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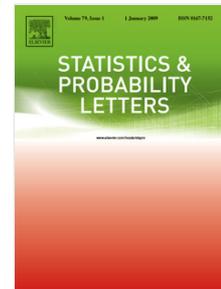
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Uniform Hölder exponent of a stationary increments Gaussian process: estimation starting from average values

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Abstract

Let $\{X(t)\}_{t \in \mathbb{R}}$ be a stationary increments Gaussian process satisfying some assumptions. By using the notion of generalized quadratic variation we build a strongly consistent and asymptotically normal estimator of the uniform Hölder exponent of X , over a compact interval. Our estimator is obtained starting from average values of the process over a regular grid.

Keywords:

Gaussian processes
Hölder exponents
Generalized increments
Quadratic variations
Central Limit Theorems

1. Introduction

Since several years, there has been a considerable interest in the statistical estimation of some indices related to Hölder regularity of sample paths of stochastic processes, we refer e.g. to Ayache and Lévy Véhel (2004), Bardet (2000), Bardet and Bertrand (2007), Benassi et al. (1998), Coeurjolly (2005), Gloter and Hoffmann (2007), Istas and Lang (1997); as for instance the uniform Hölder exponent over a compact interval. Let us recall the definition of the latter exponent. Denote by $\{X(t)\}_{t \in \mathbb{R}}$ a real-valued stochastic process and by K a fixed compact interval in \mathbb{R} ; one says that a sample path $X(\cdot, \omega) : t \mapsto X(t, \omega)$ belongs to the Hölder space $C^\gamma(K)$ where $\gamma \in \mathbb{R}_+ \setminus \mathbb{Z}_+$, if the following two conditions (i) and (ii) are satisfied:

- (i) $X(\cdot, \omega)$ is a $[\gamma]$ -times continuously differentiable function over K , where $[\cdot]$ denotes the integer part function;
- (ii) there is a constant $c = c(K, \omega) > 0$ such that for all $s_1, s_2 \in K$,

$$|X^{([\gamma])}(s_1, \omega) - X^{([\gamma])}(s_2, \omega)| \leq c|s_1 - s_2|^{\gamma - [\gamma]}. \quad (1.1)$$

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The uniform Hölder exponent of $X(\cdot, \omega)$ over K is defined as,

$$h_X(K, \omega) := \sup \{ \gamma \in \mathbb{R}_+ \setminus \mathbb{Z}_+ : X(\cdot, \omega) \in C^\gamma(K) \}.$$

From now on, we suppose that $\{X(t)\}_{t \in \mathbb{R}}$ is a centered stationary increments Gaussian process satisfying almost surely $X(0) = 0$; thus the distribution of $\{X(t)\}_{t \in \mathbb{R}}$ is completely determined by its variogram, namely the even function v defined for all $t \in \mathbb{R}$, as,

$$v(t) = 2^{-1} \mathbb{E} (X(t))^2; \quad (1.2)$$

moreover it follows from zero-one law that there is a deterministic quantity $H = H_X(K) \in [0, +\infty]$ such that one has, for almost all ω , $h_X(K, \omega) = H$. There is no restriction to assume that $K = [0, 1]$. In their seminal article Istas and Lang (1997), by using the notion of generalized quadratic variation, Istas and Lang have constructed, under some assumptions, asymptotically normal estimators of H , starting from the observation of $\{X(i\delta_N)\}_{i=0, \dots, [\delta_N^{-1}]-1}$, the true values of X over a regular grid. However, in the setting of some applications, for example when one has to model a quite fluctuating signal, it sounds to be more realistic to say that one observes average values of the process X , namely $\{\delta_N^{-1} \int_{i\delta_N}^{(i+1)\delta_N} X(s) ds\}_{i=0, \dots, [\delta_N^{-1}]-1}$. The goal of our article is to construct a strongly consistent and asymptotically normal estimator of H starting from such data. From now on, for the sake of simplicity, we assume that the discretization mesh $\delta_N = 1/N$ (N being an integer big enough) and we set

$$\{\bar{X}_{i,N}\}_{i=0, \dots, N-1} := \left\{ N \int_{i/N}^{(i+1)/N} X(s) ds \right\}_{i=0, \dots, N-1}. \quad (1.3)$$

Our results as well as their proofs are inspired by Istas and Lang (1997), however new difficulties appear in our setting; they are mainly due to the fact that

$$\text{Cov} \left(\sum_{k=0}^p a_k \bar{X}_{i+k,N}, \sum_{k'=0}^p a_{k'} \bar{X}_{j+k',N} \right),$$

is more difficult to estimate, than

$$\text{Cov} \left(\sum_{k=0}^p a_k X \left(\frac{i+k}{N} \right), \sum_{k'=0}^p a_{k'} X \left(\frac{j+k'}{N} \right) \right),$$

here $p \geq 1$ is an integer and $a = (a_0, \dots, a_p)$ denotes an arbitrary finite fixed sequence of $p+1$ real numbers satisfying Assumption (2.6).

2. Statement of the main results

Let us first precisely present the assumptions we need for obtaining our main results. Note in passing that these assumptions are nearly similar to some fundamental hypotheses in Istas and Lang (1997).

