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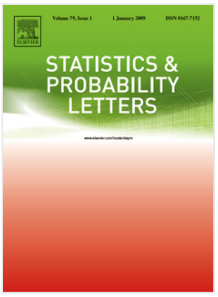

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Large-time asymptotics for an uncorrelated stochastic volatility model

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Abstract

We derive a large-time large deviation principle for the log stock price under an uncorrelated stochastic volatility model. For this we use a Donsker-Varadhan-type large deviation principle for the occupation measure of the Ornstein-Uhlenbeck process, combined with a simple application of the contraction principle and exponential tightness.

Key words: Stochastic volatility, Donsker-Varadhan large deviations for occupation measures, Large time implied volatility asymptotics.

1. The uncorrelated Ornstein-Uhlenbeck model with σ bounded

We work on a model $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ supporting two Brownian motions which satisfies the usual conditions.

Set $f(y) = \sigma^2(y)$, and assume that $0 < f_{\min} \leq f \leq f_{\max} < \infty$. We consider an uncorrelated stochastic volatility model for a log stock price process $X_t = \log S_t$ defined by the following stochastic differential equations

$$\begin{cases} dX_t = -\frac{1}{2}\sigma^2(Y_t)^2 dt + \sigma(Y_t) dW_t^1 \,, \\ dY_t = -\alpha Y_t dt + dW_t^2 \,, \end{cases} \tag{1}$$

for $\alpha > 0$, $X_0 = x_0, Y_0 = y_0$, where W_1, W_2 are two independent standard Brownian motions and Y is an Ornstein-Uhlenbeck process. We set $S_0 = 1$ (i.e. $x_0 = 0$) without loss of generality, because $X_t - x_0$ is independent of x_0 as the SDEs have no dependence on x .

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1.1. Large deviations for the occupation measure of the OU process

For each $t > 0$ and $A \in \mathcal{B}(\mathbb{R})$, let

$$\mu_t(A) = \frac{1}{t}\int_0^t 1_A(Y_s)ds \tag{2}$$

denote the proportion of time up to t that the sample path of Y spends in A. For each $t > 0$ and ω, $\mu_t(\omega, .)$ is a probability measure on $\mathbb{R}$. Let $\mathcal{P}(\mathbb{R})$ denote the space of probability measures on $\mathbb{R}$. Then $\mu_t(A)$ satisfies a large-time large deviation principle in the topology of weak convergence, with convex, lower semicontinuous rate function given by

$$I_B(\mu) = \frac{1}{2}\int_{-\infty}^{\infty} |\partial_y \sqrt{\frac{d\mu}{d\mu_\infty}}(y)|^2 \, \mu_\infty(dy) \tag{3}$$

for $\mu \in \mathcal{P}(\mathbb{R})$, where $\mu_\infty(y) = (\frac{\alpha}{\pi})^{\frac{1}{2}} \, e^{-\alpha y^2}$ is the unique stationary distribution for Y, i.e. $N(0, 1/2\alpha)$ (see Donsker and Varadhan (1975a, 1975b, 1975c, 1976), Stroock (1984) and pages 367-8 in Feng and Kurtz (2006)). If μ is not absolutely continuous with respect to μ_∞, then $I_B(\mu) = \infty$.

Remark 1.1. Clearly $I_B(\mu)$ attains it minimum value of zero at $\mu = \mu_\infty$. Moreover, any measure which makes the rate function zero is a stationary distribution, and Y has a unique stationary distribution, so μ_∞ is the unique minimizer of $I_B(\mu)$.

We note that $F : \mathcal{P}(\mathbb{R}) \mapsto [f_{\min}, f_{\max}]$ given by

$$F(\mu) = \langle f, \mu \rangle = \int_{-\infty}^{\infty} f(y)\mu(dy) \tag{4}$$

is a bounded, continuous functional.

1.2. The Prokhorov metric

Given two measures μ and ν in $\mathcal{P}(\mathbb{R})$, the Prokhorov metric is defined by $d(\mu, \nu) = \inf\{\delta > 0 : \mu(C) \leq \nu(C^\delta) + \delta, \, \nu(C) \leq \mu(C^\delta) + \delta$ for all closed $C \subset \mathbb{R}\}$. $\mathcal{P}(\mathbb{R})$ then becomes a compact metric space (note that $d(\mu, \nu) \leq 1$), so the rate function $I_B(\mu)$ is *good*. Convergence of measures in the Prokhorov metric is equivalent to weak convergence of measures.

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1.3. Contraction principle

By the contraction principle, the quantity

$$A_t = \frac{1}{t}\int_0^t f(Y_s)ds = \int_{-\infty}^{\infty} f(y)\mu_t(dy)$$

also satisfies the LDP, with good lower semicontinuous rate function given by

$$I_f(a) = \inf_{\mu\in\mathcal{P}(\mathbb{R}):\langle f,\mu\rangle = a} I_B(\mu) \quad , \quad a \in [f_{\min}, f_{\max}]\,. \tag{5}$$

Remark 1.2. $I_B(.)$ is non-negative and $I_B(\mu_\infty) = 0$, so

$$I_f(\bar{\sigma}^2) = 0\,,$$

where

$$\bar{\sigma}^2 = \langle f, \mu_\infty\rangle = \int_{-\infty}^{\infty} \sigma^2(y)\mu_\infty(y)dy\,. \tag{6}$$

Moreover, μ_∞ is the unique minimizer of I_B, so $\bar{\sigma}^2$ is the unique minimizer of I_f.

