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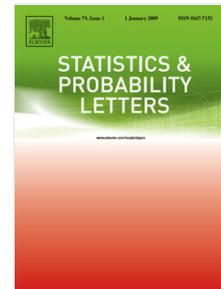
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PRICING BASKET DEFAULT SWAPS IN A TRACTABLE SHOT-NOISE MODEL

ALEXANDER HERBERTSSON, JIWOOK JANG, THORSTEN SCHMIDT

ABSTRACT. We value CDS spreads and k th-to-default swap spreads in a tractable shot noise model. The default dependence is modelled by letting the individual jumps of the default intensity be driven by a common latent factor. The arrival of the jumps is driven by a Poisson process. By using conditional independence and properties of the shot noise processes we derive tractable closed-form expressions for the default distribution and the ordered survival distributions. These quantities are then used to price k th-to-default swap spreads. We calibrate a homogeneous version of the model to the term structure on market data from the iTraxx Europe index series sampled during the period 2008-01-14 to 2010-02-11. We perform 435 calibrations in this turbulent period and almost all calibrations yields very good fits. Finally we study k th-to-default spreads in the calibrated model.

1. INTRODUCTION

In recent years the market for portfolio credit derivatives, which are derivatives with a payoff linked to the credit loss in a portfolio, has seen a rapid growth and increased liquidity. This has been followed by an intense research for understanding and modelling the main feature driving these products, namely default dependence. The current credit crisis undermines the necessity of models which can calibrate to market data on one side and also capture contagion effects on the other side. The model proposed in this paper is an affine model with a jump component. This allows to introduce a high dependence between different obligors which is of immense importance for practical applications.

Affine models have been widely used in modelling of interest rates and credit risk, but typically the jump component plays a minor role. However, the main driver of contagion effects¹ is the jump component. In this paper we concentrate on the jump component only, while the results easily can be enriched by adding a diffusion component.

As an illustration of the tractability of the model we present a number of numerical studies. We calibrate the model to market data from the iTraxx Europe index series in

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¹Contagion is the effect that a default of a company leads to an increased default probability of related firms and came most apparent after the default of Lehman Brothers.

the period 2008-01-14 to 2010-02-11. During this turbulent period the term structure and the market spreads have a very wide spectrum of different values and relationships. We perform 435 calibrations in this period and almost all calibrations yields very good fits. This demonstrates the flexibility and the robustness of this remarkably simple model.

The rest of this paper is organized as follows. In Section 2 we introduce the shot noise model that is used in the paper. Section 3 derives the formulas that are needed for pricing portfolio credit derivatives and thereafter consider the case of inhomogeneous portfolios. In Section 4 we study an explicit example of the model. Section 6 present how to price credit default swaps (CDS) and k th-to default swaps. We also show that the index-CDS spread and the CDS spread coincide in a homogeneous portfolio which will be used in the calibration. Finally, in Section 7 we calibrate the model to the term structure on market data from the iTraxx Europe index series. An analysis of k th-to default swaps in the calibrated model completes the numerical section.

2. THE MODEL

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ where the filtration \mathbb{F} satisfies the usual conditions. In this paper \mathbb{Q} is a martingale measure equivalent to the objective measure \mathbb{P} , and below all expectations are done with respect to the risk neutral measure \mathbb{Q} .

Let $\{X_{i,j}, Y_j : 1 \leq i \leq m, j \geq 1\}$ be independent nonnegative random variables where $X_{i,j}$ has distribution function F_i and Y_j has distribution function F_Y . Furthermore, let M be a Poisson process with constant intensity ρ and denote its jump times by S_1, S_2, \dots . Let $\lambda_i = (\lambda_{t,i})_{t \geq 0}$, $1 \leq i \leq m$ be m processes where λ_i satisfies the SDE

$$\begin{aligned} d\lambda_{t,i} &= -\delta_i \lambda_{t,i} dt + dC_{t,i}, \\ C_{t,i} &= \sum_{j=1}^{M_t} Y_j X_{i,j}. \end{aligned} \tag{2.1}$$

The intuitive interpretation of (2.1) is that at the each jump time S_j of M , the process λ_i jumps by the amount $Y_j X_{i,j}$. Otherwise it decays exponentially with rate δ_i . This process is a Markovian shot-noise process (compare, e.g. Dassios & Jang (2003) and Gaspar & Schmidt (2011)). Furthermore, the dependence structure of the multivariate shot noise process $(\lambda_1, \lambda_2, \dots, \lambda_m)$ is determined by the process M and the random variables $\{Y_j : j \geq 1\}$. If Y_j is a deterministic constant same for all $j \geq 1$, then $\lambda_{t,1}, \dots, \lambda_{t,m}$ are independent conditional on M .

Consider a portfolio consisting of m obligors. The default time of obligor i is denoted by τ_i . Let E_1, \dots, E_m be independent random variables, exponentially distributed with parameter one, which are also independent of the processes $\lambda_1, \dots, \lambda_m$ and assume that

$$\tau_i = \inf \left\{ t > 0 : \int_0^t \lambda_{s,i} ds \geq E_i \right\},$$

This framework is typically called intensity-based modelling, see Filipović (2009) and λ_i is called *intensity* of τ_i . With the filtration defined by $\mathcal{G}_t = \sigma(\lambda_{s,i} : 0 \leq s \leq t, 1 \leq i \leq m)$ it

follows that

$$\mathbb{Q}[\tau_i > t | \mathcal{G}_\infty] = \mathbb{Q}[\tau_i > t | \mathcal{G}_t] = \exp\left(-\int_0^t \lambda_{s,i} ds\right). \quad (2.2)$$

Lemma 2.1. *Let $H_i(x) := \delta_i^{-1}(1 - e^{-\delta_i x})$. Then*

$$\int_0^t \lambda_{s,i} ds = \lambda_{0,i} H_i(t) + \sum_{j=1}^{M_t} Y_j X_{i,j} H_i(t - S_j).$$

Proof. First, observe that the solution of the SDE (2.1) is given by

$$\lambda_{t,i} = \sum_{j=0}^{M_t} Y_j X_{i,j} \exp(-\delta_i(t - S_j)),$$

where we use $Y_0 = 1$, $X_0 = \lambda_{0,i}$ and $S_0 = 0$ to simplify the notation. Hence

$$\begin{aligned} \int_0^t \lambda_{s,i} ds &= \int_0^t \sum_{j=0}^{M_s} Y_j X_{i,j} e^{-\delta_i(s - S_j)} ds \\ &= \int_0^t \sum_{j=0}^{M_t} Y_j X_{i,j} e^{-\delta_i(s - S_j)} \mathbf{1}_{\{S_j \leq s\}} ds \\ &= \sum_{j=0}^{M_t} Y_j X_{i,j} e^{\delta_i S_j} \frac{1}{\delta_i} (e^{-\delta_i S_j} - e^{-\delta_i t}) \end{aligned}$$

and the conclusion follows. \square

Note that Lemma 2.1 purely results from the shot-noise assumptions and can easily be generalized to non-exponential decay, compare for example Gaspar & Schmidt (2011). It moreover holds for arbitrary random variables η_j replacing $Y_j X_{i,j}$.