(A1) Assumptions on the variogram function v : We assume that there exists a finite non-negative integer d such that v is $2d$ -times continuously differentiable on $[-2, 2]$ and v is not $2(d+1)$ -times continuously differentiable on this interval. We denote by $v^{(2d)}$ the derivative of v

of order $2d$, with the convention that $v^{(0)} = v$. Also we assume that there are two real numbers $c \neq 0$ and $0 < s_0 < 2$ such that for all $t \in [-2, 2]$, one has

$$v^{(2d)}(t) = v^{(2d)}(0) + c|t|^{s_0} + r(t), \quad (2.1)$$

where the remainder r satisfies the following two properties:

- For $|t|$ small enough, one has,

$$r(t) = o(|t|^{s_0}). \quad (2.2)$$

- There are two real numbers $c > 0$, $w > s_0$ and an integer $q > w + 1/2$ such that the remainder r is q -times continuously differentiable on $[-2, 2] \setminus \{0\}$ and for all $t \in [-2, 2] \setminus \{0\}$, one has

$$|r^{(q)}(t)| \leq c|t|^{w-q}. \quad (2.3)$$

□

The integers d and q are supposed to be known; in fact the unknown parameter we want to estimate, starting from the data (1.3), is s_0 . It is worth noticing that Assumption (A1) implies, see for instance Cramér and Leadbetter (1967), Ibragimov and Rozanov (1978), that the uniform Hölder exponent H satisfies,

$$H = d + \frac{s_0}{2}. \quad (2.4)$$

Though, this assumption might seem to be a bit technical, it is satisfied Istas and Lang (1997) by fractional Brownian motion (i.e. $v(t) = c|t|^{2\alpha}$ where $c > 0$ is a constant and $\alpha \in (0, 1)$ the Hurst parameter) and other, more or less, classical classes of stationary increments Gaussian processes (for example when $v(t) = 1 - \exp(-|t|^\beta)$, where $\beta \in (0, 2)$ is a parameter).

For any integer $N \geq p + 1$, the generalized increments of the average values $\bar{X}_{i,N}$, $i = 0, \dots, N - 1$ of the process X are defined as,

$$\{\Delta_a \bar{X}_{i,N}\}_{i=0, \dots, N-p-1} = \left\{ \sum_{k=0}^p a_k \bar{X}_{i+k,N} \right\}_{i=0, \dots, N-p-1}, \quad (2.5)$$

where $a = (a_0, \dots, a_p) \in \mathbb{R}^{p+1}$ is an arbitrary finite fixed sequence of $p + 1$ real numbers whose $M(a) \geq d + q/2$ first moments vanish, that is:

$$\sum_{k=0}^p k^l a_k = 0, \text{ for all } l = 0, \dots, M(a) - 1, \text{ and } \sum_{k=0}^p k^{M(a)} a_k \neq 0. \quad (2.6)$$

It worth noticing that one necessarily has $p \geq M(a)$. Also notice that (2.6) implies that for all $l \in \{0, \dots, 2M(a) - 1\}$,

$$\sum_{k=0}^p \sum_{k'=0}^p a_k a_{k'} (k - k')^l = 0. \quad (2.7)$$

The important idea of replacing usual 1-order increments by generalized increments has been initially introduced by Istas and Lang (1997). The main advantage in doing so, is that the statistical estimator of H , defined through generalized quadratic variation, is asymptotically normal whatever the value of H might be. Basically, this asymptotic normality comes from the fact that generalized increments are less correlated than usual increments.

In fact, we need to impose to the sequence a an additional assumption.

(A2) Assumption related to the generalized increments: For all $\nu \in (0, 2)$, one has

$$R(0, 1, 2d, (\cdot)^\nu) \neq 0, \quad (2.8)$$

where $R(0, 1, 0, (\cdot)^\nu)$ is defined by (3.5) and $R(0, 1, 2d, (\cdot)^\nu)$ is defined by (3.6) when $d \geq 1$. \square

It is worth noticing that standard computations allow to show that: the sequence $a^{(2)} = (1, -2, 1)$ has 2 vanishing moments (i.e. $M(a^{(2)}) = 2$) and satisfies Assumption (A2) when $d = 0$; the sequence $a^{(3)} = (1, -3, 3, -1)$ has 3 vanishing moments (i.e. $M(a^{(3)}) = 3$) and satisfies Assumption (A2) when $d \in \{0, 1\}$.

Now we are in position to state the two main results of our article.

Theorem 2.1 *Let us denote by*

$$\hat{H}_N = \frac{1}{2} \left(1 + \log_2 \left(\frac{V_N}{V_{2N}} \right) \right), \quad (2.9)$$

where V_N is the generalized quadratic variation defined as,

$$V_N = \sum_{i=0}^{N-p-1} (\Delta_a \bar{X}_{i,N})^2. \quad (2.10)$$

Then, under Assumptions (A1) and (A2), when $N \rightarrow +\infty$, \hat{H}_N converges almost surely to H ,

Theorem 2.2 *Under the same assumptions as in Theorem 2.1 and the additional assumption that*

$$r(t) = o(|t|^{s_0+1/2}), \quad (2.11)$$

a Central Limit Theorem holds, namely $N^{1/2}(\hat{H}_N - H)$ converges in law to a centered Gaussian random variable.

