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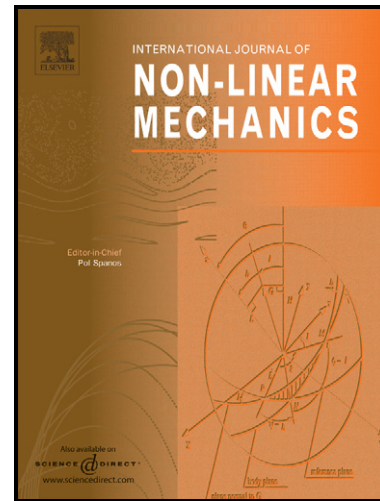
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Mathematical modelling of rate-independent pseudoelastic SMA material

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Abstract. The paper presents a procedure for the formulation of constitutive equations for rate-independent pseudoelastic SMA material models. The procedure applies a rheological scheme representing mechanical properties of the material. An additive decomposition of strains into two parts is proposed. The first part describes strains of a perfectly elastic body while the second part may be represented by a combination of a rigid perfectly elastic body and a rigid perfectly plastic body. It is demonstrated that the key problem of formulation of constitutive relationships is to derive the 1st order differential equation with respect to the tensor describing the second part of the strain field. This equation may be obtained in explicit form starting from the variational inequalities defining non-elastic parts of rheological model. The uniqueness of the obtained differential equation has been proved. A numerical implementation of the constitutive relationships of SMA material was done through the user subroutine module VUMAT within the FE commercial code ABAQUS/Explicit. As an example we analyzed the problem of vibration of a simple 3D structure made of SMA.

Keywords: SMA, Pseudoelasticity, FEM

1. Introduction

A strong non-linear response of the shape memory alloys (SMAs) is associated with the two special effects: shape memory and pseudoelasticity. The shape memory effect allows material to recover its shape upon heating. The pseudoelasticity phenomenon makes the material capable of experiencing large inelastic strains recoverable upon unloading. In the literature there is a comprehensive description of SMA behaviour associated with the stress- or temperature-induced transformations of their crystalline structure [1, 2].

The response of the structures made of SMA is characterized by the energy dissipation capabilities as well as by the shape recovery along the loading path. As a result, SMA materials are now widely used in engineering. The main non-medical applications of SMA,

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recently explored are: protection of civil constructions against earthquake damages [3-5] and control of space structures such as antennas and satellites [6].

The mathematical modelling of 1D and 3D systems based on SMA has received a great deal of attention in the literature. The model presented by Lubliner and Auricchio [7] uses the notion of theory of plasticity. An additional internal variable, so called phase fraction, was used there. The problem of numerical implementation of 3D thermomechanical SMA material was considered by Auricchio and Petrini [8]. A robust integration algorithm was adopted there for finite element method (FEM) implicit applications. Bernardini and Vestroni [9] investigated a non-linear dynamic behaviour of a pseudoelastic oscillator. Another 1D dynamical system made of the SMA bar with an additional mass was analyzed by Feng and Li [10]. Auricchio and Fugazza and DesRoches [11] presented an uniaxial rate-dependent viscous model suited for seismic applications.

The objective of our paper is to formulate the constitutive equations of pseudoelastic SMA material model suited for reproducing its hysteretic behavior. The model is purely phenomenological without considering any phase transformation properties. The problem will be defined within the notion of classical small strain theory of plasticity [12]. The rate-dependency phenomenon will not be considered. The main advantage of the method being proposed herein is that the differential equations defining the behavior of the material are of explicit type. Thus, the existence of the solution as well as its uniqueness may be proved. On the other hand, it allows straightforward implementations in the FEM commercial codes.

Another novel finding of our approach is that the 3D constitutive equations of SMA are formulated using a rheological scheme representing its mechanical properties. We demonstrate the procedure comparing the SMA scheme with the classical elastoplastic scheme (a combination of a spring element and a slider element joined in series). After introduction of an additional non-classical element describing so called rigid perfectly elastic material, the variational description is proposed. The rheological scheme being used herein was implemented previously for 1D beam's dynamics by Zbiciak [13].

It will be demonstrated that the procedure of formulation of constitutive equations needs the differential successions to be applied. Differential successions define additional relations which should be satisfied by the time derivatives of variables, describing the constitutive relation. The notion of differential successions was introduced by Grzesikiewicz [14] for dynamic analysis of discrete mechanical systems exhibiting dry friction and locking. Then, the procedure was generalized for 3D elastoplastic solids in [15].

2. Stress and strain state description

In order to describe the stress and the strain state of a body, the symmetrical 2nd order tensors are used. Let S and E be the Euclidean linear spaces, for stress and strain respectively. The elements belonging to the above spaces can be represented as a 3×3 matrixes.

The inner product of the two 2nd order tensors is denoted $\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} \in \mathbb{R}^1$ where $\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)} \in S$. It can be defined as follows

$$\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} = \sigma_{ij}^{(1)} \sigma_{ij}^{(2)}; \quad i, j = 1, 2, 3 \quad (1a)$$

The summation convention is assumed over the repeated indices. The norm of a tensor is denoted $\|\boldsymbol{\sigma}\| \in \mathbb{R}^1$ and is defined by

$$\|\boldsymbol{\sigma}\| := \sqrt{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}} = \sqrt{\sigma_{ij} \sigma_{ij}} \quad (1b)$$

The space of the stress and strain tensors may be decomposed into two orthogonal sub-spaces. The first sub-space constitutes the sub-space of spherical tensors. The second one is the sub-space of deviatoric tensors. Let us assume the following symbols to be used for the stress and strain state description

$$\boldsymbol{\sigma} = \mathbf{p} + \mathbf{s}; \quad \boldsymbol{\varepsilon} = \mathbf{a} + \mathbf{e} \quad \text{where} \quad \mathbf{p} = \frac{1}{3} \text{tr}(\boldsymbol{\sigma}) \mathbf{I} \quad \text{and} \quad \mathbf{a} = \frac{1}{3} \text{tr}(\boldsymbol{\varepsilon}) \mathbf{I} \quad (2)$$

Here \mathbf{p} and \mathbf{a} denote spherical tensors, \mathbf{s} and \mathbf{e} are deviators and \mathbf{I} denotes identity 2nd order tensor. In the above equations the operation $\text{tr} \boldsymbol{\sigma} = \sigma_{ii} \in \mathbb{R}^1$ denoting the trace of a tensor was used additionally.

