k}(r) & \text{if } j = 0, \\ R_{jk}(r) Y_{jm}(\theta) & \text{if } j \geqslant 1, \end{cases}$$

for $j \in \mathbb{N}$, $k \in \mathbb{N}^*$ and $m = 1, 2$, where (r, θ) are the usual polar coordinates. The functions $Y_{jm}(\theta)$ are defined by $Y_{j1}(\theta) = \frac{1}{\sqrt{\pi}} \cos(j\theta)$ and $Y_{j2}(\theta) = \frac{1}{\sqrt{\pi}} \sin(j\theta)$, and the functions R_{jk} are defined by

$$R_{jk}(r) = \sqrt{2} \, \frac{J_j(z_{jk} r)}{|J_j'(z_{jk})|},$$

where J_j is the Bessel function of the first kind of order j, and $z_{jk} > 0$ is the k^{th}-zero of J_j. The eigenvalues of the Dirichlet-Laplacian are given by the double sequence of $-z_{jk}^2$ and are of multiplicity 1 if $j = 0$, and 2 if $j \geqslant 1$. In Proposition 1, a no-gap result is stated in this case. Some simulations are provided on Figure 5. We observe that optimal domains are radially symmetric. This is actually an immediate consequence of the uniqueness of a maximizer for the modal approximations problem stated in Theorem 10 and of the fact that Ω is itself radially symmetric.

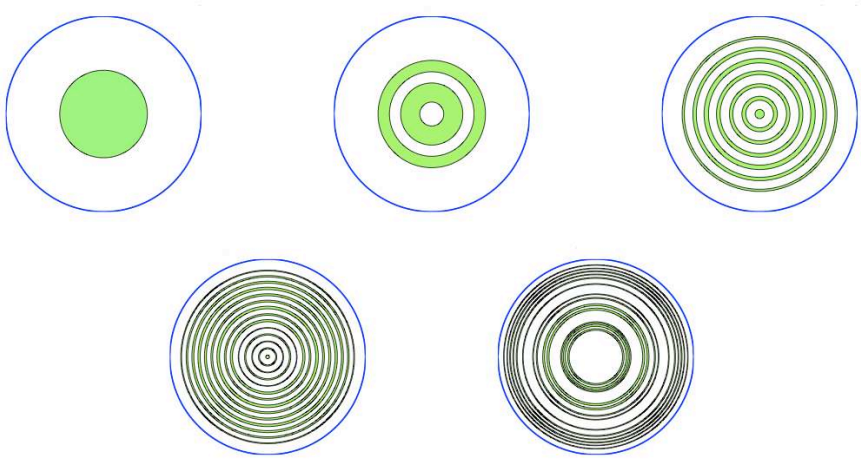

Figure 5: $\Omega = \{x \in \mathbb{R}^2 \mid |x\| \leqslant 1\}$, with Dirichlet boundary conditions, and $L = 0.2$. Optimal domain for $N = 1$ (1 eigenmode), $N = 2$ (4 eigenmodes), $N = 5$ (25 eigenmodes), $N = 10$ (100 eigenmodes) and $N = 20$ (400 eigenmodes).

4.3 A first remedy: other classes of admissible domains

According to Lemma 3, we know that, in the one-dimensional case, the problem (31) is ill-posed in the sense that it has no solution except for $L = 1/2$. In larger dimension, we expect a similar conclusion. One of the reasons is that the set $\mathcal{U}_L$ defined by (30) is not compact for the usual topologies, as discussed in Remark 6. To overcome this difficulty, a possibility consists of defining a new class of admissible sets, $\mathcal{V}_L \subset \mathcal{U}_L$, enjoying sufficient compactness properties and to replace the problem (31) with

$$\sup_{\chi_\omega \in \mathcal{V}_L} J(\chi_\omega). \tag{58}$$

Of course, now, the extremal value is not necessarily the same since the class of admissible domains has been further restricted.

To ensure the existence of a maximizer χ_{ω^*} of (58), it suffices to endow $\mathcal{V}_L$ with a topology, finer than the weak star topology of L^∞, for which $\mathcal{V}_L$ is compact. Of course in this case, one has

$$J(\chi_{\omega^*}) = \max_{\chi_\omega \in \mathcal{V}_L} J(\chi_\omega) \leqslant \sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega).$$

This extra compactness property can be guaranteed by, for instance, considering some $\alpha > 0$, and

then any of the following possibles choices

$$\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L \mid P_\Omega(\omega) \leqslant \alpha\}, \tag{59}$$

where $P_\Omega(\omega)$ is the relative perimeter of ω with respect to Ω,

$$\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L \mid \|\chi_\omega\|_{BV(\Omega)} \leqslant \alpha\}, \tag{60}$$

where $\|\cdot\|_{BV(\Omega)}$ is the $BV(\Omega)$-norm of all functions of bounded variations on Ω (see for example [2]), or

$$\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L \mid \omega \text{ satisfies the } 1/\alpha\text{-cone property}\}, \tag{61}$$

(see Section 3.4, footnote 4). Naturally, the optimal set then depends on the bound α under consideration, and numerical simulations (not reported here) show that, as α tends to $+\infty$, the family of optimal sets behaves as the maximizing sequence built in Section 4.2, in particular the number of connected components grows as α is increasing.

The point of view that we adopted in this article is however not to restrict the classes of possible subsets ω, but rather to discuss the physical relevance of the criterion under consideration. In the next subsection we rather consider a modification of the spectral criterion, based on physical remarks.

4.4 A second remedy: weighted observability inequality

We start our discussion from the remark that, in the observability inequality (12), by definition all modes (in the spectral expansion) have the same weight. It is however expected (and finally, observed) that the problem is difficult owing to the increasing complexity of the geometry of highfrequency eigenfunctions. Moreover, measuring lower frequencies is in some sense physically different from measuring highfrequencies. It seems then relevant to introduce weighted versions of the observability inequality (12), by considering the inequality

$$C^{(W)}_{T,\sigma}(\chi_\omega)\left(\|(y^0,y^1)\|^2_{L^2\times H^{-1}} + \sigma\|y^0\|^2_{H^{-1}}\right) \leqslant \int_0^T \int_\omega |y(t,x)|^2\,dx\,dt, \tag{62}$$

where $\sigma \geqslant 0$ is some weight.

This inequality hold true under GCC. Since the norm used at the left-hand side is stronger than the one of (12), it follows that $C^{(W)}_{T,\sigma}(\chi_\omega) \leqslant C^{(W)}_T(\chi_\omega)$, for every $\sigma \geqslant 0$.

From this weighted observability inequality (82), we can define as well the randomized observability constant and the time asymptotic observability constant (we do not provide the details), and we come up with the following result, which is the weighted version of Theorem 4 and of Corollary 1.

Proposition 3. *For every measurable subset ω of Ω, there holds*

$$2\,C^{(W)}_{T,\sigma,rand}(\chi_\omega) = T J_\sigma(\chi_\omega),$$

and moreover if every eigenvalue of $\triangle$ is simple, then

$$2\,C^{(W)}_{\infty,\sigma}(\chi_\omega) = J_\sigma(\chi_\omega),$$

where

$$J_\sigma(\chi_\omega) = \inf_{j\in I\!N^*} \frac{\lambda_j^2}{\sigma+\lambda_j^2}\int_\omega \phi_j(x)^2\,dx. \tag{63}$$

It is seen from this proposition that the (initial data or time) averaging procedures do not lead to the functional J defined by (27) but to the slightly different (weighted) functional J_σ defined by (63). Let us now investigate the problem

$$\sup_{\chi_\omega \in \mathcal{U}_L} J_\sigma(\chi_\omega). \tag{64}$$

We will see that the study of (63) differs significantly from the one considered previously. Note that the sequence $(\lambda_j^2/(\sigma+\lambda_j^2))_{j\in\mathbb{N}^*}$ is monotone increasing, and that $0 < \lambda_1^2/(\sigma+\lambda_1^2) \leqslant \lambda_j^2/(\sigma+\lambda_j^2) < 1$ for every $j \in \mathbb{N}^*$.

As in Section 3.1, the convexified version of this problem is defined accordingly by

$$\sup_{a\in\overline{\mathcal{U}}_L} J_\sigma(a), \tag{65}$$

where

$$J_\sigma(a) = \inf_{j\in\mathbb{N}^*} \frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega a(x)\phi_j(x)^2\,dx. \tag{66}$$

As in Sections 3.1 and 3.2, under the assumption that there exists a subsequence of $(\phi_j^2)_{j\in\mathbb{N}^*}$ converging to $\frac{1}{|\Omega|}$ in weak star L^∞ topology (L^∞-WQE property), the problem (65) has at least one solution, and $\sup_{a\in\overline{\mathcal{U}}_L} J_\sigma(a) = L$, and the supremum is reached with the constant function $a = L$.

We will next establish a no-gap result, similar to Theorem 6, but only valuable for nonsmall values of L. Actually, we will show that the present situation differs significantly from the previous one, in the sense that, if $\frac{\lambda_1^2}{\sigma+\lambda_1^2} < L < 1$ then the highfrequency modes do not play any role in the problem (64). Before coming to that result, let us first define a truncated version of the problem (64). For every $N \in \mathbb{N}^*$, we define

$$J_{\sigma,N}(a) = \inf_{1\leqslant j\leqslant N} \frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega a(x)\phi_j(x)^2\,dx. \tag{67}$$

An immediate adaptation of the proof of Theorem 10 yields the following result.

Proposition 4. *For every $N \in \mathbb{N}^*$, the problem*

$$\sup_{a\in\overline{\mathcal{U}}_L} J_{\sigma,N}(a) \tag{68}$$

has a unique solution a^N that is the characteristic function of a set ω^N. Moreover, ω^N is semi-analytic (see Footnote 8) and has a finite number of connected components.

The main result of this section is the following.

Theorem 11. *Assume that the QUE one the base and uniform L^∞-boundedness properties hold. Let $L \in (\frac{\lambda_1^2}{\sigma+\lambda_1^2}, 1)$. Then there exists $N_0 \in \mathbb{N}^*$ such that*

$$\max_{\chi_\omega\in\mathcal{U}_L} J_\sigma(\chi_\omega) = \max_{\chi_\omega\in\mathcal{U}_L} J_{\sigma,N}(\chi_\omega) \leqslant \frac{\lambda_1^2}{\sigma+\lambda_1^2} < L, \tag{69}$$

for every $N \geqslant N_0$. In particular, the problem (64) has a unique solution $\chi_{\omega^{N_0}}$, and moreover the set ω^{N_0} is semi-analytic and has a finite number of connected components.

Proof. Using the same arguments as in Lemma 1, it is clear that the problem (65) has at least one solution, denoted by a^∞. Let us first prove that there exists $N_0 \in \mathbb{N}^*$ such that $J_\sigma(a^\infty) = J_{\sigma,N_0}(a^\infty)$. Let $\varepsilon \in (0, L - \frac{\lambda_1^2}{\sigma+\lambda_1^2})$. It follows from the L^∞-QUE property that there exists $N_0 \in \mathbb{N}^*$ such that

$$\frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega a^\infty(x)\phi_j(x)^2\,dx \geqslant L - \varepsilon, \tag{70}$$

for every $j > N_0$. Therefore,

$$\begin{aligned} J_\sigma(a^\infty) &= \inf_{j\in\mathbb{N}^*} \frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega a^\infty(x)\phi_j(x)^2\,dx \\ &= \min\left(\inf_{1\leqslant j\leqslant N_0} \frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega a^\infty(x)\phi_j(x)^2\,dx,\ \inf_{j>N_0} \frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega a^\infty(x)\phi_j(x)^2\,dx \right) \\ &\geqslant \min\left(J_{\sigma,N_0}(a^\infty), L-\varepsilon\right) = J_{\sigma,N_0}(a^\infty), \end{aligned}$$

since $L - \varepsilon > \frac{\lambda_1^2}{\sigma+\lambda_1^2}$ and $J_{\sigma,N_0}(a^\infty) \leqslant \frac{\lambda_1^2}{\sigma+\lambda_1^2}$. It follows that $J_\sigma(a^\infty) = J_{\sigma,N_0}(a^\infty)$.