3. PRICING CREDIT DERIVATIVES IN A HOMOGENEOUS MODEL

Consider a portfolio consisting of m obligors with default times $\tau_1, \tau_2, \dots, \tau_m$ and identical recovery rates $\phi_1 = \phi_2 = \dots = \phi_m = \phi$. The credit loss L_t for this portfolio at time t , in percent of the nominal portfolio value at $t = 0$, is given by

$$L_t = \frac{1 - \phi}{m} \sum_{i=1}^m \mathbf{1}_{\{\tau_i \leq t\}} = \frac{1 - \phi}{m} N_t \quad (3.1)$$

where $N_t := \sum_{i=1}^m \mathbf{1}_{\{\tau_i \leq t\}}$ counts the number of defaults in the portfolio.

It is well known that in order to price portfolio credit derivatives – such as basket default swaps or CDO tranches – on portfolios with homogeneous recoveries, it is enough to find the distribution $\{\mathbb{Q}(N_t = k)\}_{k=0}^m$ at different time points t , see e.g. Herbertsson (2008).

In order to simplify computations further, one often assumes that the portfolio is *homogeneous*, which means that all default times are exchangeable. To this regard, consider

$$\mathbb{Q}(N_t = k) = \sum_{M \subset \{1, \dots, m\}, |M|=k} \mathbb{Q}(\tau_i \leq t : i \in M, \tau_i > t : i \notin M).$$

Exchangeability means that the probability on the r.h.s. does only depend on the number of defaults being smaller than t . Then

$$\mathbb{Q}(N_t = k) = \binom{m}{k} \mathbb{Q}\left(\bigcap_{i=1}^k \{\tau_i \leq t\}, \bigcap_{i=k+1}^m \{\tau_i > t\}\right). \quad (3.2)$$

Therefore we need to compute the probabilities on the r.h.s in (3.2). In this section we consider exchangeable portfolios which is equivalent to the following assumption regarding (2.1)

$$\lambda_{i,0} = \lambda_0, \quad \delta_i = \delta \quad \text{and} \quad F_i = F \quad \text{for} \quad 1 \leq i \leq m. \quad (3.3)$$

Denote $H(x) = \delta^{-1}(1 - e^{-\delta x})$. We can then state the following useful lemma.

Lemma 3.1. *Under (3.3) we have that*

$$\mathbb{Q}(N_t = k) = \binom{m}{m-k} G(m-k) - \binom{m}{m-k+1} G(m-k+1) \quad (3.4)$$

where

$$G(k) := \mathbb{E}\left(e^{-\sum_{i=1}^k \int_0^t \lambda_{s,i} ds}\right). \quad (3.5)$$

Proof. First, observe that $N_t \leq m - k$ implies that at least k companies have survived up to time t . Hence, the homogeneity assumption (3.3) then yields

$$\mathbb{Q}(N_t \leq m - k) = \binom{m}{k} \mathbb{E}\left(e^{-\sum_{i=1}^k \int_0^t \lambda_{s,i} ds}\right) \quad (3.6)$$

and since $\mathbb{Q}(N_t = k) = \mathbb{Q}(N_t \leq k) - \mathbb{Q}(N_t \leq k - 1)$ we conclude the lemma. \square

Thus, in order to find $\mathbb{Q}(N_t = k)$ it is sufficient to compute $G(k)$ for any $k = 1, \dots, m$. Throughout we denote by X a prototype for $X_{i,j}$, for example $X_{1,j}$ and similarly Y for Y_j . Furthermore, denote by $\varphi_X(z) = \mathbb{E}(e^{-zX})$ the Laplace transform of the non-negative random variable X . The following result gives the necessary quantities for finding $\{G(t, j)\}$.

Proposition 3.2. *Under (3.3) we have that*

$$G(k) = e^{-k\lambda_0 H(t) - \rho t} \cdot \exp\left(\rho t \int_{\mathbb{R}} \int_0^1 (\varphi_X(yH(tz)))^k dz F_Y(dy)\right). \quad (3.7)$$

Proof. First, by Lemma 2.1

$$\mathbb{E}\left(\exp\left(-\sum_{i=1}^k \int_0^t \lambda_{s,i} ds\right)\right) = \mathbb{E}\left(\exp\left(-\sum_{i=1}^k (\lambda_{0,i} H(t) + \sum_{j=1}^{M_t} Y_j X_{i,j} H(t - S_j))\right)\right). \quad (3.8)$$

To compute the right hand side in (3.8) we use the following observations: Conditional on $M_t = \ell$ the jump times $\{S_i\}_{i=1}^\ell$ are distributed like the order statistics of uniform random variables over the interval, see for example p.502 in Rolski, Schmidli, Schmidt & Teugels (1999). More precisely, let $\eta_1, \eta_2, \dots, \eta_\ell$ be ℓ independent random variables all with distribution $U[0, t]$, then $\mathcal{L}(S_1, \dots, S_\ell | M_t = \ell) = \mathcal{L}(\eta_{1:\ell}, \dots, \eta_{\ell:\ell})$ where $\{\eta_{n:\ell}\}_{n=1}^\ell$ is the ordering of $\{\eta_n\}_{n=1}^\ell$. Thus,

$$\begin{aligned} \mathbb{E} \left(e^{-\sum_{i=1}^k \sum_{j=1}^{M_t} Y_j X_{i,j} H(t-S_j)} \middle| M_t = \ell \right) &= \mathbb{E} \left(e^{-\sum_{i=1}^k \sum_{j=1}^\ell Y_j X_{i,j} H(t(1-\eta_{j:\ell}))} \right) \\ &= \mathbb{E} \left(e^{-\sum_{i=1}^k \sum_{j=1}^\ell Y_j X_{i,j} H(t(1-\eta_j))} \right) \end{aligned} \quad (3.9)$$

where the last equality follows because all $Y_j, X_{i,j}$ are independent of η_1, \dots, η_j and since all $X_{i,j}$ are exchangeable as they are independent and have identical distributions. By (3.3) we have that $\lambda_{0,i} = \lambda_0$ and thus,

$$\begin{aligned} \mathbb{E} \left(\exp \left(- \sum_{i=1}^k \left(\lambda_{0,i} H(t) + \sum_{j=1}^{M_t} Y_j X_{i,j} H(t-S_j) \right) \right) \middle| M_t = \ell \right) \\ = e^{-k\lambda_0 H(t)} \mathbb{E} \left(\exp \left(- \sum_{i=1}^k \sum_{j=1}^\ell Y_j X_{i,j} H(t(1-\eta_j)) \right) \right). \end{aligned} \quad (3.10)$$