Mathematical description of elastic properties of isotropic bodies can be formulated separately for the spherical and deviatoric sub-spaces. For example the Hooke's law may be written in the form of two linear equations

$$\mathbf{p} = 3K \mathbf{a}; \quad \mathbf{s} = 2G \mathbf{e} \quad (3)$$

where K and G denote bulk modulus and shear modulus respectively.

Analyzing the constitutive properties of elastoplastic material, the hypothesis stating that the deviatoric part of the strain tensor is decomposed into two components may be assumed

$$\mathbf{e} = \mathbf{e}_{el} + \mathbf{e}_o \quad (4a)$$

where the deviators \mathbf{e}_{el} and \mathbf{e}_o are related to the elastic and the plastic part of the strain state.

The decomposition rule given by Eq. (4a) may be visualized using classical rheological scheme shown in Fig. 1a. The spring element represents elastic properties (Hooke body) while the slider represents so called rigid perfectly plastic model (Saint Venant body).

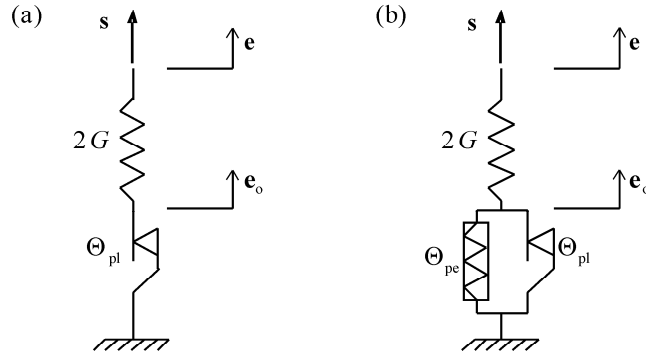


Fig. 1. Rheological models of elastoplastic material (a) and SMA material (b) (deviatoric sub-spaces).

The deviatoric stress state in elastic perfectly plastic material model shown in Fig. 1a is described as follows (see Eq. (3))

$$\mathbf{s} = 2G(\mathbf{e} - \mathbf{e}_o) \quad (4b)$$

The constitutive properties of the plastic pressure-independent material may be described in the form of the following inclusion and variational inequality [16]

$$\begin{aligned} \mathbf{s} &\in \Theta_{pl} \\ \dot{\mathbf{e}}_o \cdot (\mathbf{s} - \tilde{\mathbf{s}}) &\geq 0 \quad \forall \tilde{\mathbf{s}} \in \Theta_{pl} \end{aligned} \quad (5)$$

where the set Θ_{pl} determines admissible stresses in deviatoric sub-space and the superposed dot denotes differentiation with respect to the time coordinate. The Eqs. (5) should be treated as constitutive definition of the slider element. The relationship (5)₂ is well-known as Drucker's stability postulate or Hill's principle of maximum plastic work.

Based on (3), (4) and (5) the following relation between the strains and strain rates can be obtained

$$\dot{\mathbf{e}}_o = f_{ep}(\mathbf{e}_o, \mathbf{e}, \dot{\mathbf{e}}) \quad (6)$$

The detailed form of the mapping f_{ep} depends on the description of the set Θ_{pl} . Taking into account the Huber-Mises-Hencky's (HMH) yield criterion the set Θ_{pl} is described as

$$\Theta_{pl} := \left\{ \mathbf{s} \in S : \|\mathbf{s}\| \leq \sqrt{2} k_{pl} \right\} \quad (7)$$

Here k_{pl} denotes the value of yield limit obtained via pure shear test.

As a consequence of introduction Eq. (7), the relations (5) may be rewritten in the following form

$$\begin{aligned} \dot{\mathbf{e}}_o &= \lambda_{pl} \mathbf{s} \\ \|\mathbf{s}\|^2 &\leq 2 k_{pl}^2; \\ \lambda_{pl} &\geq 0; \quad \lambda_{pl} (\|\mathbf{s}\|^2 - 2 k_{pl}^2) = 0 \end{aligned} \quad (8)$$

where the Eq. (8)₁ is well-known associated flow rule while the relations (8)₂ and (8)₃ are loading/unloading or Kuhn-Tucker conditions. The scalar λ_{pl} denotes so called Lagrange multiplier. The procedure of evaluation of Lagrange multiplier for elastic-perfectly plastic material is a classical problem [12]. Thus, it will not be explained in this section. In the next section, analyzing the constitutive relationships of SMA material the similar problem will occur. In that case we will use the notion of differential successions in order to evaluate Lagrange multipliers.

Finally, based on Eqs. (4b) and (8) the mapping f_{ep} becomes

$$f_{ep}(\mathbf{e}_o, \mathbf{e}, \dot{\mathbf{e}}) = \begin{cases} \mathbf{0} & \text{if } \|\mathbf{s}\| < \sqrt{2} k_{pl} \\ \frac{[\mathbf{s} \cdot \dot{\mathbf{e}}]^+}{2k_{pl}^2} \mathbf{s} & \text{if } \|\mathbf{s}\| = \sqrt{2} k_{pl} \end{cases} \quad (9a)$$

where the function $[\bullet]^+ \in \mathbb{R}^1$ denotes projection onto the set of non-negative numbers

$$[z]^+ := \begin{cases} z & \text{if } z > 0 \\ 0 & \text{otherwise} \end{cases} \quad (9b)$$

The constitutive relations (4) and (6) and (9) are the well-known Prandtl-Reuss equations for elastic-perfectly plastic material.

3. SMA material model

3.1. Rheological scheme

A rheological model of SMA material is shown in Fig. 1b. Comparing the models shown in Fig. 1, we can see an additional element which represents so called rigid perfectly elastic material (Hencky body). This element plays a crucial role in the model. Thanks to the Hencky element, the model does not exhibit any permanent strains after unloading (see Fig. 2).

The Hencky element is characterized by the set of admissible stresses Θ_{pe} . Let us assume the constitutive description of the Hencky material in the following form [16]

$$\begin{aligned} \mathbf{s}_{pe} &\in \Theta_{pe} \\ \mathbf{e}_o \cdot (\mathbf{s}_{pe} - \tilde{\mathbf{s}}) &\geq 0 \quad \forall \tilde{\mathbf{s}} \in \Theta_{pe} \end{aligned} \quad (10)$$

where \mathbf{e}_o and \mathbf{s}_{pe} denote strain and stress deviators respectively.

