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# Existence of global strong solution for the compressible Navier-Stokes system and the Korteweg system in two-dimension

Boris Haspot \*

## Abstract

This paper is dedicated to the study of viscous compressible barotropic fluids in dimension  $N = 2$ . We address the question of the global existence of strong solutions with large initial data for compressible Navier-Stokes system and Korteweg system. In the first case we are interested by slightly extending a famous result due to V. A. Vaigant and A. V. Kazhikhov in [32] concerning the existence of global strong solution in dimension two for a suitable choice of viscosity coefficient ( $\mu(\rho) = \mu > 0$  and  $\lambda(\rho) = \lambda\rho^\beta$  with  $\beta > 3$ ) in the torus. We are going to weaken the condition on  $\beta$  by assuming only  $\beta > 2$  essentially by taking profit of commutator estimates introduced by Coifman et al in [6] and using a notion of *effective velocity* as in [32]. In the second case we study the existence of global strong solution with large initial data in the sense of the scaling of the equations for Korteweg system with degenerate viscosity coefficient and with friction term.

## 1 Introduction

The motion of a general barotropic compressible fluid with capillary tensor is described by the following system, which can be derived from a Cahn-Hilliard free energy (see the pioneering work by J.- E. Dunn and J. Serrin in [8] and also in [2, 5, 11]):

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho)Du) - \nabla(\lambda(\rho)\operatorname{div}u) + \nabla P(\rho) + a\rho u = \operatorname{div}K, \end{cases} \quad (1.1)$$

where  $\operatorname{div}K$  is the capillary tensor which reads as follows:

$$\operatorname{div}K = \nabla(\rho\kappa(\rho)\Delta\rho + \frac{1}{2}(\kappa(\rho) + \rho\kappa'(\rho))|\nabla\rho|^2) - \operatorname{div}(\kappa(\rho)\nabla\rho \otimes \nabla\rho). \quad (1.2)$$

The term  $\operatorname{div}K$  allows to describe the variation of density at the interfaces between two phases, generally a mixture liquid-vapor.  $P$  is a general increasing pressure term that we assume in the sequel under the form  $P(\rho) = b\rho^\gamma$  with  $b > 0$  and  $\gamma \geq 1$ ,  $a\rho u$  is a friction

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term with  $a > 0$  (see [25]).  $D(u) = \frac{1}{2}[\nabla u + {}^t \nabla u]$  being the stress tensor,  $\mu$  and  $\lambda$  are the two Lamé viscosity coefficients depending on the density  $\rho$  and satisfying:

$$\mu > 0 \text{ and } 2\mu + N\lambda \geq 0.$$

In the present paper, we are interested in dealing with two different situations:

- The case of the compressible Navier-Stokes system where we assume no capillarity,  $\kappa(\rho) = 0$  and where  $\mu(\rho) = 1$  is a constant and  $\lambda(\rho) = \lambda\rho^\beta$  with  $\beta \geq 2$ .
- The case of Korteweg system with the viscosity coefficients and the capillarity coefficient such that:

$$\mu(\rho) = \mu\rho \quad \lambda(\rho) = 0 \quad \text{and} \quad \kappa(\rho) = \frac{\kappa}{\rho} \quad \text{with} \quad \kappa > 0, \quad \mu > 0.$$

In the first case we would like to extend the famous result of global strong solution in two dimension in the torus discovered by V. A. Vaigant and A. V. Kazhikhov in [32]. Indeed in [32], the authors assume that  $\lambda(\rho) = \lambda\rho^\beta$  with  $\beta > 3$  and  $\lambda > 0$  and  $\mu(\rho) = 1$ . Let us emphasize that a such choice on the viscosity coefficients allows to exhibits two different phenomena; the first one concerns the notion of *effective velocity* introduced by Lions in [24] which is crucial in this context for getting a priori estimates on the divergence and the rotational of the velocity; the second reason to choose such coefficient concerns the possibility to obtain  $L_T^\infty(L^p(\mathbb{T}^2))$  estimates for any  $p > 1$  on the density and any  $T > 0$ . Indeed the viscosity coefficient  $\lambda(\rho)$  offers enough weight in order to derivate such estimates on the density (see the p1115 "Second a priori estimate for the density" in [32]). In the first part of this paper we wish to improve this result by assuming only  $\lambda(\rho) = \lambda\rho^\beta$  with  $\beta > 2$  and  $\lambda > 0$ . The key point will consist in using commutator estimates for dealing with term of the form  $[R_{ij}, u_j](\rho u_i)$  (we refer to [6] for such estimates but also to ?? where Lions prove global existence of weak solution for compressible Navier-Stokes equations by introducing this kind of ingredient).