Let us now prove that $J_\sigma(a^\infty) = J_{\sigma,N_0}(a^{N_0})$, where a^{N_0} is the unique maximizer of J_{σ,N_0} (see Proposition 4). By definition of a maximizer, one has $J_\sigma(a^\infty) = J_{\sigma,N_0}(a^\infty) \leqslant J_{\sigma,N_0}(a^{N_0})$. By contradiction, assume that $J_{\sigma,N_0}(a^\infty) < J_{\sigma,N_0}(a^{N_0})$. Let us then design an admissible perturbation $a_t \in \overline{\mathcal{U}}_L$ of a^∞ such that $J_\sigma(a_t) > J_\sigma(a^\infty)$, which raises a contradiction with the optimality of a^∞. For every $t \in [0,1]$, set $a_t = a^\infty + t(a^{N_0} - a^\infty)$. Since J_{σ,N_0} is concave, one gets

$$J_{\sigma,N_0}(a_t) \geqslant (1-t)J_{\sigma,N_0}(a^\infty) + tJ_{\sigma,N_0}(a^{N_0}) > J_{\sigma,N_0}(a^\infty),$$

for every $t \in (0,1]$, which means that

$$\inf_{1\leqslant j\leqslant N_0} \frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega a_t(x)\phi_j(x)^2\,dx > \inf_{1\leqslant j\leqslant N_0} \frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega a^\infty(x)\phi_j(x)^2\,dx \geqslant J_\sigma(a^\infty), \tag{71}$$

for every $t \in (0,1]$. Besides, since $a^{N_0}(x) - a^\infty(x) \in (-2,2)$ for almost every $x \in \Omega$, it follows from (70) that

$$\begin{aligned} \frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega a_t(x)\phi_j^2(x)\,dx &= \frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega a^\infty(x)\phi_j(x)^2\,dx + t\frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega (a^{N_0}(x) - a^\infty(x))\phi_j(x)^2\,dx \\ &\geqslant L - \varepsilon - 2t, \end{aligned}$$

for every $j \geqslant N_0$. Let us choose t such that $0 < t < \frac{1}{2}(L - \varepsilon - \frac{\lambda_1^2}{\sigma+\lambda_1^2})$, so that the previous inequality yields

$$\frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega a_t(x)\phi_j(x)^2\,dx > \frac{\lambda_1^2}{\sigma+\lambda_1^2} \geqslant \frac{\lambda_1^2}{\sigma+\lambda_1^2} \int_\Omega a^\infty(x)\phi_1(x)^2\,dx \geqslant J_\sigma(a^\infty). \tag{72}$$

for every $j \geqslant N_0$. Combining the low modes estimate (71) with the high modes estimate (72), we conclude that

$$J_\sigma(a_t) = \inf_{j\in\mathbb{N}^*} \frac{\lambda_j^2}{\sigma+\lambda_j^2} \int_\Omega a_t(x)\phi_j(x)^2\,dx > J_\sigma(a^\infty),$$

which contradicts the optimality of a^∞.

Therefore $J_{\sigma,N_0}(a^\infty) = J_\sigma(a^\infty) = J_{\sigma,N_0}(a^{N_0})$, and the result follows. □

Remark 25. Under the assumptions of the theorem, there is no gap between the problem (64) and its convexified formulation (65), as well as before. But, in contrast to the previous results, here there always exists a maximizer in the class of characteristic functions whenever L is larger than a threshold value, and moreover, this optimal set can be computed from a truncated formulation (67) for a certain value of N. In other words, the maximizing sequence $(\chi_{\omega^N})_{N\in\mathbb{N}^*}$ resulting from Proposition 4 is stationary. Here, the high modes play no role, whereas in the previous results all modes had the same impact. This result is due to the fact that we added in the left hand-side of the observability inequalities the weight $\sigma \geqslant 0$. It can be noted that the threshold value $\frac{\lambda_1^2}{\sigma+\lambda_1^2}$, accounting for the existence of an optimal set, is as smaller as σ is larger. This is in accordance with what could be physically expected.

Remark 26. Here, if L is not too small then there exists an optimal set (sharing nice regularity properties) realizing the largest possible time asymptotic and randomized observability constants. The optimal value of these constants is known to be less than L but its exact value is not known. It is related to solving a finite dimensional numerical optimization problem.

Remark 27. In the case where $L \leqslant \frac{\lambda_1^2}{\sigma+\lambda_1^2}$, we do not know whether there is a gap or not between the problem (64) and its convexified formulation (65). Adapting shrewdly the proof of Theorems 6 or 7 does not seem to allow one to derive a no-gap result. Nevertheless, one can prove using these arguments that $\sup_{\chi_\omega\in\mathcal{U}_L} J_\sigma(\chi_\omega) \geqslant \frac{\lambda_1^2}{\sigma+\lambda_1^2}L$.

Remark 28. We formulate the following two open questions.

- Under the assumptions of Theorem 11, does the conclusion hold true for every $L \in (0,1)$?
- Does the statement of Theorem 11 still hold true under weaker ergodicity assumptions, for instance is it possible to weaken QUE into WQE?

Remark 29. The QUE assumption made in Theorem 11 is very strong, as already discussed. It is true in the one-dimensional case but up to now no example of a multi-dimensional domain satisfying QUE is known.

Anyway, we are able to prove that the conclusion of Theorem 11 holds true in a domain which is a tensorized version of a one-dimensional domain. Indeed, consider the Euclidean n-dimensional square $\Omega = (0,\pi)^n$. The normalized eigenfunctions of $\triangle$ are then $\phi_{j_1\dots j_n}(x_1,\dots,x_n) = \left(\frac{2}{\pi}\right)^{n/2}\prod_{k=1}^n \sin(j_k x_k)$, for all $(j_1,\dots,j_n) \in (\mathbb{N}^*)^n$, for every $x \in (0,\pi)$. Obviously, Ω does not satisfy QUE (nor QE), but satisfies WQE, and moreover the eigenfunctions $\phi_{j_1\dots j_n}$ are uniformly bounded in $L^\infty(\Omega)$. Let us prove however that the equality (69) holds. More precisely, one has the following result.

Proposition 5. *Assume that $\Omega = (0,\pi)^n$. There exists $L_0 \in (0,1)$ and $N_0 \in \mathbb{N}^*$ such that*

$$\max_{\chi_\omega\in\mathcal{U}_L} J_\sigma(\chi_\omega) = \max_{\chi_\omega\in\mathcal{U}_L} J_{\sigma,N}(\chi_\omega), \tag{73}$$

for every $L \in [L_0,1)$ and every $N \geqslant N_0$.

Proof. The proof follows the same lines as the one of Theorem 11. Nevertheless, the inequality (70) may not hold whenever QUE is not satisfied and has to be questioned. In the specific case under consideration, (70) is replaced with the following assertion: there exists $N_0 \in \mathbb{N}^*$ such that for $\varepsilon > 0$, there exists $L_0 \in (0,1)$ such that

$$\frac{\lambda_{j_1\dots j_n}^2}{\sigma+\lambda_{j_1\dots j_n}^2}\int_\Omega a^\infty(x)\phi_{j_1\dots j_n}(x)^2\,dx \geqslant L-\varepsilon,$$

for every $L \in [L_0, 1)$ and for all $(j_1, \dots, j_n) \in (\mathbb{N}^*)^n$ such that $\min(j_1, \dots, j_n) \geqslant N_0$. This assertion indeed follows from the following general lemma.

Lemma 5. *Let $\rho \in L^\infty(\Omega, \mathbb{R}_+)$ be such that $\int_\Omega \rho(x)\,dx > 0$. Then*

$$\inf_{(j_1,\dots,j_n)\in(\mathbb{N}^*)^n} \int_\Omega \rho(x)\phi_{j_1\dots j_n}(x)^2\,dx \geqslant F^{[n]}\left(\int_\Omega \rho(x)\,dx\right) > 0,$$

where $F(x) = \frac{1}{\pi}(x - \sin x)$ for every $x \in (0, \pi)$ and $F^{[n]} = F \circ \cdots \circ F$ (n times).

This lemma itself easily follows from [48, Lemma 6] (case $n = 1$) and from an induction argument. □

We end this section by providing several numerical simulations based on the modal approximation of this problem for the Euclidean square $\Omega = (0, \pi)^2$. Note that we are then in the framework of Remark 29, and hence the conclusion of Proposition 5 holds true. As in Section 4.2, we use an interior point line search filter method to solve the spectral approximation of the problem $\sup_{\chi_\omega \in \mathcal{U}_L} J_{N,\sigma}(\chi_\omega)$, with $\sigma = 1$. Some numerical simulations are provided on Figures 6, where the optimal domains are represented for $L \in \{0.2, 0.4, 0.6, 0.9\}$ (by row). In the three first cases, the number of connected components of the optimal set seems to increase with N. On the last row ($L = 0.9$), the numerical results illustrate the conclusion of Proposition 5, showing clear evidence of the stationarity feature of the maximizing sequence proved in this proposition.

5 Generalization to wave and Schrödinger equations on manifolds with various boundary conditions

In this section we show how all the results previously derived can be generalized to wave and Schrödinger equations posed on any bounded connected subset of a Riemannian manifold, with various possible boundary conditions. For each step of our analysis we explain what are the modifications that have to be taken into account.