It should be strongly emphasized that the variational inequality (10)₂ relates stresses and strains \mathbf{e}_o while in Eq. (5)₂ the applied variables are the stress and the strain rate $\dot{\mathbf{e}}_o$.

Let us analyze the complete set of relations describing the properties of the SMA rheological model shown in Fig. 1b. Based on these relations the constitutive equations of SMA can be established in the form of a relationship between the stress tensor $\boldsymbol{\sigma}$ and the

strain state $\boldsymbol{\varepsilon}$ and its rate $\dot{\boldsymbol{\varepsilon}}$. In order to describe the strain state of the model, the tensors $\boldsymbol{\varepsilon}$ and \mathbf{e}_o will be used (see Fig. 1b). The strain tensor $\boldsymbol{\varepsilon}$ can be decomposed into spherical and deviatoric part as it was presented in Eq. (2). The stress state of the model will be described using the tensors \mathbf{p} , \mathbf{s} , \mathbf{s}_{pl} and \mathbf{s}_{pe} , where the \mathbf{p} and the \mathbf{s} determine the spherical and deviatoric part of the stress tensor, that is $\boldsymbol{\sigma} = \mathbf{p} + \mathbf{s}$. The tensors \mathbf{s}_{pl} and \mathbf{s}_{pe} denote the stresses related to the parts: rigid perfectly plastic and rigid perfectly elastic.

There is only one relation describing the constitutive properties of the SMA model in the spherical sub-space

$$\mathbf{p} = 3K\mathbf{a} \quad (11a)$$

On the other hand in the deviatoric sub-space we should write

$$\mathbf{s} = 2G(\mathbf{e} - \mathbf{e}_o) \quad (11b)$$

$$\mathbf{s} = \mathbf{s}_{pl} + \mathbf{s}_{pe} \quad (11c)$$

$$\mathbf{s}_{pl} \in \Theta_{pl}; \quad \dot{\mathbf{e}}_o \cdot (\mathbf{s}_{pl} - \tilde{\mathbf{s}}) \geq 0 \quad \forall \tilde{\mathbf{s}} \in \Theta_{pl} \quad (11d)$$

$$\mathbf{s}_{pe} \in \Theta_{pe}; \quad \mathbf{e}_o \cdot (\mathbf{s}_{pe} - \tilde{\mathbf{s}}) \geq 0 \quad \forall \tilde{\mathbf{s}} \in \Theta_{pe} \quad (11e)$$

Further investigations deal with the determination of the constitutive relations of the SMA in the following form, based on the relations (11)

$$\dot{\mathbf{e}}_o = f_{SMA}(\mathbf{e}_o, \mathbf{e}, \dot{\mathbf{e}}) \quad (12)$$

The mapping f_{SMA} is determined using the sets of admissible stresses Θ_{pl} and Θ_{pe} in the form

$$\Theta_{pl} := \left\{ \mathbf{s} \in S : \|\mathbf{s}_{pl}\| \leq \sqrt{2} k_{pl} \right\} \quad (13)$$

$$\Theta_{pe} := \left\{ \mathbf{s} \in S : \|\mathbf{s}_{pe}\| \leq \sqrt{2} k_{pe} \right\} \quad (14)$$

The parameters k_{pl} and k_{pe} denote limit stresses in the rigid perfectly plastic and rigid perfectly elastic parts respectively, and can be determined based on a pure shear loading/unloading test (see Fig. 2).

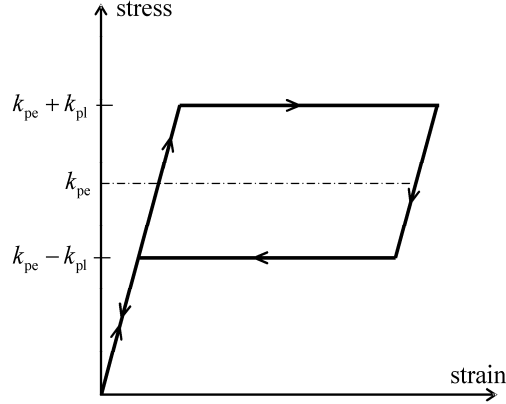


Fig. 2. Pure shear test idealization to determine the values of limit stresses k_{pl} and k_{pe}

Taking into account the presented above form of the admissible stresses, the relations (11b)÷(11e) become

$$\mathbf{s}_{pl} + \mathbf{s}_{pe} = 2G(\mathbf{e} - \mathbf{e}_o) \quad (15a)$$

$$\mathbf{e}_o = \lambda_{pe} \mathbf{s}_{pe}; \quad \lambda_{pe} \geq 0; \quad \|\mathbf{s}_{pe}\|^2 \leq 2k_{pe}^2; \quad \lambda_{pe} (\|\mathbf{s}_{pe}\|^2 - 2k_{pe}^2) = 0 \quad (15b)$$

$$\dot{\mathbf{e}}_o = \lambda_{pl} \mathbf{s}_{pl}; \quad \lambda_{pl} \geq 0; \quad \|\mathbf{s}_{pl}\|^2 \leq 2k_{pl}^2; \quad \lambda_{pl} (\|\mathbf{s}_{pl}\|^2 - 2k_{pl}^2) = 0 \quad (15c)$$

where λ_{pe} and λ_{pl} denote scalar multipliers.

The detailed form of the mapping f_{SMA} is specified on the basis of additional theorems and conclusions resulting from (15) presented in the next section.

3.2. Theorems and auxiliary conclusions

The stress and strain tensors, we use in order to analyze the problem, are time-dependent functions. Let us denote these functions as right-hand continuous and possessing right-hand derivate being continuous as well.

