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# Convergence to stable laws in the space $\mathcal{D}$

François Roueff\*

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## Abstract

We study the convergence of centered and normalized sums of i.i.d. random elements of the space  $\mathcal{D}$  of càdlàg functions endowed with Skorohod's  $J_1$  topology, to stable distributions in  $\mathcal{D}$ . Our results are based on the concept of regular variation on metric spaces and on point process convergence. We provide some applications, in particular to the empirical process of the renewal-reward process.

## 1 Introduction and main results

The main aim of this paper is to study the relation between regular variation in the space  $\mathcal{D}_I$  and convergence to stable processes in  $\mathcal{D}_I$ . Let us first describe the framework of regular variation on metric spaces introduced by [Hult and Lindskog \(2005, 2006\)](#). Let  $I$  be a nonempty closed subinterval of  $\mathbb{R}$ . We denote by  $\mathcal{D}_I$  the set of real valued càdlàg functions defined on  $I$ , endowed with the  $J_1$  topology. Let  $\mathcal{S}_I$  be the subset of  $\mathcal{D}_I$  of functions  $x$  such that

$$\|x\|_I = \sup_{t \in I} |x(t)| = 1 .$$

A random element  $X$  in  $\mathcal{D}_I$  is said to be regularly varying if there exists  $\alpha > 0$ , an increasing sequence  $a_n$  and a probability measure  $\nu$  on  $\mathcal{S}_I$ , called the spectral measure, such that

$$\lim_{n \rightarrow \infty} n \mathbb{P} \left( \|X\|_I > a_n x , \frac{X}{\|X\|_I} \in A \right) = x^{-\alpha} \nu(A) , \quad (1)$$

for any Borel set  $A$  of  $\mathcal{S}_I$  such that  $\nu(\partial A) = 0$  where  $\partial A$  is the topological boundary of  $A$ . Then  $\|X\|_I$  has a regularly varying right-tail, the sequence  $a_n$  is regularly varying at infinity with index  $1/\alpha$  and satisfies  $\mathbb{P}(\|X\|_I > a_n) \sim 1/n$ . [Hult and Lindskog \(2006, Theorem 10\)](#) states that (1) is equivalent to the regular variation of the finite dimensional marginal distributions of the process  $X$  together with a certain tightness criterion.

In the finite dimensional case, it is well-known that if  $\{X_n\}$  is an i.i.d. sequence of finite dimensional vectors whose common distribution is multivariate regularly varying, then the sum  $\sum_{i=1}^n X_i$ , suitably centered and normalized converge to an  $\alpha$ -stable distribution. In statistical applications, such sums appear to evaluate the asymptotic behavior of an empirical estimator around its mean. Therefore we will consider centered sums and we shall always

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assume that  $1 < \alpha < 2$ . The case  $\alpha \in (0, 1)$  is actually much simpler. Very general results in the case  $\alpha \in (0, 1)$  can be found in [Davydov et al. \(2008\)](#). In this case no centering is needed to ensure the absolute convergence of the series representation of the limiting process. In contrast if  $\alpha \in (1, 2)$ , the centering raises additional difficulties. This can be seen in [Resnick \(1986\)](#), where the point process of exceedances has been first introduced for deriving the asymptotic behavior of the sum  $S_n = \sum_{i=1}^n X_{i,n}$ , for  $X_{i,n} = Y_i 1_{[i/n, 1]}$  with the  $Y_i$ 's i.i.d. regularly varying in a finite-dimensional space. A thinning of the point process has to be introduced to deal with the centering. In this contribution, we also rely on the point process of exceedances for more general random elements  $X_{i,n}$  valued in  $\mathcal{D}_I$ . Our results include the case treated in ([Resnick, 1986](#), Proposition 3.4), see [Section 3.1](#). However they do not require the centered sum  $S_n - \mathbb{E}[S_n]$  to be a Martingale and the limit process that we obtain is not a Lévy process in general, see the other two examples treated in [Section 3](#). Hence Martingale type arguments as in [Jacod and Shiryaev \(2003\)](#) cannot be used. Our main result is the following.

**Theorem 1.1.** *Let  $\{X_i\}$  be a sequence of i.i.d. random elements of  $\mathcal{D}_I$  with the same distribution as  $X$  and assume that (1) holds with  $1 < \alpha < 2$ . Assume moreover*

(A-i) *For all  $t \in I$ ,  $\nu(\{x \in \mathcal{S}_I, t \in \text{Disc}(x)\}) = 0$ .*

(A-ii) *For all  $\eta > 0$ , we have*

$$\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \left\| \sum_{i=1}^n \left( X_i 1_{\{\|X_i\|_I \leq a_n \epsilon\}} - \mathbb{E}[X 1_{\{\|X\|_I \leq a_n \epsilon\}}] \right) \right\|_I > a_n \eta \right) = 0. \quad (2)$$

*Then  $a_n^{-1} \sum_{i=1}^n \{X_i - \mathbb{E}[X]\}$  converges weakly in  $(\mathcal{D}_I, J_1)$  to an  $\alpha$ -stable process  $\aleph$ , that admits the integral representation*

$$\aleph(t) = c_\alpha \int_{\mathcal{S}_I} w(t) \, dM(w), \quad (3)$$

*where  $M$  is an  $\alpha$ -stable independently scattered random measure on  $\mathcal{S}_I$  with control measure  $\nu$  and skewness intensity  $\beta \equiv 1$  (totally skewed to the right) and  $c_\alpha^\alpha = \Gamma(1 - \alpha) \cos(\pi\alpha/2)$ .*

**Remark 1.2.** *For  $x \in \mathcal{D}_I$ , let the sets of discontinuity points of  $x$  be denoted by  $\text{Disc}(x)$ . If  $x$  and  $y$  are two functions in  $\mathcal{D}_I$ , then, for the  $J_1$  topology, addition may not be continuous at  $(x, y)$  if  $\text{Disc}(x) \cap \text{Disc}(y) \neq \emptyset$ . Condition (A-i) means that if  $W$  is a random element of  $\mathcal{S}_I$  with distribution  $\nu$  then, for any  $t \in I$ ,  $\mathbb{P}(t \in \text{Disc}(W)) = 0$ ; i.e.  $W$  has no fixed jumps. See [Kallenberg \(2002, p. 286\)](#). Condition (A-i) also implies that  $\nu \otimes \nu$ -almost all  $(x, y) \in \mathcal{S}_I \times \mathcal{S}_I$ ,  $x$  and  $y$  have no common jumps. Equivalently, if  $W$  and  $W'$  are i.i.d. random elements of  $\mathcal{S}_I$  with distribution  $\nu$ , then, almost surely,  $W$  and  $W'$  have no common jump. This implies that if  $W_1, \dots, W_n$  are i.i.d. with distribution  $\nu$ , then, almost surely, addition is continuous at the point  $(W_1, \dots, W_n)$  in  $(\mathcal{D}_I, J_1)^n$ . Cf. [Whitt \(1980, Theorem 4.1\)](#).*

It will be useful to extend slightly [Theorem 1.1](#) by considering triangular arrays of independent multivariate càdlàg processes.

To deal with  $\ell$ -dimensional càdlàg functions, for some positive integer  $\ell$ , we endow  $\mathcal{D}_I^\ell$  with  $J_1^\ell$ , the product  $J_1$ -topology, sometimes referred to as the weak product topology (see [Whitt](#)

(2002)). We then let  $\mathcal{S}_{I,\ell}$  be the subset of  $\mathcal{D}_I^\ell$  of functions  $x = (x_1, \dots, x_\ell)$  such that

$$\|x\|_{I,\ell} = \max_{i=1,\dots,\ell} \sup_{t \in I} |x_i(t)| = 1 .$$

Note that in the multivariate setting, we have

$$\text{Disc}(x) = \bigcup_{i=1,\dots,\ell} \text{Disc}(x_i) .$$

We will prove the following slightly more general result.

**Theorem 1.3.** *Let  $(m_n)$  be a nondecreasing sequence of integers tending to infinity. Let  $\{X_{i,n}, 1 \leq i \leq m_n\}$  be an array of independent random elements of  $\mathcal{D}_I^\ell$ . Assume that there exists  $\alpha \in (1, 2)$  and a probability measure  $\nu$  on the Borel sets of  $(\mathcal{S}_{I,\ell}, J_1^\ell)$  such that  $\nu$  satisfies Condition (A-i) and, for all  $x > 0$  and Borel sets  $A$  such that  $\nu(\partial A) = 0$ ,*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{m_n} \mathbb{P} \left( \|X_{i,n}\|_{I,\ell} > x, \frac{X_{i,n}}{\|X_{i,n}\|_{I,\ell}} \in A \right) = x^{-\alpha} \nu(A) , \quad (4)$$

$$\lim_{n \rightarrow \infty} \max_{i=1,\dots,m_n} \mathbb{P} \left( \|X_{i,n}\|_{I,\ell} > x \right) = 0 , \quad (5)$$

$$\lim_{x \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^{m_n} \mathbb{E} \left[ \|X_{i,n}\|_{I,\ell} 1_{\{\|X_{i,n}\|_{I,\ell} > x\}} \right] = 0 . \quad (6)$$

Suppose moreover that, for all  $\eta > 0$ , we have

$$\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \left\| \sum_{i=1}^{m_n} \left( X_{i,n} 1_{\{\|X_{i,n}\|_{I,\ell} \leq \epsilon\}} - \mathbb{E}[X_{i,n} 1_{\{\|X_{i,n}\|_{I,\ell} \leq \epsilon\}}] \right) \right\|_{I,\ell} > \eta \right) = 0 . \quad (7)$$

Then  $\sum_{i=1}^{m_n} \{X_{i,n} - \mathbb{E}[X_{i,n}]\}$  converges weakly in  $(\mathcal{D}_I^\ell, J_1^\ell)$  to an  $\ell$ -dimensional  $\alpha$ -stable process  $\aleph$ , that admits the integral representation given by (3) with  $\mathcal{S}_I$  replaced by  $\mathcal{S}_{I,\ell}$ .

**Remark 1.4.** If  $m_n = n$  and  $\|X_{i,n}\|_{I,\ell} = Y_i/a_n$  where  $\{Y_i, i \geq 1\}$  is an i.i.d. sequence and Condition (4) holds, then the common distribution of the random variables  $Y_i$  has a regularly varying right tail with index  $\alpha$ . It follows that (5) trivially holds and (6) holds by Karamata's Theorem. Note also that, obviously, if moreover  $X_{i,n} = X_i/a_n$  with  $\{X_i, i \geq 1\}$  an i.i.d. sequence valued in  $\mathcal{D}_I^\ell$ , then Condition (4) is equivalent to the regular variation of the common distribution of the  $X_i$ s.

Hence Theorem 1.1 is a special case of Theorem 1.3 which shall be proved in Section 2.6.

We conclude this section with some comments about the  $\alpha$ -stable limit appearing in Theorem 1.1 (or Theorem 1.3). Its finite dimensional distributions are defined by the integral representation (3) and only depend on the probability measure  $\nu$ . If  $X/\|X\|_I$  is distributed according to  $\nu$  and is independent of  $\|X\|_I$ , as in Section 3.2, then Assumption (1) holds straightforwardly and, provided that the negligibility condition (A-ii) holds, a byproduct of Theorem 1.1 is that the integral representation (3) admits a version in  $\mathcal{D}_I$ . The existence of càdlàg versions of  $\alpha$ -stable processes is also a byproduct of the convergence in  $\mathcal{D}_I$  of series representations as recently investigated by Davydov and Dombry (2012) and Basse-O'Connor and Rosiński (2011). We will come back later to this question in Section 3.2 below. For now, let us state an interesting consequence of the Itô-Nisio theorem proved in Basse-O'Connor and Rosiński (2011).