3. Proof of the main results

First it is convenient to notice that the stationarity of the increments of the process X , implies that for all $N \geq p + 1$ and $i \in \{0, \dots, N - p - 1\}$, one has

$$\sigma_{a,N}^2 := \text{Var}(\Delta_a \bar{X}_{0,N}) = \text{Var}(\Delta_a \bar{X}_{i,N}). \quad (3.1)$$

The proof of Theorem 2.1 mainly relies on the following proposition.

Proposition 3.1 *Under Assumptions (A1) and (A2), there exist two constants $c_1 > 0$ and $c_2 > 0$, such that the following two equalities hold for all $N \geq p + 1$:*

$$\sigma_{a,N}^2 = c_1 N^{-2H} + o(N^{-2H}), \quad (3.2)$$

and

$$s_N^2 := \text{Var} \left(\sum_{i=0}^{N-p-1} \frac{(\Delta_a \bar{X}_{i,N})^2}{\sigma_{a,N}^2} \right) = c_2 N + o(N). \quad (3.3)$$

Now, let us focus on the proof of Proposition 3.1. In order to show that the latter proposition holds, we need several preliminary results. The following lemma (whose proof has been omitted since it is more or less similar to that of Lemma 1 in Gloter and Hoffmann (2004)) gives a nice expression of $\text{Cov}(\Delta_a \bar{X}_{i,N}, \Delta_a \bar{X}_{j,N})$, in terms of the variogram function v .

Lemma 3.1 *For all integer $N \geq p + 1$ and all $i, j \in \{0, \dots, N - p - 1\}$, the following equality holds:*

$$\mathbb{E}(\Delta_a \bar{X}_{i,N} \Delta_a \bar{X}_{j,N}) = -N^2 \sum_{0 \leq k, l \leq p} a_k a_l \left(\int_0^{\frac{1}{N}} \int_0^{\frac{1}{N}} v \left(\frac{|i-j|}{N} + s - s' + \frac{k-l}{N} \right) ds' ds \right). \quad (3.4)$$

Next we will use (3.4) and (2.1) for estimating $\mathbb{E}(\Delta_a \bar{X}_{i,N} \Delta_a \bar{X}_{j,N})$; to this end, we need to introduce some notations. Let h and g be two real-valued Borel functions defined on the real line, let $N \geq p + 1$ and $u \geq 1$ be two integers and let $x \in \mathbb{R}$, we set

$$R(x, N, 0, h) = \sum_{0 \leq k, l \leq p} a_k a_l \int_0^{\frac{1}{N}} \int_0^{\frac{1}{N}} h \left(x + s - s' + \frac{k-l}{N} \right) ds' ds, \quad (3.5)$$

and

$$\begin{aligned} R(x, N, u, g) &= \sum_{0 \leq k, l \leq p} a_k a_l (k-l)^u \int_0^{\frac{1}{N}} \int_0^{\frac{1}{N}} \int_0^1 \frac{(1-\eta)^{u-1}}{(u-1)!} \\ &\quad \times g \left(x + s - s' + \frac{(k-l)\eta}{N} \right) d\eta ds' ds. \end{aligned} \quad (3.6)$$

Of course we assume that h, g, N, u and x have been chosen in such a way that all the integrals in (3.5) and (3.6) are well-defined and finite. Observe that in view of Lemma 3.1, one has

$$\mathbb{E}(\Delta_a \bar{X}_{i,N} \Delta_a \bar{X}_{j,N}) = -N^2 R \left(\frac{|i-j|}{N}, N, 0, v \right). \quad (3.7)$$

In the sequel we set $m = |i - j|$. Let us now give a nice property of R .

Lemma 3.2 *Let $x \in \mathbb{R}$ and u be an integer such that $0 \leq u \leq 2M(a)$. Assume that h is u -times continuously differentiable on the interval $[x - (p+1)/N, x + (p+1)/N]$. Then, for all integer u' , satisfying $0 \leq u' \leq u$, $R(x, N, u', h^{(u')})$ is well-defined and one has*

$$R(x, N, u', h^{(u')}) = N^{-(u-u')} R(x, N, u, h^{(u)}). \quad (3.8)$$

PROOF OF LEMMA 3.2: First observe that, Lemma 3.2 clearly holds when $u = 0$, so from now on we assume that $u \geq 1$. Next observe that for all $(s, s') \in [0, 1/N]^2$ and all $(k, l) \in \{0, \dots, p\}^2$, h is a C^u function on the compact interval of extremities $x + s - s'$ and $x + s - s' + (k-l)/N$. By applying Taylor formula to h on this interval, one has

$$\begin{aligned} h \left(x + s - s' + \frac{k-l}{N} \right) &= \sum_{m=0}^{u-1} \frac{h^{(m)}(x + s - s')}{m!} \left(\frac{k-l}{N} \right)^m \\ &\quad + \int_0^1 \frac{(1-\eta)^{u-1}}{(u-1)!} h^{(u)} \left(x + s - s' + \frac{(k-l)\eta}{N} \right) d\eta \times \left(\frac{k-l}{N} \right)^u. \end{aligned}$$

Then using (3.5), (2.7) and (3.6), it follows that

$$R(x, N, 0, h) = N^{-u} R(x, N, u, h^{(u)}). \quad (3.9)$$

By replacing in (3.9) u by u' , one also has

$$R(x, N, 0, h) = N^{-u'} R(x, N, u', h^{(u')}). \quad (3.10)$$

Finally combining (3.9) with (3.10) one obtains (3.8). \square

The following remark is a consequence of (3.8), (3.5), (3.6), (2.1) and (2.7).