Lemma 1. *If the relations (15a) and (15b) are satisfied and if $\|\mathbf{e}_o\| \neq 0$, then the following equations are true*

$$\mathbf{s}_{pe} = \frac{\sqrt{2} k_{pe}}{\|\mathbf{e}_o\|} \mathbf{e}_o \quad (16)$$

$$\mathbf{s}_{pl} = 2G(\mathbf{e} - \mathbf{e}_o) - \frac{\sqrt{2} k_{pe}}{\|\mathbf{e}_o\|} \mathbf{e}_o \quad (17)$$

Proof. If $\|\mathbf{e}_o\| \neq 0$, then applying (15b) gives

$$\lambda_{pe} \geq 0, \quad \|\mathbf{s}_{pe}\| = \sqrt{2} k_{pe}, \quad \|\mathbf{e}_o\| = \lambda_{pe} \sqrt{2} k_{pe}.$$

Thus, we obtain

$$\lambda_{pe} = \frac{\|\mathbf{e}_o\|}{\sqrt{2} k_{pe}} \quad \text{and} \quad \mathbf{s}_{pe} = \frac{1}{\lambda_{pe}} \mathbf{e}_o.$$

This gives the Eq. (16). Using Eqs. (15a) and (16) we obtain the Eq. (17). ■

Lemma 2. *If $\|\mathbf{s}_{pl}\| = \sqrt{2} k_{pl}$ and the relations (15c) are satisfied, then the following relations constitute the differential successions of (15c)*

$$\dot{\mathbf{e}}_o = \lambda_{pl} \mathbf{s}_{pl}; \quad \lambda_{pl} \geq 0; \quad \mathbf{s}_{pl} \cdot \dot{\mathbf{s}}_{pl} \leq 0; \quad \lambda_{pl} \mathbf{s}_{pl} \cdot \dot{\mathbf{s}}_{pl} = 0 \quad (18)$$

Proof. The relations (18)₁ and (18)₂ are the same as (15c)₁ and (15c)₂. The third condition (18)₃ $\mathbf{s}_{pl} \cdot \dot{\mathbf{s}}_{pl} \leq 0$ constitutes the succession of $\mathbf{s}_{pl} \cdot \mathbf{s}_{pl} \leq 2k_{pl}^2$ if $\mathbf{s}_{pl} \cdot \mathbf{s}_{pl} = 2k_{pl}^2$. The last relation (18)₄ is obvious if $\mathbf{s}_{pl} \cdot \dot{\mathbf{s}}_{pl} = 0$. On the other hand, if $\mathbf{s}_{pl} \cdot \dot{\mathbf{s}}_{pl} < 0$ then considering an infinitesimal time increment τ we obtain

$$\|\mathbf{s}_{pl}(t + \tau)\|^2 < 2k_{pl}^2 \quad \text{and} \quad \lambda_{pl}(t + \tau) = 0.$$

Thus, the assumption about the right-hand continuity of the analyzed functions gives $\lambda_{pl}(t) = 0$ what finally shows that the relation (18)₄ is satisfied. ■

Lemma 3. *If the relations (15) are satisfied and $\|\mathbf{e}_o\| \neq 0$ and $\|\mathbf{s}_{pl}\| = \sqrt{2} k_{pl}$ then*

$$\lambda_{pl} = \begin{cases} 0 & \text{if } \mathbf{s}_{pl} \cdot \dot{\mathbf{e}} \leq 0 \\ \frac{\mathbf{s}_{pl} \cdot \dot{\mathbf{e}}}{2k_{pl}^2} \left[1 + \frac{\sqrt{2}k_{pe}}{2G\|\mathbf{e}_o\|} \left[1 - \left(\frac{\mathbf{s}_{pe} \cdot \mathbf{s}_{pl}}{2k_{pe}k_{pl}} \right)^2 \right] \right]^{-1} & \text{if } \mathbf{s}_{pl} \cdot \dot{\mathbf{e}} > 0 \end{cases} \quad (19)$$

Proof. Differentiating Eq. (16) with respect to time we obtain

$$\dot{\mathbf{s}}_{pe} = \sqrt{2}k_{pe} \left(\frac{\dot{\mathbf{e}}_o}{\|\mathbf{e}_o\|} - \frac{\mathbf{e}_o \cdot \dot{\mathbf{e}}_o}{\|\mathbf{e}_o\|^3} \mathbf{e}_o \right)$$

and using (15c) gives

$$\dot{\mathbf{s}}_{pe} = \sqrt{2}k_{pe} \lambda_{pl} \left(\frac{\mathbf{s}_{pl}}{\|\mathbf{e}_o\|} - \frac{\mathbf{e}_o \cdot \mathbf{s}_{pl}}{\|\mathbf{e}_o\|^3} \mathbf{e}_o \right)$$

what may be transformed into

$$\mathbf{s}_{pl} \cdot \dot{\mathbf{s}}_{pe} = \sqrt{2}k_{pe} \lambda_{pl} \left(\frac{\|\mathbf{s}_{pl}\|^2}{\|\mathbf{e}_o\|} - \frac{(\mathbf{e}_o \cdot \mathbf{s}_{pl})^2}{\|\mathbf{e}_o\|^3} \right).$$

Then, using (15b) the last above equation takes finally the form

$$\mathbf{s}_{pl} \cdot \dot{\mathbf{s}}_{pe} = 2 \frac{\lambda_{pl}}{\lambda_{pe}} k_{pl}^2 \left[1 - \left(\frac{\mathbf{s}_{pe} \cdot \mathbf{s}_{pl}}{2k_{pe}k_{pl}} \right)^2 \right].$$

Let the equation (15a) be differentiated with respect to time

$$2G\dot{\mathbf{e}}_o + \dot{\mathbf{s}}_{pl} + \dot{\mathbf{s}}_{pe} = 2G\dot{\mathbf{e}}.$$

Calculating the inner product with the tensor \mathbf{s}_{pl} over each component of the above equation

and using the formulated before formula for $\mathbf{s}_{pl} \cdot \dot{\mathbf{s}}_{pe}$, we finally get

$$\lambda_{pl} \left\{ 2G\|\mathbf{s}_{pl}\|^2 + \frac{2k_{pl}^2}{\lambda_{pe}} \left[1 - \left(\frac{\mathbf{s}_{pe} \cdot \mathbf{s}_{pl}}{2k_{pe}k_{pl}} \right)^2 \right] \right\} + \mathbf{s}_{pl} \cdot \dot{\mathbf{s}}_{pl} = 2G\mathbf{s}_{pl} \cdot \dot{\mathbf{e}} \quad (20)$$

Additionally, using relations (18) $\lambda_{pl} \geq 0$, $\mathbf{s}_{pl} \cdot \dot{\mathbf{s}}_{pl} \leq 0$, $\lambda_{pl} \mathbf{s}_{pl} \cdot \dot{\mathbf{s}}_{pl} = 0$ we finally obtain the relation (19) defining the multiplier λ_{pl} . ■

Now, let us formulate two theorems to be used in order to determine the strain rate tensor $\dot{\mathbf{e}}_o$.