**Lemma 1.5.** *Let  $\alpha \in (1, 2)$ ,  $\nu$  be a probability measure on  $\mathcal{S}_I$  and  $\aleph$  be a process in  $\mathcal{D}_I$  which admits the integral representation (3). Let  $\{\Gamma_i, i \geq 1\}$  be the points of a unit rate homogeneous Poisson point process on  $[0, \infty)$  and  $\{W, W_i, i \geq 1\}$  be a sequence of i.i.d. random elements of  $\mathcal{S}_I$  with common distribution  $\nu$ , independent of  $\{\Gamma_i\}$ . Then  $\mathbb{E}[W]$  defined by  $\mathbb{E}[W](t) = \mathbb{E}[W(t)]$  for all  $t \in I$  is in  $\mathcal{D}_I$  and the series  $\sum_{i=1}^{\infty} \{\Gamma_i^{-1/\alpha} W_i - \mathbb{E}[\Gamma_i^{-1/\alpha}] \mathbb{E}[W]\}$  converges uniformly almost surely in  $\mathcal{D}_I$  to a limit having the same finite dimensional distribution as  $\aleph$ .*

*Proof.* The fact that  $\mathbb{E}[W]$  is in  $\mathcal{D}_I$  follows from dominated convergence and  $\|w\|_I = 1$  a.s. The finite dimensional distributions of  $S_n = \sum_{i=1}^n \{\Gamma_i^{-1/\alpha} W_i - \mathbb{E}[\Gamma_i^{-1/\alpha}] \mathbb{E}[W_i]\}$  converge to those of  $\aleph$  as a consequence of [Samorodnitsky and Taqqu \(1994, Theorem 3.9\)](#). Hence, to obtain the result, it suffices to show that the series  $\sum_{i=1}^{\infty} \{\Gamma_i^{-1/\alpha} W_i - \mathbb{E}[\Gamma_i^{-1/\alpha}] \mathbb{E}[W_i]\}$  converges uniformly a.s. Note that the series  $\sum_{i=1}^{\infty} \{\Gamma_i^{-1/\alpha} - \mathbb{E}[\Gamma_i^{-1/\alpha}]\}$  converges almost surely, thus, writing

$$\sum_{i=1}^{\infty} \{\Gamma_i^{-1/\alpha} W_i - \mathbb{E}[\Gamma_i^{-1/\alpha}] \mathbb{E}[W]\} = \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} \{W_i - \mathbb{E}[W]\} + \mathbb{E}[W] \sum_{i=1}^{\infty} \{\Gamma_i^{-1/\alpha} - \mathbb{E}[\Gamma_i^{-1/\alpha}]\},$$

we can assume without loss of generality that  $\mathbb{E}[W] \equiv 0$ . Define  $T_n = \sum_{i=1}^n i^{-1/\alpha} W_i$ . By Kolmogorov's three series theorem (see [Kallenberg \(2002, Theorem 4.18\)](#)), since  $\sum_{i=1}^{\infty} i^{-2/\alpha} < \infty$  and  $\text{var}(W_i(t)) \leq 1$ , for all  $t \in I$ ,  $T_n(t)$  converges a.s. to a limit, say,  $T_{\infty}(t)$ .

Arguing as in [Davydov and Dombry \(2012\)](#), we apply [Samorodnitsky and Taqqu \(1994, Lemma 1.5.1\)](#) to obtain that the series  $\sum_{i=1}^{\infty} |\Gamma_i^{-1/\alpha} - i^{-1/\alpha}|$  is summable. This implies that the series  $\Delta = \sum_{i=1}^{\infty} (\Gamma_i^{-1/\alpha} - i^{-1/\alpha}) W_i$  is uniformly convergent. Hence  $S_n - T_n$  converges uniformly a.s. to  $\Delta$  and  $\Delta \in \mathcal{D}_I$ . Thus, for all  $t \in I$ ,  $S_n(t)$  converges a.s. to  $\Delta(t) + T_{\infty}(t)$ . Since the finite distributions of  $S_n$  converge weakly to those of  $\aleph$ , which belongs to  $\mathcal{D}_I$  by assumption, we conclude that  $\Delta + T_{\infty}$  has a version in  $\mathcal{D}_I$ . Hence  $T_{\infty}$  also has a version in  $\mathcal{D}_I$ .

We can now apply [Basse-O'Connor and Rosiński \(2011, Theorem 2.1 \(ii\)\)](#) and obtain that, suitably centered,  $T_n$  converges uniformly a.s. Moreover, for each  $t$ , we have  $\mathbb{E}[T_n(t)] = \mathbb{E}[T(t)] = 0$  and

$$\mathbb{E}[|T(t)|^2] = \mathbb{E}[|W(t)|^2] \sum_{i=1}^{\infty} i^{-2/\alpha} \leq \sum_{i=1}^{\infty} i^{-2/\alpha}.$$

Hence  $\{T(t), t \in I\}$  is uniformly integrable. Then [Basse-O'Connor and Rosiński \(2011, Theorem 2.1 \(iii\)\)](#) shows that  $T_n$  converges uniformly a.s. without centering. Thus  $S_n$  also converges almost surely uniformly.  $\square$

**Corollary 1.6.** *The process  $\aleph$  defined in [Theorem 1.1](#) also admits the series representation*

$$\aleph(t) = \sum_{i=1}^{\infty} \{\Gamma_i^{-1/\alpha} W_i - \mathbb{E}[\Gamma_i^{-1/\alpha}] \mathbb{E}[W_1]\}, \quad (8)$$

where  $\{\Gamma_i, W_i, i \geq 1\}$  are as in [Lemma 1.5](#). This series is almost surely uniformly convergent.

It seems natural to conjecture that the limit process in [Theorem 1.1](#) or the sum of the series in [Lemma 1.5](#) is regularly varying with spectral measure  $\nu$  (the distribution of the process  $W$ ). However, such a result is not known to hold generally. It is proved in [Davis and Mikosch \(2008, Section 4\)](#) under the assumption that  $W$  has almost surely continuous paths. Under an additional tightness condition, we obtain the following result.

**Lemma 1.7.** Let  $\alpha \in (1, 2)$ ,  $\nu$  be a probability measure on  $\mathcal{S}_I$  and  $W$  be a random element of  $\mathcal{S}_I$  with distribution  $\nu$ . Assume that  $\mathbb{E}[W]$  is continuous on  $I$  and there exist  $p \in (\alpha, 2]$ ,  $\gamma > 1/2$  and a continuous increasing function  $F$  such that, for all  $s < t < u$ ,

$$\mathbb{E}[|\bar{W}(s, t)|^p] \leq \{F(t) - F(s)\}^\gamma, \quad (9a)$$

$$\mathbb{E}[|\bar{W}(s, t) \bar{W}(t, u)|^p] \leq \{F(u) - F(s)\}^{2\gamma}, \quad (9b)$$

where  $\bar{W}(s, t) = W(t) - W(s) - \mathbb{E}[W(t) - W(s)]$ . Then the stable process  $\aleph$  defined by the integral representation (3) admits a version in  $\mathcal{D}_I$  which is regularly varying in the sense of (1), with spectral measure  $\nu$ .

**Remark 1.8.** Our assumptions on  $W$  are similar to those of [Basse-O'Connor and Rosiński \(2011, Theorem 4.3\)](#) and [Davydov and Dombrý \(2012, Theorem 1\)](#), with a few minor differences. For instance Conditions (9a) and (9b) are expressed on a non-centered  $W$  in these references. Here we only require  $\mathbb{E}[W]$  to be continuous, which, under (9a), is equivalent to Condition (A-i). Indeed take a random element  $W$  in  $\mathcal{S}_I$ . Then, by dominated convergence,  $\mathbb{E}[W]$  is in  $\mathcal{D}_I$ . Condition (9a) implies that  $W - \mathbb{E}[W]$  has no pure jump. Thus under (9a), the process  $W$  has no pure jump if and only if  $\mathbb{E}[W]$  is continuous on  $I$ .

*Proof.* A straightforward adaptation of the arguments of the proof of [Davydov and Dombrý \(2012, Theorem 1\)](#) to the present context shows that the stable process  $\aleph$  defined by (3) admits a version in  $\mathcal{D}_I$ . This fact is also a consequence of [Proposition 3.1](#) below, so we omit the details of the adaptation.

Therefore, we only have to prove that the càdlàg version of  $\aleph$ , still denoted  $\aleph$ , is regularly varying in the sense of (1), with spectral measure  $\nu$ . By [Corollary 1.6](#),  $\aleph$  can be represented as the almost surely uniformly convergent series

$$\sum_{i=1}^{\infty} \{\Gamma_i^{-1/\alpha} W_i - \mathbb{E}[\Gamma_i^{-1/\alpha} W_i]\} = \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} \bar{W}_i + \mathbb{E}[W] \sum_{i=1}^{\infty} \{\Gamma_i^{-1/\alpha} - \mathbb{E}[\Gamma_i^{-1/\alpha}]\},$$

where  $\bar{W}_i = W_i - \mathbb{E}[W]$ . For  $k \geq 1$ , define  $\Sigma_k = \sum_{i=k}^{\infty} \Gamma_i^{-1/\alpha} \bar{W}_i$ . We proceed as in the proof of [Corollary 2.15](#). Conditioning on the Poisson process and applying Burkholder's inequality, we have for any  $t \in I$ , since  $\|W\|_I = 1$ ,

$$\mathbb{E}[|\Sigma_4(t)|^p] \leq C_p \sum_{i=4}^{\infty} \mathbb{E}[\Gamma_i^{-p/\alpha}] < \infty. \quad (10)$$

Similarly, using the conditions (9a) and (9b), we have for some constants  $C$  and  $C'$  only depending on  $p$  and  $\alpha$ , for all  $s < t < u$ ,

$$\begin{aligned} & \mathbb{E}[|\Sigma_4(t) - \Sigma_4(s)|^p \mid \Sigma_4(u) - \Sigma_4(t)] \\ & \leq C \mathbb{E} \left[ \left( \sum_{i=4}^{\infty} \Gamma_i^{-2/\alpha} \right)^p \right] \mathbb{E}[|\bar{W}(s, t) \bar{W}(t, u)|^p] \\ & \quad + C \mathbb{E} \left[ \left( \sum_{i=4}^{\infty} \Gamma_i^{-p/\alpha} \right)^2 \right] \mathbb{E}[|\bar{W}(s, t)|^p] \mathbb{E}[|\bar{W}(t, u)|^p] \\ & \leq C' \{F(u) - F(s)\}^{2\gamma}. \end{aligned} \quad (11)$$

In the first inequality we used the bounds

$$\begin{aligned}
& \mathbb{E} \left[ \left| \sum_{i=4}^{\infty} \Gamma_i^{-2/\alpha} \bar{W}_i(s, t) \bar{W}_i(t, u) \right|^p \right] \\
& \leq 2^p \sum_{i=4}^{\infty} \mathbb{E}[\Gamma_i^{-2p/\alpha}] \mathbb{E} \left[ \left| \bar{W}(s, t) \bar{W}(t, u) - \mathbb{E}[\bar{W}(s, t) \bar{W}(t, u)] \right|^p \right] \\
& \quad + 2^p \mathbb{E} \left[ \left( \sum_{i=4}^{\infty} \Gamma_i^{-2/\alpha} \right)^p \right] \mathbb{E} \left[ \left| \bar{W}_i(s, t) \bar{W}_i(t, u) \right|^p \right] \\
& \leq 2^{2p+2} \mathbb{E} \left[ \left( \sum_{i=4}^{\infty} \Gamma_i^{-2/\alpha} \right)^p \right] \mathbb{E} \left[ \left| \bar{W}(s, t) \bar{W}(t, u) \right|^p \right] , \\
& \mathbb{E} \left[ \left| \sum_{i \neq j \geq 4} \Gamma_i^{-1/\alpha} \Gamma_j^{-1/\alpha} \bar{W}_i(s, t) \bar{W}_j(t, u) \right|^p \right] \\
& \leq \sum_{i \neq j \geq 4} \mathbb{E} \left[ \Gamma_i^{-p/\alpha} \Gamma_j^{-p/\alpha} \right] \mathbb{E} \left[ \left| \bar{W}(s, t) \right|^p \right] \mathbb{E} \left[ \left| \bar{W}(t, u) \right|^p \right] .
\end{aligned}$$

In the second inequality we used [Lemma A.1](#).

The bound (11) and (10) imply that  $\mathbb{E}[\|\Sigma_4\|_I^p] < \infty$ , see ([Billingsley, 1968](#), Chapter 15). Moreover, since  $p/\alpha < 2$  we have, for  $i = 2, 3$ ,  $\mathbb{E}[\Gamma_i^{-p/\alpha}] < \infty$ . Using  $\|W_i\|_I \leq 1$  for  $i = 2, 3$  we finally get that  $\mathbb{E}[\|\Sigma_2\|_I^p] < \infty$  and  $Z$  can be represented as

$$\Gamma_1^{-1/\alpha} \bar{W}_1 + \Sigma_2 + \mathbb{E}[W] \sum_{i=1}^{\infty} \{\Gamma_i^{-1/\alpha} - \mathbb{E}[\Gamma_i^{-1/\alpha}]\} = \Gamma_1^{-1/\alpha} W_1 + T ,$$

where  $T = \Sigma_2 + \mathbb{E}[W] \sum_{i=2}^{\infty} \{\Gamma_i^{-1/\alpha} - \mathbb{E}[\Gamma_i^{-1/\alpha}]\} - \mathbb{E}[\Gamma_1^{-1/\alpha} W_1]$  satisfies  $\mathbb{E}[\|T\|_I^p] < \infty$ . Observe that  $p > \alpha$ . Since  $\Gamma_1^{-1/\alpha}$  has a Frechet distribution with index  $\alpha$ , it holds that  $\Gamma_1^{-1/\alpha} W_1$  is regularly varying with spectral measure  $\nu$ , which concludes the proof.  $\square$

In the next section, we prove some intermediate results needed to prove Theorems [1.1](#) and [1.3](#). In particular, we give a condition for the convergence in  $\mathcal{D}_I$  of the sequence of expectations. This is not obvious, since the expectation functional is not continuous in  $\mathcal{D}_I$ . We provide a criterion for the negligibility condition (7) and for the sake of completeness, we recall the main tools of random measure theory we need. In section [3](#), we give some applications of Theorem [1.1](#).