Next, we compute the expectation in (3.10). First,

$$\begin{aligned} \mathbb{E} \left(e^{-\sum_{i=1}^k \sum_{j=1}^\ell Y_j X_{i,j} H(t(1-\eta_j))} \middle| Y_1 = y_1, \dots, Y_\ell = y_\ell, \eta_1 = z_1, \dots, \eta_\ell = z_\ell \right) \\ = \mathbb{E} \left(\prod_{i=1}^k \prod_{j=1}^\ell e^{-y_j X_{i,j} H(t(1-z_j))} \right) = \prod_{i=1}^k \prod_{j=1}^\ell \mathbb{E} \left(e^{-y_j X_{i,j} H(t(1-z_j))} \right) \\ = \prod_{i=1}^k \prod_{j=1}^\ell \varphi_X(y_j H(t(1-z_j))) = \prod_{j=1}^\ell \left(\varphi_X(y_j H(t(1-z_j))) \right)^k, \end{aligned} \quad (3.11)$$

where we used that $\{Y_j\}$ and $\{\eta_j\}$ are independent of $\{X_{i,j}\}$. Hence, by (3.11)

$$\begin{aligned} \mathbb{E} \left(e^{-\sum_{i=1}^k \sum_{j=1}^\ell Y_j X_{i,j} H(t(1-\eta_j))} \right) &= \mathbb{E} \left(\prod_{j=1}^\ell \left(\varphi_X(Y_j H(t(1-\eta_j))) \right)^k \right) \\ &= \left[\mathbb{E} \left(\left(\varphi_X(Y_1 H(t(1-\eta_1))) \right)^k \right) \right]^\ell \\ &= \left[\int_{\mathbb{R}} \int_0^1 \left(\varphi_X(y H(tz)) \right)^k dz F_Y(dy) \right]^\ell \end{aligned}$$

where the first and second equality follows from (3.11) and the last equality is due to the fact that $1-\eta$ is uniformly distributed on $[0, 1]$. Finally, using the above results together

with (3.10) and the definition of $G(k)$ in (3.5) and (3.8) we get

$$\begin{aligned}
G(k) &= \mathbb{E} \left(e^{-\sum_{i=1}^k \int_0^t \lambda_{s,i} ds} \right) \\
&= \sum_{\ell=0}^{\infty} \mathbb{E} \left(e^{-\sum_{i=1}^k \int_0^t \lambda_{s,i} ds} \mid M_t = \ell \right) \mathbb{Q}(M_t = \ell) \\
&= \sum_{\ell=0}^{\infty} e^{-\rho t} \frac{(\rho t)^\ell}{\ell!} e^{-k\lambda_0 H(t)} \left[\int_{\mathbb{R}} \int_0^1 (\varphi_X(yH(tz)))^k dz F_Y(dy) \right]^\ell \\
&= e^{-\rho t - k\lambda_0 H(t)} \cdot \exp \left(\rho t \int_{\mathbb{R}} \int_0^1 (\varphi_X(yH(tz)))^k dz F_Y(dy) \right)
\end{aligned}$$

which concludes the proposition. \square

As already mentioned, the quantity $\mathbb{Q}(N_t = k)$ is central for pricing portfolio credit derivatives, and the fact that we are able to derive $\mathbb{Q}(N_t = k)$ explicitly up to the quantity $\int \int_0^1 (\varphi_X(yH(tz)))^k dz F_Y(dy)$ is highly remarkable. Depending on φ_X this quantity can be computed explicitly.

4. AN EXPLICIT EXAMPLE

In this section we give an tractable and explicit example of the model presented in (2.1) under the assumption (3.3). To be more specific, we assume that

$$Y \in \{y_1, y_2, \dots, y_K\} \quad \text{where} \quad \mathbb{Q}(Y = y_j) = q_j \quad \text{and} \quad X \sim \chi^2(2) \quad (4.1)$$

where $K \geq 2$ is an integer and $y_j, q_j \geq 0$ for each $j = 1, \dots, K$ and $\sum_{j=1}^K q_j = 1$. Hence, Y is a K -point distributed random variable and X has chi-squared distribution with 2 degrees of freedom. This result can be generalized in a number of ways. For example, any distribution for X with has an closed form expression for its Laplace transform still leads to tractable formulas, e.g. a Gamma distribution. We chose the stated formulation for simplicity and it is remarkable that it provides a good fit in our numerical examples.

Proposition 4.1. *Under (3.3) and (4.1) we have that*

$$G(k) = \exp \left(-k\lambda_0 H(t) + \rho t \left[\sum_{j=1}^K q_j I(y_j, k, t) - 1 \right] \right) \quad (4.2)$$

where

$$I(y, k, t) := \int_0^1 \frac{1}{(1 + 2y\delta^{-1}(1 - e^{-\delta tz}))^k} dz. \quad (4.3)$$

Furthermore,

$$\mathbb{Q}(\tau_i > t) = e^{-\lambda_0 H(t) + ct} \prod_{j=1}^K [1 + 2y_j \delta^{-1} (1 - e^{-\delta t})]^{\frac{\rho q_j}{\delta + 2y_j}} \quad (4.4)$$

where c is given by

$$c = \rho \left(\sum_{j=1}^K \frac{q_j}{1 + 2y_j \delta^{-1}} - 1 \right).$$

Proof. Recall that if $X \sim \chi^2(2)$ then $\varphi_X(s) = (1 + 2s)^{-1}$ and since $H(x) = \delta^{-1}(1 - e^{-\delta x})$ we have

$$\varphi_X(yH(tz)) = \frac{1}{1 + 2y\delta^{-1}(1 - e^{-\delta tz})}. \quad (4.5)$$

Since Y is a K -point distributed random variable with $\{y_1, \dots, y_K\}$ and $\mathbb{Q}(Y = y_j) = q_j$ we get

$$\int_{\mathbb{R}} \int_0^1 \left(\varphi_X(yH(tz)) \right)^k dz F_Y(dy) = \sum_{j=1}^K q_j I(y_j, k, t) \quad (4.6)$$

where we define $I(y, k, t)$ as in (4.3). Hence, plugging (4.6) into (3.7) in Proposition 3.2 yields (4.2). It is possible to obtain analytical expressions for the integrals $I(y, k, t)$, however as k increases these become quite long and tedious. In practice we evaluate $I(y, k, t)$ using numerical quadrature. However, for $k = 1$, we can simplify (4.2). To see this, note that (2.2) and (3.5) imply

$$\mathbb{Q}(\tau_i > t) = \mathbb{E} \left(\exp \left(- \int_0^t \lambda_{s,i} ds \right) \right) = G(1)$$

and by (4.2) with $k = 1$ we have

$$G(1) = e^{-\lambda_0 H(t)} \exp \left(\rho t \left[\sum_{j=1}^K q_j I(y_j, 1, t) - 1 \right] \right) \quad (4.7)$$

where $I(y, 1, t)$ is given by (4.3) with $k = 1$. Furthermore, note that (see e.g. p.171 in Råde & Westergren (1995))

$$\int \frac{1}{b + ce^{az}} dz = \frac{z}{b} - \frac{1}{ab} \ln |b + ce^{az}|$$

and this observation with (4.3) and $k = 1$ yields

$$I(y, 1, t) = \frac{1}{1 + 2y\delta^{-1}} + \frac{1}{t(\delta + 2y)} \ln [1 + 2y\delta^{-1}(1 - e^{-\delta t})]. \quad (4.8)$$