Remark 3.1 For all integer $N \geq p + 1$ and for all $m \in \{0, \dots, N - p - 1\}$, one has

$$R\left(\frac{m}{N}, N, 0, v\right) = N^{-2d} \left(cR\left(\frac{m}{N}, N, 2d, |\cdot|^{s_0}\right) + R\left(\frac{m}{N}, N, 2d, r\right) \right). \quad (3.11)$$

Now our goal will be to estimate $R(m/N, N, 2d, |\cdot|^{s_0})$ and $R(m/N, N, 2d, r)$.

The following lemma can be obtained by setting in the integrals in (3.5) and (3.6), $(y, y') = (Ns, Ns')$, and then by using the fact that $|\cdot|^{s_0}$ is a homogeneous function of degree s_0 .

Lemma 3.3 For all integers $N \geq p + 1$, $m \in \{0, \dots, N - p - 1\}$ and $u \geq 0$, one has

$$R\left(\frac{m}{N}, N, u, |\cdot|^{s_0}\right) = N^{-2-s_0} R(m, 1, u, |\cdot|^{s_0}). \quad (3.12)$$

Lemma 3.4 There is a constant $c > 0$, only depending on d , q and a , such that one has for all $m \in \{p + 2, \dots, N - p - 1\}$,

$$|R(m, 1, 2d, |\cdot|^{s_0})| \leq c(m - p - 1)^{s_0 - q}. \quad (3.13)$$

PROOF OF LEMMA 3.4: First notice that for all $s, s' \in [0, 1]$, $m \in \{p + 2, \dots, N - p - 1\}$, $k, l \in \{0, \dots, p\}$ and $\eta \in [0, 1]$, one has

$$m + s - s' + (k - l)\eta \geq m - p - 1 \geq 1 > 0. \quad (3.14)$$

Then by using the definition of R ((3.5) and (3.6)) as well as (3.14), one gets

$$R(m, 1, 2d, |\cdot|^{s_0}) = R(m, 1, 2d, (\cdot)^{s_0}). \quad (3.15)$$

Moreover, denoting by $((\cdot)^{s_0})^{(q)}$ the derivative of order q of the function $z \mapsto z^{s_0}$, it follows from Lemma 3.2 that

$$\begin{aligned} R(m, 1, 2d, (\cdot)^{s_0}) &= R(m, 1, 2d + q, ((\cdot)^{s_0})^{(q)}) \\ &= c_1 R(m, 1, 2d + q, (\cdot)^{s_0 - q}), \end{aligned} \quad (3.16)$$

where c_1 is a constant only depending on s_0 and q . Since $2d + q \geq 1$, again by using (3.6), one has

$$\begin{aligned} R(m, 1, 2d + q, (\cdot)^{s_0 - q}) &= \sum_{0 \leq k, l \leq p} a_k a_l (k - l)^{2d+q} \int_0^1 \frac{(1 - \eta)^{2d+q-1}}{(2d + q - 1)!} \\ &\quad \times \int_0^1 \int_0^1 (m + s - s' + (k - l)\eta)^{s_0 - q} ds' ds d\eta. \end{aligned} \quad (3.17)$$

Then, it results from (3.17), the triangle inequality, (3.14) and the inequality $s_0 - q < 0$, that

$$\left| R(m, 1, u + q, (\cdot)^{s_0 - q}) \right| \leq c_2(m - p - 1)^{s_0 - q}, \quad (3.18)$$

where c_2 is a constant only depending on a , d and q . Finally, putting together (3.15), (3.16) and (3.18) one obtains the lemma. \square

Lemma 3.5 *For all $N \geq p + 1$ and $u \in \mathbb{Z}_+$, let us set*

$$r_{u,N}^* = \max_{0 \leq m \leq p+1} \left| R\left(\frac{m}{N}, N, u, r\right) \right|. \quad (3.19)$$

Then one has, when N is big enough

$$r_{u,N}^* = o(N^{-s_0 - 2}). \quad (3.20)$$

PROOF OF LEMMA 3.5: We will only show that the lemma holds in the case where $u = 0$, since it can be proved similarly in the case where $u \geq 1$. First observe that, by using the triangle inequality, one has that, for all integer $0 \leq m \leq p + 1$, all $s, s' \in [0, 1/N]$ and $k, l \in \{0, \dots, p\}$,

$$\left| \frac{m}{N} + s - s' + \frac{k-l}{N} \right| \leq \frac{m}{N} + |s - s'| + \frac{|k-l|}{N} \leq \frac{2p+2}{N}. \quad (3.21)$$