## 2 Some results on convergence in $\mathcal{D}_I$ and proof of the main results

### 2.1 Convergence of the expectation in $\mathcal{D}_I$

It may happen that a uniformly bounded sequence  $(X_n)$  converges weakly to  $X$  in  $(\mathcal{D}_I, J_1)$  but  $\mathbb{E}[X_n]$  does not converge to  $\mathbb{E}[X]$  in  $(\mathcal{D}_I, J_1)$ . Therefore, to deal with the centering, we will need the following lemma.

**Lemma 2.1.** *Suppose that  $X_n$  converges weakly to  $X$  in  $(\mathcal{D}_I, J_1)$ . Suppose moreover that there exists  $m > 0$  such that  $\sup_n \|X_n\|_I \leq m$  a.s. and  $X$  has no fixed jump, i.e. for all  $t \in I$ ,*

$$\mathbb{P}(t \in \text{Disc}(X)) = 0 .$$

*Then the maps  $\mathbb{E}[X_n] : t \rightarrow \mathbb{E}[X_n(t)]$  and  $\mathbb{E}[X] : t \rightarrow \mathbb{E}[X(t)]$  are in  $\mathcal{D}_I$ ,  $\mathbb{E}[X]$  is continuous on  $I$  and  $\mathbb{E}[X_n]$  converges to  $\mathbb{E}[X]$  in  $(\mathcal{D}_I, J_1)$ .*

*Proof.* Since we have assumed that  $\sup_{n \geq 0} \|X_n\|_I \leq m$ , almost surely, it also holds that  $\|X\|_I \leq m$  almost surely. The fact that  $\mathbb{E}[X_n]$  and  $\mathbb{E}[X]$  are in  $\mathcal{D}_I$  follows by bounded convergence. Because  $X$  has no fixed jump, we also get that  $\mathbb{E}[X]$  is continuous on  $I$ .

By Skorokhod's representation theorem, we can assume that  $X_n$  converges to  $X$  almost surely in  $\mathcal{D}_I$ . By the definition of Skorokhod's metric (see e.g. Billingsley (1968)), there exists a sequence  $(\lambda_n)$  of random continuous strictly increasing functions mapping  $I$  onto itself such that  $\|\lambda_n - \text{id}_I\|_I$  and  $\|X_n - X \circ \lambda_n\|_I$  converge almost surely to 0. By bounded convergence, it also holds that  $\lim_{n \rightarrow \infty} \mathbb{E}[\|X_n - X \circ \lambda_n\|_I] = 0$ . Write now

$$\|\mathbb{E}[X_n] - \mathbb{E}[X]\|_I \leq \|\mathbb{E}[X_n - X \circ \lambda_n]\|_I + \|\mathbb{E}[X \circ \lambda_n - X]\|_I .$$

The first term on the right-hand side converges to zero so we only consider the second one. Denote the oscillation of a function  $x$  on a set  $A$  by

$$\text{osc}(x; A) = \sup_{t \in A} x(t) - \inf_{t \in A} x(t) . \quad (12)$$

Let the open ball centered at  $t$  with radius  $r$  be denoted by  $B(t, r)$ . Since  $X$  is continuous at  $t$  with probability one, it holds that  $\lim_{r \rightarrow 0} \text{osc}(X; B(t, r)) = 0$ , almost surely. Since  $\|X\|_I \leq m$  almost surely, by dominated convergence, for each  $t \in I$ , we have

$$\lim_{r \rightarrow 0} \mathbb{E}[\text{osc}(X; B(t, r))] = 0 .$$

Let  $\eta > 0$  arbitrary. For each  $t \in I$  there exists  $r(t, \eta) \in (0, \eta) > 0$  such that

$$\mathbb{E}[\text{osc}(X; B(t, r(t, \eta)))] \leq \eta .$$

Since  $I$  is compact, it admits a finite covering by balls  $B(t_i, \epsilon_i)$ ,  $i = 1, \dots, p$  with  $\epsilon_i = r(t_i, \eta)/2$ . Fix some  $\zeta \in (0, \min_{1 \leq i \leq p} \epsilon_i)$ . Then, for  $s \in B(t_i, \epsilon_i)$  and by choice of  $\zeta$ , we have

$$\begin{aligned} |\mathbb{E}[X \circ \lambda_n(s)] - \mathbb{E}[X(s)]| &\leq \mathbb{E}[|X \circ \lambda_n(s) - X(s)| 1_{\{\|\lambda_n - \text{id}_I\|_I \leq \zeta\}}] + 2m\mathbb{P}(\|\lambda_n - \text{id}_I\|_I > \zeta) \\ &\leq \mathbb{E}[\text{osc}(X; B(t_i, r(t_i, \eta)))] + 2m\mathbb{P}(\|\lambda_n - \text{id}_I\|_I > \zeta) \\ &\leq \eta + 2m\mathbb{P}(\|\lambda_n - \text{id}_I\|_I > \zeta) . \end{aligned}$$

The last term does not depend on  $s$ , thus

$$\|\mathbb{E}[X \circ \lambda_n] - \mathbb{E}[X]\|_I \leq \eta + 2m\mathbb{P}(\|\lambda_n - \text{id}_I\|_I > \zeta) .$$

Since  $\|\lambda_n - \text{id}_I\|_I$  converges almost surely to zero, we obtain that

$$\limsup_{n \rightarrow \infty} \|\mathbb{E}[X \circ \lambda_n] - \mathbb{E}[X]\|_I \leq \eta .$$

Since  $\eta$  is arbitrary, this concludes the proof.  $\square$



**Remark 2.2.** Observe that, since  $\mathbb{E}[X]$  is continuous, the convergence  $\mathbb{E}[X_n] \rightarrow \mathbb{E}[X]$  in the  $J_1$  topology implies the uniform convergence so it is not surprising that we did obtain uniform convergence in the proof.

**Remark 2.3.** Under the stronger assumption that  $X_n$  converges uniformly to  $X$ , [Lemma 2.1](#) trivially holds since

$$\|\mathbb{E}[X_n - X]\|_I \leq \mathbb{E}[\|X_n - X\|_I] ,$$

and the result then follows from dominated convergence. If  $X$  is a.s. continuous the uniform convergence follows from the convergence in the  $J_1$  topology. If  $X_n$  is a sum of independent variables converging weakly in the  $J_1$  topology to  $X$  with no pure jumps then the convergence in the  $J_1$  topology again implies the uniform convergence. See [Basse-O'Connor and Rosiński \(2011, Corollary 2.2\)](#). However, under our assumptions, the uniform convergence does not always hold, see [Example 2.5](#) below.

**Remark 2.4.** We can rephrase [Lemma 2.1](#). Let  $m > 0$ . Consider the closed subset of  $\mathcal{D}_I$

$$\mathcal{B}_I(m) = \{x \in \mathcal{D}(I), \|x\|_I \leq m\} ,$$

i.e. the closed ball centered at zero with radius  $m$  in the uniform metric. Let  $\mathcal{M}_0$  be the set of probability measures  $\xi$  on  $\mathcal{B}_I(m)$  such that, for all  $t \in I$ ,  $x$  is continuous at  $t$   $\xi$ -a.s., i.e.

$$\forall t \in I, \quad \xi(\{x \in \mathcal{B}_I(m), t \in \text{Disc}(x)\}) = 0 .$$

Then [Lemma 2.1](#) means that the map  $\xi \mapsto \int x \xi(dx)$  defined on the set of probability measures on  $\mathcal{B}_I(m)$  endowed with the topology of weak convergence takes its values in  $\mathcal{D}_I$  (endowed with the  $J_1$  topology), and is continuous on  $\mathcal{M}_0$ . In other words, if  $\{\xi_n\}$  is a sequence of probability measures on  $(\mathcal{D}_I, J_1)$  which converges weakly to  $\xi$ , such that  $\xi_n(\mathcal{B}_I(m)) = 1$  and  $\xi \in \mathcal{M}_0$ , then  $\int x \xi_n(dx)$  converges to  $\int x \xi(dx)$  in  $(\mathcal{D}_I, J_1)$ .

The continuity of the map  $\xi \mapsto \int x \xi(dx)$  is not true out of  $\mathcal{M}_0$ , see [Example 2.6](#) and [Example 2.7](#) below.

**Example 2.5.** For  $I = [0, 1]$ , set  $X_n = 1_{[U(n-1)/n, 1]}$  and  $X = 1_{[U, 1]}$  with  $U$  uniform on  $[0, 1]$ . Then the assumptions of [Lemma 2.1](#) hold. However,  $X_n$  converges a.s. to  $X$  in the  $J_1$  topology but not uniformly.

Let us now provide counter examples in the case where the assumption of [Lemma 2.1](#) on the limit  $X$  is not satisfied.

**Example 2.6.** Let  $I = [0, 1]$ ,  $X = 1_{[1/2, 1]}$  and  $X_n = 1_{[U_n, 1]}$  where  $U_n$  is drawn uniformly on  $[1/2 - 1/n, 1/2]$ . Then  $X_n \rightarrow X$  a.s. in  $\mathcal{D}_I$  but  $\mathbb{E}[X_n]$  does not converge to  $\mathbb{E}[X] = X$  in the  $J_1$ -topology, though it does converge in the  $M_1$ -topology.

**Example 2.7.** Set  $X_n = 1_{[u_n, 1]}$  for all  $n$  with probability  $1/2$  and  $X_n = -1_{[v_n, 1]}$  for all  $n$  with probability  $1/2$ , where  $u_n = 1/2 - 1/n$  and  $v_n = 1/2 - 1/(2n)$ . In the first case  $X_n \rightarrow 1_{[1/2, 1]}$  in  $\mathcal{D}_I$  with  $I = [0, 1]$  and in the second case,  $X_n \rightarrow -1_{[1/2, 1]}$  in  $\mathcal{D}_I$ . Hence  $X_n \rightarrow X$  a.s. in  $\mathcal{D}_I$  for  $X$  well chosen. On the other hand, we have  $\mathbb{E}[X_n] = 1_{[u_n, v_n]}$  which converges uniformly to the null function on  $[0, u] \cup [1/2, 1]$  for all  $u \in (0, 1/2)$ , but whose sup on  $I = [0, 1]$  does not converge to 0; hence  $\mathbb{E}[X_n]$  cannot converge in  $\mathcal{D}_I$  endowed with  $J_1$ , nor with the other usual distances on  $\mathcal{D}_I$  such as the  $M_1$  distance.

The assumption that  $\sup_n \|X_n\|_I \leq m$  a.s. can be replaced by a uniform integrability assumption. Using a truncation argument, the following corollary is easily proved. The extension of the univariate case to the multivariate one is obvious in the product topology so we state the result in a multivariate setting.

**Corollary 2.8.** *Suppose that  $X_n$  converges weakly to  $X$  in  $(\mathcal{D}_I^\ell, J_1^\ell)$ . Suppose moreover that  $X$  has no fixed jump and  $\{\|X_n\|_{I,\ell}, n \geq 1\}$  is uniformly integrable, that is,*

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left[ \|X_n\|_{I,\ell} 1_{\{\|X_n\|_{I,\ell} > M\}} \right] = 0 .$$

*Then the maps  $\mathbb{E}[X_n] : t \rightarrow \mathbb{E}[X_n(t)]$  and  $\mathbb{E}[X] : t \rightarrow \mathbb{E}[X(t)]$  are in  $\mathcal{D}_I^\ell$ ,  $\mathbb{E}[X]$  is continuous on  $I$  and  $\mathbb{E}[X_n]$  converges to  $\mathbb{E}[X]$  in  $(\mathcal{D}_I^\ell, J_1^\ell)$ .*

## 2.2 Weak convergence of random measures

Let  $\mathcal{X}$  be a complete separable metric space (CSMS). Let  $\mathcal{M}(\mathcal{X})$  denote the set of boundedly finite nonnegative Borel measures  $\mu$  on  $\mathcal{X}$ , i.e. such that  $\mu(A) < \infty$  for all bounded Borel sets  $A$ . A sequence  $(\mu_n)$  of elements of  $\mathcal{M}(\mathcal{X})$  is said to converge weakly to  $\mu$ , noted by  $\mu_n \rightarrow_{w\#} \mu$ , if  $\lim_{n \rightarrow \infty} \mu_n(f) = \mu(f)$  for all continuous functions  $f$  with bounded support in  $\mathcal{X}$ . The weak convergence in  $\mathcal{M}(\mathcal{X})$  is metrizable in such a way that  $\mathcal{M}(\mathcal{X})$  is a CSMS, see [Daley and Vere-Jones \(2003, Theorem A2.6.III\)](#). We denote by  $\mathcal{B}(\mathcal{M}(\mathcal{X}))$  the corresponding Borel sigma-field.