Theorem 1. If $\|\mathbf{e}_o\| \neq 0$, then

i) if $\left\| 2G(\mathbf{e} - \mathbf{e}_o) - \frac{\sqrt{2}k_{pe}}{\|\mathbf{e}_o\|} \mathbf{e}_o \right\| > \sqrt{2}k_{pl}$, then the equation (15a) cannot be satisfied, so

the analyzed problem does not possess a solution,

ii) if $\left\| 2G(\mathbf{e} - \mathbf{e}_o) - \frac{\sqrt{2}k_{pe}}{\|\mathbf{e}_o\|} \mathbf{e}_o \right\| < \sqrt{2}k_{pl}$, then $\dot{\mathbf{e}}_o = \mathbf{0}$,

iii) if $\left\| 2G(\mathbf{e} - \mathbf{e}_o) - \frac{\sqrt{2}k_{pe}}{\|\mathbf{e}_o\|} \mathbf{e}_o \right\| = \sqrt{2}k_{pl}$, then $\dot{\mathbf{e}}_o = \lambda_{pl} \left[2G(\mathbf{e} - \mathbf{e}_o) - \frac{\sqrt{2}k_{pe}}{\|\mathbf{e}_o\|} \mathbf{e}_o \right]$.

Proof. The propositions (i) and (ii) result directly from (15a) and (15c). The proposition (iii) results from Lemma 3 along with the conditions (15c). ■

Theorem 2. If $\|\mathbf{e}_o\| = 0$ and the conditions described by (15) are satisfied, then

$$\dot{\mathbf{e}}_o = \begin{cases} \mathbf{0} & \text{if } 2G\|\mathbf{e}\| < \sqrt{2}(k_{pl} + k_{pe}) \\ \frac{[\mathbf{e} \cdot \dot{\mathbf{e}}]^+}{\|\mathbf{e}\|^2} \mathbf{e} & \text{if } 2G\|\mathbf{e}\| = \sqrt{2}(k_{pl} + k_{pe}) \end{cases} \quad (21)$$

and if $2G\|\mathbf{e}\| > \sqrt{2}(k_{pl} + k_{pe})$, then the solution of the problem does not exist.

Proof. Taking into account the assumption $\|\mathbf{e}_o\| = 0$, the equation (15a) becomes

$$\mathbf{s}_{pl} + \mathbf{s}_{pe} = 2G\mathbf{e} \quad (22a)$$

The differential succession of (15b) gives

$$\|\mathbf{s}_{pe}\| \leq \sqrt{2}k_{pe}; \quad \dot{\lambda}_{pe} \geq 0; \quad \dot{\lambda}_{pe} (\|\mathbf{s}_{pe}\| - \sqrt{2}k_{pe}) \quad (22b)$$

$$\dot{\mathbf{e}}_o = \dot{\lambda}_{pe} \mathbf{s}_{pe} \quad (22c)$$

where the relation (22c) results from the formula $\mathbf{e}_o = \lambda_{pe} \mathbf{s}_{pe}$ in the case $\|\mathbf{e}_o\| = 0$. Using a truncated Taylor series expansion gives

$$0 + \tau \dot{\mathbf{e}}_o + o(\tau) = \lambda_{pe}(t + \tau) \mathbf{s}_{pe}(t + \tau)$$

and can be transformed into

$$\dot{\mathbf{e}}_o = \left[\lim_{\tau \rightarrow 0} \frac{\lambda_{pe}(t + \tau)}{\tau} \right] \mathbf{s}_{pe}(t) \text{ and finally } \dot{\mathbf{e}}_o = \dot{\lambda}_{pe} \mathbf{s}_{pe}.$$

Let us note that $\mathbf{s}_{pl} + \mathbf{s}_{pe} = \mathbf{s}$ (see Eq. (11c)). Thus the (15c) and (22c) gives

$$\dot{\mathbf{e}}_o = \lambda_{pl}(\mathbf{s} - \mathbf{s}_{pe}) \text{ and } \dot{\mathbf{e}}_o = \dot{\lambda}_{pe} \mathbf{s}_{pe}.$$

Using the last two expressions we obtain

$$\mathbf{s}_{pe} = \frac{\lambda_{pl}}{\lambda_{pl} + \dot{\lambda}_{pe}} \mathbf{s}.$$

Now, the equation describing the strain rate $\dot{\mathbf{e}}_o$ has the following form

$$\dot{\mathbf{e}}_o = \lambda_{int} \mathbf{s}, \text{ where } \lambda_{int} := \frac{\lambda_{pl} \dot{\lambda}_{pe}}{\lambda_{pl} + \dot{\lambda}_{pe}}.$$

Taking into account the assumption $\|\mathbf{e}_o\| = 0$, the relations (15) become

$$\mathbf{s} = 2G\mathbf{e} \tag{23a}$$

$$\dot{\mathbf{e}}_o = \lambda_{int} \mathbf{s}; \quad \lambda_{int} \geq 0; \quad \|\mathbf{s}\|^2 \leq 2(k_{pl} + k_{pe})^2; \quad \lambda_{int} \left[\|\mathbf{s}\|^2 - 2(k_{pl} + k_{pe})^2 \right] = 0 \tag{23b}$$

If $\|\mathbf{s}\| = \sqrt{2}(k_{pl} + k_{pe})$, then using (15a) and (23b) the following differential successions are obtained

$$\dot{\mathbf{s}} = 2G(\dot{\mathbf{e}} - \dot{\mathbf{e}}_o), \text{ what gives } \mathbf{s} \cdot \dot{\mathbf{s}} = 2G(\mathbf{s} \cdot \dot{\mathbf{e}} - \mathbf{s} \cdot \dot{\mathbf{e}}_o) \tag{24a}$$

$$\dot{\mathbf{e}}_o = \lambda_{int} \mathbf{s}; \quad \mathbf{s} \cdot \dot{\mathbf{s}} \leq 0; \quad \lambda_{int} \geq 0; \quad \lambda_{int} \mathbf{s} \cdot \dot{\mathbf{s}} = 0 \tag{24b}$$