Next, some calculations renders

$$\rho t \left(\sum_{j=1}^K q_j I(y_j, 1, t) - 1 \right) = \rho t \sum_{j=1}^K \frac{q_j}{1 + 2y_j \delta^{-1}} - \rho t + \sum_{j=1}^K \frac{\rho q_j}{\delta + 2y_j} \ln [1 + 2y_j \delta^{-1}(1 - e^{-\delta t})]$$

and plugging this into (4.7) yields (4.4). \square

Proposition 4.1 will be used in Section 7 to calibrate the model to market data.

5. A HOMOGENEOUS GROUP APPROACH

It is sometimes natural to consider extensions of the homogeneous framework presented in Section 3. One example is to obtain a higher degree of heterogeneity in the model. This can easily be achieved by splitting the portfolio into several different homogeneous subportfolios and then apply the results of Section 3 to each subportfolio. To show this in more detail, we here follow the outline presented in Papageorgiou & Sircar (2010). Assume that the m obligors consist of g homogeneous groups, and each firm belongs to one group only. The number of firms in group j is denoted by $m_j > 0$. We assume there exists a common factor W , such that given W , the intensities are independent and they are exchangeable in each group. Denoting by $N_{t,j}$ the number of losses in group j , we obtain as in Lemma 3.1 that

$$\mathbb{Q}(N_{t,j} = k | W = w) = \binom{m_j}{m_j - k} G^j(m_j - k) - \binom{m_j}{m_j - k + 1} G^j(m_j - k + 1) \quad (4.1)$$

where $G^j = G^j(w)$ can be determined as in Proposition 3.2. From (2.2) we obtain the probability of having k defaults in the whole portfolio as

$$\mathbb{Q}(N_t = k) = \sum_{k_1=0}^k \cdots \sum_{k_g=0}^{k-k_1-\dots-k_{g-1}} \mathbb{Q}(N_{t,1} = k_1, \dots, N_{t,g} = k_g)$$

and

$$\mathbb{Q}(N_{t,1} = k_1, \dots, N_{t,g} = k_g | W) = \mathbb{Q}(N_{t,1} = k_1 | W) \cdots \mathbb{Q}(N_{t,g} = k_g | W).$$

Computing the sum in (4.1) is quite intensive, and following Papageorgiou & Sircar (2010) it may be improved by using Fourier-inversion methods. To this note that

$$\mathbb{E}(e^{\theta N_{t,1}}) = \int_0^\infty \mathbb{Q}(N_{t,1} \geq \theta^{-1} \ln(x)) dx = (e^\theta - 1) \sum_{k \geq 0} e^\theta k \mathbb{Q}(N_{t,1} \geq k),$$

with an analogous expression for the expectation conditional on W . The probability may be obtained analogously to Lemma 3.1 as by (3.6) we have

$$\mathbb{Q}(N_{t,j} \leq k) = \binom{m_j}{m_j - k} G^j(m_j - k).$$

As the calibration procedure in Section 7 shows a surprisingly good fit of the homogeneous model we, however, leave the application of these results to future research.

6. PRICING CDS AND BASKET DEFAULT SWAPS

In this section we give a short account on single-name CDS and k th-to default swaps. As previously, all computations are done under a risk-neutral martingale measure \mathbb{Q} . Further, we assume that the risk-free interest rate is a deterministic constant given by r .

6.1. Single-name CDS. Consider an obligor \mathbf{C} with default time τ and recovery rate R . A single-name CDS with maturity T where the reference entity is obligor \mathbf{C} , is a bilateral contract between two counterparties, \mathbf{A} and \mathbf{B} , where \mathbf{B} promises to pay \mathbf{A} the credit losses $(1 - R)$ at τ if the obligor defaults before time T . As compensation for this, \mathbf{A} pays $S\Delta$ to the protection seller \mathbf{B} , at $t_1 < t_2 < \dots < t_N = T$, at most until τ . We assume that $\Delta = t_n - t_{n-1}$ for any n . The CDS spread S is determined so that expected discounted cashflows between \mathbf{A} and \mathbf{B} are equal when the CDS contract is settled at $t = 0$. Assuming a constant interest rate r and deterministic recovery rate implies that S is given by

$$S(T) = \frac{(1 - R) \int_0^T e^{-rs} dF(s)}{\Delta \sum_{n=1}^{n_T} e^{-rt_n} (1 - F(t_n))} \quad (6.1.1)$$

where $F(t) = \mathbb{Q}(\tau \leq t)$ is the distribution functions of the default time for the obligor \mathbf{C} .

6.2. Pricing k th-to-default swaps. A k th-to default swap offers protection against the k th default in the portfolio. To be more specific, consider a basket of m bonds each with notional N , issued by m obligors with default times $\tau_1, \tau_2, \dots, \tau_m$ and recovery rates R_1, R_2, \dots, R_m . Further, let $T_1 < \dots < T_m$ be the ordering of $\tau_1, \tau_2, \dots, \tau_m$. A k th-to-default swap with maturity T on this basket is a bilateral contract between two counterparties, \mathbf{A} and \mathbf{B} , where \mathbf{B} promises \mathbf{A} to pay the credit losses that \mathbf{B} suffers at T_k if $T_k < T$. Just as in the CDS, \mathbf{A} pays \mathbf{B} a fee up to the default time T_k or until T , whichever comes first. The payments dates are identical to those in the CDS case and the fee is $S^{(k)}\Delta$ where Δ is as previously $t_n - t_{n-1}$ and we assumed equidistant payment dates. The main difference lies in the default payment at T_k . If $T_k < T$, \mathbf{B} pays \mathbf{A} $N(1 - R_i)$ if it was obligor i which defaulted at time T_k . However, if

$$R_1 = R_2 = \dots = R_m = R, \quad (6.2.1)$$

the payment at T_k is always $N(1 - R)$ if $T_k < T$. The k th-to-default spread $S^{(k)}$ is expressed in bp per annum and determined so that the expected discounted cash-flows between \mathbf{A} and \mathbf{B} coincide at $t = 0$. Assuming the same conditions as in the CDS, we therefore have

$$S^{(k)}(T) = \frac{(1 - R) \int_0^T e^{-rs} dF_k(s)}{\Delta \sum_{n=1}^{n_T} e^{-rt_n} (1 - F_k(t_n))}.$$