Let  $(M_n)$  be a sequence of random elements of  $(\mathcal{M}(\mathcal{X}), \mathcal{B}(\mathcal{M}(\mathcal{X})))$ . Then, by [Daley and Vere-Jones \(2008, Theorem 11.1.VII\)](#),  $M_n$  converges weakly to  $M$ , noted  $M_n \Rightarrow M$ , if and only if

$$(M_n(A_1), \dots, M_n(A_k)) \Rightarrow (M(A_1), \dots, M(A_k)) \quad \text{in } \mathbb{R}^k$$

for all  $k = 1, 2, \dots$ , and all bounded sets  $A_1, \dots, A_k$  in  $\mathcal{B}(\mathcal{M}(\mathcal{X}))$  such that  $M(\partial A_i) = 0$  a.s. for all  $i = 1, \dots, k$ . As stated in [Daley and Vere-Jones \(2008, Proposition 11.1.VIII\)](#), this is equivalent to the pointwise convergence of the Laplace functional of  $M_n$  to that of  $M$ , that is,

$$\lim_{n \rightarrow \infty} \mathbb{E}[e^{-M_n(f)}] = \mathbb{E}[e^{-M(f)}] , \tag{13}$$

for all bounded continuous function  $f$  with bounded support.

A point measure in  $\mathcal{M}(\mathcal{X})$  is a measure which takes integer values on the bounded Borel sets of  $\mathcal{X}$ . A point process in  $\mathcal{M}(\mathcal{X})$  is a random point measure in  $\mathcal{M}(\mathcal{X})$ . In particular a Poisson point process has an intensity measure in  $\mathcal{M}(\mathcal{X})$ . In the following, we shall denote by  $\mathcal{N}(\mathcal{X})$  the set of point measures in  $\mathcal{M}(\mathcal{X})$  and by  $\mathcal{M}_f(\mathcal{X})$  the set of finite measures in  $\mathcal{M}(\mathcal{X})$ .

Consider now the space  $\mathcal{D}_I$  endowed with the  $J_1$  topology. Let  $\delta$  be a bounded metric generating the  $J_1$  topology on  $\mathcal{D}_I$  and which makes it a CSMS, see [Billingsley \(1968, Section 14\)](#). From now on we denote by  $\mathcal{X}_I = (\mathcal{D}_I, \delta \wedge 1)$  this CSMS, all the Borel sets of which are bounded, since we chose  $\delta$  bounded. We further let  $\mathcal{N}^*(\mathcal{X}_I)$  be the subset of point measures  $m$  such that, for all distinct  $x$  and  $y$  in  $\mathcal{D}_I$  such that  $\text{Disc}(x) \cap \text{Disc}(y) \neq \emptyset$ ,  $m(\{x, y\}) < 2$ . In other words,  $m$  is simple (the measure of all singletons is at most 1) and the elements of the (finite) support of  $m$  have disjoint sets of discontinuity points.

**Lemma 2.9.** *Let  $\phi : \mathcal{N}(\mathcal{X}_I) \rightarrow \mathcal{D}_I$  be defined by*

$$\phi(m) = \int w \, m(dw) .$$

*Let  $\mathcal{N}(\mathcal{X}_I)$  be endowed with the  $w^\#$  topology and  $\mathcal{D}_I$  with the  $J_1$  topology. Then  $\phi$  is continuous on  $\mathcal{N}^*(\mathcal{X}_I)$ .*

This result follows from [Whitt \(1980, Theorem 4.1\)](#), which establishes the continuity of the summation on the subset of all  $(x, y) \in \mathcal{D}_I \times \mathcal{D}_I$  (endowed with the product  $J_1$  topology) such that  $\text{Disc}(x) \cap \text{Disc}(y) = \emptyset$ . However it requires some adaptation to the setting of finite point measures endowed with the  $w^\#$  topology. We provide a detailed proof for the sake of completeness.

*Proof.* Let  $(\mu_n)$  be a sequence in  $\mathcal{N}(\mathcal{X}_I)$  and  $\mu \in \mathcal{N}^*(\mathcal{X}_I)$  such that  $\mu_n \rightarrow_{w^\#} \mu$ . We write  $\mu = \sum_{k=1}^p \delta_{y_k}$ , where  $\delta_x$  denotes the unit point mass measure at  $x$  and  $p = \mu(\mathcal{D}_I)$ . Since  $\mu$  is simple, we may find  $r > 0$  such that,  $\mu(B(y_k, r)) = 1$  for all  $k = 1, \dots, p$ , where  $B(x, r) = \{y \in \mathcal{D}_I : \delta(x, y) < r\}$ . Let  $g_k$  be the mapping defined on  $\mathcal{N}(\mathcal{X}_I)$  with values in  $\mathcal{D}_I$  defined by

$$g_k(m) = \begin{cases} 0 & \text{if } m(B(y_k, r/2)) \neq 1, \\ x & \text{otherwise,} \end{cases}$$

where, in the second case,  $x$  is the unique point in the intersection of  $B(y_k, r/2)$  with the support of  $m$ .

Now, for any  $r' \in (0, r)$  and all  $k = 1, \dots, p$ , since  $\partial B(y_k, r') \subset B(y_k, r) \setminus \{y_k\}$ , we have  $\mu(\partial B(y_k, r)) = 0$  and thus

$$\lim_{n \rightarrow \infty} \mu_n(B(y_k, r')) = \mu(B(y_k, r')) = 1 . \quad (14)$$

On the other hand, since  $\mathcal{D}_I$  is endowed with a bounded metric, the definition of the  $w^\#$  convergence implies that

$$\lim_{n \rightarrow \infty} \mu_n(\mathcal{D}_I) = \mu(\mathcal{D}_I) = p .$$

It follows that, for  $n$  large enough,

$$\phi(\mu_n) = \sum_{k=1}^p g_k(\mu_n) .$$

By [Whitt \(1980, Theorem 4.1\)](#), to conclude, it only remains to show that each term of this sum converges to its expected limit, that is, for all  $k = 1, \dots, p$ ,  $g_k(\mu_n) \rightarrow y_k$  in  $\mathcal{X}_I$  as  $n \rightarrow \infty$ .

Using (14), we deduce that, for all  $r' \in (0, r/2)$ ,

$$\limsup_{n \rightarrow \infty} \delta(g_k(\mu_n), y_k) \leq r' ,$$

which shows the continuity of  $g_k$  at  $\mu$  and achieves the proof.  $\square$

To deal with multivariate functions, we endow  $\mathcal{X}_I^\ell$  with the metric

$$\delta_\ell((x_1, \dots, x_\ell), (x'_1, \dots, x'_\ell)) = \sum_{i=1}^{\ell} \delta(x_i, x'_i) ,$$

so that the corresponding topology is the product topology denoted by  $J_1^\ell$ . We immediately get the following result.

**Corollary 2.10.** *Let  $\Phi : \mathcal{N}(\mathcal{X}_I^\ell) \rightarrow \mathcal{D}_I^\ell$  be defined by*

$$\Phi(m) = \int w \, m(dw) .$$

*Let  $\mathcal{N}(\mathcal{X}_I^\ell)$  be endowed with the  $w^\#$  topology and  $\mathcal{D}_I^\ell$  with the  $J_1^\ell$  topology. Then  $\Phi$  is continuous on  $\mathcal{N}^*(\mathcal{X}_I^\ell)$ .*

*Proof.* Let us write  $\Phi = (\Phi_1, \dots, \Phi_\ell)$  with each  $\Phi_i : \mathcal{N}(\mathcal{X}_I^\ell) \rightarrow \mathcal{D}_I$ . Since  $J_1^\ell$  is the product topology, it amounts to show that each component  $\Phi_i$  is continuous on  $\mathcal{N}(\mathcal{X}_I^\ell)$ . We shall do it for  $i = 1$ . Observe that the mapping  $m \mapsto m_1$  defined from  $\mathcal{N}(\mathcal{X}_I^\ell)$  to  $\mathcal{N}(\mathcal{X}_I)$ , by  $m_1(A) = m(A \times \mathcal{D}_I^{\ell-1})$  for all Borel set  $A$  in  $\mathcal{X}_I$  is continuous for the  $\rightarrow_{w^\#}$  topology. Moreover we have  $\Phi_1(m) = \phi(m_1)$  and  $m \in \mathcal{N}^*(\mathcal{X}_I^\ell)$  implies  $m_1 \in \mathcal{N}^*(\mathcal{X}_I)$ . Hence, by [Lemma 2.9](#),  $\Phi_1$  is continuous on  $\mathcal{N}^*(\mathcal{X}_I^\ell)$ , which concludes the proof.  $\square$

We now consider the space  $(0, \infty] \times \mathcal{S}_{I,\ell}$ , which, endowed with the metric

$$d((r, x), (r', x')) = |1/r - 1/r'| + \delta_\ell(x, x') ,$$

is also a CSMS. For convenience, we shall denote the corresponding metric space by

$$\mathcal{Y}_{I,\ell} = ((0, \infty] \times \mathcal{S}_{I,\ell}, d) .$$

In this case the necessary and sufficient condition [\(13\)](#) must be checked for all bounded continuous function  $f$  defined on  $\mathcal{Y}_{I,\ell}$  and vanishing on  $(0, \eta) \times \mathcal{S}_{I,\ell}$  for some  $\eta > 0$ .

The following consequence of [Corollary 2.10](#) will be useful.

**Corollary 2.11.** *Let  $\mu \in \mathcal{M}(\mathcal{Y}_{I,\ell})$  and  $\epsilon > 0$  be such that*

(D-i) *for all  $t \in I$ ,  $\mu(\{(y, x) \in \mathcal{Y}_{I,\ell}, t \in \text{Disc}(x)\}) = 0$ ,*

(D-ii)  *$\mu(\{\epsilon, \infty\} \times \mathcal{S}_{I,\ell}) = 0$ .*

*Let  $M$  be a Poisson point process on  $\mathcal{Y}_{I,\ell}$  with control measure  $\mu$ . Let  $\{M_n\}$  be a sequence of point processes in  $\mathcal{M}(\mathcal{Y}_{I,\ell})$  which converges weakly to  $M$  in  $\mathcal{M}(\mathcal{Y}_{I,\ell})$ . Then, the weak convergence*

$$\int_{(\epsilon, \infty)} \int_{\mathcal{S}_{I,\ell}} yw \, M_n(dy, dw) \Rightarrow \int_{(\epsilon, \infty)} \int_{\mathcal{S}_{I,\ell}} yw \, M(dy, dw) \quad (15)$$

*holds in  $(\mathcal{D}_I^\ell, J_1^\ell)$  and the limit has no pure jump.*

*Proof.* Let us define the mapping  $\psi : \mathcal{Y}_{I,\ell} \rightarrow \mathcal{X}_I^\ell$  by

$$\psi(y, w) = \begin{cases} yw & \text{if } y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Let further  $\Psi : \mathcal{M}(\mathcal{Y}_{I,\ell}) \rightarrow \mathcal{M}(\mathcal{X}_I^\ell)$  be the mapping defined by

$$[\Psi(m)](A) = m(\psi^{-1}(A) \cap ((\epsilon, \infty) \times \mathcal{S}_{I,\ell})) ,$$

for all Borel subsets  $A$  in  $\mathcal{M}(\mathcal{X}_I^\ell)$ . Since

$$\partial(A \cap ((\epsilon, \infty) \times \mathcal{S}_{I,\ell})) \subset \partial A \cap (\{\epsilon, \infty\} \times \mathcal{S}_{I,\ell}) ,$$

we have that  $m \mapsto m(\cdot \cap ((\epsilon, \infty) \times \mathcal{S}_{I,\ell}))$  is continuous from  $\mathcal{M}(\mathcal{Y}_{I,\ell})$  to  $\mathcal{M}(\mathcal{Y}_{I,\ell})$  on the set

$$\mathcal{A}_1 = \{\mu \in \mathcal{M}(\mathcal{Y}_{I,\ell}) : \mu(\{\epsilon, \infty\} \times \mathcal{S}_{I,\ell}) = 0\} .$$

Using the continuity of  $\psi$  on  $(0, \infty) \times \mathcal{S}_{I,\ell}$ , it is easy to show that  $m \mapsto m \circ \psi^{-1}$  is continuous on

$$\mathcal{A}_2 = \{\mu \in \mathcal{M}(\mathcal{Y}_{I,\ell}) : \exists M > 0, \mu([M, \infty] \times \mathcal{S}_{I,\ell}) = 0\} .$$

Hence  $\Psi$  is continuous on  $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2$ . With [Corollary 2.10](#), we conclude that

$$m \mapsto \int_{(\epsilon, \infty)} \int_{\mathcal{S}_{I,\ell}} yw m(dy, dw) = \int w \Psi(m)(dw)$$

is continuous as a mapping from  $\Psi^{-1}(\mathcal{N}(\mathcal{X}_I))$  endowed with the  $\rightarrow_{w\#}$  topology to  $(\mathcal{D}_I^\ell, J_1^\ell)$  on the set  $\Psi^{-1}(\mathcal{N}^*(\mathcal{X}_I)) \cap \mathcal{A}$ . Thus the weak convergence [\(15\)](#) follows from the continuous mapping theorem, and by observing that the sequence  $(M_n)$  belongs to  $\Psi^{-1}(\mathcal{N}(\mathcal{X}_I))$  for all  $n$  and that, by Conditions [\(D-i\)](#) and [\(D-ii\)](#),  $M$  belongs to  $\Psi^{-1}(\mathcal{N}^*(\mathcal{X}_I)) \cap \mathcal{A}$  a.s. (see [Remark 1.2](#)).