General framework. Let (M, g) be a smooth n-dimensional Riemannian manifold, $n \geqslant 1$. Let T be a positive real number and Ω be an open bounded connected subset of M. We consider both the wave equation

$$\partial_{tt} y = \triangle_g y, \tag{74}$$

and the Schrödinger equation

$$i\partial_t y = \triangle_g y, \tag{75}$$

in $(0, T) \times \Omega$. Here, $\triangle_g$ denotes the usual Laplace-Beltrami operator on M for the metric g. If the boundary $\partial\Omega$ of Ω is nonempty, then we consider boundary conditions

$$By = 0 \quad \text{on } (0, T) \times \partial\Omega, \tag{76}$$

where B can be either:

- the usual Dirichlet trace operator, $By = y_{|\partial\Omega}$,
- or Neumann, $By = \frac{\partial y}{\partial n}_{|\partial\Omega}$, where $\frac{\partial}{\partial n}$ is the outward normal derivative on the boundary $\partial\Omega$,
- or mixed Dirichlet-Neumann, $By = \chi_{\Gamma_0} y_{|\partial\Omega} + \chi_{\Gamma_1} \frac{\partial y}{\partial n}_{|\partial\Omega}$, where $\partial\Omega = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$, and χ_{Γ_i} is the characteristic function of Γ_i, $i = 0, 1$,

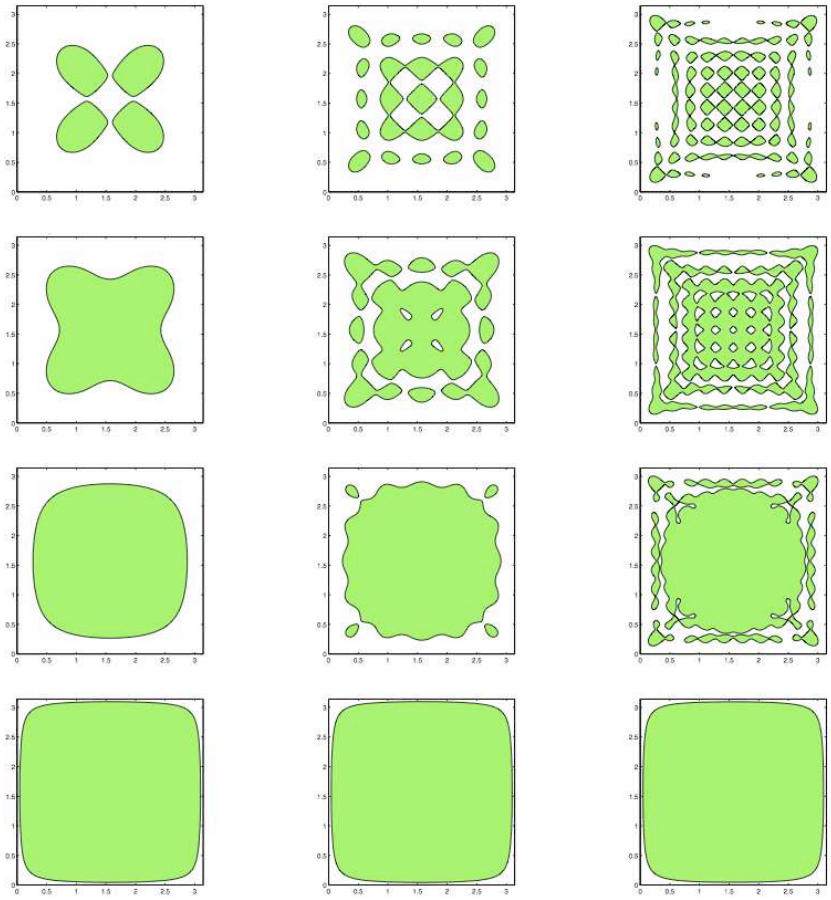

Figure 6: $\Omega = (0,\pi)^2$, with Dirichlet boundary conditions. Row 1: $L = 0.2$; row 2: $L = 0.4$; row 3: $L = 0.6$.; row 4: $L = 0.9$. From left to right: $N = 2$ (4 eigenmodes), $N = 5$ (25 eigenmodes), $N = 10$ (100 eigenmodes).

- or Robin, $By = \frac{\partial y}{\partial n}_{|\partial\Omega} + \beta y_{|\partial\Omega}$, where β is a nonnegative bounded measurable function defined on $\partial\Omega$, such that $\int_{\partial\Omega} \beta > 0$.

Our study encompasses the case where $\partial\Omega = \emptyset$: in this case, (76) is unnecessary and Ω is a compact connected n-dimensional Riemannian manifold. The canonical Riemannian volume on M is denoted by V_g, inducing the canonical measure dV_g. Measurable sets[9] are considered with respect to the measure dV_g.

In the boundaryless or in the Neumann case, the Laplace-Beltrami operator is not invertible on $L^2(\Omega,\mathbb{C})$ but is invertible in

$$L^2_0(\Omega,\mathbb{C}) = \{y \in L^2(\Omega,\mathbb{C}) \mid \int_\Omega y(x)\,dx = 0\}.$$

In what follows, the notation X stands for the space $L^2_0(\Omega,\mathbb{C})$ in the boundaryless or in the Neumann case and for the space $L^2(\Omega,\mathbb{C})$ otherwise. We denote by $A = -\triangle_g$ the Laplace operator

[9] If M is the usual Euclidean space $\mathbb{R}^n$ then $dV_g = dx$ is the usual Lebesgue measure.

defined on $D(A) = \{y \in X \mid Ay \in X \text{ and } By = 0\}$ with one of the above boundary conditions whenever $\partial\Omega \neq \emptyset$. Note that A is a selfadjoint positive operator.

For all $(y^0, y^1) \in D(A^{1/2}) \times X$, there exists a unique solution y of the wave equation (74) in the space $C^0(0,T;D(A^{1/2})) \cap C^1(0,T;X)$ such that $y(0,\cdot) = y^0(\cdot)$ and $\partial_t y(0,\cdot) = y^1(\cdot)$.

Let ω be an arbitrary measurable subset of Ω of positive measure. The equation (74) is said to be observable on ω in time T if there exists $C_T^{(W)}(\chi_\omega) > 0$ such that

$$C_T^{(W)}(\chi_\omega)\|(y^0, y^1)\|^2_{D(A^{1/2})\times X} \leqslant \int_0^T \int_\omega |\partial_t y(t,x)|^2 \, dV_g \, dt, \tag{77}$$

for all $(y^0, y^1) \in D(A^{1/2}) \times X$. This observability inequality holds if (ω, T) satisfies the GCC in Ω.

A similar observability problem can be formulated for the Schrödinger equation (75). For every $y^0 \in D(A)$, there exists a unique solution y of (75) in the space $C^0(0,T;D(A))$ such that $y(0,\cdot) = y^0(\cdot)$. The equation (75) is said to be observable on ω in time T if there exists $C_T^{(S)}(\chi_\omega) > 0$ such that

$$C_T^{(S)}(\chi_\omega)\|y^0\|^2_{D(A)} \leqslant \int_0^T \int_\omega |\partial_t y(t,x)|^2 \, dV_g \, dt, \tag{78}$$

for every $y^0 \in D(A)$. If (ω, T^*) satisfies the GCC then the observability inequality (78) holds for every $T > 0$ (see [38]). Indeed the Schrödinger equation can be viewed as a wave equation with an infinite speed of propagation. We refer to [37] for a thorough discussion of the problem of obtaining necessary and sufficient conditions ensuring the observability inequality, which is a widely open problem.

Remark 30. These inequalities can be formulated in different ways by adequate choices of the functional spaces. For instance, the observability inequality (12) is equivalent to

$$C_T^{(W)}(\chi_\omega)\|(y^0, y^1)\|^2_{X\times(D(A^{1/2}))'} \leqslant \int_0^T \int_\omega |y(t,x)|^2 \, dV_g \, dt, \tag{79}$$

for all $(y^0, y^1) \in X \times (D(A^{1/2}))'$, with the same observability constants. Here the dual is considered with respect to the pivot space X. For instance if A is the negative of the Dirichlet-Laplacian as it has been considered previously, then the observability inequality (79) exactly coincides with (12): we then recover the observability inequality that we considered up to now throughout the paper for wave equations with Dirichlet boundary conditions.

Similarly, the observability inequality (78) is equivalent to

$$C_T^{(S)}(\chi_\omega)\|y^0\|^2_X \leqslant \int_0^T \int_\omega |y(t,x)|^2 \, dV_g \, dt, \tag{80}$$

for every $y^0 \in X$.

Spectral expansions. We fix an an orthonormal Hilbertian basis $(\phi_j)_{j\in\mathbb{N}^*}$ of X consisting of eigenfunctions of A on Ω, associated with the positive eigenvalues $(\lambda_j^2)_{j\in\mathbb{N}^*}$.

Remark 31. Note that, in the Neumann case or in the case $\partial\Omega = \emptyset$, one has $X = L^2_0(\Omega)$. Otherwise if we would consider $X = L^2(\Omega)$ in those cases, then we would have $\lambda_1 = 0$ (simple eigenvalue) and $\phi_1 = 1/\sqrt{|\Omega|}$. The fact that in those cases we define $X = L^2_0(\Omega)$ permits to keep a uniform presentation for all boundary conditions considered here.

Remark 32. There holds

$$D(A) = \{y \in X \mid \sum_{j=1}^{+\infty} \lambda_j^4 \langle y, \phi_j\rangle_{L^2}^2 < +\infty\}, \quad D(A^{1/2}) = \{y \in X \mid \sum_{j=1}^{+\infty} \lambda_j^2 \langle y, \phi_j\rangle_{L^2}^2 < +\infty\}.$$

In the case of Dirichlet boundary conditions, and if $\partial\Omega$ is C^2 then one has $D(A) = H^2(\Omega,\mathbb{C}) \cap H_0^1(\Omega,\mathbb{C})$ and $D(A^{1/2}) = H_0^1(\Omega,\mathbb{C})$. For Neumann boundary conditions, one has $D(A) = \{y \in H^2(\Omega,\mathbb{C}) \mid \frac{\partial y}{\partial n}_{|\partial\Omega} = 0$ and $\int_\Omega y(x)\,dx = 0\}$ and $D(A^{1/2}) = \{y \in H^1(\Omega,\mathbb{C}) \mid \int_\Omega y(x)\,dx = 0\}$. In the mixed Dirichlet-Neumann case (with $\Gamma_0 \neq \emptyset$), one has $D(A) = \{y \in H^2(\Omega,\mathbb{C}) \mid y_{|\Gamma_0} = \frac{\partial y}{\partial n}_{|\Gamma_1} = 0\}$ and $D(A^{1/2}) = H^1_{\Gamma_0}(\Omega,\mathbb{C}) = \{y \in H^1(\Omega,\mathbb{C}) \mid y_{|\Gamma_0} = 0\}$ (see e.g. [36]).

Let us briefly show for the Schrödinger equation how the solutions can be expanded. For every $y^0 \in D(A)$, the solution $y \in C^0(0,T;D(A))$ of (75) such that $y(0,\cdot) = y^0(\cdot)$ can be expanded as

$$y(t,x) = \sum_{j=1}^{+\infty} c_j e^{i\lambda_j^2 t}\phi_j(x),$$

where the sequence $(\lambda_j^2 c_j)_{j\in\mathbb{N}^*}$ belongs to $\ell^2(\mathbb{C})$ and is determined in terms of y^0 by $c_j = \int_\Omega y^0(x)\phi_j(x)\,dV_g$, for every $j \in \mathbb{N}^*$. Moreover, $\|y^0\|_{D(A)}^2 = \sum_{j=1}^{+\infty} \lambda_j^4 |c_j|^2$. It follows that

$$\int_0^T \int_\omega |\partial_t y(t,x)|^2\,dV_g\,dt = \sum_{j,k=1}^{+\infty} \lambda_j^2\lambda_k^2\alpha_{jk}\int_\omega \phi_j(x)\phi_k(x)\,dV_g,$$

with

$$\alpha_{jk} = c_j\bar{c}_k\int_0^T e^{i(\lambda_j^2-\lambda_k^2)t}\,dt = \frac{2c_j\bar{c}_k}{\lambda_j^2-\lambda_k^2}\sin\left((\lambda_j^2-\lambda_k^2)\frac{T}{2}\right)e^{i(\lambda_j^2-\lambda_k^2)\frac{T}{2}},$$

whenever $j \neq k$, and $\alpha_{jj} = |c_j|^2 T$ whenever $j = k$. The observability constant is given by

$$C_T^{(S)}(\chi_\omega) = \inf_{\substack{(\lambda_j^2 c_j)\in\ell^2(\mathbb{C}) \\ \sum_{j=1}^{+\infty}\lambda_j^4|c_j|^2=1}} \int_0^T\int_\omega \left|\sum_{j=1}^{+\infty}\lambda_j^2 c_j e^{i\lambda_j^2 t}\phi_j(x)\right|^2 dV_g\,dt.$$

Making as in Section 2.3 a random selection of all possible initial data for the Schrödinger equation (75) leads to define its randomized version as

$$C_{T,\mathrm{rand}}^{(S)}(\chi_\omega) = \inf_{\substack{(\lambda_j^2 c_j)\in\ell^2(\mathbb{C}) \\ \sum_{j=1}^{+\infty}\lambda_j^4|c_j|^2=1}} \mathbb{E}\left(\int_0^T\int_\omega \left|\sum_{j=1}^{+\infty}\lambda_j^2\beta_j^\nu c_j e^{i\lambda_j^2 t}\phi_j(x)\right|^2 dV_g\,dt\right),$$

where $(\beta_j^\nu)_{j\in\mathbb{N}^*}$ denotes a sequence of independent Bernoulli random variables on a probability space $(\mathcal{X},\mathcal{A},\mathbb{P})$. This corresponds to considering the randomized observability inequality

$$C_T^{(S)}(\chi_\omega)\|y^0\|_{D(A)}^2 \leqslant \mathbb{E}\left(\int_0^T\int_\omega |\partial_t y_\nu(t,x)|^2\,dV_g\,dt\right),$$

for every $y^0(\cdot) \in D(A)$, where y_ν denotes the solution of the Schrödinger equation with the random initial data $y_\nu^0(\cdot)$ determined by its Fourier coefficients $c_j^\nu = \beta_j^\nu c_j$.