Moreover, the assumption $r(x) = o(|x|^{s_0})$ implies that, there is a sequence $\{e_N\}_{N \geq p+1}$ of positive real-numbers converging to 0, such that for all real number x satisfying $|x| \leq (2p+2)/N \leq 2$, one has

$$|r(x)| \leq e_N |x|^{s_0}. \quad (3.22)$$

It follows from (3.5), (3.21) and (3.22) that

$$\begin{aligned} & \left| R\left(\frac{m}{N}, N, 0, r\right) \right| \\ &= \left| \sum_{0 \leq k, l \leq p} a_k a_l \int_0^{\frac{1}{N}} \int_0^{\frac{1}{N}} r\left(\frac{m}{N} + s - s' + \frac{k-l}{N}\right) ds ds' \right| \\ &\leq e_N \sum_{0 \leq k, l \leq p} |a_k a_l| \int_0^{\frac{1}{N}} \int_0^{\frac{1}{N}} \left(\frac{m}{N} + |s - s'| + \frac{|k-l|}{N}\right)^{s_0} ds ds' \\ &= c e_N N^{-s_0 - 2}, \end{aligned} \quad (3.23)$$

where $c > 0$ is a constant only depending on a . \square

Lemma 3.6 *For all $N \geq p + 1$, there is a constant $c > 0$, only depending on a , r and d , such that one has for each $m \in \{p + 2, \dots, N - p - 1\}$,*

$$\left| R\left(\frac{m}{N}, N, 2d, r\right) \right| \leq c N^{-w-2} (m - p - 1)^{w-q}. \quad (3.24)$$

PROOF OF LEMMA 3.6: First observe that, in view of Assumption (A1), r is q -times continuously differentiable on the interval $[m/N - (p + 1)/N, m/N + (p + 1)/N] \subset [1/N, 1]$. Therefore we are allowed to use Lemma 3.2 and we obtain that,

$$R\left(\frac{m}{N}, N, 2d, r\right) = N^{-q} R\left(\frac{m}{N}, N, 2d + q, r^{(q)}\right). \quad (3.25)$$

Moreover (3.6) implies that,

$$\begin{aligned} R\left(\frac{m}{N}, N, 2d+q, r^{(q)}\right) &= \sum_{0 \leq k, l \leq p} a_k a_l (k-l)^{2d+q} \int_0^{\frac{1}{N}} \int_0^{\frac{1}{N}} \int_0^1 \frac{(1-\eta)^{2d+q-1}}{(2d+q-1)!} \\ &\quad \times r^{(q)}\left(\frac{m}{N} + s - s' + \frac{(k-l)\eta}{N}\right) d\eta ds' ds. \end{aligned} \quad (3.26)$$

Then, it follows from (3.26), the triangle inequality, (2.3) and the inequality $w - q < -1/2 < 0$ that

$$\begin{aligned} \left| R\left(\frac{m}{N}, N, u, r^{(q)}\right) \right| &\leq c_1 \sum_{0 \leq k, l \leq p} |a_k a_l| |k-l|^{2d+q} \\ &\quad \times \int_0^{\frac{1}{N}} \int_0^{\frac{1}{N}} \int_0^1 \frac{|m/N + s - s' + (k-l)\eta/N|^{w-q}}{(2d+q-1)!} d\eta ds' ds \\ &\leq c_2 N^{-w+q-2} (m-p-1)^{w-q}, \end{aligned} \quad (3.27)$$

where $c_1 > 0$ is the constant c in (2.3) and $c_2 > 0$ is a constant only depending on a, r and d . Finally, putting together (3.25), (3.26) and (3.27) we get the lemma. \square

Let us now recall a useful result concerning centered, 2-D, Gaussian random vectors.

Lemma 3.7 *Let (Z, Z') be a centered, 2-D Gaussian random vector such that the variances of Z and Z' are equal; we denote by λ their common value. Then*

$$\mathbb{E}((Z^2 - \lambda)(Z'^2 - \lambda)) = 2(\text{Cov}(Z, Z'))^2. \quad (3.28)$$

Now we are in position to prove Proposition 3.1.

PROOF OF PROPOSITION 3.1: First observe that it follows from (3.7) and (3.11) that for all integers $N \geq p+1$ and $i, j \in \{0, \dots, N-p-1\}$, one has

$$\mathbb{E}(\Delta_a \bar{X}_{i,N} \Delta_a \bar{X}_{j,N}) = -N^{2-2d} \left(cR\left(\frac{|i-j|}{N}, N, 2d, |\cdot|^{s_0}\right) + R\left(\frac{|i-j|}{N}, N, 2d, r\right) \right). \quad (3.29)$$

Taking $i = j$ in (3.29) and using (3.1), we get

$$\sigma_{a,N}^2 = \text{Var}(\Delta_a \bar{X}_{i,N}) = -N^{2-2d} (cR(0, N, 2d, |\cdot|^{s_0}) + R(0, N, 2d, r)). \quad (3.30)$$

Then (3.2) results from (3.30), Lemma 3.3, (2.4), (2.8) and Lemma 3.5.