The fact that the limit has no pure jump also follows from Condition [\(D-i\)](#).  $\square$

### 2.3 Convergence in $\mathcal{D}_I^\ell$ based on point process convergence

The truncation approach is usual in the context of regular variation to exhibit  $\alpha$ -stable approximations of the empirical mean of an infinite variance sequence of random variables.

The proof relies on separating small jumps and big jumps and rely on point process convergence. In the following result, we have gathered the main steps of this approach. To our knowledge, such a result is not available in this degree of generality.

**Theorem 2.12.** *Let  $\{N_n, n \geq 1\}$  be a sequence of finite point processes on  $\mathcal{X}$  and  $N$  be a Poisson point process on  $\mathcal{Y}_{I,\ell}$  with mean measure  $\mu$ . Define, for all  $n \geq 1$  and  $\epsilon > 0$ ,*

$$\begin{aligned} S_n &= \int_{(0, \infty)} \int_{\mathcal{S}_{I,\ell}} yw N_n(dy, dw) , \quad S_n^{<\epsilon} = \int_{(0, \epsilon]} \int_{\mathcal{S}_{I,\ell}} yw N_n(dy, dw) , \\ Z_\epsilon &= \int_{(\epsilon, \infty)} \int_{\mathcal{S}_{I,\ell}} yw N(dy, dw) , \end{aligned}$$

*which are well defined in  $\mathcal{D}_I^\ell$  since  $N$  and  $N_n$  have finite supports in  $(\epsilon, \infty) \times \mathcal{S}_{I,\ell}$  and  $(0, \infty) \times \mathcal{S}_{I,\ell}$ , respectively. Assume that the following assertions hold.*

(B-i)  $N_n \Rightarrow N$  in  $(\mathcal{M}(\mathcal{Y}_{I,\ell}), \mathcal{B}(\mathcal{M}(\mathcal{Y}_{I,\ell})))$ .

(B-ii) For all  $t \in I$ ,  $\mu(\{(y, x) \in \mathcal{Y}_{I,\ell}, t \in \text{Disc}(x)\}) = 0$  and  $\mu(\{\infty\} \times \mathcal{S}_{I,\ell}) = 0$ .

(B-iii)  $\int_{(0,1]} y^2 \mu(dy, \mathcal{S}_{I,\ell}) < \infty$ .

(B-iv) For each  $\epsilon > 0$ , the sequence  $\left\{ \int_{(\epsilon, \infty)} y N_n(dy, \mathcal{S}_{I,\ell}), n \geq 1 \right\}$  is uniformly integrable.

(B-v) The following negligibility condition holds : for all  $\eta > 0$ ,

$$\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \|S_n^{<\epsilon} - \mathbb{E}[S_n^{<\epsilon}] \|_{I,\ell} > \eta \right) = 0, \quad (16)$$

Then the following assertions hold.

(C1) For each  $\epsilon > 0$ ,  $Z_\epsilon \in \mathcal{D}_I^\ell$ ,  $\mathbb{E}[Z_\epsilon] \in \mathcal{D}_I^\ell$  and  $Z_\epsilon - \mathbb{E}[Z_\epsilon]$  converges weakly in  $(\mathcal{D}_I^\ell, J_1^\ell)$  to a process  $\bar{Z}$  as  $\epsilon \rightarrow 0$ .

(C2)  $S_n - \mathbb{E}[S_n]$  converges weakly in  $(\mathcal{D}_I^\ell, J_1^\ell)$  to  $\bar{Z}$ .

*Proof.* For  $\epsilon > 0$ , we define

$$S_n^{>\epsilon} = \int_{(\epsilon, \infty)} \int_{\mathcal{S}_{I,\ell}} y w N_n(dy, dw), \quad \bar{S}_n^{>\epsilon} = S_n^{>\epsilon} - \mathbb{E}[S_n^{>\epsilon}], \quad \bar{S}_n^{<\epsilon} = S_n^{<\epsilon} - \mathbb{E}[S_n^{<\epsilon}],$$

which are random elements of  $\mathcal{D}_I^\ell$ . By [Corollary 2.11](#) and Conditions (B-i) and (B-ii), we have that  $S_n^{>\epsilon}$  converges weakly in  $(\mathcal{D}_I^\ell, J_1^\ell)$  to  $Z_\epsilon$ , provided that  $\mu(\{\epsilon\} \times \mathcal{S}_{I,\ell}) = 0$ , and  $Z_\epsilon$  has no pure jump.

Since  $\|S_n^{>\epsilon}\|_{I,\ell} \leq \int_{(\epsilon, \infty)} y N_n(dy, \mathcal{S}_{I,\ell})$ , by Condition (B-iv), we get that  $\{\|S_n^{>\epsilon}\|_{I,\ell}, n \geq 1\}$  is uniformly integrable. Applying [Corollary 2.8](#), we get that  $\mathbb{E}[S_n^{>\epsilon}]$  converges to  $\mathbb{E}[Z_\epsilon]$  in  $(\mathcal{D}_I^\ell, J_1^\ell)$  and that  $\mathbb{E}[Z_\epsilon]$  is continuous on  $I$ . Thus addition is continuous at  $(Z_\epsilon, \mathbb{E}[Z_\epsilon])$ . See [Whitt \(2002, p. 84\)](#). We obtain that, for all  $\epsilon > 0$ , as  $n \rightarrow \infty$ ,

$$\bar{S}_n^{>\epsilon} \Rightarrow Z_\epsilon - \mathbb{E}[Z_\epsilon] \quad \text{in } \mathcal{D}_I. \quad (17)$$

Define  $\bar{S}_n = S_n - \mathbb{E}[S_n]$ . Then

$$\bar{S}_n = \bar{S}_n^{>\epsilon} + \bar{S}_n^{<\epsilon}, \quad (18)$$

and (16) can be rewritten as

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\|\bar{S}_n - \bar{S}_n^{>\epsilon}\|_{I,\ell} > \eta) = 0. \quad (19)$$

By [Billingsley \(1968, Theorem 4.2\)](#), Assertion (C1) and (19) imply Assertion (C2). Hence, to conclude the proof, it remains to prove Assertion (C1), that is,  $\bar{Z}_\epsilon$  converges weakly in  $(\mathcal{D}_I^\ell, J_1^\ell)$  to a process  $\bar{Z}$ . For all  $t \in I$  and  $0 < \epsilon < \epsilon'$ , we have

$$Z_\epsilon(t) - Z_{\epsilon'}(t) = \int_{(\epsilon, \epsilon']} \int_{\mathcal{S}_{I,\ell}} y w(t) N(dy, dw),$$

where  $N$  is a Poisson process with intensity measure  $\mu$ . Thus, denoting by  $|a|$  the Euclidean norm of vector  $a$  and by  $\text{Tr}(A)$  the trace of matrix  $A$ , we have

$$\begin{aligned}\mathbb{E}[|\bar{Z}_\epsilon(t) - \bar{Z}_{\epsilon'}(t)|^2] &= \text{Tr}(\text{Cov}(Z_\epsilon(t) - Z_{\epsilon'}(t))) \\ &= \int_{(\epsilon, \epsilon']} \int_{\mathcal{S}_{I, \ell}} y^2 |w(t)|^2 \mu(dy, dw) \\ &\leq \ell \int_{(\epsilon, \epsilon']} y^2 \mu(dy, \mathcal{S}_{I, \ell}).\end{aligned}$$

We deduce from (B-iii) that  $\bar{Z}_\epsilon(t) - \bar{Z}_1(t)$  converges in  $L^2$  as  $\epsilon$  tends to 0. Thus there exists a process  $\bar{Z}$  such that  $\bar{Z}_\epsilon$  converges to  $\bar{Z}$  pointwise in probability, hence in the sense of finite dimensional distributions. To obtain the convergence in  $(\mathcal{D}_I^\ell, J_1^\ell)$ , since we use the product topology in  $\mathcal{D}_I^\ell$ , it only remains to show the tightness of each component. Thus, hereafter we assume that  $\ell = 1$ . Denote, for  $x \in \mathcal{D}_I$  and  $\delta > 0$ ,

$$w''(x, \delta) = \sup\{|x(t) - x(s)| \wedge |x(u) - x(t)| ; s \leq t \leq u \in I, |u - s| \leq \delta\}. \quad (20)$$

By Billingsley (1968, Theorem 15.3), it is sufficient to prove that, for all  $\eta > 0$ ,

$$\lim_{A \rightarrow \infty} \sup_{0 < \epsilon \leq 1} \mathbb{P}(\|\bar{Z}_\epsilon\|_I > A) = 0, \quad (21)$$

$$\lim_{\delta \downarrow 0} \limsup_{\epsilon \downarrow 0} \mathbb{P}(w''(\bar{Z}_\epsilon, \delta) > \eta) = 0, \quad (22)$$

$$\lim_{\delta \downarrow 0} \limsup_{\epsilon \downarrow 0} \mathbb{P}(\text{osc}(\bar{Z}_\epsilon; [a, a + \delta]) > \eta) = 0, \quad (23)$$

$$\lim_{\delta \downarrow 0} \limsup_{\epsilon \downarrow 0} \mathbb{P}(\text{osc}(\bar{Z}_\epsilon; [b - \delta, b]) > \eta) = 0, \quad (24)$$

where  $I = [a, b]$  and  $\text{osc}$  is defined in (12). We start by proving (21). For any  $\epsilon_0 \in (0, 1]$  and  $\epsilon \in [\epsilon_0, 1]$ , we have  $\|\bar{Z}_\epsilon\|_I \leq \int_{(\epsilon_0, \infty)} y N(dy, \mathcal{S}_I)$ , whence

$$\sup_{\epsilon_0 \leq \epsilon \leq 1} \mathbb{P}(\|\bar{Z}_\epsilon\|_I > A) \leq A^{-1} \mathbb{E} \left[ \int_{(\epsilon_0, \infty)} y N(dy, \mathcal{S}_I) \right],$$

which is finite by Condition (B-iv).