Here $F_k(t) = \mathbb{Q}(T_k \leq t)$ is the distribution functions of the ordered default times. The rest of the notation are the same as in the CDS contract. Under (6.2.1) the k th-to-default spread is completely determined by the distribution for N_t : recall that

$$\mathbb{Q}(T_k > t) = \mathbb{Q}(N_t < k) = \sum_{j=0}^{k-1} \mathbb{Q}(N_t = j)$$

where $\mathbb{Q}(N_t = j)$ is given in Lemma 3.1. Furthermore, note that for $k \leq m - 1$ we have

$$\mathbb{Q}(T_{k+1} > t) = \mathbb{Q}(T_k > t) + \mathbb{Q}(N_t = k)$$

which is useful from computational point of view when finding the survival distribution $F_k(t)$ for several $k = 1, 2, \dots, \ell$ where $\ell \leq m$.

6.3. The index-CDS spread and its relation to the CDS-spread in a homogeneous portfolio. The financial instrument that constitutes a index-CDS spread with maturity T on a portfolio with m defaultable obligors, is a bilateral contract where the protection seller \mathbf{B} agrees to pay the protection buyer \mathbf{A} , all losses that occurs in the portfolio up to time T , that is L_T defined as in Equation (3.1). The payments are made at the corresponding default times, if they arrive before T , and at T the contract ends. As compensation for this, \mathbf{A} pays \mathbf{B} a periodic fee proportional to the current outstanding (possible reduced due to losses) value of the portfolio up to time T . To make our discussion more formal, the payments from \mathbf{A} to \mathbf{B} and from \mathbf{B} to \mathbf{A} are given by

$$\mathbb{E} \left(\int_0^T e^{-rs} dL_s \right) \quad \text{and} \quad S_{ind}(T) \sum_{n=1}^{n_t} e^{-rt_n} \left(1 - \frac{1}{m} \mathbb{E}(N_{t_n}) \right) \Delta \quad (6.3.1)$$

where $L_t = \frac{1-R}{m} N_t$ for $N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$. The rest of the notation are the same as in the CDS contract. The index CDS spread $S_{ind}(T)$ is determined so that the expected discounted payments between \mathbf{A} and \mathbf{B} are equal at the inception time 0. This observation together with the quantities in (6.3.1) implies

$$S_{ind}(T) = \frac{\mathbb{E} \left(\int_0^T e^{-rs} dL_s \right)}{\sum_{n=1}^{n_t} e^{-rt_n} \left(1 - \frac{1}{m} \mathbb{E}(N_{t_n}) \right) \Delta}. \quad (6.3.2)$$

The spread $S_{ind}(T)$ is quoted in bp per annum and is independent of the nominal size of the portfolio.

We can now state the following lemma for a homogeneous portfolio.

Lemma 6.1. *Consider a homogeneous credit portfolio. Then, with notation as above, $S_{ind}(T) = S(T)$, that is the index-CDS spread is equal to the individual CDS-spread.*

Proof. Since $L_t = \frac{1-R}{m} N_t$ where $N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ we have that

$$\int_0^T e^{-rs} dL_s = \frac{1-R}{m} \int_0^T e^{-rs} dN_s = \frac{1-R}{m} \sum_{i=1}^m e^{-r\tau_i} 1_{\{\tau_i \leq T\}}$$

so

$$\mathbb{E} \left(\int_0^T e^{-rs} dL_s \right) = \frac{1-R}{m} \sum_{i=1}^m \mathbb{E} \left(e^{-r\tau_i} 1_{\{\tau_i \leq T\}} \right) = (1-R) \int_0^T e^{-rs} dF(s) \quad (6.3.3)$$

where $F(t) = \mathbb{Q}(\tau_i \leq t)$ is the distribution functions of the default time for the obligor i , and the last equality in (6.3.3) is due to the exchangeability in the portfolio. Furthermore, we also have that

$$\frac{1}{m} \mathbb{E}(N_{t_n}) = \frac{1}{m} \sum_{i=1}^m \mathbb{Q}(\tau_i \leq t_n) = F(t_n) \quad (6.3.4)$$

where the last equality in (6.3.4) again is due to the exchangeability in the portfolio. Inserting (6.3.3) and (6.3.4) into (6.3.2) yields that

$$S_{ind}(T) = \frac{(1-R) \int_0^T e^{-rs} dF(s)}{\Delta \sum_{n=1}^{n_T} e^{-rt_n} (1-F(t_n))}$$

and comparing this expression with the formula for the CDS-spread in Equation (6.1.1) shows that $S_{ind}(T) = S(T)$ which proves the lemma. \square

From Lemma 6.1 we conclude that if we want to calibrate the CDS-spread in a homogeneous model, then we can instead calibrate the index-CDS spread. This is very useful since there exists very liquidly traded index-CDS spreads on standardized portfolios such as iTraxx Europe, iTraxx Asia, iTraxx Europe financial, the North American CDX series etc.

7. SOME NUMERICAL EXAMPLES

In this section we use the model (4.1) presented in Section 4 with $K = 2$ and calibrate it to the term structure on market data from the iTraxx Europe index series sampled during the period 2008-01-14 to 2010-02-11. We then compute k th-to defaults swaps in the calibrated model. The data set used in the calibration was collected from Reuters with a daily sample frequency and the term structure is given by $T = 3, 5, 7, 10$ years. However, since several dates in the period 2008-01-14 to 2010-02-11 did not contain the market quotes $S_{M,ind}(T)$ for all four maturities, $T = 3, 5, 7, 10$, the sample-days with incomplete term-structure quotes were removed from the data. This removal reduced the time-series to 435 observations of quadruples $\{S_{M,ind}(T)\}_{T \in \{3,5,7,10\}}$ sampled at different days during the period 2008-01-14 to 2010-02-11, which are displayed in Figure 1.