Let us now prove that Relation (3.3) holds. We denote by $\rho_N(|i-j|)$ the correlation coefficient between $\Delta_a \bar{X}_{i,N}$ and $\Delta_a \bar{X}_{j,N}$, that is,

$$\rho_N(|i-j|) = \frac{\mathbb{E}(\Delta_a \bar{X}_{i,N} \Delta_a \bar{X}_{j,N})}{\sigma_{a,N}^2}. \quad (3.31)$$

On the other hand, using the definition of a variance, one has,

$$\begin{aligned}
s_N^2 &:= \text{Var} \left(\sum_{i=0}^{N-p-1} \frac{(\Delta_a \bar{X}_{i,N})^2}{\sigma_{a,N}^2} \right) \\
&= \text{Var} \left(\sum_{i=0}^{N-p-1} \left(\frac{(\Delta_a \bar{X}_{i,N})^2}{\sigma_{a,N}^2} - 1 \right) \right) \\
&= \sum_{i=0}^{N-p-1} \sum_{j=0}^{N-p-1} \mathbb{E} \left(\left(\frac{(\Delta_a \bar{X}_{i,N})^2}{\sigma_{a,N}^2} - 1 \right) \left(\frac{(\Delta_a \bar{X}_{j,N})^2}{\sigma_{a,N}^2} - 1 \right) \right).
\end{aligned} \tag{3.32}$$

Then, it follows from (3.32), Lemma 3.7 and (3.31), that

$$\begin{aligned}
s_N^2 &= 2 \sum_{i=0}^{N-p-1} \sum_{j=0}^{N-p-1} \rho_N^2(i-j) \\
&= 2 \sum_{|j| \leq N-p-1} (N-p-|j|) \rho_N^2(|j|).
\end{aligned} \tag{3.33}$$

Let us now split s_N^2 in two parts: $|j| \leq p+1$ and $p+2 \leq |j| \leq N-p-1$; more precisely we express s_N^2 as,

$$s_N^2 = F_N + G_N, \tag{3.34}$$

where

$$F_N = 2 \sum_{|j| \leq p+1} (N-p-|j|) \rho_N^2(|j|), \tag{3.35}$$

and

$$G_N = 2 \sum_{p+2 \leq |j| \leq N-p-1} (N-p-|j|) \rho_N^2(|j|). \tag{3.36}$$

Thus for finishing our proof, it remains to show that there exist two non-vanishing constants c_1 and c_2 , such that

$$F_N = c_1 N + o(N), \tag{3.37}$$

and

$$G_N = c_2 N + o(N). \tag{3.38}$$

By using (3.31), (3.29) and Lemma 3.3, one gets

$$\rho_N(|j|) = \frac{cR(|j|, 1, 2d, |\cdot|^{s_0}) + N^{2+s_0} R(|j|/N, N, 2d, r)}{cR(0, 1, 2d, |\cdot|^{s_0}) + N^{2+s_0} R(0, N, 2d, r)}. \tag{3.39}$$

Thanks to Lemma 3.5, the two terms $N^{2+s_0} R(|j|/N, N, 2d, r)$ (for $|j| \leq p+1$) and $N^{2+s_0} R(0, N, 2d, r)$ converge to 0. This fact together with (3.39) implies

$$\lim_{N \rightarrow +\infty} \max_{|j| \leq p+1} |\rho_N(|j|) - C(|j|)| = 0, \tag{3.40}$$

where for all $0 \leq |j| \leq p+1$, we have set,

$$C(|j|) = \frac{R(|j|, 1, 2d, |\cdot|^{s_0})}{R(0, 1, 2d, |\cdot|^{s_0})}.$$

Then, combining (3.35) with (3.40), one obtains that

$$\begin{aligned} \frac{F_N - 2 \sum_{|j| \leq p+1} (C(|j|))^2 N}{N} &= \frac{-2 \sum_{|j| \leq p+1} (p+|j|) C(|j|)^2}{N} \\ &\quad + \frac{2 \sum_{|j| \leq p+1} (N-p-|j|) (\rho_N^2(|j|) - (C(|j|))^2)}{N} \\ &\xrightarrow{N \rightarrow +\infty} 0, \end{aligned}$$

which proves that (3.37) holds; observe that (2.8) implies that

$$c_1 := 2 \sum_{|j| \leq p+1} (C(|j|))^2 \neq 0.$$

Let us now show that (3.38) is satisfied. For the sake of simplicity, from now on, for all $|j| \in \{p+2, \dots, N-p-1\}$, we set

$$L(|j|) = R(|j|, 1, 2d, (\cdot)^{s_0}). \quad (3.41)$$

First, we will show that there is a non-vanishing constant c_3 such that

$$\sum_{p+2 \leq |j| \leq N-p-1} \left| \rho_N^2(|j|) - c_3 L^2(|j|) \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (3.42)$$

By using (3.31), (3.29) and the fact that for $|j| \geq p+1$, one has,

$$\rho_N(|j|) = \frac{-N^{2-2d}}{\sigma_{a,N}^2} \left(cR\left(\frac{|j|}{N}, N, 2d, (\cdot)^{s_0}\right) + R\left(\frac{|j|}{N}, N, 2d, r\right) \right). \quad (3.43)$$