Theorem 4 then still holds in this general framework, and one has

$$2\,C^{(W)}_{T,\mathrm{rand}}(\chi_\omega) = C^{(S)}_{T,\mathrm{rand}}(\chi_\omega) = T \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\,dV_g = TJ(\chi_\omega),$$

for every measurable subset ω of Ω, where J is defined as before by (27).

The time asymptotic observability constant is defined accordingly for the Schrödinger equation by

$$C^{(S)}_\infty(\chi_\omega) = \inf\left\{ \lim_{T\to+\infty} \frac{1}{T} \frac{\int_0^T \int_\omega |\partial_t y(t,x)|^2\,dV_g\,dt}{\|y^0\|^2_{D(A)}} \;\middle|\; y^0 \in D(A)\setminus\{0\} \right\}. \tag{81}$$

Corollary 1 holds as well, stating that $2\,C^{(W)}_\infty(\chi_\omega) = C^{(S)}_\infty(\chi_\omega) = J(\chi_\omega)$ whenever every eigenvalue of A is simple. Note that the spectrum of the Neumann-Laplacian is known to consist of simple eigenvalues for many choices of Ω: for instance, it is proved in [26] that this property holds for almost every polygon of $\mathbb{R}^2$ having N vertices.

Main results under quantum ergodicity assumptions. Theorems 6 and 7 are unchanged in this general framework.

Spectral approximation. It must be noted that the third point of Theorem 10 holds true only if M is an analytic Riemannian manifold and if Ω has a nontrivial boundary.

This assumption is used at the end of the proof of this theorem, when showing by contradiction that the function $x \mapsto \sum_{j=1}^N \alpha_j^N \phi_j(x)^2$ is never constant on any subset of positive measure. Indeed at this step we have to distinguish between the different boundary conditions under consideration. For Neumann boundary conditions, we infer that $\triangle_g(\sum_{j=1}^N \alpha_j^N \phi_j(x)^2) = 0$ on $\partial\Omega$ (by continuity), and therefore $\sum_{j=1}^N \alpha_j^N \lambda_j \phi_j(x)^2 = 0$ on $\partial\Omega$, whence the contradiction since the coefficients α_j^N are nonnegative for every $j \in \{1,\cdots,N\}$ and $\sum_{j=1}^N \alpha_j^N = 1$. For the other boundary conditions, we infer that the function $x \mapsto \sum_{j=1}^N \alpha_j^N \phi_j(x)^2$ vanishes on $\bar{\Omega}$, which is a contradiction.

It follows from this fact and from (57) that there exists $\lambda^N > 0$ such that $a^N(x) = 1$ if $\sum_{j=1}^N \alpha_j^N \phi_j(x)^2 \geqslant \lambda^N$, and $a^N(x) = 0$ otherwise, for almost every $x \in \Omega$. Hence there exists $\omega^N \in \mathcal{U}_L$ such that $a^N = \chi_{\omega^N}$. Since the eigenfunctions ϕ_j are analytic in Ω (by analytic hypoellipticity), it follows that ω^N is semi-analytic (see Footnote 8) and has a finite number of connected components.

Numerical simulations. Some results are provided on Figure 7 in the case $\Omega = (0,\pi)^2$ with Neumann boundary conditions. They illustrate as well the non-stationarity feature of the maximizing sequence of optimal sets ω^N.

Remedy: weighted observability inequalities In the general framework, the weighted versions (as discussed in Section 4.4) of the observability inequalities (77) and (78) are

$$C^{(W)}_{T,\sigma}(\chi_\omega)\left(\|(y^0,y^1)\|^2_{D(A^{1/2})\times X} + \sigma\|y^0\|^2_X\right) \leqslant \int_0^T \int_\omega |\partial_t y(t,x)|^2\,dx\,dt \tag{82}$$

in the case of the wave equation, and

$$C^{(S)}_{T,\sigma}(\chi_\omega)\left(\|y^0\|^2_{D(A)} + \sigma\|y^0\|^2_X\right) \leqslant \int_0^T \int_\omega |\partial_t y(t,x)|^2\,dx\,dt \tag{83}$$

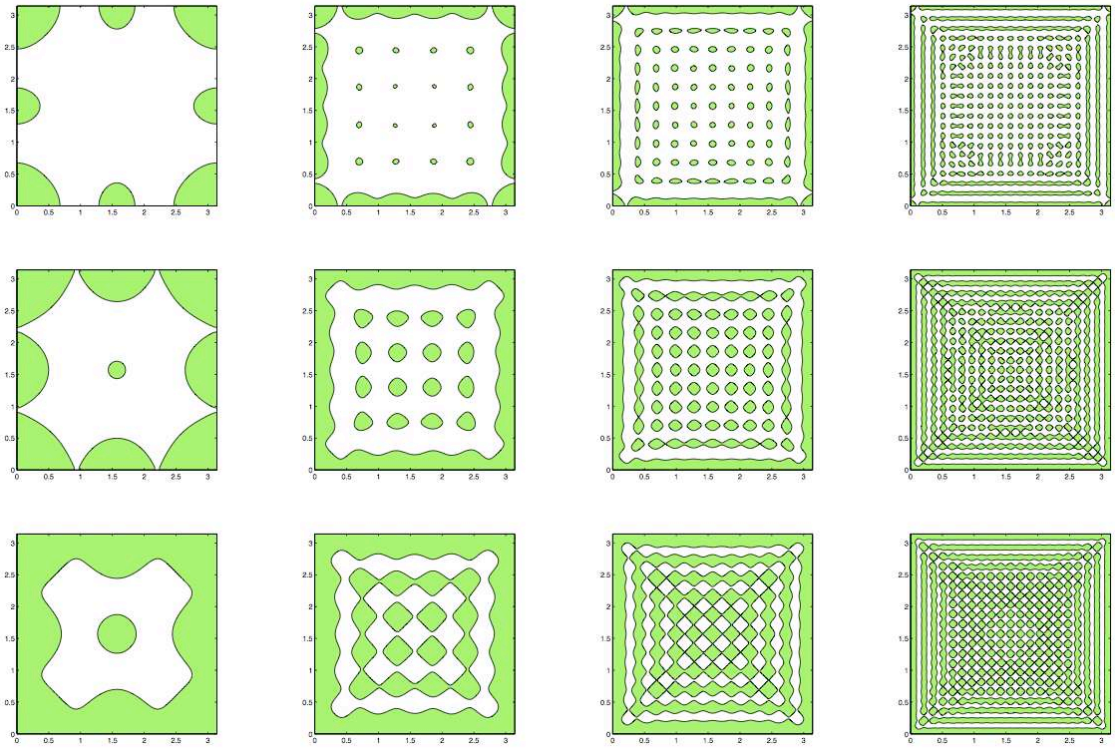

Figure 7: $\Omega = (0,\pi)^2$, with Neumann boundary conditions. Row 1: $L = 0.2$; row 2: $L = 0.4$; row 3: $L = 0.6$. From left to right: $N = 2$ (4 eigenmodes), $N = 5$ (25 eigenmodes), $N = 10$ (100 eigenmodes), $N = 20$ (400 eigenmodes).

in the case of the Schrödinger equation, where $\sigma \geqslant 0$.

Note that, in the Dirichlet case, if $\sigma = 1$ then the inequality (82) corresponds to replacing the H^1_0 norm with the full H^1 norm defined by $\|f\|_{H^1(\Omega,\mathbb{C})} = (\|f\|^2_{L^2(\Omega,\mathbb{C})} + \|\nabla f\|^2_{L^2(\Omega,\mathbb{C})})^{1/2}$.

Clearly, there holds $C^{(W)}_{T,\sigma}(\chi_\omega) \leqslant C^{(W)}_T(\chi_\omega)$ and $C^{(S)}_{T,\sigma}(\chi_\omega) \leqslant C^{(S)}_T(\chi_\omega)$, for every $\sigma \geqslant 0$.

Proposition (3) remains unchanged, stating that $2\,C^{(W)}_{T,\sigma,\mathrm{rand}}(\chi_\omega) = C^{(S)}_{T,\sigma,\mathrm{rand}}(\chi_\omega) = TJ_\sigma(\chi_\omega)$ for every measurable subset ω of Ω, and that $2\,C^{(W)}_{\infty,\sigma}(\chi_\omega) = C^{(S)}_{\infty,\sigma}(\chi_\omega) = J_\sigma(\chi_\omega)$ if moreover every eigenvalue of A is simple, where J_σ is defined by (63).

Theorem 11 remains in force as well. Therefore, in the general framework, considering the averaged versions of these weighted observability inequalities constitutes a physically relevant remedy to ensure the existence and uniqueness of an optimal set

For the completeness let us provide a numerical simulation illustrating this result. In Remark 29, the domain $\Omega = \mathbb{T}^n$ (flat torus) can be considered as well, or we can also consider the domain $\Omega = (0,\pi)^n$ with Dirichlet boundary conditions, or mixed Dirichlet-Neumann boundary conditions with either Dirichlet or Neumann condition on every full edge of the hypercube. Then the ϕ_j's consist either or sine or of cosine functions, and it is easy to see that the conclusion of Proposition 5 holds true in these more general cases.