Finally, applying the conditions (23) and (24) gives

$$\lambda_{\text{int}} = \begin{cases} 0 & \text{if } 2G\|\mathbf{e}\| < \sqrt{2}(k_{\text{pl}} + k_{\text{pe}}) \\ \frac{[\mathbf{e} \cdot \dot{\mathbf{e}}]^+}{2G\|\mathbf{e}\|^2} & \text{if } 2G\|\mathbf{e}\| < \sqrt{2}(k_{\text{pl}} + k_{\text{pe}}) \end{cases} \quad (25)$$

what in fact states the proposition of the theorem 2. ■

3.3. Constitutive relationships

Based on the theorems we proved above, the following form of the mapping f_{SMA} can be formulated

$$f_{\text{SMA}}(\mathbf{e}_o, \mathbf{e}, \dot{\mathbf{e}}) = \begin{cases} \mathbf{0} & \text{if } \|\mathbf{e}_o\| = 0 \text{ and } 2G\|\mathbf{e}\| < \sqrt{2}(k_{\text{pl}} + k_{\text{pe}}) \\ \mathbf{0} & \text{if } \|\mathbf{e}_o\| \neq 0 \text{ and } \|\mathbf{s}_{\text{pl}}\| < \sqrt{2}k_{\text{pl}} \\ \frac{[\mathbf{e} \cdot \dot{\mathbf{e}}]^+}{\|\mathbf{e}\|^2} \mathbf{e} & \text{if } \|\mathbf{e}_o\| = 0 \text{ and } 2G\|\mathbf{e}\| = \sqrt{2}(k_{\text{pl}} + k_{\text{pe}}) \\ \lambda_{\text{pl}} \mathbf{s}_{\text{pl}} & \text{if } \|\mathbf{e}_o\| \neq 0 \text{ and } \|\mathbf{s}_{\text{pl}}\| = \sqrt{2}k_{\text{pl}} \end{cases} \quad (26)$$

where

$$\begin{aligned} \lambda_{\text{pl}} &= \frac{[\mathbf{s}_{\text{pl}} \cdot \dot{\mathbf{e}}]^+}{2k_{\text{pl}}^2} \left\{ 1 + \frac{\sqrt{2}k_{\text{pe}}}{2G\|\mathbf{e}_o\|} \left[1 - \left(\frac{\mathbf{s}_{\text{pe}} \cdot \mathbf{s}_{\text{pl}}}{2k_{\text{pe}}k_{\text{pl}}} \right)^2 \right] \right\}^{-1}; \\ \mathbf{s}_{\text{pl}} &= 2G(\mathbf{e} - \mathbf{e}_o) - \frac{\sqrt{2}k_{\text{pe}}}{\|\mathbf{e}_o\|} \mathbf{e}_o; \\ \mathbf{s}_{\text{pe}} &= \frac{\sqrt{2}k_{\text{pe}}}{\|\mathbf{e}_o\|} \mathbf{e}_o. \end{aligned} \quad (27)$$

Finally, the following description of the analyzed SMA material model is formulated

$$\begin{aligned} \mathbf{p} &= 3K\mathbf{a} \\ \mathbf{s} &= 2G(\mathbf{e} - \mathbf{e}_o) \\ \dot{\mathbf{e}}_o &= f_{\text{SMA}}(\mathbf{e}_o, \mathbf{e}, \dot{\mathbf{e}}) \end{aligned} \quad (28)$$

Thus, for given strain tensor $\boldsymbol{\varepsilon} = \mathbf{a} + \mathbf{e}$ and its rate $\dot{\boldsymbol{\varepsilon}} = \dot{\mathbf{a}} + \dot{\mathbf{e}}$, the stress state described by $\boldsymbol{\sigma} = \mathbf{p} + \mathbf{s}$ can be determined using the equations (28).

4. Numerical simulations

The constitutive model of SMA has been implemented in the Finite Element code ABAQUS/Explicit which is suited for non-linear transient dynamic problems. Although the ABAQUS constitutive library is extensive the model of SMA is missing. Additional material models can be introduced to the system via user subroutine VUMAT. The subroutine is typically written in Fortran language.

As an example we present a 3D cantilever beam made of Nitinol alloy subjected to a uniform impulsive loading at the top surface. The load intensity equals to 4 MPa. With the total analysis time of 0.06 sec. and the loading acting during 0.03 sec. the unloading process was also considered. The values of the limit stresses in the plastic and perfectly elastic parts (see Table 1) were specified in the way to make the material capable of experiencing the total equivalent HMH stress $\sigma_{eq} = \sqrt{3/2} \|\mathbf{s}\|$ in the range of 150 MPa and 350 MPa, when $\|\mathbf{e}_o\| \neq 0$. The material parameters were assumed partially based on experiments presented in the literature [17].

Table 1
Material parameters of SMA

| Density (kg/m ³) | K (GPa) | G (GPa) | k_{pe} (MPa) | k_{pl} (MPa) |
|------------------------------|-----------|-----------|----------------|----------------|
| 6500 | 41.67 | 19.23 | 144.34 | 57.74 |

The beam was divided into 3000 brick elements C3D8 as shown in Fig. 3. The contours of equivalent stress at the chosen instant of time are visualised in Fig. 3.

The time history of the equivalent stress and the norm of the non-elastic strain $\|\mathbf{e}^o\|$ for an element at the clamped section are visualised in Figs. 4 and 5. It can be observed that the maximum value of the equivalent stress does not exceed the plastic limit (350 MPa). On the other hand, if $\|\mathbf{e}_o\| \neq 0$, the equivalent stress may not be lower than 150 MPa. This phenomenon can be observed comparing Figs. 4 and 6a.

The hysteretic loop for an element located in the vicinity of the clumped zone (see Fig. 3) is shown in Fig. 6. The variables selected for the hysteretic loop construction are the

equivalent strain versus equivalent stress (left diagram) and the longitudinal strain versus longitudinal stress (right diagram).

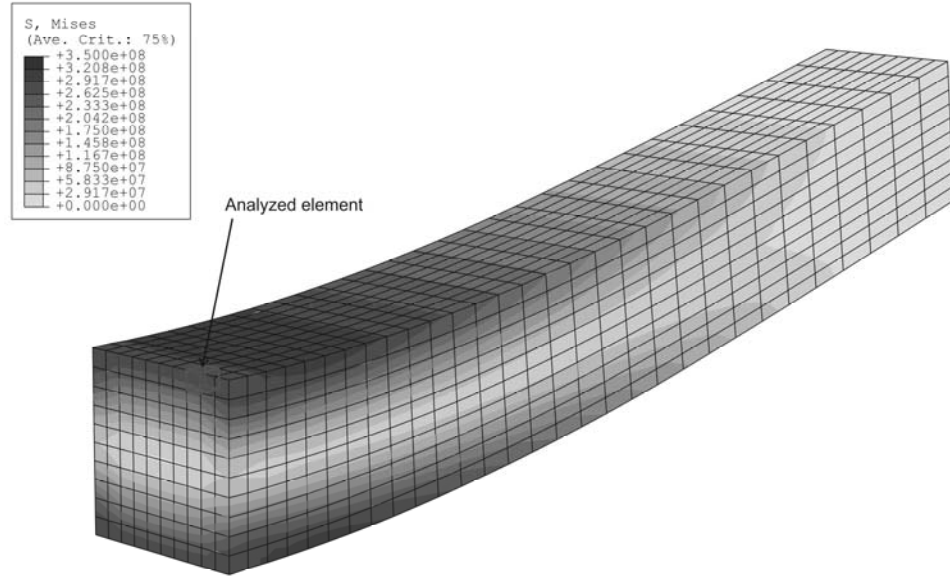


Fig. 3. Contours of equivalent HMH stress and beam's deformation at chosen instant of time $t = 4.75 \text{ E} - 2 \text{ sec}$. (scale deformation factor 5).