It follows from (3.43), (2.4), (3.12) and (3.41) that,

$$\rho_N(|j|) = -cL(|j|) \frac{N^{-2H}}{\sigma_{a,N}^2} - \frac{N^{2-2d}}{\sigma_{a,N}^2} R\left(\frac{|j|}{N}, N, 2d, r\right). \quad (3.44)$$

On the other hand, assuming that c_1 is the constant introduced in (3.2), the following relation holds:

$$\left(\frac{c}{c_1}\right) L(|j|) = cL(|j|) \frac{N^{-2H}}{\sigma_{a,N}^2} + cL(|j|) M(N), \quad (3.45)$$

where

$$M(N) = \frac{1}{c_1} - \frac{N^{-2H}}{\sigma_{a,N}^2} = \frac{\sigma_{a,N}^2 - c_1 N^{-2H}}{c_1 \sigma_{a,N}^2} = o(1). \quad (3.46)$$

Next using the inequality $|x^2 - y^2| = |(x + y)^2 - 2(x + y)y| \leq (x + y)^2 + 2|(x + y)y|$ for all real numbers x and y , (3.44) and (3.45), one gets

$$\begin{aligned}
 \left| \rho_N^2(lj) - \left(\frac{c}{c_1}\right)^2 L^2(j) \right| &\leq \left| \rho_N(lj) + \left(\frac{c}{c_1}\right)L(j) \right|^2 + 2 \left| \rho_N(lj) + \left(\frac{c}{c_1}\right)L(j) \right| \left| \left(\frac{c}{c_1}\right)L(lj) \right| \\
 &= \left| \frac{N^{2-2d}}{\sigma_{a,N}^2} R\left(\frac{|j|}{N}, N, 2d, r\right) - cL(lj)M(N) \right|^2 \\
 &\quad + 2 \left| \frac{N^{2-2d}}{\sigma_{a,N}^2} R\left(\frac{|j|}{N}, N, 2d, r\right) - cL(lj)M(N) \right| \left| \left(\frac{c}{c_1}\right)L(lj) \right|.
 \end{aligned} \tag{3.47}$$

It follows from (3.47), (3.2), (2.4), Lemma 3.6 and Lemma 3.4 that there exists a constant $c_4 > 0$, non depending on j and N , such that,

$$\begin{aligned}
 &\left| \rho_N^2(lj) - \left(\frac{c}{c_1}\right)^2 L^2(j) \right| \\
 &\leq c_4 \left((N^{s_0-w}(|j| - p - 1)^{w-q} + |M(N)|(|j| - p - 1)^{s_0-q})^2 \right. \\
 &\quad \left. + (N^{s_0-w}(|j| - p - 1)^{w-q} + |M(N)|(|j| - p - 1)^{s_0-q})(|j| - p - 1)^{s_0-q} \right) \\
 &\leq 2c_4 \left(N^{2(s_0-w)}(|j| - p - 1)^{2(w-q)} + (M(N))^2(|j| - p - 1)^{2(s_0-q)} \right. \\
 &\quad \left. + (N^{s_0-w} + |M(N)|)(|j| - p - 1)^{w+s_0-2q} \right).
 \end{aligned} \tag{3.48}$$

Next, setting $c_3 = c^2/c_1^2$, it follows from (3.48), (3.46) and the inequalities $s_0 - w < 0$, $s_0 - q < -1/2$, $w - q < -1/2$, that (3.42) is satisfied.

Now we are in position to show that (3.38) holds. On one hand, (3.43) and (3.42) imply

$$\begin{aligned}
 &\frac{|G_N - 2c_3 \sum_{p+2 \leq |j| \leq N-p-1} (N-p-|j|)L^2(lj)|}{N} \\
 &\leq 2 \sum_{p+2 \leq |j| \leq N-p-1} \frac{(N-p-|j|)}{N} \left| \rho_N^2(lj) - c_3 L^2(lj) \right| \\
 &\leq 4 \sum_{p+2 \leq |j| \leq N-p-1} \left| \rho_N^2(j) - c_3 L^2(lj) \right| \\
 &\xrightarrow[N \rightarrow +\infty]{} 0.
 \end{aligned} \tag{3.49}$$

On the other hand, standard computations allow to show that the sequence

$$\left(\frac{\sum_{p+2 \leq |j| \leq N-p-1} (N-p-|j|)L^2(lj)}{N} \right)_{N \geq p+1}$$

is increasing and bounded. Therefore this sequence converges to finite positive limit denoted by c_5 . Then setting $c_2 = 2c_3c_5$, (3.49) implies that (3.38) is satisfied. Finally, (3.3) results from (3.34), (3.37) and (3.38). \square

The previous proposition will play a crucial role in the proof of Theorem 2.1; the proof of this theorem will also make use of the following lemma, which, roughly speaking, means that, almost surely, the generalized quadratic variation V_N behaves like cN^{1-2H} when N goes to infinity.

Lemma 3.8 *There exists a constant $c > 0$ such that*

$$W_N := \frac{V_N}{cN^{1-2H}} - 1 \xrightarrow[N \rightarrow +\infty]{a.s.} 0.$$

Lemma 3.8 is a straightforward consequence of (3.2) and the following result.