Some numerical simulations are provided on Figure 8 (with the weight $\sigma = 1$), again clearly

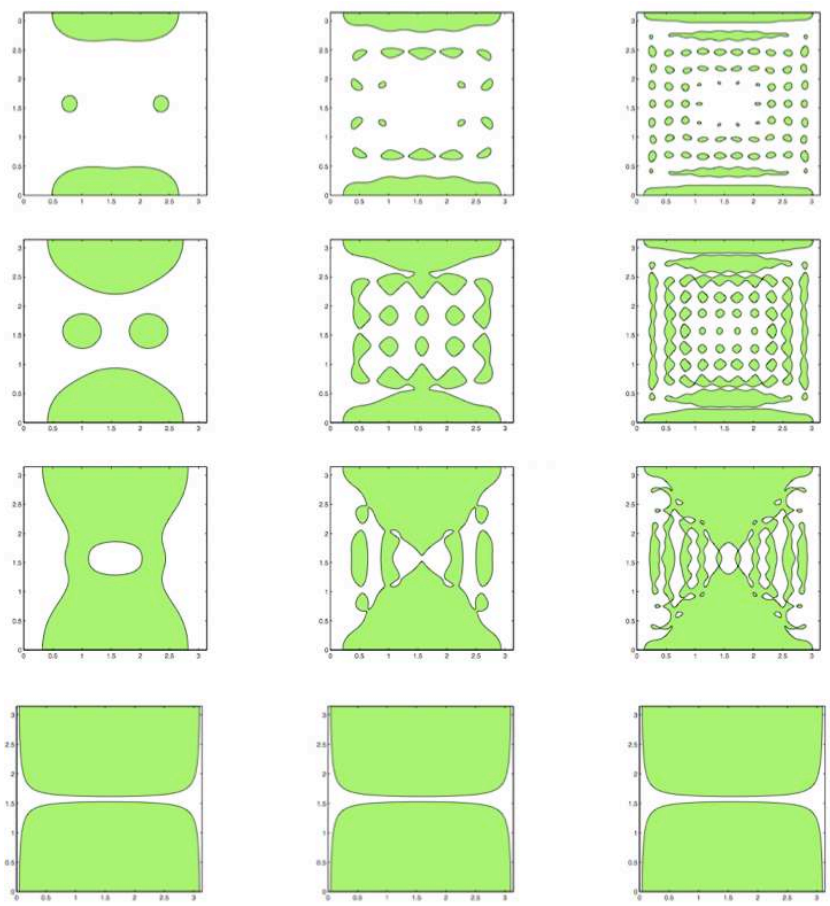

Figure 8: $\Omega = (0,\pi)^2$, with Dirichlet boundary conditions on $\partial\Omega \cap (\{x_2 = 0\} \cup \{x_2 = \pi\}$ and Neumann boundary conditions on the rest of the boundary. Row 1: $L = 0.2$; row 2: $L = 0.4$; row 3: $L = 0.6$; row 4: $L = 0.9$. From left to right: $N = 2$ (4 eigenmodes), $N = 5$ (25 eigenmodes), $N = 10$ (100 eigenmodes).

illustrating the stationarity feature of the maximizing sequence, as soon as L is large enough.

6 Further comments

In Section 6.1, we show how our results for the second problem can be extended to a natural variant of observability inequality for Neumann boundary conditions or in the boundaryless case. In Section 6.2 we show how the problem of maximizing the observability constant is equivalent to an optimal design of a control problem and, namely, to that of controllability in which solutions are driven to rest in final time by means of a suitable control function. Section 6.3 is devoted to comment on several open issues.

6.1 Further remarks for Neumann boundary conditions or in the boundaryless case

In the Neumann case, or in the case $\partial\Omega = \emptyset$, there is a problem with the constants, as explained in Footnote 31. In this section, let us show that, if instead of considering the observability inequalities (77) and (78), we consider the inequalities

$$C_T^{(W)}(\chi_\omega)\|(y^0,y^1)\|^2_{H^1(\Omega,\mathbb{C})\times L^2(\Omega,\mathbb{C})} \leqslant \int_0^T \int_\omega \left(|\partial_t y(t,x)|^2 + |y(t,x)|^2\right)\, dV_g\, dt \tag{84}$$

in the case of the wave equation, and

$$C_T^{(S)}(\chi_\omega)\|y^0\|^2_{H^2(\Omega,\mathbb{C})} \leqslant \int_0^T \int_\omega \left(|\partial_t y(t,x)|^2 + |y(t,x)|^2\right)\, dV_g\, dt \tag{85}$$

in the case of the Schrödinger equation (see [56, Chapter 11] for a survey on these problems), then all results remain unchanged.

Indeed, consider initial data $(y^0,y^1) \in H^1(\Omega,\mathbb{C}) \times L^2(\Omega,\mathbb{C})$. The corresponding solution y can still be expanded as (14), except that now $(\phi_j)_{j\in\mathbb{N}^*}$ consists of the eigenfunctions of the Neumann-Laplacian or of the Laplace-Beltrami operator in the boundaryless case, associated with the eigenvalues $(-\lambda_j^2)_{j\in\mathbb{N}^*}$, with $\lambda_1 = 0$ and ϕ_1 which is constant, equal to $1/\sqrt{|\Omega|}$. The relation (16) does not hold any more and is replaced with

$$\|(y^0,y^1)\|^2_{H^1(\Omega,\mathbb{C})\times L^2(\Omega,\mathbb{C})} = \sum_{j=1}^{+\infty} \left(2\lambda_j^2|a_j|^2 + 2\lambda_j^2|b_j|^2 + |a_j+b_j|^2\right). \tag{86}$$

Following Section 2.3, we define the time asymptotic observability constant $C_\infty^{(W)}(\chi_\omega)$ as the largest possible nonnegative constant for which the time asymptotic observability inequality

$$C_\infty^{(W)}(\chi_\omega)\|(y^0,y^1)\|^2_{H^1(\Omega,\mathbb{C})\times L^2(\Omega,\mathbb{C})} \leqslant \lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_\omega \left(|\partial_t y(t,x)|^2 + |y(t,x)|^2\right)\, dV_g\, dt \tag{87}$$

holds, for all $(y^0,y^1) \in H^1(\Omega,\mathbb{C}) \times L^2(\Omega,\mathbb{C})$. Similarly, we define the randomized observability constant $C^{(W)}_{T,\mathrm{rand}}(\chi_\omega)$ as the largest possible nonnegative constant for which the randomized observability inequality

$$C^{(W)}_{T,\mathrm{rand}}(\chi_\omega)\|(y^0,y^1)\|^2_{H^1(\Omega,\mathbb{C})\times L^2(\Omega,\mathbb{C})} \leqslant \mathbb{E}\left(\int_0^T \int_\omega \left(|\partial_t y_\nu(t,x)|^2 + |y_\nu(t,x)|^2\right)\, dV_g\, dt\right) \tag{88}$$

holds, for all $y^0(\cdot) \in H^1(\Omega,\mathbb{C})$ and $y^1(\cdot) \in L^2(\Omega,\mathbb{C})$, where y_ν is defined as before by (26). The time asymptotic and randomized observability constants are defined accordingly for the Schrödinger equation. We have the following result, showing that we recover the same criterion as before.

Theorem 12. *Let ω be a measurable subset of Ω.*

1. *If the domain Ω is such that every eigenvalue of the Neumann-Laplacian is simple, then $2\,C_\infty^{(W)}(\chi_\omega) = C_\infty^{(S)}(\chi_\omega) = J(\chi_\omega)$.*
2. *There holds $2\,C^{(W)}_{T,\mathrm{rand}}(\chi_\omega) = C^{(S)}_{T,\mathrm{rand}}(\chi_\omega) = TJ(\chi_\omega)$.*

Proof. Following the same lines as those in the proofs of Theorems 4 and 5, we obtain $C^{(W)}_{T,\mathrm{rand}}(\chi_\omega) = TC^{(W)}_\infty(\chi_\omega) = T\,\Gamma$, with

$$\Gamma = \inf_{((a_j),(b_j))\in(\ell^2(\mathbb{C}))^2\setminus\{0\}} \frac{\sum_{j=1}^{+\infty}(1+\lambda_j^2)(|a_j|^2+|b_j|^2)\int_\omega \phi_j(x)^2\,dV_g}{\sum_{j=1}^{+\infty}\left(2\lambda_j^2(|a_j|^2+|b_j|^2)+|a_j+b_j|^2\right)}.$$

Let us prove that $\Gamma = \frac{1}{2}J(\chi_\omega)$. First of all, it is easy to see that, in the definition of Γ, it suffices to consider the infimum over real sequences (a_j) and (b_j). Next, setting $a_j = \rho_j\cos\theta_j$ and $b_j = \rho_j\sin\theta_j$, since $|a_j+b_j|^2 = \rho_j^2(1+\sin(2\theta_j))$, to reach the infimum one has to take $\theta_j = \pi/4$ for every $j\in\mathbb{N}^*$. It finally follows that

$$\Gamma = \inf_{\substack{(\rho_j)\in\ell^2(\mathbb{R})\\ \sum_{j=1}^{+\infty}\rho_j^2=1}} \frac{1}{2}\sum_{j=1}^{+\infty}\rho_j^2\int_\omega \phi_j(x)^2\,dV_g = \frac{1}{2}J(\chi_\omega).$$

□

6.2 Optimal shape and location of internal controllers

In this section, we investigate the question of determining the shape and location of the control domain for wave or Schrödinger equations that minimizes the L^2 norm of the controllers realizing null controllability. In particular, we explain why this optimization problem is exactly equivalent to the problem of maximizing the observability constant. For the sake of simplicity, we will only deal with the wave equation, the Schrödinger case being easily adapted from that case. Also, without loss of generality we restrict ourselves to Dirichlet boundary conditions.

Consider the internally controlled wave equation on Ω with Dirichlet boundary conditions

$$\begin{array}{ll} \partial_{tt}y(t,x) - \triangle_g y(t,x) = h_\omega(t,x), & (t,x)\in(0,T)\times\Omega,\\ y(t,x) = 0, & (t,x)\in[0,T]\times\partial\Omega,\\ y(0,x) = y^0(x),\ \partial_t y(0,x) = y^1(x), & x\in\Omega, \end{array} \tag{89}$$

where h_ω is a control supported in $[0,T]\times\omega$ and ω is a measurable subset of Ω. Note that the Cauchy problem (89) is well posed for all initial data $(y^0,y^1)\in H_0^1(\Omega,\mathbb{C})\times L^2(\Omega,\mathbb{C})$ and every $h_\omega\in L^2((0,T)\times\Omega,\mathbb{C})$, and its solution y belongs to $C^0(0,T;H_0^1(\Omega,\mathbb{C}))\cap C^1(0,T;L^2(\Omega,\mathbb{C}))\cap C^2(0,T;H^{-1}(\Omega,\mathbb{C}))$. The exact null controllability problem settled in these spaces consists of finding a control h_ω steering the control system (89) to $y(T,\cdot) = \partial_t y(T,\cdot) = 0$. It is well known that, for every subset ω of Ω of positive measure, the exact null controllability problem is by duality equivalent to the fact that the observability inequality

$$C\|(\phi^0,\phi^1)\|^2_{L^2(\Omega,\mathbb{C})\times H^{-1}(\Omega,\mathbb{C})} \leqslant \int_0^T\int_\omega |\phi(t,x)|^2\,dV_g\,dt, \tag{90}$$

holds, for all $(\phi^0,\phi^1)\in L^2(\Omega,\mathbb{C})\times H^{-1}(\Omega,\mathbb{C})$, for a positive constant C (only depending on T and ω), where ϕ is the (unique) solution of the adjoint system

$$\begin{array}{ll} \partial_{tt}\phi(t,x) - \triangle_g\phi(t,x) = 0, & (t,x)\in(0,T)\times\Omega,\\ \phi(t,x) = 0, & (t,x)\in[0,T]\times\partial\Omega,\\ \phi(0,x) = \phi^0(x),\ \partial_t\phi(0,x) = \phi^1(x), & x\in\Omega. \end{array} \tag{91}$$

The Hilbert Uniqueness Method (HUM, see [40, 41]) provides a way to design the unique control solving the above exact null controllability problem and having moreover a minimal $L^2((0,T)\times$

$\Omega, \mathbb{C})$ norm. This control is referred to as the HUM control and is characterized as follows. Define the HUM functional $\mathcal{J}_\omega$ by

$$\mathcal{J}_\omega(\phi^0, \phi^1) = \frac{1}{2}\int_0^T \int_\omega \phi(t,x)^2 \, dV_g \, dt - \langle \phi^1, y^0 \rangle_{H^{-1}, H_0^1} + \langle \phi^0, y^1 \rangle_{L^2}. \tag{92}$$