In the calibration of our model we use Lemma 6.1 which states that in a homogeneous model, the index-CDS spread will coincide with the individual CDS-spread. Hence, for each sample date the quoted iTraxx Europe index spreads $\{S_{M,ind}(T)\}_{T \in \{3,5,7,10\}}$ will be a proxy to the “market” CDS-spread $\{S_M(T)\}_{T \in \{3,5,7,10\}}$ of one obligor in the homogeneous portfolio (recall that all obligors by hypothesis are exchangeable). Thus, the “market” CDS-spread $S_M(T)$ is defined as $S_M(T) = S_{M,ind}(T)$ for $T = 3, 5, 7, 10$. For each sample time in our dataset displayed in Figure 1, we calibrate our model so that the model spread $S(T)$ will coincide with the corresponding market spread $S_M(T)$ for $T = 3, 5, 7, 10$ at the time point where the data was retrieved. To be more specific, at each sample point in the time series the parameters $\theta = (y_1, y_2, q, \delta, \rho, \lambda_0)$ are obtained according to

$$\theta = \underset{\hat{\theta}}{\operatorname{argmin}} \sum_{T \in \{3,5,7,10\}} \frac{(S(T; \hat{\theta}) - S_M(T))^2}{S_M(T)^2} \quad (7.1)$$

with the constraint that all elements in θ are nonnegative. In $S(T; \theta)$ we have emphasized that the model spreads are functions of $\theta = (y_1, y_2, q, \delta, \rho, \lambda_0)$ but suppressed the dependence of interest rate, payment frequency, etc. In the calibration, we used a interest rate

of 3%, the payments in the premium leg were quarterly and the integral in the default leg was discretized on a quarterly mesh.

We performed this calibration for our 435 quadruples $\{S_{M,ind}(T)\}_{T \in \{3,5,7,10\}}$ and the calibrated parameters $(y_1, y_2, q, \delta, \rho, \lambda_0)$ are displayed in Figure 2. At each sample date, the initial guess parameters used in the calibration (7.1) was chosen to be the calibrated parameters from the previous sample date.

Furthermore, the corresponding average absolute errors $\frac{1}{4} \sum_{T \in \{3,5,7,10\}} |S(T; \boldsymbol{\theta}) - S_M(T)|$ and average relative (i.e. mean) errors $\frac{1}{4} \sum_{T \in \{3,5,7,10\}} \frac{|S(T; \boldsymbol{\theta}) - S_M(T)|}{S_M(T)}$ are displayed in Figure 3. From the lower graph in Figure 3 we conclude that a substantial majority of the calibrations will yield an average relative error which is smaller than 4%. In fact, a careful study of the average relative errors in the lower graph of Figure 3 reveals that out of the 435 calibrations, 398 have a mean relative error less than 4%, 331 calibrations have an average relative error less than 3%, 196 less than 2% and 63 less than 1%. Thus, we can therefore speak of very good fits and a general robustness of the model. Furthermore, it is also interesting to note that the calibrations are performed against term structures $\{S_{M,ind}(T)\}_{T \in \{3,5,7,10\}}$ of market spreads which have a wide spectrum of different relationships, as can be seen from Figure 1. Note especially that we have an reverse term structure, i.e. $S_{M,ind}(3) > S_{M,ind}(5) > S_{M,ind}(7) > S_{M,ind}(10)$ for a period from November 2008 to May 2009, while the rest of time the market data has the much more common appearance of an increasing term-structure, that is $S_{M,ind}(3) < S_{M,ind}(5) < S_{M,ind}(7) < S_{M,ind}(10)$.

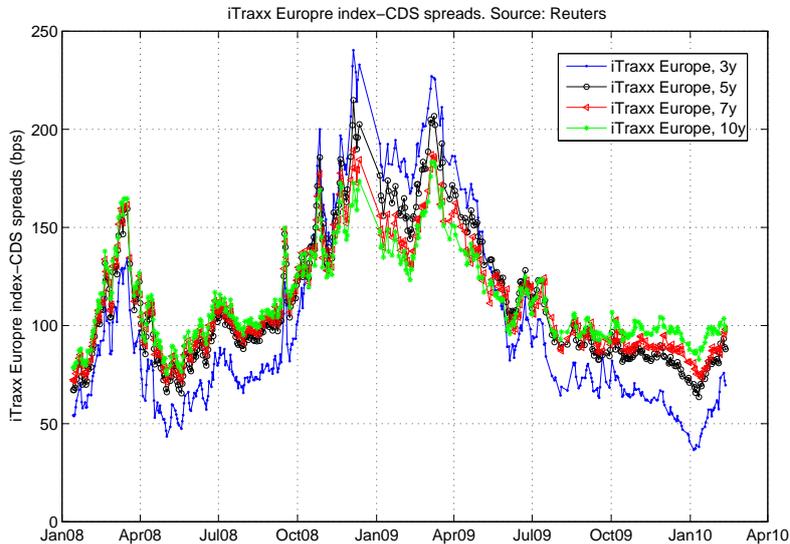


Figure 1. The market spreads for the iTraxx Europe index-CDS with $T = 3, 5, 7$ and 10 years maturity in the period 2008-01-14 to 2010-02-11.

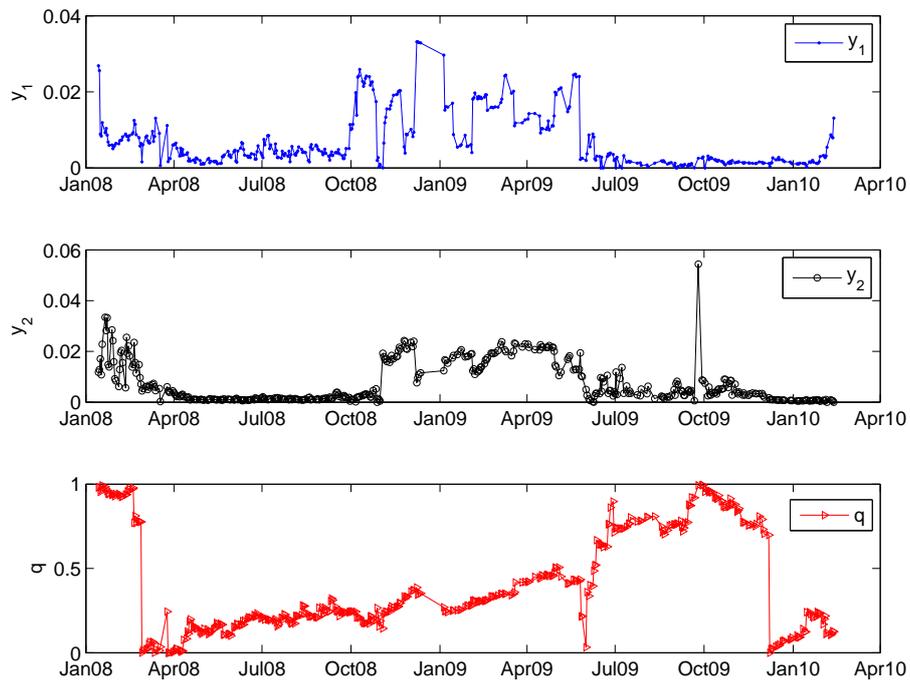
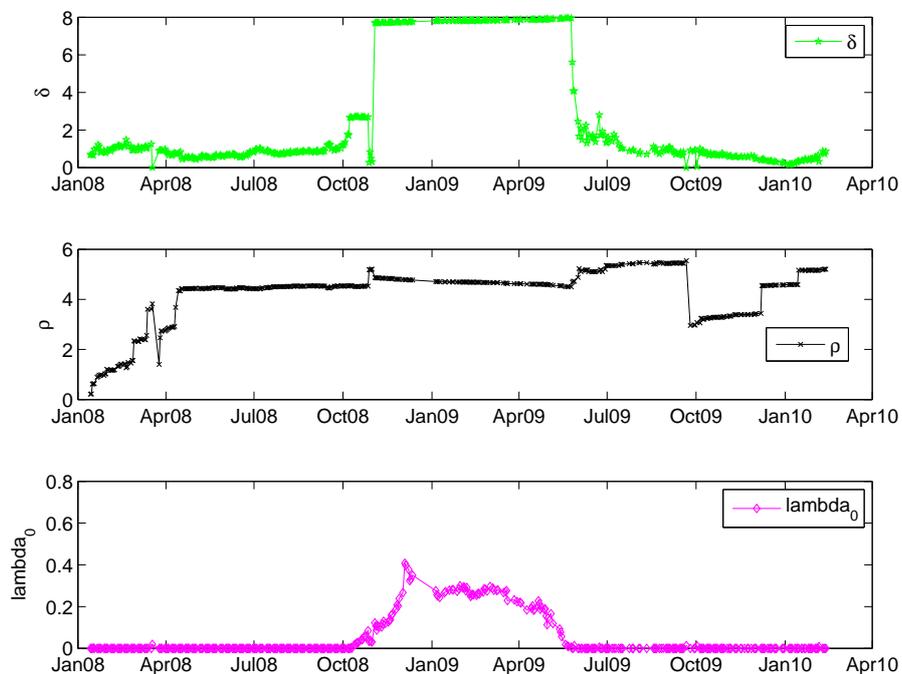
(a) The calibrated parameters y_1 , y_2 and q .(b) The calibrated parameters δ , ρ and λ_0 .