This yields that  $\lim_{A \rightarrow \infty} \sup_{\epsilon_0 \leq \epsilon \leq 1} \mathbb{P}(\|W_\epsilon\|_I > A) = 0$  and to conclude the proof of (21), we only need to show that, for any  $\eta > 0$ ,

$$\lim_{\epsilon_0 \downarrow 0} \sup_{0 < \epsilon < \epsilon_0} \mathbb{P}(\|\bar{Z}_\epsilon - \bar{Z}_{\epsilon_0}\|_I > \eta) = 0. \quad (25)$$

The arguments leading to (17) can be used to show that, for all  $0 < \epsilon < \epsilon_0$ ,

$$\bar{S}_n^{>\epsilon_0} - \bar{S}_n^{>\epsilon} \Rightarrow \bar{Z}_{\epsilon_0} - \bar{Z}_\epsilon \quad \text{in } \mathcal{D}_I. \quad (26)$$

(although the latter is not a consequence of (17) because  $Z_{\epsilon_0}$  and  $Z_\epsilon$  have common jumps). By definition (see (18)), we have  $\bar{S}_n^{<\epsilon_0} - \bar{S}_n^{<\epsilon} = \bar{S}_n^{>\epsilon} - \bar{S}_n^{>\epsilon_0}$ . By (26) and the continuous

mapping theorem, we get that  $\|\bar{S}_n^{<\epsilon_0} - \bar{S}_n^{<\epsilon}\|_I \Rightarrow \|\bar{Z}_{\epsilon_0} - \bar{Z}_\epsilon\|_I$ . Thus, by the Portmanteau Theorem, for all  $\eta > 0$ ,

$$\begin{aligned} \mathbb{P}(\|\bar{Z}_{\epsilon_0} - \bar{Z}_\epsilon\|_I \geq \eta) &= \limsup_{n \rightarrow \infty} \mathbb{P}(\|\bar{S}_n^{<\epsilon_0} - \bar{S}_n^{<\epsilon}\|_I \geq \eta) \\ &\leq \limsup_{n \rightarrow \infty} \mathbb{P}(\|\bar{S}_n^{<\epsilon_0}\|_I \geq \eta/2) + \limsup_{n \rightarrow \infty} \mathbb{P}(\|\bar{S}_n^{<\epsilon}\|_I \geq \eta/2) . \end{aligned}$$

We conclude by applying Condition **(B-v)** which precisely states that both terms in the right-hand side tend to zero as  $\epsilon_0$  tends to 0, for any  $\eta > 0$ . This yields (25) and (21) follows.

Define now the modulus of continuity of a function  $x \in \mathcal{D}_I$  by

$$w(x, \delta) = \sup\{|x(t) - x(s)|, s, t \in I, |t - s| \leq \delta\} .$$

We shall rely on the fact that, for any  $x, y \in \mathcal{D}_I$ ,

$$w''(x + y, \delta) \leq w''(x, \delta) + w(y, \delta) .$$

Note that this inequality is no longer true if  $w(y, \delta)$  is replaced by  $w''(y, \delta)$ . We get that, for any  $0 < \epsilon < \epsilon_0$  and  $\delta > 0$ ,

$$\begin{aligned} w''(\bar{Z}_\epsilon, \delta) &\leq w''(\bar{Z}_{\epsilon_0}, \delta) + w(\bar{Z}_\epsilon - \bar{Z}_{\epsilon_0}, \delta) \\ &\leq w''(\bar{Z}_{\epsilon_0}, \delta) + 2 \|\bar{Z}_\epsilon - \bar{Z}_{\epsilon_0}\|_I . \end{aligned} \tag{27}$$

Since  $\bar{Z}_{\epsilon_0}$  is in  $\mathcal{D}_I$ , we have, for any fixed  $\epsilon_0 > 0$ ,

$$\lim_{\delta \rightarrow 0} \mathbb{P}(w''(\bar{Z}_{\epsilon_0}, \delta) > \eta) = 0 .$$

Hence, with (25), we conclude that (22) holds. Similarly, since, for each subinterval  $T$ , we have

$$\text{osc}(\bar{Z}_\epsilon; T) \leq \text{osc}(\bar{Z}_{\epsilon_0}; T) + 2 \|\bar{Z}_\epsilon - \bar{Z}_{\epsilon_0}\|_I ,$$

so we obtain (23) and (24). This concludes the proof.  $\square$

## 2.4 Regular variation in $\mathcal{D}$ and point process convergence

Let now  $\{X_{i,n}, 1 \leq i \leq m_n\}$  be an array of independent random elements in  $\mathcal{D}_I$  and define the point process of exceedances  $N_n$  on  $(0, \infty] \times \mathcal{S}_{I,\ell}$  by

$$N_n = \sum_{i=1}^{m_n} \delta_{\|X_{i,n}\|_{I,\ell}, \frac{X_{i,n}}{\|X_{i,n}\|_I}} , \tag{28}$$

with the convention that  $\delta_{0,0/0}$  is the null mass. If the processes  $X_{n,i}, 1 \leq i \leq m_n$  are i.i.d. for each  $n$ , then it is shown in [de Haan and Lin \(2001, Theorem 2.4\)](#) that Condition (1) implies the convergence of the sequence of point processes  $N_n$  to a Poisson point process on  $\mathcal{D}_I$ . We slightly extend here this result to triangular arrays of vector valued processes.

Let  $N$  be a Poisson point process on  $\mathcal{Y}_{I,\ell} = (0, \infty] \times \mathcal{S}_{I,\ell}$  (see [Section 2.2](#)) with mean measure  $\mu_\alpha$  defined by  $\mu_\alpha(dydw) = \alpha y^{-\alpha-1} dy \nu(dw)$ .



**Proposition 2.13.** *Conditions (4) and (5) in Theorem 1.3 imply the weak convergence of  $N_n$  to  $N$  in  $(\mathcal{M}(\mathcal{Y}_{I,\ell}), \mathcal{B}(\mathcal{M}(\mathcal{Y}_{I,\ell})))$ .*

*Proof.* As explained in Section 2.2, we only need to check the convergence

$$\lim_{n \rightarrow \infty} \mathbb{E}[e^{-N_n(f)}] = \mathbb{E}e^{-N(f)} \quad (29)$$

for all bounded continuous functions  $f$  on  $(0, \infty] \times \mathcal{S}_{I,\ell}$  and vanishing on  $(0, \epsilon) \times \mathcal{S}_{I,\ell}$  for some  $\epsilon > 0$ . Consider such a function  $f$ . We have

$$\log \mathbb{E}[e^{-N_n(f)}] = \sum_{i=1}^{m_n} \log \left( 1 + \mathbb{E} \left[ g(\|X_{i,n}\|_{I,\ell}, X_{i,n}/\|X_{i,n}\|_{I,\ell}) \right] \right),$$

where we denoted  $g : (y, w) \mapsto e^{-f(y,w)} - 1$ , which is continuous and bounded with the same support as  $f$ . Moreover  $g$  is lower bounded by some constant  $A > -1$ . It follows that there exists a positive constant  $C$  such that

$$\left| \log \left( 1 + \mathbb{E} \left[ g(\|X_{i,n}\|_{I,\ell}, X_{i,n}/\|X_{i,n}\|_{I,\ell}) \right] \right) - \mathbb{E} \left[ g(\|X_{i,n}\|_{I,\ell}, X_{i,n}/\|X_{i,n}\|_{I,\ell}) \right] \right| \leq C \mathbb{P}^2(\|X_{i,n}\|_{I,\ell} > \epsilon).$$

Define  $\delta_n = \max_{i=1, \dots, n} \mathbb{P}(\|X_{i,n}\|_{I,\ell} > \epsilon)$ . Then

$$\left| \log \mathbb{E}[e^{-N_n(f)}] - \sum_{i=1}^{m_n} \mathbb{E} \left[ g(\|X_{i,n}\|_{I,\ell}, X_{i,n}/\|X_{i,n}\|_{I,\ell}) \right] \right| \leq C \delta_n \sum_{i=1}^{m_n} \mathbb{P}(\|X_{i,n}\|_{I,\ell} > \epsilon).$$

Since  $\delta_n = o(1)$  by (5) and  $\sum_{i=1}^{m_n} \mathbb{P}(\|X_{i,n}\|_{I,\ell} > \epsilon) = O(1)$  by (4), we obtain

$$\log \mathbb{E}[e^{-N_n(f)}] = \sum_{i=1}^{m_n} \mathbb{E} \left[ g(\|X_{i,n}\|_{I,\ell}, X_{i,n}/\|X_{i,n}\|_{I,\ell}) \right] + o(1).$$

We conclude by applying (4) again.  $\square$

## 2.5 A criterion for negligibility

Condition (B-v) is a negligibility condition in the sup-norm. It can be checked separately on each component of  $S_n^{<\epsilon} - \mathbb{E}[S_n^{<\epsilon}]$ . We give here a sufficient condition based on a tightness criterion.

**Lemma 2.14.** *Let  $\{U_{\epsilon,n}, \epsilon > 0, n \geq 1\}$  be a collection of random elements in  $\mathcal{D}_I$  such that, for all  $t \in I$ ,*

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \text{var}(U_{\epsilon,n}(t)) = 0, \quad (30)$$

*and, for all  $\eta > 0$ ,*

$$\lim_{\delta \rightarrow 0} \sup_{0 < \epsilon \leq 1} \limsup_{n \rightarrow \infty} \mathbb{P}(w''(U_{\epsilon,n}, \delta) > \eta) = 0, \quad (31)$$

*where  $w''$  is defined in (20). Then  $\{U_{\epsilon,n}, \epsilon > 0, n \geq 1\}$  satisfies the negligibility condition (B-v), that is, for all  $\eta > 0$ ,*

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\|U_{\epsilon,n}\|_I > \eta) = 0. \quad (32)$$

*Proof.* By (30) and the Bienaimé-Chebyshev inequality, we get that, for all  $\eta > 0$  and  $t \in I$ ,

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(|U_{\epsilon,n}(t)| > \eta) = 0 .$$

It follows that, for any  $p \geq 1$ ,  $t_1 < \dots < t_p$  and  $\eta > 0$ ,

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\max_{k=1,\dots,p} |U_{\epsilon,n}(t_k)| > \eta) = 0 . \quad (33)$$

Fix some  $\zeta > 0$ . By Condition (31), we can choose  $\delta > 0$  such that  $\limsup_{n \rightarrow \infty} \mathbb{P}(w''(U_{\epsilon,n}, \delta) > \eta) \leq \zeta$  for all  $\epsilon \in (0, 1]$ . Note now that, as in (Billingsley, 1968, Proof of Theorem 15.7, Page 131), for any  $\delta > 0$ , we may find an integer  $m \geq 1$  and  $t_1 < t_1 < \dots < t_m$ , such that for all  $x \in \mathcal{D}$

$$\|x\|_I \leq w''(x, \delta) + \max_{k=1,\dots,m} |x(t_k)| . \quad (34)$$

Thus, by (33), we obtain

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\|U_{\epsilon,n}\|_I > \eta) \\ & \leq \sup_{0 < \epsilon \leq 1} \limsup_{n \rightarrow \infty} \mathbb{P}(w''(U_{\epsilon,n}, \delta) > \eta) + \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\max_{k=1,\dots,p} |U_{\epsilon,n}(t_k)| > \eta/2) \leq \zeta , \end{aligned}$$

which concludes the proof since  $\zeta$  is arbitrary.  $\square$

In order to obtain (31), we can use the tightness criteria of Billingsley (1968, Chapter 15). We then get the following corollary.

**Corollary 2.15.** *Let  $X, X_i, i \geq 1$ , be i.i.d. random elements in  $\mathcal{D}_I$  such that  $\|X\|_I$  is regularly varying with index  $\alpha \in (1, 2)$ . Let  $\{a_n\}$  be an increasing sequence such that  $\lim_{n \rightarrow \infty} n\mathbb{P}(\|X\|_I > a_n) = 1$ . Assume that there exist  $p \in (\alpha, 2]$ ,  $\gamma > 1/2$  and a continuous increasing function  $F$  on  $I$  and a sequence of increasing functions  $F_n$  that converges pointwise (hence uniformly) to  $F$  such that*

$$\sup_{0 < \epsilon \leq 1} na_n^{-p} \mathbb{E}[|\bar{X}_{\epsilon,n}(s, t)|^p] \leq \{F_n(t) - F_n(s)\}^\gamma , \quad (35a)$$

$$\sup_{0 < \epsilon \leq 1} n^2 a_n^{-2p} \mathbb{E}[|\bar{X}_{\epsilon,n}(s, t)|^p |\bar{X}_{\epsilon,n}(t, u)|^p] \leq \{F_n(u) - F_n(s)\}^{2\gamma} , \quad (35b)$$

where

$$\bar{X}_{\epsilon,n}(s, t) = \{X(t) - X(s)\} 1_{\|X\|_I \leq a_n \epsilon} - \mathbb{E} \left[ \{X(t) - X(s)\} 1_{\|X\|_I \leq a_n \epsilon} \right] .$$

Then the negligibility condition (32) holds with

$$U_{\epsilon,n} = a_n^{-1} \sum_{i=1}^n \left\{ X_i 1_{\|X_i\|_I \leq a_n \epsilon} - \mathbb{E} \left[ X_i 1_{\|X_i\|_I \leq a_n \epsilon} \right] \right\} .$$

*Proof.* We apply Lemma 2.14. By the regular variation of  $\|X\|_I$ , it holds that

$$\limsup_{n \rightarrow \infty} \text{var}(U_{\epsilon,n}(t)) \leq \limsup_{n \rightarrow \infty} na_n^{-2} \mathbb{E}[\|X\|_I^2 1_{\{\|X\|_I \leq a_n \epsilon\}}] = O(\epsilon^{2-\alpha}) ,$$

which yields (30).