The notation $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1}$ stands for the duality bracket between $H^{-1}(\Omega, \mathbb{C})$ and $H_0^1(\Omega, \mathbb{C})$, and the notation $\langle \cdot, \cdot \rangle_{L^2}$ stands for the usual scalar product of $L^2(\Omega, \mathbb{C})$. If (90) holds then the functional $\mathcal{J}_\omega$ has a unique minimizer (still denoted (ϕ^0, ϕ^1)) in the space $L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$, for all $(y^0, y^1) \in H_0^1(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{C})$. The HUM control h_ω steering (y^0, y^1) to $(0,0)$ in time T is then given by

$$h_\omega(t,x) = \chi_\omega(x)\phi(t,x), \tag{93}$$

for almost all $(t,x) \in (0,T) \times \Omega$, where ϕ is the solution of (91) with initial data (ϕ^0, ϕ^1) minimizing $\mathcal{J}_\omega$. The HUM operator Γ_ω is then defined by

$$\begin{array}{rccc} \Gamma_\omega : & H_0^1(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{C}) & \longrightarrow & L^2((0,T) \times \Omega, \mathbb{C}) \\ & (y^0, y^1) & \longmapsto & h_\omega \end{array}$$

With this definition, it is a priori natural to model the problem determining the best control domain as the problem of minimizing the norm of the operator Γ_ω,

$$\|\Gamma_\omega\| = \sup \left\{ \frac{\|h_\omega\|_{L^2((0,T)\times\Omega,\mathbb{C})}}{\|(y^0,y^1)\|_{H_0^1(\Omega,\mathbb{C})\times L^2(\Omega,\mathbb{C})}} \mid (y^0,y^1) \in H_0^1(\Omega,\mathbb{C}) \times L^2(\Omega,\mathbb{C}) \setminus \{(0,0)\} \right\}, \tag{94}$$

over the set $\mathcal{U}_L$. We have the following result (generalizing [48] where similar issues were investigated in the one-dimensional case).

Proposition 6. *Let $T > 0$ and let ω be measurable subset of Ω. If $C_T^{(W)}(\chi_\omega) > 0$ then*

$$\|\Gamma_\omega\| = \frac{1}{C_T^{(W)}(\chi_\omega)},$$

and if $C_T^{(W)}(\chi_\omega) = 0$, then $\|\Gamma_\omega\| = +\infty$.

Proof. Denote by ϕ_ω the adjoint state solution of (91) whose initial data minimize the functional $\mathcal{J}_\omega$. Then ϕ_ω can be expanded as

$$\phi_\omega(t,x) = \sum_{j=1}^{+\infty} \left(A_j^\omega e^{i\lambda_j t} + B_j^\omega e^{-i\lambda_j t} \right) \phi_j(x),$$

where the sequences $A = (A_j^\omega)_{j \in \mathbb{N}^*}$ and $B = (B_j^\omega)_{j \in \mathbb{N}^*}$ belong to $\ell^2(\mathbb{C})$ and are determined in function of the initial data $(\phi_\omega^0, \phi_\omega^1)$ minimizing $\mathcal{J}_\omega$. Since $\mathcal{J}_\omega$ is convex, the first-order optimality conditions for the problem of minimizing $\mathcal{J}_\omega$ over $L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$ are necessary and sufficient. In terms of Fourier coefficients, they are written as $\Lambda_\omega(A,B) = C$, where the operator $\Lambda_\omega : (\ell^2(\mathbb{C}))^2 \to (\ell^2(\mathbb{C}))^2$ is defined by

$$\Lambda_\omega(A,B)_j = \int_0^T \int_\omega \sum_{k=1}^{+\infty} (A_k e^{i\lambda_k t} + B_k e^{-i\lambda_k t}) \phi_k(x) \phi_j(x) \overline{\begin{pmatrix} e^{i\lambda_j t} \\ e^{-i\lambda_j t} \end{pmatrix}} \, dV_g \, dt, \tag{95}$$

for every $j \in \mathbb{N}^*$, with the notation $\Lambda_\omega(A,B) = (\Lambda_\omega(A,B)_j)_{j \in \mathbb{N}^*}$, and where

$$C_j = \begin{pmatrix} -\langle \phi_j, \phi^1 \rangle_{L^2, L^2} \\ \lambda_j \langle \phi_j, \phi^0 \rangle_{H^{-1}, H_0^1} \end{pmatrix},$$

for every $j \in \mathbb{N}^*$. For all $(A, B) \in [\ell^2(\mathbb{C})]^2$, one has

$$\langle \Lambda_\omega(A,B),(A,B)\rangle_{(\ell^2(\mathbb{C}))^2} = \int_0^T \int_\omega \left| \sum_{k=1}^{+\infty} (A_k e^{i\lambda_k t} + B_k e^{-i\lambda_k t})\phi_k(x) \right|^2 dV_g\, dt,$$

and it follows that

$$C_T^{(W)}(\chi_\omega) \leqslant \frac{\langle \Lambda_\omega(A,B),(A,B)\rangle_{(\ell^2(\mathbb{C}))^2}}{\|(A,B)\|^2_{(\ell^2(\mathbb{C}))^2}} \leqslant 2T.$$

Indeed, we obtain the left-hand side inequality by definition of the observability constant. The right-hand side one is easily obtained, writing that the integral of a nonnegative function over ω is lower than the integral of the same function over Ω, which permits to use the orthogonality properties of the ϕ_j's. By duality, we deduce that Λ_ω is a continuous symmetric invertible operator from $(\ell^2(\mathbb{C}))^2$ to $(\ell^2(\mathbb{C}))^2$. Note that

$$\|\Gamma_\omega\| = \sup_{C\in(\ell^2(\mathbb{R}))^2\setminus\{0\}} \frac{\langle \Lambda_\omega^{-1}(C), C\rangle_{(\ell^2(\mathbb{R}))^2}}{\|C\|^2_{(\ell^2(\mathbb{R}))^2}} = \sup_{C\in(\ell^2(\mathbb{R}))^2\setminus\{0\}} \frac{\|\Lambda_\omega^{-1/2}(C)\|^2_{(\ell^2(\mathbb{R}))^2}}{\|C\|^2_{(\ell^2(\mathbb{R}))^2}},$$

where $\Lambda_\omega^{-1/2}$ denotes the square root of the operator Λ_ω^{-1}. Setting $\varphi = \Lambda_\omega^{-1/2}(C)$, one computes

$$\begin{aligned}\|\Gamma_\omega\| &= \sup_{\varphi\in(\ell^2(\mathbb{R}))^2\setminus\{0\}} \frac{\|\varphi\|^2_{(\ell^2(\mathbb{R}))^2}}{\|\Lambda_\omega^{1/2}(\varphi)\|^2_{(\ell^2(\mathbb{R}))^2}} = \frac{1}{\inf\left\{ \frac{\|\Lambda_\omega^{1/2}(\varphi)\|^2_{(\ell^2(\mathbb{R}))^2}}{\|\varphi\|^2_{(\ell^2(\mathbb{R}))^2}} \mid \varphi \in (\ell^2(\mathbb{R}))^2 \setminus \{0\}\right\}} \\ &= \frac{1}{\inf\left\{ \frac{\langle \Lambda_\omega(\varphi),\varphi\rangle_{(\ell^2(\mathbb{R}))^2}}{\|\varphi\|^2_{(\ell^2(\mathbb{R}))^2}} \mid \varphi \in (\ell^2(\mathbb{R}))^2 \setminus \{0\}\right\}} = \frac{1}{C_T^{(W)}(\chi_\omega)}.\end{aligned}$$

The conclusion follows. □

This result illustrates the well known duality between controllability and observability, but says moreover that, for the optimal design control problem, one has

$$\inf_{\chi_\omega\in\mathcal{U}_L} \|\Gamma_\omega\| = \left(\sup_{\chi_\omega\in\mathcal{U}_L} C_T^{(W)}(\chi_\omega) \right)^{-1}, \tag{96}$$

and therefore the problem of minimizing $\|\Gamma_\omega\|$ is equivalent to the problem of maximizing the observability constant over $\mathcal{U}_L$. As discussed previously in the article, it is more relevant to maximize rather the randomized observability constant $C_{T,\mathrm{rand}}(\chi_\omega)$ defined by (24) (see Section 2.3). If doing so then all considerations done in this article can be applied to the optimal design control problem as well. The interpretation of this problem in terms of random initial data is however not clear (see next section).

6.3 Open problems

We provide here a list of open problems.

Optimal shape and location of internal controllers with random initial data. In Section 6.2 we have obtained the relation (96), establishing a clear duality relation between the problem of optimal control domain and the problem of optimal observation domain, in their classical, deterministic versions. In this article we have defined the randomized observability constant $C_{T,\mathrm{rand}}(\chi_\omega)$

and shown its relevance in the problem of shape and location of sensors. However the problem of maximizing $C_{T,\mathrm{rand}}(\chi_\omega)$ does not have a nice interpretation in terms of controlling to zero random initial data in time T. The reason is that the randomization procedure does not commute with the operator Λ_ω defined by (95). Actually the set of random initial data that can be steered to 0 in a random way is the image through Λ_ω of the random laws used in the randomization procedure. Since this mapping is viewed as an infinite dimensional symmetric full (i.e., non sparse) matrix, it is not clear then to show that the resulting random laws have nice probability properties.

An alternative way to model the problem of optimal shape and location of internal controllers is to use, instead of the HUM approach, the well known moment method, which leads to define a relevant problem that can be interpreted in terms of random initial data (see the ongoing work[50]).

Optimal stabilization domain. Similar important problems can be addressed as well for stabilization issues. When considering the wave equation with a local damping,

$$\partial_{tt} y = \triangle y - 2k\chi_\omega y_t, \tag{97}$$

with $k > 0$, one can address the question of determining the best possible damping domain ω (in the class $\mathcal{U}_L$), achieving for instance (if possible) the largest possible exponential decay rate. This question was investigated in [23] in the one-dimensional case, on the base of the two following remarks. First, if k tends to $+\infty$ then the decay rate tends to 0 (overdamping phenomenon). Second, it is proved in [15] that (in 1D), if the set ω has a finite number of connected components and if k is small enough, then at the first order the decay rate is equivalent to $k \inf_{j\in\mathbb{N}^*} \int_\omega \sin^2(jx)\,dx$. Therefore, in this 1D case, for k small maximizing the decay rate is then equivalent to the problem (31) in 1D (however, with the additional restriction that the subsets ω consist only of a finite union of intervals and thus cannot be any measurable sets).

Note that, even in 1D, for k not small and not too large the problem of maximizing the decay rate over $\mathcal{U}_L$ is a completely open problem. As is well known, the exponential stability property of (97) is equivalent to the observability property of the corresponding conservative wave equation (1) (see [20]), and by duality this problem is similar to the problem of maximizing the (classical) observability constant $C_T^{(W)}(\chi_\omega)$ over $\mathcal{U}_L$ (see below).

In the multi-dimensional case the situation is much more intricate. Indeed, as proved in [39], the exponential decay rate $\tau(\omega)$ does not coincide in general with the negative of the spectral abscissa $S(\omega)$: it is the minimum of this real number and of a geometric quantity giving an account for the average time spent by geodesics crossing ω (see [22] for a study of this geometric quantity in the square). It is an interesting open problem to study the maximization of this geometric criterion over the set $\mathcal{U}_L$.