Figure 2. The calibrated parameters y_1 , y_2 , q , δ , ρ and λ_0 when calibrating the model against the term-structure of market spreads for the iTraxx Europe index-CDS with $T = 3, 5, 7, 10$ years in the period 2008-01-14 to 2010-02-11.

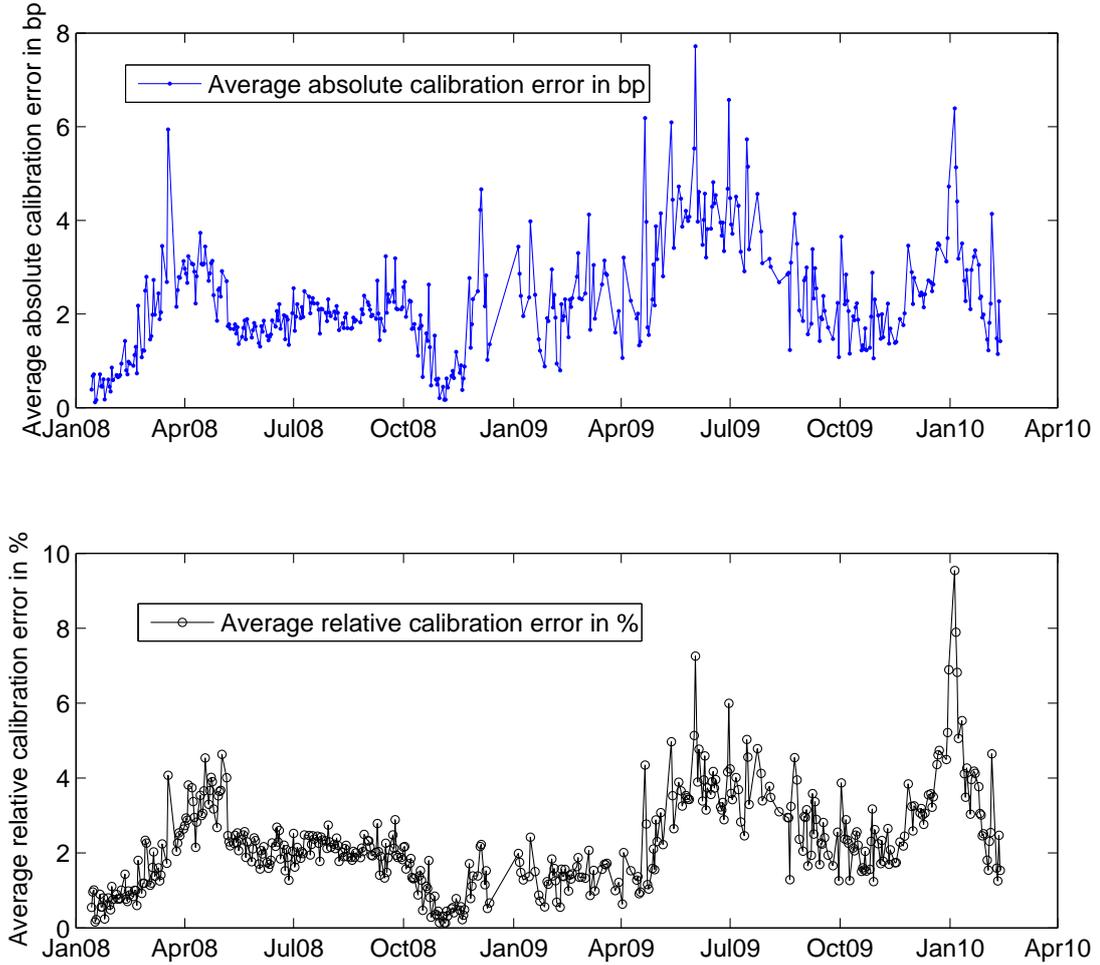


Figure 3. The average absolute calibration errors and average relative calibration errors when calibrating the model against the market spreads for the iTraxx Europe index-CDS with $T = 3, 5, 7$ and 10 years maturity in the period 2008-01-14 to 2010-02-11.

To give some more detailed results of the calibrations, we have picked out three different dates where the market spreads differ substantially. These dates are 2008-03-12, 2008-12-10 and 2009-10-09. The calibrated model spread and the absolute and relative errors are displayed in Table 1 while the calibrated parameters are given in Table 2. The mean of the relative calibration error for these three samples are 1.412%, 0.521256% and 1.26162%. Furthermore, note that market spreads at 2008-12-10 have a reverse term-structure.

Table 1. The market and model spreads (in bps) and their corresponding calibration errors at three different time points.