Lemma 3.9 *The generalized quadratic variation V_N satisfies*

$$\frac{V_N}{N\sigma_{a,N}^2} - 1 \xrightarrow[N \rightarrow +\infty]{a.s.} 0,$$

where $\sigma_{a,N}$ has been defined in (3.1).

The proof of Lemma 3.9 is rather similar to that of Proposition 2, equation (25), in Benassi et al. (1998), this is why we will not give it. Now, we are in position to prove Theorem 2.1.

PROOF OF THEOREM 2.1: Let $c_1 = c$ and W_N be respectively the constant and the random variable which have been introduced in Lemma 3.8. Therefore, one has

$$V_N = c_1 N^{1-2H} (W_N + 1).$$

Thus, it follows from (2.9) that

$$\begin{aligned} \hat{H}_N &= \frac{1}{2} \left(1 + \log_2 \left(\frac{V_N}{V_{2N}} \right) \right) \\ &= \frac{1}{2} \left(1 + \log_2 \left(\frac{c_1 N^{1-2H} (W_N + 1)}{c_1 (2N)^{1-2H} (W_{2N} + 1)} \right) \right) \\ &= H + \frac{1}{2} \left(\log_2 (W_N + 1) - \log_2 (W_{2N} + 1) \right). \end{aligned}$$

Finally applying Lemma 3.8 one obtains the theorem. \square

From now on, our goal will be to show that Theorem 2.2 holds. The proof of this theorem mainly relies on the following lemma, which, roughly speaking, means that a Central Limit Theorem holds for the generalized quadratic variation V_N .

Lemma 3.10 *For any $N \geq p + 1$ let us set,*

$$\tau_N^2 := \text{Var} \left(\frac{\sum_{i=0}^{N-p-1} (\Delta_a \bar{X}_{i,N})^2}{(N-p)(\sigma_{a,N})^2} \right) = \frac{s_N^2}{(N-p)^2}. \quad (3.50)$$

Then, there is a constant $c > 0$ such that one has

$$\tau_N = cN^{-1/2} + o(N^{-1/2}). \quad (3.51)$$

Moreover,

$$\frac{1}{\tau_N} \left(\frac{\sum_{i=0}^{N-p-1} (\Delta_a \bar{X}_{i,N})^2}{(N-p)\sigma_{a,N}^2} - 1 \right) \xrightarrow[N \rightarrow +\infty]{d} \mathcal{N}(0, 1).$$

The proof of Lemma 3.10 mainly relies on the following two lemmas. It is rather similar to that of Proposition 2, equation (26), in Benassi et al. (1998), this is why we will not give it.

Lemma 3.11 (Csörgo and Révész (1981)) Consider the sequence of random variables $\{S_N\}_{N \geq p+1}$ defined by $S_N = \sum_{j=0}^{N-p-1} \lambda_{j,N}(\epsilon_{j,N}^2 - 1)$, where for all $N \geq p + 1$, $\{\epsilon_{j,N}\}_{j=0, \dots, N-p-1}$ is a finite sequence of i.i.d. standard Gaussian random variables and $\{\lambda_{j,N}\}_{j=0, \dots, N-p-1}$ a finite sequence of positive real numbers. Let $\lambda_N = \max_{j=0, \dots, N-p-1} \lambda_{j,N}$, if $\lambda_N = o((\text{Var}(S_N))^{1/2})$, then

$$\frac{S_N}{(\text{Var}(S_N))^{1/2}} \xrightarrow[N \rightarrow +\infty]{\text{d}} \mathcal{N}(0, 1).$$

This lemma is in fact a straightforward consequence of Lindeberg-Féllér Central Limit Theorem.

Lemma 3.12 (Luenberger (1979), Chapter 6.2, Page 194) For all integer $n \geq 1$, let $C = (C_{ij})_{n \times n}$ be a symmetric positive definite matrix and let λ be its largest eigenvalue. Then one has,

$$\lambda \leq \max_{1 \leq i \leq n} \sum_{j=1}^n |C_{ij}|.$$

Lemma 3.13 Let V_N be the generalized quadratic variation defined in (2.10) and let τ_N be the quantity defined in (3.50). Then, under the additional assumption (2.11), there is $c > 0$ a constant non depending on N , such that

$$\frac{1}{\tau_N} \left(\frac{V_N}{cN^{1-2H}} - 1 \right) \xrightarrow[N \rightarrow +\infty]{\text{d}} \mathcal{N}(0, 1).$$

Lemma 3.13 can easily be proved by using Lemma 3.10, (3.2) and the following classical lemma.

Lemma 3.14 Let (Z_N) be a sequence of random variables which converges in distribution to a standard Gaussian random variable. Let (a_N) and (b_N) be two arbitrary sequences of real numbers satisfying $\lim_{N \rightarrow +\infty} a_N = 1$ and $\lim_{N \rightarrow +\infty} b_N = 0$. Then, the sequence of random variables $(a_N Z_N + b_N)$ converges in distribution to a standard Gaussian random variable.

At last let us give the proof of Theorem 2.2.

PROOF OF THEOREM 2.2: This Theorem can be obtained by using Lemma 3.13, the δ -method (see for instance Theorem 3.3.11 in Dacunha-Castelle (1983)) and Relation (3.51). \square

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