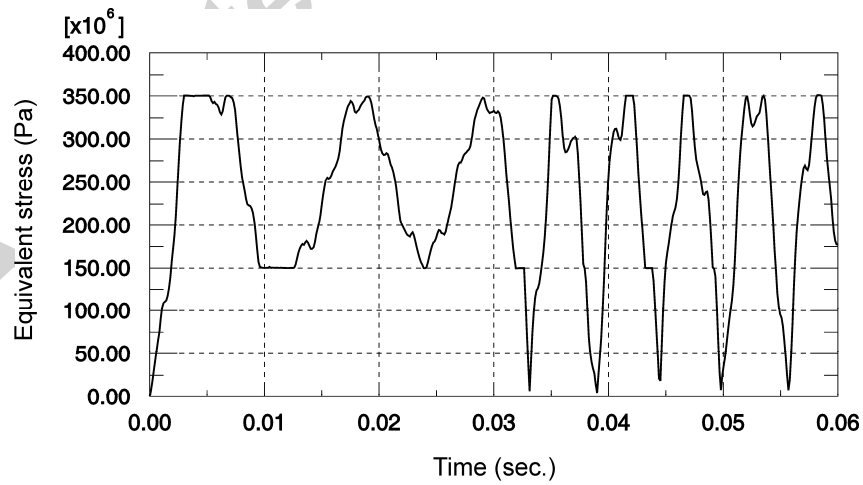


Fig. 4. Time history of the equivalent stress for an element at the clamped section

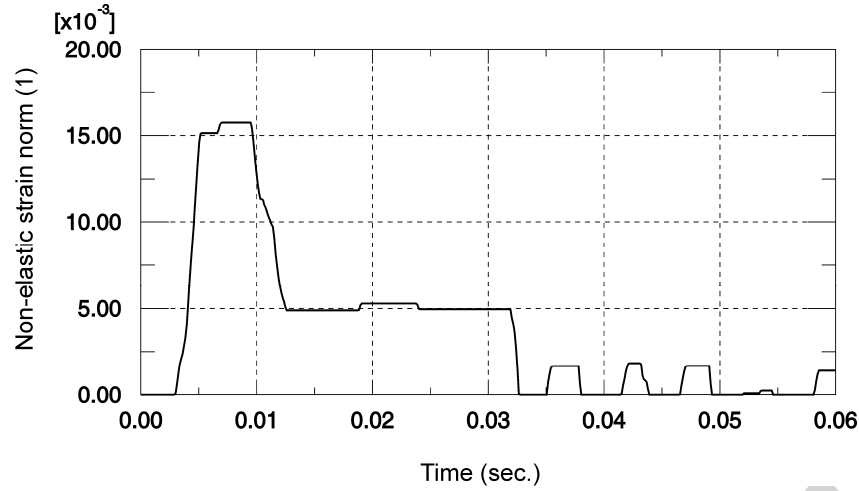


Fig. 5. Time history of the norm of the non-elastic strain $\|e_o\|$ in the clamped section element

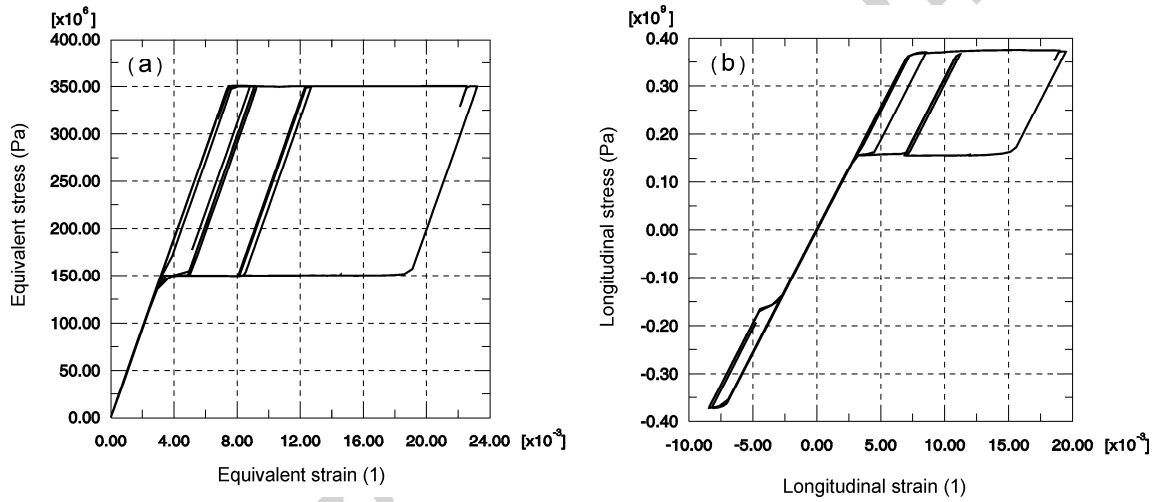


Fig. 6. Hysteretic loops in the clamped section element

5. Final remarks

The presented numerical results show validity of the proposed formulation. The main objective of this example was to demonstrate the possibility of implementation of SMA relationships within commercial FEM software. In this paper small-strain theory was considered. Such limitation seems to be reasonable observing results of experimental tests showing that the pseudoelasticity phenomenon occurs within the range of strain to be less than 5% [18]. Considering a finite strain problem, one has to replace additive decomposition of strains $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{el} + \boldsymbol{\varepsilon}_o$ by multiplicative decomposition of deformation gradients $\mathbf{F} = \mathbf{F}_{el} \mathbf{F}_o$ [19]. The simplest approximation to capture the behaviour of SMA material can be achieved

using hyperelastic model. This becomes necessary if the material is subjected to large strains.