It can be noted that the fact that $\tau(\omega) \leqslant -S(\omega)$ and that in multi-D the strict inequality may hold, is similar to the fact, underlined in Remark 4, that $C_T^{(W)}(\chi_\omega) \leqslant C_{T,\mathrm{rand}}^{(W)}(\chi_\omega)$ and that the strict inequality may hold. As recalled above, the exponential stability property of (97) is equivalent to the observability property of the corresponding conservative wave equation (1), and establishing such an equivalence in a randomized context (in the sense of what we developed in Section 2.3) is an open problem. This could give another way to model the optimal stabilization domain, and to make precise some possible relations between $C_{T,\mathrm{rand}}^{(W)}(\chi_\omega)$ and the abovementioned geometric quantities.

Maximization of the deterministic observability constant. As discussed in Section 2.3, the problem of maximizing the (usual) deterministic observability constant $C_T^{(W)}(\chi_\omega)$ (defined by (13)) over $\mathcal{U}_L$ is open, and is difficult due to the crossed terms appearing in the spectral expansion. Although less relevant than the one we considered throughout, this problem is however natural and

interesting. It can be noted that the convexified version of this problem, consisting of maximizing $C_T^{(W)}(a)$ over $\overline{\mathcal{U}}_L$, obviously has some solutions, and again here the question of a gap, and the question of knowing whether the supremum is reached over $\mathcal{U}_L$ (existence of a classical optimal set) are open. Even a truncated version of this criterion is an open problem, that is, the problem of maximizing the lowest eigenvalue of the Gramian matrix whose element row j and column k is $\int_\omega \phi_j(x)\phi_k(x)\,dx$. An interesting problem consists of investigating theoretically or numerically the sequence of maximizing subsets for this truncated problem.

Even in 1D, this problem is open.

As it was noted in Remark 1, in the one-dimensional case and if T is an integer multiple of 2π then the crossed terms disappear and the Gramian matrix is diagonal, but if T is not an integer multiple of 2π then owing to the crossed terms the functional cannot be handled easily. Similar difficulties due to crossed terms are encountered in the open problem of determining the best constants in Ingham's inequality (see [28]), according to which, for every $\gamma > 0$ and every $T > \frac{2\pi}{\gamma}$, there exist $C_1(T,\gamma) > 0$ and $C_2(T,\gamma) > 0$ such that for every sequence $(\lambda_n)_{n\in\mathbb{N}^*}$ of real numbers satisfying $|\lambda_{n+1} - \lambda_n| \geqslant \gamma$ for every $n \in \mathbb{N}^*$, there holds

$$C_1(T,\gamma) \sum_{n\in\mathbb{N}^*} |a_n|^2 \leqslant \int_0^T \left| \sum_{n\in\mathbb{N}^*} a_n \mathrm{e}^{i\lambda_n t} \right|^2 dt \leqslant C_2(T,\gamma) \sum_{n\in\mathbb{N}^*} |a_n|^2,$$

for every $(a_n)_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{C})$ (see, e.g., [29, 31, 33, 59]).

Dependence on time. Instead of maximizing the observability constant over $\mathcal{U}_L$, for a fixed time T, one can think of running the optimization also over the time.

Before setting this problem, let us make the following remark in 1D. Setting $\Omega = [0,\pi]$ (with Dirichlet boundary conditions), it is clear that if $T \geqslant 2\pi$ then the observability inequality (12) is satisfied for every subset ω of positive measure. However 2π is not the smallest possible time for a given specific choice of ω. For instance if ω is a subinterval of $[0,\pi]$ then the smallest possible time for which the observability inequality holds is $2\,\mathrm{diam}((0,\pi)\setminus\omega)$. The question of determining this minimal time is nontrivial if, instead of an interval, the set ω is a fractal set. We settle the following open problem (not only in 1D but also in general): given $L \in (0,1)$, does there exist a time $T_L > 0$ such that the observability inequality (12) holds for every $\omega \in \mathcal{U}_L$ and every $T \geqslant T_L$?

Having in mind this open question, it is interesting to investigate the problem of maximizing the functional $(\chi_\omega, T) \mapsto C_T(\chi_\omega)$ over the set $\mathcal{U}_L \times (0,+\infty)$. Similar questions arise can as well be addresses when the unknown (χ_ω, T) runs over a class which is not necessarily cylindrical but is rather a measurable space-time set having a certain fixed measure. For such problems note that the existence of a maximizer is easy to derive when considering their convexified version, but then the question of proving a no-gap result is nontrivial and has not been studied. Also, it is interesting to investigate whether or not the supremum is reached in the class of classical sets.

Nonexistence of an optimal set. In Section 4.1, using harmonic analysis we have proved that, in 1D, the supremum of J over $\mathcal{U}_L$ is reached if and only if $L = 1/2$ (and there is an infinite number of optimal sets). In the Euclidean square the question is open, however if the supremum is considered only over sets to Cartesian products of 1D subsets, then it is reached if and only if $L \in \{1/4, 1/2, 3/4\}$. In general, the question of the existence of an optimal set is completely open, and we expect that the supremum is not reached for generic domains Ω and generic values of L.

This conjecture is in accordance with the observed increasing complexity of the sequence of optimal sets ω^N solutions of the problem of maximizing the truncated spectral criterion J_N. An interesting question occurs here. In the (certainly) nongeneric case where an optimal set does exist (like in 1D for $L = 1/2$ where there is an infinite number of optimal sets), what is the limit of

the sets ω^N? More precisely, can it happen that ω^N converges to a set (if it does) which is of fractal type? The study of [49], done for fixed initial data, indicates that it might be the case. The question is however completely open.

Note that the spillover phenomenon was proved to occur in 1D for L sufficiently small, according to which the optimal set ω^N maximizing J_N is, loosely speaking, the worst possible one for the problem of maximizing J_{N+1}. Proving this fact in a more general context is an open problem.

Besides, note that J_N is defined as a truncation of the functional J, keeping the N first modes. It would be interesting to consider similar optimal design problems running for instance over initial data whose Fourier coefficients satisfy a uniform exponential decreasing property. Another possibility is to truncate the Fourier series and keep only the modes whose index is between two integers N_1 and N_2.

Weighted observability inequalities as a remedy. In view of providing a physically relevant remedy to the problem of nonexistence of an optimal set, in Section 4.4 we introduced a weighted version of the observability inequality, which is however equivalent to the classical one. We proved that, if $L > \lambda_1^2/(\sigma + \lambda_1^2)$ then there exists a unique optimal set, which is moreover the limit of the stationary sequence of optimal set ω^N of the truncated criterion. Our simulations indicate that this threshold in L is sharp. It is an open question to investigate the situation where $L \leqslant \lambda_1^2/(\sigma + \lambda_1^2)$: is there a gap or not between the problem and its convexified version? Is the supremum over $\mathcal{U}_L$ reached or not?

Quantum ergodicity assumptions. Theorem 6 has been established under WQE on the base and uniform L^∞ boundedness assumptions. Up to our knowledge, WQE on the base is a new concept and has not been investigated in mathematical physics. It is interesting to study whether this property frequently occurs or not. The uniform L^∞ boundedness is a strong assumption but as already discussed nothing is known on this property in general. We recall that it is conjectured that flat tori are the sole compact manifolds without boundary for which this property holds true.

In Theorem 7, we assumed the stronger QUE on the base, and uniform L^p boundedness. As discussed in Section 3.3, except in 1D up to now no domain is known where these assumptions hold true. The property QUE is attached to a well known conjecture in mathematical physics. With the example of the disk (Proposition 1), we have seen that these assumptions are however not sharp.

Theorem 11, providing the existence of an optimal set for the weighted version of the problem, holds true under L^∞-QUE on the base. The example of the hypercube (Proposition 5) shows that these assumptions are not sharp.

Weakening the sufficient assumptions of these three results is a completely open problem.

Besides of that, note that, concerning the quantum ergodicity assumptions that we used, and the discussion we made in Section 3.3, we used the current state of the art in mathematical physics. The model that we used throughout, based on averaging either with respect to time or with respect to random initial data, leads to a spectral criterion whose solving requires a good knowledge on quantum ergodicity properties which are in the current state of the art not well known. The question is open to look for more robust models in which the solving of an optimal design problem would not require such a fine knowledge of the eigenelements. For instance it is likely that the microlocal methods used in [3] in order to provide an almost necessary and sufficient condition for the observability to hold (the Geometric Control Condition) in terms of geometric rays, should allow one to identify classes of domains where the constant is governed by a finite number of modes.

In brief, it is an open question to model the optimal design problems under consideration (possibly, based on the notion of geometric rays as discussed above) in such a way that the resulting

problem will be both physically and mathematically relevant, and will not require, for its solving, such strong sufficient assumptions than the ones considered here.

Other models. In this article we have modeled and studied the optimal observability problem for wave and Schrödinger equations. It can be noted that, using the randomization procedure or the time averaging procedure that we have introduced on the observability inequalities, the spectral criterion J considered throughout can be derived as well for many other conservative models, however then nothing is known on the probability measures $\mu_j = \phi_j^2\,dx$ where the ϕ_j are the eigenfunctions of the underlying operator. As we have seen, even for the Laplacian the quantum ergodicity properties are widely unknown, and then the situation is even more open for other operators.

For parabolic models the situation seems to go differently. The randomization leads to a weighted spectral criterion similar to J_σ, but where the sequence of weights σ_j is an increasing sequence tending to $+\infty$ (whereas, here, it was an increasing sequence converging to 1). Because of that, in contrast to the results of the present article, it is expected that an optimal set does exist, only under slight assumptions. We refer to [50] for results in that direction.

Also, for such other models, the previous raised questions – optimal shape and location of internal controllers; maximization of the deterministic observability constant – can be as well settled as open problems.

A Appendix: proof of Theorem 5 and of Corollary 1

For the convenience of the reader, we first prove Theorem 5 in the particular case where all the eigenvalues of $\triangle$ are simple (it corresponds exactly to the proof of Corollary 1) and we then give the generalization to the case of multiple eigenvalues.

From (14), we have $y(t,x) = \sum_{j=1}^{+\infty} y_j(t,x)$ with

$$y_j(t,x) = (a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t})\phi_j(x). \tag{98}$$

Without loss of generality, we consider initial data $(y^0, y^1) \in L^2(\Omega,\mathbb{C}) \times H^{-1}(\Omega,\mathbb{C})$ such that $\|(y^0,y^1)\|^2_{L^2\times H^{-1}} = 2$, in other words such that $\sum_{j\in\mathbb{N}^*}(|a_j|^2 + |b_j|^2) = 1$ (using (16)).

Setting

$$\Sigma_T(a,b) = \frac{1}{T}\frac{\int_0^T\int_\omega |y(t,x)|^2\,dx\,dt}{\|(y^0,y^1)\|^2_{L^2\times H^{-1}}} = \frac{1}{2T}\int_0^T\int_\omega |y(t,x)|^2\,dxdt,$$

we write for an arbitrary $N \in \mathbb{N}^*$,

$$\Sigma_T(a,b) = \frac{1}{T}\int_0^T\int_\omega \left(\left|\sum_{j=1}^N y_j(t,x)\right|^2 + \left|\sum_{k=N+1}^{+\infty} y_k(t,x)\right|^2 + 2\Re e\left(\sum_{j=1}^N y_j(t,x)\sum_{k=N+1}^{+\infty}\bar{y}_k(t,x)\right)\right)dx\,dt. \tag{99}$$

Using the assumption that the spectrum of $\triangle$ consists of simple eigenvalues, we have the following result.