Calibration of iTraxx Europe index-CDS at 2008-03-12				
Term Structure	$T = 3$	$T = 5$	$T = 7$	$T = 10$
Market CDS-spread	121.3	146.5	152.4	156.4
Model CDS-spread	122.7	142.7	153.0	158.9
Absolute error in bp	1.427	3.828	0.4017	2.493
Rel. error in percent	1.177	2.613	0.2636	1.594
Calibration of iTraxx Europe index-CDS at 2008-12-10				
Term Structure	$T = 3$	$T = 5$	$T = 7$	$T = 10$
Market CDS-spread	225.2	195.8	179.3	169.0
Model CDS-spread	226.2	193.7	179.8	169.5
Absolute error in bp	1.005	2.055	0.5444	0.4816
Rel. error in percent	0.4464	1.050	0.3036	0.2849
Calibration of iTraxx Europe index-CDS at 2009-10-09				
Term Structure	$T = 3$	$T = 5$	$T = 7$	$T = 10$
Market CDS-spread	73.41	89.93	96.81	99.62
Model CDS-spread	74.07	88.62	95.74	101.2
Absolute error in bp	0.6570	1.307	1.068	1.589
Rel. error in percent	0.8950	1.453	1.104	1.595

Table 2. The calibrated parameters for the three dates in Table 1.

Date	y_1	y_2	q	δ	ρ	λ_0
2008-03-12	0.008254	0.005994	3.327e-011	1.016	2.564	0
2008-12-10	0.03299	0.01097	0.3500	7.756	4.771	0.3320
2009-10-09	0.002531	0.003801	0.9507	0.8482	3.203	0

After the calibration of the model on the dates 2008-03-12, 2008-12-10 and 2009-10-09 we compute the k -th-to default swap spread for $1 \leq k \leq 7$ where $T = 5$ and $T = 10$ years in a portfolio with $m = 10$ obligors. We do this by using the results presented in Section 3, Section 4 and Subsection 6.1. Given the huge differences in the index-CDS spreads Table 1, it is not surprising that the k th-to default swap spreads greatly differ between these dates, as can be seen in Table 3. Furthermore, it is interesting to note that for 2008-12-10, the 5-year First-to-default (FtD) spread is bigger than the 10-year FtD-spread. Intuitively, this is consistent with the fact that the market spreads at 2008-12-10 used in the calibration, have a reverse term-structure, which means that the 5-year CDS-spread is bigger than the the 10-year CDS-spread.

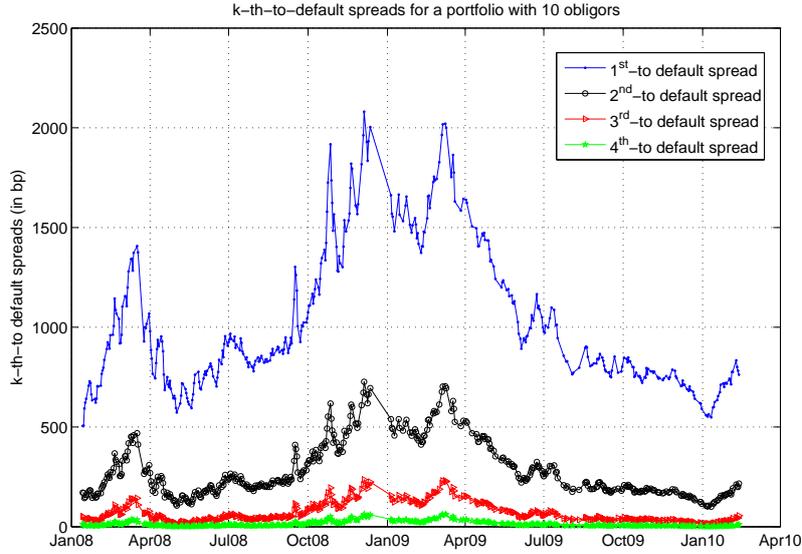
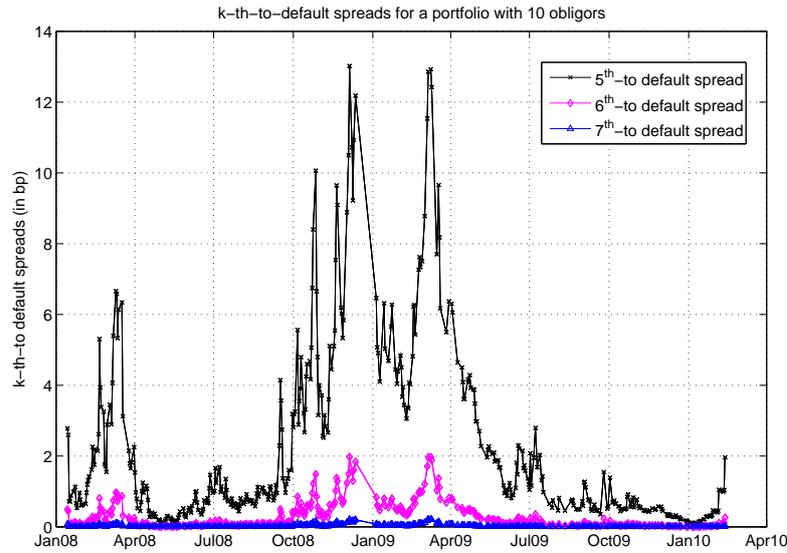
(a) The k th-to-default spreads for $k \leq 4$.(b) The k th-to-default spreads for $k = 5, 6, 7$.

Figure 4. The k th-to-default spreads for $1 \leq k \leq 7$ and $T = 5$ in a portfolio of 10 obligors, after calibrating the single-name model CDS-spread against the term-structure of market spreads for the iTraxx Europe index-CDS with $T = 3, 5, 7, 10$ in the period 2008-01-14 to 2010-02-11.

We complement the results in Table 3 by also computing the k th-to default swap spreads ($1 \leq k \leq 7$ and $T = 5$) in a portfolio with 10 obligors, for all 435 calibrations of the quoted market spreads displayed in Figure 1. The computed k th-to default swap spreads follows the same trend as the market spreads on the iTraxx Europe index series, which is clearly shown in Figure 4.

Table 3. The k th-to default swap spreads (in bps) for $T = 5$ and $T = 10$ years, computed in the three calibrated models in Table 1. The portfolio consist of 10 obligors.

		2008-03-12						
		$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$T = 5$		1284	420.0	124.8	29.48	5.332	0.7253	0.07227
$T = 10$		1372	611.2	287.3	119.3	40.45	10.61	2.057

		2008-12-10						
		$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$T = 5$		1932	663.2	210.6	54.86	10.93	1.607	0.1693
$T = 10$		1820	721.5	306.3	118.8	37.82	9.232	1.640

		2009-10-09						
		$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$T = 5$		828.9	200.6	40.04	5.971	0.6597	0.05376	0.003166
$T = 10$		916.5	341.3	122.5	35.74	7.981	1.322	0.1577

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