Besides of this limitation, it is clear from the present study that the theory of plasticity used herein, is able to model pseudoelastic behaviour of rate-independent SMAs. Also, the computational ease of such type of formulation needs to be emphasized. The rate-dependency phenomenon may also be considered based on the methodology presented in the paper. Simple modifications of rheological scheme shown in Fig. 1b, by adding viscous elements, lead to the models similar to that analyzed in viscoplasticity theory.

The model presented in this study is suited for simulation of the stress-induced partial phase transformation behaviour. The analysis of the fully transformed material can be carried out taking into account the rheological scheme shown in Fig. 7a where an additional locking element was employed (compare Fig. 1b). Thanks to this element the non-elastic strains are limited as it is shown in Fig. 7b.

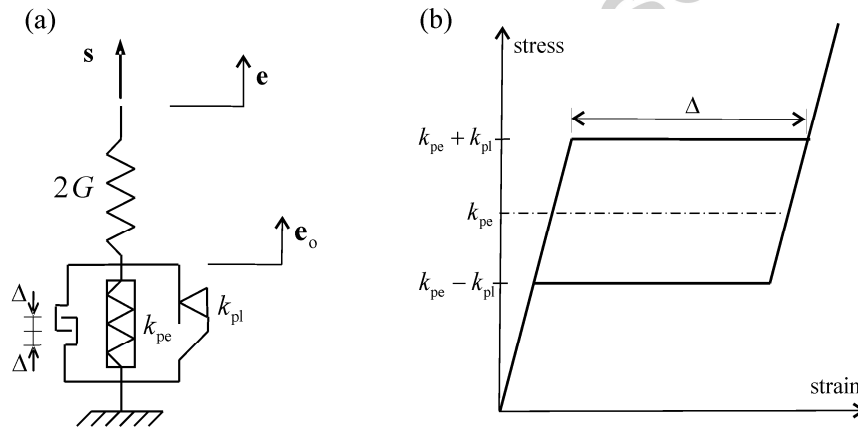


Fig. 7. Rheological model of fully transformed SMA (a) and pure shear test idealization to determine the values of limit stresses k_{pl} and k_{pe} and limit strain Δ (b).

The mathematical description of the locking element is as follows

$$\begin{aligned} \mathbf{e}_o &\in \Omega_{loc} \\ \mathbf{s}_{loc} \cdot (\mathbf{e}_o - \tilde{\mathbf{e}}_o) &\geq 0 \quad \forall \tilde{\mathbf{e}}_o \in \Omega_{loc} \\ \Omega_{loc} &:= \left\{ \mathbf{e}_o \in E : \|\mathbf{e}_o\| \leq \sqrt{2} \Delta \right\} \end{aligned} \quad (29a)$$

where Ω_{loc} denotes the set of admissible strains, Δ denotes limit deviatoric strain (see Fig. 7b) and the \mathbf{s}_{loc} is the stress in locking network.

Because of the complexity of the mathematical description of the fully transformed SMA material we did not analyse this problem in details. The work in this area is currently underway.

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References

- [1] H. Funakubo, Shape Memory Alloys, Gordon and Breach Science Publishers, London, 1987.
- [2] Z. Bojarski, H. Morawiec, Shape Memory Alloys [in Polish], PWN, Warsaw, 2000.
- [3] R. DesRoches, M. Delemont, Seismic retrofit of simply supported bridges using shape memory alloys, *Engineering Structures*, 24 (2002) 325-332.
- [4] G. Song, N. Ma, H.-N Li, Applications of shape memory alloys in civil structures, *Engrg. Struct.*, 28 (2006) 1266-1274.
- [5] H. Tamai, Y. Kitagawa, Pseudoelastic behaviour of shape memory alloy wire and its application to seismic resistance member for building, *Comp. Mater. Sci.*, 25 (2002) 218-227.
- [6] J.V. Humbeeck, Non-medical applications of shape memory alloys, *Materials Science and Engineering*, A273-275 (1999) 134-148.
- [7] J. Lubliner, F. Auricchio, Generalized plasticity and shape-memory alloys, *Int. J. Solids Structures*, 33 (1996) 991-1003.
- [8] F. Auricchio, L. Petrini, Improvements and algorithmical considerations on a recent three-dimensional model describing stress-induced solid phase transformations, *Int. J. Numer. Meth. Engng.*, 55 (2002) 1255-1284.
- [9] D. Bernardini, F. Vestroni, Non-isothermal oscillations of pseudoelastic devices, *Int. J. Non-Linear Mechanics*, 38 (2003) 1297-1313.
- [10] Z.C. Feng, D.Z. Li, Dynamic of a mechanical system with a shape memory alloy bar, *J. Intell. Mater. Syst. Struct.*, 7 (1996) 399-410.
- [11] F. Auricchio, D. Fugazza, R. DesRoches, A 1D viscous constitutive model for superelastic shape memory alloys: formulation and comparison with experiment data, *Smart Mater. Struct.* 16 (2007) 39-50.
- [12] A. S. Khan, S. Huang, *Continuum Theory of Plasticity*, Wiley, New York, 1995.
- [13] A. Zbiciak, Dynamic analysis of pseudoelastic SMA beam. *Int. J. Mech. Sciences*, 52 (2010) 56-64.
- [14] W. Grzesikiewicz, Dynamics of mechanical systems with constraints [in Polish], *Transactions of Warsaw Univ. of Technology, Mechanics*, 117, Warsaw, 1990.
- [15] W. Grzesikiewicz, W. Wojewódzki, A. Zbiciak, Non-smooth dynamic problem formulation for elastic-perfectly plastic solid, *Theoretical Foundations of Civil Engineering*, 11 (2003) 339-350.

- [16] P.D. Panagiotopoulos, Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions, Birkhauser, Basel, 1985.
- [17] J.M. Mc Naney, V.Y. Jung, P. Papadopoulos, R.O. Ritchie, An experimental study of the superelastic effect in a shape-memory Nitinol alloy under biaxial loading, *Mechanics of Materials*, 35 (2003) 969-986.
- [18] R. Lammering, I. Schmidt, Experimental investigations on the damping capacity of NiTi components, *Smart Mater. Struct.*, 10 (2001) 853-859.
- [19] V. A. Lubarda, *Elastoplasticity Theory*, CRC, Boca Raton, 2002.

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