2. A joint large deviation principle for $(X_t/t, A_t)$

Recall the definition of $A_t = \frac{1}{t}\int_0^t f(Y_s)ds$. The following proposition estbalishes a joint LDP for X_t/t and A_t in the large-time limit.

Proposition 2.1. *$(X_t/t, A_t)$ satisfies a joint LDP as $t \to \infty$ with good rate function $I(x,a) = \frac{(x+\frac{1}{2}a)^2}{2a} + I_f(a)$.*

Proof. See Appendix. □

From this we obtain the following proposition:

Proposition 2.2. *(X_t/t) satisfies the LDP as $t \to \infty$ with a good rate function given by*

$$I(x) = \inf_{a\in[f_{\min},f_{\max}]} [\frac{(x+\frac{1}{2}a)^2}{2a} + I_f(a)] \le \frac{(x+\frac{1}{2}\bar{\sigma}^2)^2}{2\bar{\sigma}^2}$$

and I attains its minimum value of zero at $x = -\frac{1}{2}\bar{\sigma}^2$.

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Proof. The LDP with a good rate function just follows from the contraction principle. Setting $a = \bar{\sigma}^2$ defined in 6 and using that $I_f(\bar{\sigma}^2) = 0$, we see that $I(-\frac{1}{2}\bar{\sigma}^2) = 0$. □

Remark 2.1 For non-zero correlation and/or unbounded σ, the approach outlined here will not work. However, we can transform the problem to a *small-noise, fast mean-reverting regime*, which is the same scaling used in the recent paper by Feng et al. (2010), aside from the fact that the drift of the log Stock price process is not small in this case. This problem then falls into the class of homogenization and averaging problems for nonlinear HJB type equations, where the fast volatility variable lives on a non-compact space. The Feng et al. argument based on viscosity solutions can be easily adapted to the large-time regime, using Bryc's lemma combined with exponential tightness to prove a large deviation principle. The leading order term is the unique viscosity solution to a HJB equation where the Hamiltonian is given in terms of the limiting log mgf for the integrated variance; this will be dealt with in a sequel article. For the well known SABR model with $\beta = 1$, we can derive large-time asymptotics for the correlated case using the Willard mixing formula, see Forde (2010).

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A. Proof of Proposition 2.1

Let $Z_t = X_t/t$. We first note that $(Z_t, A_t) \stackrel{d}{=} (W_{tA_t} - \frac{1}{2}A_t, A_t)$. We first assume $x + \frac{1}{2}a > 0$. Now choose δ so that $0 < \delta < x + \frac{1}{2}a$. Then

$$\begin{aligned}\mathbb{P}(|Z_t - x| < \frac{\delta}{\sqrt{2}}, |A_t - a| < \frac{\delta}{\sqrt{2}}) &\leq \mathbb{P}(\|(Z_t, A_t) - (x, a)\| < \delta) \\ &\leq \mathbb{P}(|Z_t - x| < \delta, |A_t - a| < \delta)\,.\end{aligned}$$

From the Gärtner-Ellis theorem, we can verify that $\frac{W^1_{at} - \frac{1}{2}at}{t}$ satisfies the LDP as $t \to \infty$ with rate $\frac{(x+\frac{1}{2}a)^2}{2a}$. Then $\forall \epsilon > 0$, conditioning on A_t and using the LDPs for A_t and $\frac{W^1_{at} - \frac{1}{2}at}{t}$, we see that there exists a $t = t^*(\epsilon, \delta)$ such that

$$\mathbb{P}(|Z_t - x| < \delta, |A_t - a| < \delta) \leq e^{-t[-\epsilon + \inf_{y \in B_\delta(x)} \frac{(y+\frac{1}{2}(a-\delta))^2}{2(a+\delta)}]}\, e^{-t[-\epsilon + \inf_{a_1 \in \bar{B}_a(\delta)} I_f(a_1)]}\,.$$

Then

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(|Z_t - x| < \delta, |A_t - a| < \delta) \leq -\inf_{y \in B_\delta(x)} \frac{(y + \frac{1}{2}(a-\delta))^2}{2(a+\delta)} - \inf_{a_1 \in \bar{B}_a(\delta)} I_f(a_1)\,,$$

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and by the lower semicontinuity of $I_f(a)$ we have $\lim_{\delta\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(|Z_t - x| < \delta, |A_t - a| < \delta) \leq -[\frac{(x+\frac{1}{2}a)^2}{2a} + I_f(a)]$. Using a similar argument for the lower bound, we replace the limsup here by a genuine limit, so (Z_t, A_t) satisfies the weak LDP with rate function $\frac{(x+\frac{1}{2}a)^2}{2a} + I_f(a)$. $I_f(a)$ is good, so (A_t) is exponentially tight; hence for all $R > 0, a > 0$, there exists a compact $K_a \subset \mathbb{R}$ such that $\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}((Z_t, A_t) \in [-R, R] \times K_a^c) \leq \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(A_t \in K_a^c) \leq -a$, so (Z_t, A_t) is exponentially tight, hence (Z_t, A_t) satisfies the full LDP and the rate function is good. We proceed similarly for $x + \frac{1}{2}a \leq 0$.