By Burkholder's inequality, (Hall and Heyde, 1980, Theorem 2.10), and conditions (35a) and (35b), we have, for some constant  $C > 0$ , for any  $\epsilon \in (0, 1]$ ,

$$\begin{aligned} \mathbb{E}[|U_{\epsilon,n}(s, t)|^p |U_{\epsilon,n}(t, u)|^p] &\leq C n a_n^{-2p} \mathbb{E}[|\bar{X}_{\epsilon,n}(s, t)|^p |\bar{X}_{\epsilon,n}(t, u)|^p] \\ &\quad + C n^2 a_n^{-2p} \mathbb{E}[|\bar{X}_{\epsilon,n}(s, t)|^p] \mathbb{E}[|\bar{X}_{\epsilon,n}(t, u)|^p] \\ &\leq 2C \{F_n(u) - F_n(s)\}^{2\gamma} . \end{aligned}$$

By Markov's inequality, this yields

$$\mathbb{P}(|U_{\epsilon,n}(s, t)| > \lambda, |U_{\epsilon,n}(t, u)| > \lambda) \leq C \lambda^{-2p} \{F_n(u) - F_n(s)\}^{2\gamma} .$$

Arguing as in the proof of Billingsley (1968, Theorem 15.6) (see also the proof of the Theorem of Genest et al. (1996, p. 335)), we obtain, for  $\delta > 0$  and  $\eta > 0$

$$\mathbb{P}(w''(\bar{S}_n^{<\epsilon}, \delta) > \eta) \leq C' \eta^{-2p} \sum_{i=1}^K \{F_n(t_i) - F_n(t_{i-1})\}^{2\gamma} ,$$

where  $C'$  is a constant which depends neither on  $\epsilon \in (0, 1]$  nor on  $\delta > 0$ ,  $K > 1/(2\delta)$  and  $t_1 < \dots < t_K$  are such that  $|t_i - t_{i-1}| \leq \delta$  and  $I \subset \cup_{i=1}^n [t_{i-1}, t_i]$ . Thus

$$\sup_{0 < \epsilon \leq 1} \limsup_{n \rightarrow \infty} \mathbb{P}(w''(\bar{S}_n^{<\epsilon}, \delta) > \eta) \leq C'' \eta^{-2p} \{w(F, \delta)\}^{2\gamma-1} ,$$

where  $C''$  does not depend on  $\delta$ . Since  $F$  is continuous and  $\gamma > 1/2$ , this yields (31).  $\square$

## 2.6 Proof of Theorem 1.3

We apply Theorem 2.12 to the point processes  $N_n$  and  $N$  defined in Section 2.4 and the measure  $\mu_\alpha$  in lieu of  $\mu$ . By Proposition 2.13, we have that  $N_n$  converges weakly to  $N$  in  $\mathcal{M}(\mathcal{Y}_{I,\ell})$ , i.e. Condition (B-i) holds. Condition (A-i) and the definition of  $\mu_\alpha$  imply (B-ii). Condition (7) corresponds to Condition (B-v). Condition (B-iii) is a consequence of the definition of  $\mu_\alpha$ :

$$\int_{(0,1]} y^2 \mu_\alpha(dy, \mathcal{S}_{I,\ell}) = \int_0^1 \alpha y^{2-\alpha-1} dt = \frac{\alpha}{2-\alpha} .$$

For  $0 < \epsilon < x$ , define  $Y_n = \int_{(\epsilon,\infty)} y N_n(dy, \mathcal{S}_{I,\ell})$  and  $Y = \int_{(\epsilon,\infty)} y N(dy, \mathcal{S}_{I,\ell})$ . The weak convergence of  $N_n$  to  $N$  implies that of  $N_n(\cdot \times \mathcal{S}_{I,\ell})$  to  $N(\cdot \times \mathcal{S}_{I,\ell})$ . In turn, by continuity of the map  $m \mapsto \int_\epsilon^\infty y m(dy)$  on the set of point measures on  $(0, \infty]$  without mass on  $\{\epsilon, \infty\}$ , the weak convergence of  $N_n(\cdot \times \mathcal{S}_{I,\ell})$  to  $N(\cdot \times \mathcal{S}_{I,\ell})$  implies that of  $Y_n$  to  $Y$ .

Let  $\sigma_n$  be the measure on  $(0, \infty]$  defined by  $\sigma_n((x, \infty]) = \sum_{i=1}^{m_n} \mathbb{P}(\|X_{i,n}\|_{I,\ell} > x)$ . We have

$$\mathbb{E}[Y_n] = \int_{(\epsilon,x]} y \sigma_n(dy) + \sum_{i=1}^{m_n} \mathbb{E} \left[ \|X_{i,n}\|_I 1_{\|X_{i,n}\|_{I,\ell} > x} \right] .$$

Condition (4) implies that  $\sigma_n$  converges vaguely on  $(0, \infty]$  to the measure with density  $\alpha x^{-\alpha-1}$  with respect to Lebesgue's measure. Thus

$$\lim_{n \rightarrow \infty} \int_{(\epsilon, x]} y \sigma_n(dy) = \int_{(\epsilon, x]} y \alpha y^{-\alpha-1} dy .$$

The last two displays and Condition (6) imply that  $\mathbb{E}[Y_n]$  converges to  $\mathbb{E}[Y]$ . and  $\mathbb{E}[Y] < \infty$ . Since  $Y_n, Y$  are non-negative random variables, this implies the uniform integrability of  $\{Y_n\}$ . We have proved (B-iv) for  $\epsilon$  such that  $\mu(\{\epsilon\} \times \mathcal{S}_{I, \ell}) = 0$ . By monotony with respect to  $\epsilon$ , this actually holds for any  $\epsilon > 0$ .

Finally, the representation (3) follows from Samorodnitsky and Taqqu (1994, Theorem 3.12.2).

### 3 Applications

The usual way to prove the weak convergence of a sum of independent regularly varying functions in  $\mathcal{D}_I$  is to establish the convergence of finite-dimensional distributions (which follows from the finite-dimensional regular variation) and a tightness criterion. We consider here another approach, based on functional regular variation. It has been proved in Section 2.4 that functional regular variation implies the convergence of the point process of (functional) exceedances. Thus, in order to apply Theorem 1.1 or Theorem 1.3, an asymptotic negligibility condition (such as (2) or (7), respectively) must be proved. Since the functional regular variation condition takes care of the “big jumps”, the negligibility condition concerns only the “small jumps”, i.e. we must only prove the tightness of sum of truncated terms. This can be conveniently done by computing moments of any order  $p > \alpha$ , even though they are infinite for the original series. We provide in this section some examples where this new approach can be fully carried out.

#### 3.1 Invariance principle

We start by proving that the classical invariance principle is a particular case of Theorem 2.12. Let  $\{z_i\}$  be a sequence of i.i.d. random variables in the domain of attraction of an  $\alpha$ -stable law, with  $\alpha \in (1, 2)$ . Let  $a_n$  be the  $1/n$ -th quantile of the distribution of  $|z_1|$  and define the partial sum process  $S_n$  by

$$S_n(t) = a_n^{-1} \sum_{k=1}^{[nt]} (z_k - \mathbb{E}[z_1]) .$$

For  $u \in [0, 1]$ , denote by  $w_u$  the indicator function of the interval  $[u, 1]$  i.e.  $w_u(t) = 1_{[u, 1]}(t)$  and define  $X_{k,n} = a_n^{-1} z_k w_{k/n}$ . Then we can write  $S_n = \sum_{k=1}^n (X_{k,n} - \mathbb{E}[X_{k,n}])$ . We will apply Theorem 1.3 to prove the convergence of  $S_n$  to a stable process in  $\mathcal{D}(I)$  with  $I = [0, 1]$ . Note that  $\|X_{k,n}\|_I = z_k/a_n$ . Thus, by Remark 1.4, we only need to prove that (4) holds with a measure  $\nu$  that satisfies Condition (A-i) and the negligibility condition (7). Let  $\nu$  be the probability measure defined on  $\mathcal{S}_I$  by  $\nu(\cdot) = \int_0^1 \delta_{w_u}(\cdot) du$ ,  $\mu_\alpha$  be defined on  $(0, \infty) \times \mathcal{S}_I$  by  $\mu_\alpha((r, \infty] \times \cdot) = r^{-\alpha} \nu(\cdot)$  and  $\mu_n$  be the measure in the left-hand side of (4). Since

$\|X_{k,n}\|_I = z_k/a_n$ , and the random variables  $z_k$  are i.i.d. and  $w_{k/n}$  are deterministic, we have, for all  $r > 0$  and Borel subsets  $A$  of  $\mathcal{S}_I$ ,

$$\mu_n((r, \infty] \times A) = (n\mathbb{P}(z_1 > a_nr)) \times \left( \frac{1}{n} \sum_{k=1}^n 1_{\{w_{\frac{k}{n}} \in A\}} \right).$$

By the regular variation of  $z_1$ , the first term of this product converges to  $r^{-\alpha}$ . The second term of this product can be written as  $P_n \circ \phi^{-1}(A)$ , where  $P_n = n^{-1} \sum_{k=1}^n \delta_{k/n}$  is seen as a probability measure on the Borel sets of  $[0, 1]$  and  $\phi : [0, 1] \rightarrow \mathcal{D}_I$  is defined by  $\phi(u) = w_u$ . Since  $\phi$  is continuous (with  $\mathcal{D}_I$  endowed by  $J_1$ ) and  $P_n$  converges weakly to the Lebesgue measure on  $[0, 1]$ , denoted by  $\text{Leb}$ , by the continuous mapping theorem, we have that  $P_n \circ \phi^{-1}$  converges weakly to  $\text{Leb} \circ \phi^{-1} = \nu$ . This proves that (4) holds.

To prove Condition (7), note that

$$\|S_n^{<\epsilon}\|_I = a_n^{-1} \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (z_i 1_{\{|z_i| \leq a_n \epsilon\}} - \mathbb{E}[z_i 1_{\{|z_i| \leq a_n \epsilon\}}]) \right|,$$

where  $S_n^{<\epsilon}$  denotes the sum appearing in the left-hand side of (7). By Doob's inequality (Hall and Heyde, 1980, Theorem 2.2), we obtain

$$\mathbb{E}[\|S_n^{<\epsilon}\|_\infty^2] \leq 2 \text{var} \left( a_n^{-1} \sum_{i=1}^n z_i 1_{\{|z_i| \leq a_n \epsilon\}} \right) \leq n a_n^{-2} \mathbb{E}[z_1^2 1_{\{|z_1| \leq a_n \epsilon\}}] = O(\epsilon^{2-\alpha}),$$

by regular variation of  $z_1$ . This bound and Markov's inequality yield (7).

### 3.2 Stable processes

Applying Corollary 2.15, we obtain a criterion for the convergence of partial sums of a sequence of i.i.d. processes that admit the representation  $RW$ , where  $R$  is a Pareto random variable and  $W \in \mathcal{S}_I$ . This type of process is sometimes referred to as (generalized) Pareto processes. See Ferreira et al. (2012).

**Proposition 3.1.** *Let  $\{R, R_i\}$  be a sequence of i.i.d. real valued random variables in the domain of attraction of an  $\alpha$ -stable law, with  $1 < \alpha < 2$ . Let  $\{W, W_i, i \geq 1\}$  be an i.i.d. sequence in  $\mathcal{S}_I$  with distribution  $\nu$  satisfying the assumptions of Lemma 1.7, and independent of the sequence  $\{R_i\}$ . Then, defining  $a_n$  as an increasing sequence such that by  $\mathbb{P}(R > a_n) \sim 1/n$ ,  $a_n^{-1} \sum_{i=1}^n \{R_i W_i - \mathbb{E}[R] \mathbb{E}[W]\}$  converges weakly in  $\mathcal{D}_I$  to a stable process  $Z$  which admits the representation (3).*

**Remark 3.2.** By Lemma 1.5, the stable process  $Z$  also admits the series representation (8), which is almost surely convergent in  $\mathcal{D}_I$  and by Lemma 1.7 it is regularly varying in the sense of (1), with spectral measure  $\nu$ . As mentioned in the proof of Lemma 1.7, the proof we give here of the existence of a version of  $Z$  in  $\mathcal{D}_I$  is different from the proof of Davydov and Dombry (2012) or Basse-O'Connor and Rosiński (2011).