Lemma 6. *With the notations above,*

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T\int_\omega\left|\sum_{j=1}^N y_j(t,x)\right|^2 dx\,dt = \sum_{j=1}^N(|a_j|^2+|b_j|^2)\int_\omega \phi_j(x)^2\,dx.$$

Proof of Lemma 6. Since the sum is finite we can invert the infimum (which is a minimum) and the limit. Now, we write

$$\frac{1}{T}\int_0^T\int_\omega\left|\sum_{j=1}^N y_j(t,x)\right|^2 dx\,dt = \frac{1}{T}\sum_{j=1}^N \alpha_{jj}\int_\omega \phi_j(x)^2\,dx + \frac{1}{T}\sum_{j=1}^N\sum_{\substack{k=1\\k\neq j}}^N \alpha_{jk}\int_\omega \phi_j(x)\phi_k(x)\,dx,$$

where α_{jk} is defined by (18). Using (19) and (20), we get

$$\lim_{T\to+\infty}\frac{\alpha_{jj}}{T} = |a_j|^2+|b_j|^2,$$

for every $j\in\mathbb{N}^*$ and, using that the spectrum of $\triangle$ consists of simple eigenvalues,

$$|\alpha_{jk}| \leqslant \frac{4\max_{1\leqslant j,k\leqslant N}(\lambda_j,\lambda_k)}{|\lambda_j^2-\lambda_k^2|}, \tag{100}$$

whenever $j\neq k$. The conclusion follows easily. □

Let us now estimate the remaining terms

$$R = \frac{1}{T}\int_0^T\int_\omega\left|\sum_{j=N+1}^{+\infty} y_j(t,x)\right|^2 dx\,dt$$

and

$$\delta = \frac{1}{T}\Re e\left(\int_0^T\int_\omega\sum_{j=1}^N y_j(t,x)\sum_{k=N+1}^{+\infty}\bar{y}_k(t,x)\,dx\,dt\right)$$

of the right-hand side of (99).

Estimate of R. Using the fact that the ϕ_j's form a Hilbertian basis, we get

$$\begin{aligned} R &\leqslant \frac{1}{T}\int_0^T\int_\Omega\left|\sum_{j=N+1}^{+\infty} y_j(t,x)\right|^2 dx\,dt \\ &= \frac{1}{T}\sum_{j=N+1}^{+\infty}\int_0^T |a_j e^{i\lambda_j t}-b_j e^{-i\lambda_j t}|^2\,dt \\ &= \frac{1}{T}\sum_{j=N+1}^{+\infty}\left(T(|a_j|^2+|b_j|^2)-\frac{1}{\lambda_j}\Re e\left(a_j\bar{b}_j\frac{e^{2i\lambda_j T}-1}{i}\right)\right) \end{aligned}$$

and finally

$$R\leqslant\left(1+\frac{1}{\lambda_N T}\right)\sum_{j=N+1}^{+\infty}(|a_j|^2+|b_j|^2). \tag{101}$$

Estimate of δ. Using (19) and the fact that $\lambda_j \neq \lambda_k$ for every $j \in \{1, \cdots, N\}$ and every $k \geqslant N+1$, we have

$$|\delta| \leqslant \frac{2}{T}(S_1^N + S_2^N + S_3^N + S_4^N),$$

with

$$\begin{aligned}
S_1^N &= \left| \sum_{j=1}^{N} \sum_{k=N+1}^{+\infty} \frac{1}{\lambda_j - \lambda_k} a_j \bar{a}_k e^{i(\lambda_j - \lambda_k)\frac{T}{2}} \sin\left((\lambda_j - \lambda_k)\frac{T}{2}\right) \int_\omega \phi_j(x)\phi_k(x)\, dx \right|, \\
S_2^N &= \left| \sum_{j=1}^{N} \sum_{k=N+1}^{+\infty} \frac{1}{\lambda_j + \lambda_k} a_j \bar{b}_k e^{i(\lambda_j + \lambda_k)\frac{T}{2}} \sin\left((\lambda_j + \lambda_k)\frac{T}{2}\right) \int_\omega \phi_j(x)\phi_k(x)\, dx \right|, \\
S_3^N &= \left| \sum_{j=1}^{N} \sum_{k=N+1}^{+\infty} \frac{1}{\lambda_j + \lambda_k} b_j \bar{a}_k e^{-i(\lambda_j + \lambda_k)\frac{T}{2}} \sin\left((\lambda_j + \lambda_k)\frac{T}{2}\right) \int_\omega \phi_j(x)\phi_k(x)\, dx \right|, \\
S_4^N &= \left| \sum_{j=1}^{N} \sum_{k=N+1}^{+\infty} \frac{1}{\lambda_j - \lambda_k} b_j \bar{b}_k e^{-i(\lambda_j - \lambda_k)\frac{T}{2}} \sin\left((\lambda_j - \lambda_k)\frac{T}{2}\right) \int_\omega \phi_j(x)\phi_k(x)\, dx \right|.
\end{aligned}$$

Let us estimate S_1^N. We write

$$S_1^N = \left| \sum_{j=1}^{N} a_j \int_\omega \phi_j(x) \sum_{k=N+1}^{+\infty} \frac{\bar{a}_k}{\lambda_j - \lambda_k} e^{i(\lambda_j - \lambda_k)\frac{T}{2}} \sin\left((\lambda_j - \lambda_k)\frac{T}{2}\right) \phi_k(x)\, dx \right|,$$

and, using the Cauchy-Schwarz inequality and the fact that the integral of a nonnegative function over ω is lower than the integral of the same function over Ω, one gets

$$\begin{aligned}
S_1^N &\leqslant \sum_{j=1}^{N} |a_j| \left(\int_\Omega \left| \sum_{k=N+1}^{+\infty} \frac{\bar{a}_k}{\lambda_j - \lambda_k} e^{i(\lambda_j - \lambda_k)\frac{T}{2}} \sin\left((\lambda_j - \lambda_k)\frac{T}{2}\right) \phi_k(x) \right|^2 dx \right)^{1/2} \\
&= \sum_{j=1}^{N} |a_j| \left(\sum_{k=N+1}^{+\infty} \frac{|a_k|^2}{(\lambda_j - \lambda_k)^2} \sin^2\left((\lambda_j - \lambda_k)\frac{T}{2}\right) \right)^{1/2}.
\end{aligned}$$

The last equality is established by expanding the square of the sum inside the integral, and by using the fact that the ϕ_k's are orthonormal in $L^2(\Omega)$. Since the spectrum of $\triangle$ consists of simple eigenvalues (assumed to form an increasing sequence), we infer that $\lambda_k - \lambda_j \geqslant \lambda_{N+1} - \lambda_N$ for all $j \in \{1, \cdots, N\}$ and $k \geqslant N+1$, and since $\sum_{j=1}^{+\infty} |a_j|^2 \leqslant 1$, it follows that

$$S_1^N \leqslant \frac{1}{\lambda_{N+1} - \lambda_N} \sum_{j=1}^{N} |a_j| \left(\sum_{k=N+1}^{+\infty} |a_k|^2 \right)^{1/2} \leqslant \frac{N}{\lambda_{N+1} - \lambda_N}.$$

The same arguments lead to the estimates

$$S_2^N \leqslant \frac{N}{\lambda_N}, \qquad S_3^N \leqslant \frac{N}{\lambda_N}, \qquad S_4^N \leqslant \frac{N}{\lambda_{N+1} - \lambda_N},$$

and therefore,

$$|\delta| \leqslant \frac{4N}{T}\left(\frac{1}{\lambda_N} + \frac{1}{\lambda_{N+1} - \lambda_N}\right). \tag{102}$$

Now, combining Lemma 6 with the estimates (101) and (102) yields that for every $\varepsilon > 0$, there exist $N_\varepsilon \in \mathbb{N}^*$ and $T(\varepsilon, N_\varepsilon) > 0$ such that, if $N \geqslant N_\varepsilon$ and $T \geqslant T(\varepsilon, N_\varepsilon)$, then

$$\left| \Sigma_T(a,b) - \sum_{j=1}^{N} (|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 \, dx \right| \leqslant \varepsilon.$$

As an immediate consequence, and using the obvious fact that, for every $\eta > 0$, there exists $N_\eta \in \mathbb{N}^*$ such that, if $N \geqslant N_\eta$ then

$$\left| \sum_{j=1}^{+\infty} (|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 \, dx - \sum_{j=1}^{N} (|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 \, dx \right| \leqslant \eta,$$

one deduces that

$$\lim_{T \to +\infty} \Sigma_T(a,b) = \sum_{j=1}^{+\infty} (|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 \, dx.$$

At this step, we have proved the following lemma, which improves the statement of Lemma 6.

Lemma 7. *Denoting by a_j and b_j the Fourier coefficients of (y^0, y^1) defined by (15), there holds*

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt = \sum_{j=1}^{+\infty} (|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 \, dx.$$

Corollary 1 follows, noting that

$$\inf_{\substack{(a_j),(b_j) \in \ell^2(\mathbb{C}) \\ \sum_{j=1}^{+\infty} (|a_j|^2 + |b_j|^2) = 1}} \sum_{j=1}^{+\infty} (|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 \, dx = \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx.$$

To finish the proof, we now explain how the arguments above can be generalized to the case of multiple eigenvalues. In particular, the statement of Lemma 1 is adapted in the following way.

Lemma 8. *Using the previous notations, one has*

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \int_\omega \left| \sum_{j=1}^{N} y_j(t,x) \right|^2 dx \, dt = \sum_{\substack{\lambda \in U \\ \lambda \leqslant \lambda_N}} \int_\omega \left(\left| \sum_{k \in I(\lambda)} \lambda_k a_k \phi_k(x) \right|^2 + \left| \sum_{k \in I(\lambda)} \lambda_k b_k \phi_k(x) \right|^2 \right) dx.$$

Proof of Lemma 8. Following the proof of Lemma 6, simple computations show that

$$\begin{aligned} \frac{1}{T} \int_0^T \int_\omega \left| \sum_{j=1}^{N} y_j(t,x) \right|^2 dx \, dt \quad &= \quad \frac{1}{T} \sum_{\lambda \in U} \sum_{(j,k) \in I(\lambda)^2} \alpha_{jk} \int_\omega \phi_j(x) \phi_k(x) \, dx \\ &\quad + \frac{1}{T} \sum_{\substack{(\lambda,\mu) \in U^2 \\ \lambda \neq \mu}} \sum_{\substack{j \in I(\lambda) \\ k \in I(\mu)}} \alpha_{jk} \int_\omega \phi_j(x) \phi_k(x) \, dx, \end{aligned}$$

where

$$\lim_{T\to+\infty}\frac{\alpha_{jk}}{T}=\begin{cases} a_j\bar{a}_k+b_j\bar{b}_k & \text{if } (j,k)\in I(\lambda)^2,\\ 0 & \text{if } j\in I(\lambda),\ k\in I(\mu),\ \text{with } (\lambda,\mu)\in U^2 \text{ and } \lambda\neq\mu.\end{cases}$$

The conclusion of the lemma follows. □

Noting that the previous estimates on R and δ are still valid and that

$$\inf_{\substack{(a_j),(b_j)\in\ell^2(\mathbb{C})\\ \sum_{j=1}^{+\infty}(|a_j|^2+|b_j|^2)=1}}\sum_{\substack{\lambda\in U\\ \lambda\leqslant\lambda_N}}\int_\omega\left(\left|\sum_{k\in I(\lambda)}a_k\phi_k(x)\right|^2+\left|\sum_{k\in I(\lambda)}b_k\phi_k(x)\right|^2\right)dx$$

$$=\inf_{\substack{(c_k)_{j\in\mathbb{N}^*}\in\ell^2(\mathbb{C})\\ \sum_{k=1}^{+\infty}|c_k|^2}}\int_\omega\sum_{\lambda\in U}\left|\sum_{k\in I(\lambda)}c_k\phi_k(x)\right|^2dx,$$

Theorem 5 follows.

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