*Proof of Proposition 3.1.* We apply Theorem 1.1 to  $X_i = R_i W_i$ . The regular variation condition (1) holds trivially since  $\|X\|_I = R$  is independent of  $X/\|X\|_I = W$ . Condition (9b)

implies that  $W$  has no fixed jump, i.e. Condition (A-i) holds. Thus we only need to prove that the negligibility condition (A-ii) holds. Write  $S_n^{<\epsilon} = a_n^{-1} \sum_{i=1}^n \{R_i 1_{\{R_i \leq \epsilon a_n\}} W_i - \mathbb{E}[R 1_{\{R \leq a_n \epsilon\}}] \mathbb{E}[W]\}$  and  $r_{n,i} = a_n^{-1} R_i 1_{\{R_i \leq a_n \epsilon\}}$ . Then,

$$S_n^{<\epsilon} = \sum_{i=1}^n r_{n,i} \{W_i - \mathbb{E}[W]\} + \mathbb{E}[W] \sum_{i=1}^n \{r_{n,i} - \mathbb{E}[r_{n,i}]\}. \quad (36)$$

Since  $\|\mathbb{E}[W]\|_I \leq 1$ , the second term's infinite norm on  $I$  can be bounded using the Bienaymé-Chebyshev inequality and the regular variation of  $R$  which implies, for any  $p > \alpha$ , as  $n \rightarrow \infty$ ,

$$\mathbb{E}[|r_{n,i}|^p] \sim \frac{\alpha}{p-\alpha} \epsilon^{p-\alpha} n^{-1}. \quad (37)$$

Hence we only need to deal with the first term in the right-hand side of (36), which is hereafter denoted by  $\tilde{S}_n^{<\epsilon}$ . Since  $R$  is independent of  $W$ , Conditions (35a) and (35b) are straightforward consequences of (9a) and (9b). Thus Condition (A-ii) holds by Corollary 2.15. The last statement follows from Lemma 1.7  $\square$

### 3.3 Renewal reward process

Consider a renewal process  $N$  with i.i.d. interarrivals  $\{Y_i, i \geq 1\}$  with common distribution function  $F$ , in the domain of attraction of a stable law with index  $\alpha \in (1, 2)$ . Let  $a_n$  be a norming sequence defined by  $a_n = F^{\leftarrow}(1 - 1/n)$ . Then, for all  $x > 0$ ,

$$\lim_{n \rightarrow \infty} n \bar{F}(a_n x) = x^{-\alpha}.$$

Consider a sequence of rewards  $\{W_i, i \geq 1\}$  with distribution function  $G$  and define the renewal reward process  $R$  by

$$R(t) = W_{N(t)}.$$

let  $\phi$  be a measurable function and define  $A_T(\phi)$  by

$$A_T(\phi) = \int_0^T \phi(R(s)) \, ds.$$

We are concerned with the functional weak convergence of  $A_T$ . We moreover assume that the sequence  $\{(Y, W), (Y_i, W_i), i \geq 1\}$  is i.i.d. and that  $Y$  and  $W$  are asymptotically independent in the sense of Maulik et al. (2002), i.e.

$$\lim_{n \rightarrow \infty} n \mathbb{P} \left( \left( \frac{Y}{a_n}, W \right) \in \cdot \right) \xrightarrow{v} \mu_\alpha \otimes G^* \quad (38)$$

on  $]0, \infty] \times \mathbb{R}$ , where  $G^*$  is a probability measure on  $\mathbb{R}$ . This assumption is obviously satisfied when  $Y$  and  $W$  are independent, with  $G^* = G$  in that case.

When  $Y$  and  $W$  are independent and  $\mathbb{E}[|\phi(W)|^\alpha] < \infty$ , it has been proved by Taqqu and Levy (1986) that  $a_T^{-1} \{A_T(\phi) - \mathbb{E}[A_T(\phi)]\}$  converges weakly to a stable law.

Define  $\lambda = (\mathbb{E}[Y])^{-1}$  and

$$F_0(w) = \lambda \mathbb{E}[Y 1_{\{W \leq w\}}].$$

Then  $F_0$  is the steady state marginal distribution of the renewal reward process and  $\lim_{t \rightarrow \infty} \mathbb{P}(R(t) \leq w) = F_0(w)$ . For  $w \in \mathbb{R}$ , consider the functions  $1_{\{\cdot \leq w\}}$ , which yields the usual one-dimensional empirical process:

$$E_T(w) = a_T^{-1} \int_0^T \{1_{\{R(s) \leq w\}} - F_0(w)\} ds .$$

**Theorem 3.3.** *Assume that (38) holds with  $G^*$  continuous. The sequence of processes  $E_T$  converges weakly in  $\mathcal{D}(\mathbb{R})$  endowed with the  $J_1$  topology as  $T$  tends to infinity to the process  $E^*$  defined by*

$$E^*(w) = \int_{-\infty}^{\infty} \{1_{\{x \leq w\}} - F_0(w)\} M(dx) ,$$

where  $M$  is a totally skewed to the right stable random measure with control measure  $G^*$ , i.e.

$$\log \mathbb{E} \left[ e^{it \int_{-\infty}^{\infty} \phi(w) M(dw)} \right] = -|t|^\alpha \lambda c_\alpha \mathbb{E}[|\phi(W^*)|^\alpha] \{1 + i \operatorname{sign}(t) \beta(\phi) \tan(\pi\alpha/2)\} ,$$

where  $W^*$  is a random variable with distribution  $G^*$ ,  $c_\alpha^\alpha = \Gamma(1 - \alpha) \cos(\pi\alpha/2)$  and  $\beta(\phi) = \mathbb{E}[\phi_+^\alpha(W^*)]/\mathbb{E}[|\phi(W^*)|^\alpha]$ .

**Remark 3.4.** *Equivalently,  $E^*$  can be expressed as  $E^* = Z \circ G^* - F_0 \cdot Z(1)$ , where  $Z$  is a totally skewed to the right Lévy  $\alpha$ -stable process with characteristic function  $\mathbb{E}[e^{itZ(1)}] = \exp\{-|t|^\alpha \lambda c_\alpha \{1 + i \operatorname{sign}(t) \tan(\pi\alpha/2)\}\}$ . If moreover  $Y$  and  $W$  are independent, then the marginal distribution of  $R(0)$  is  $G$ ,  $G^* = G$  and the limiting distribution can be expressed as  $Z \circ G - GZ(1)$  and thus the law of  $\sup_{w \in \mathbb{R}} E^*(w)$  is independent of  $G$ .*

*Proof.* Write

$$\begin{aligned} E_T(w) &= a_T^{-1} \sum_{i=0}^{N(T)} Y_i 1_{\{W_i \leq w\}} + a_T^{-1} \{T - S_{N(T)}\} 1_{\{W_{N(T)} \leq w\}} - a_T^{-1} \lambda T \mathbb{E}[Y 1_{\{W \leq w\}}] \\ &= a_T^{-1} \sum_{i=0}^{N(T)} \{Y_i 1_{\{W_i \leq w\}} - \mathbb{E}[Y 1_{\{W \leq w\}}]\} - a_T^{-1} \{S_{N(T)} - \lambda^{-1} N(T)\} F_0(w) \end{aligned} \quad (39a)$$

$$- a_T^{-1} \{S_{N(T)} - T\} \{1_{\{W_{N(T)} \leq w\}}\} - \lambda \mathbb{E}[Y 1_{\{W \leq w\}}] \} . \quad (39b)$$

The term in (39b) is  $o_P(1)$ , uniformly with respect to  $w \in \mathbb{R}$ . Define  $U_i = G^*(W_i)$  and  $U = G^*(W)$ . Define the sequence of bivariate processes  $S_n$  on  $I = [0, 1]$  by

$$S_n(t) = a_n^{-1} \sum_{i=1}^n (Y_i [1_{\{U_i \leq t\}}, 1]' - \mathbb{E}[Y [1_{\{U_i \leq t\}}, 1]']) ,$$

where  $x'$  denotes the transpose of a vector  $x \in \mathbb{R}^2$ . Then the term in (39a) can be expressed as the scalar product  $[1, -F_0(w)] S_{N(T)}(G^*(w))$ . Using that  $N(T)/T$  converges almost surely to  $\lambda$ , we can relate the asymptotic behavior of  $S_{N(T)}$  to that of  $S_n$ . The latter is obtained by applying Theorem 1.3. We prove that the assumptions hold in two steps.

- (a) Let  $\phi$  be the mapping  $(y, w) \mapsto y [1_{[G^*(w), 1]}, 1_{[0, 1]}]'$ . This mapping is continuous from  $(0, \infty) \times \mathbb{R}$  to  $\mathcal{D}_I^2$ . Thus, (38) implies that the distribution of  $Y[1_{\{U \leq t\}}, 1]'$  is regularly varying with index  $\alpha$  in  $\mathcal{D}_I^\ell$  with  $\ell = 2$  and  $\nu$  defined by

$$\nu(\cdot) = \mathbb{P}((1_{[U^*, 1]}, 1_{[0, 1]})' \in \cdot)$$

where  $U^*$  is uniformly distributed on  $[0, 1]$ . Conditions (4), (5) and (6) then follow by Remark 1.4.

- (b) We must next prove the asymptotic negligibility condition (7). It suffices to prove it for the first marginal  $X = Y1_{[U, 1]}$ . For  $\epsilon > 0$  and  $n \geq 1$ , define  $G_{n, \epsilon}(t) = na_n^{-2} \epsilon^\alpha \mathbb{E}[Y^2 1_{\{Y \leq a_n \epsilon\}} 1_{\{U \leq t\}}]$ . It follows that (35a) and (35b) hold with  $p = 2$ ,  $\gamma = 1$  and  $F_n = \sup_{0 < \epsilon \leq 1} G_{n, \epsilon} = G_{n, 1}$ . Moreover, by Assumption (38) and Karamata's Theorem, we have  $\lim_{n \rightarrow \infty} G_{n, 1}(t) = t$ . Thus we can apply Corollary 2.15 to obtain (7).

By Theorem 1.3, the previous steps imply that  $S_n$  converges weakly in  $(\mathcal{D}, J_1)$  to a bivariate stable process which can be expressed as  $[Z, Z(1)]$ , where  $Z$  is a totally skewed to the right  $\alpha$ -stable Lévy process.  $\square$

For completeness, we state the following result.

**Proposition 3.5.** *Under Assumption (38),  $a_T^{-1} \{A_T(\phi) - \mathbb{E}[A_T(\phi)]\}$  converges weakly to a stable law which can be expressed as  $\int_{-\infty}^{\infty} \{\phi(w) - \lambda \mathbb{E}[Y\phi(W)]\} M(dw)$ , where  $M$  is a totally skewed to the right stable random measure with control measure  $G^*$ , i.e.*

$$\log \mathbb{E} \left[ e^{it \int_{-\infty}^{\infty} \phi(w) M(dw)} \right] = -|t|^\alpha \lambda c_\alpha \mathbb{E}[|\phi(W^*)|^\alpha] \{1 + i \operatorname{sign}(t) \beta(\phi) \tan(\phi\alpha/2)\}, \quad (40)$$

where  $W^*$  is a random variable with distribution  $G^*$ ,  $\beta(\phi) = \mathbb{E}[\phi_+^\alpha(W^*)]/\mathbb{E}[|\phi(W^*)|^\alpha]$ .

## A A useful lemma

**Lemma A.1.** *Let  $\{\Gamma_i, i \geq 1\}$  be the points of a unit rate homogeneous Poisson point process on  $[0, \infty)$ . Then for any  $1 < \alpha < p \leq 2$ , we have*

$$\mathbb{E} \left[ \left( \sum_{i=4}^{\infty} \Gamma_i^{-p/\alpha} \right)^2 \right] < \infty.$$

*Proof.* Observe that

$$\sum_{i=4}^{\infty} \Gamma_i^{-p/\alpha} \leq \sum_{i=1}^{\infty} \Gamma_i^{-p/\alpha} 1_{\{\Gamma_i \geq 1\}} + \sum_{i=4}^{\infty} \Gamma_i^{-p/\alpha} 1_{\{\Gamma_i < 1\}}.$$

The first term has finite second moment since  $\int_1^\infty x^{-kp/\alpha} dx < \infty$  for  $k = 1, 2$ . The second term satisfies

$$\sum_{i=4}^{\infty} \Gamma_i^{-p/\alpha} 1_{\{\Gamma_i < 1\}} \leq \Gamma_4^{-p/\alpha} \sum_{i=4}^{\infty} 1_{\{\Gamma_i < 1\}} \leq \Gamma_4^{-p/\alpha} \left( 1 + \sum_{i=5}^{\infty} 1_{\{\Gamma_i - \Gamma_4 < 1\}} \right).$$

Since  $2p/\alpha < 4$  we have  $\mathbb{E}[\Gamma_4^{-2p/\alpha}] < \infty$ . Since  $\sum_{i=5}^{\infty} 1_{\{\Gamma_i - \Gamma_4 < 1\}}$  is a Poisson variable independent of  $\Gamma_4$ , the proof is concluded.  $\square$



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