In the second case we are interested in proving global existence of strong solution for the Korteweg system with friction term when the physical coefficients verify:

$$\kappa(\rho) = \frac{\kappa}{h}, \quad \kappa = \mu^2 \quad \text{and} \quad b = a\mu, \tag{1.3}$$

with  $\mu > 0$ . This system without friction has been widely studied this last year in particular concerning the existence of global weak solution and global strong solution with small initial data. We refer in particular to the following works [4, 7, 15, 16, 17, 18, 19, 20]. Let us start with explaining a other notion of *effective velocity* used in particular in [15] which allows us to simplify the system (1.6). Indeed by computation (see [15]), we obtain the simplified system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) - \mu \Delta \rho = 0, \\ \rho \partial_t v + \rho u \cdot \nabla v - \operatorname{div}(\mu \rho \nabla v) + a \rho v = 0, \end{cases} \tag{1.4}$$

with  $v = u + \mu \nabla \ln \rho$  the *effective velocity*. For more details on the computation, we refer to [16]. When we write the system (1.4) in function of the momentum  $m = \rho v$ , the

system reads as follows:

$$\begin{cases} \partial_t \rho + \operatorname{div} m - \mu \Delta \rho = 0, \\ \partial_t m + \operatorname{div} \left( \frac{m}{\rho} \otimes m \right) - \mu \Delta m + r m = 0, \end{cases} \quad (1.5)$$

In particular we observe that  $(\rho, -\mu \nabla \ln \rho)$  when:

$$\partial_t \rho - \mu \Delta \rho = 0,$$

is a particular global solution of the system (1.6).

**Remark 1** *Let us mention that we can choose initial density which admits vacuum that we wish. In general it is not always possible to obtain global strong solution with initial density close from the vacuum.*

In the sequel we will be interested in working around this global particular solution (see remark 1) in order to prove the existence of global strong solution with large initial data on the irrotational part. More precisely we shall wish obtain global strong solution in critical space for the scaling of the equations. Let us briefly recall the notion of invariance by scaling of the equation and by what we mean by supercritical smallness on the data. By critical, we mean that we want to solve the system (1.6) in functional spaces with invariant norm by the natural changes of scales which leave (1.6) invariant. More precisely in our case, the following transformation:

$$(\rho(t, x), u(t, x)) \longrightarrow (\rho(l^2 t, lx), lu(l^2 t, lx)), \quad l \in \mathbb{R}, \quad (1.6)$$

verify this property, provided that the pressure term has been changed accordingly. In particular we can observe that  $\dot{H}^{\frac{N}{2}} \times \dot{H}^{\frac{N}{2}-1}$  is a space invariant for the scaling of the equation, more generally such Besov spaces:

$$(\rho_0 - \bar{\rho}) \in B_{p,1}^{\frac{N}{p}}, \quad u_0 \in B_{p_1,1}^{\frac{N}{p_1}-1},$$

with  $(p, p_1) \in [1, +\infty[$  are also available.

## 1.1 Results

Let us state the two main result of this paper. The first one is an improvement of the results of Vaigant and Kazhikhov [32].

**Theorem 1.1** *Let us assume the following hypothesis on the viscosity coefficients:*

$$\mu(\rho) = 1 \quad \text{and} \quad \lambda(\rho) = \lambda \rho^\beta \quad \text{with } \beta > 2.$$

Let  $u_0 \in H^2(\mathbb{T}^2)$ ,  $\rho_0 \in W^{1,q}(\mathbb{T}^2)$  with  $q > 2$  and:

$$0 < c \leq \rho_0(x) \leq m < +\infty \quad \forall x \in \mathbb{T}^2,$$

then it exists a unique global strong solution to (1.6) such that:

$$u \in W_2^{2,1}(Q_T) \quad \text{and} \quad \rho \in W_{q,\infty}^{1,1}(Q_T) \quad \forall T > 0,$$

with  $Q_T = (0, T) \times \mathbb{T}^2$ .

**Remark 2** *As mentioned above, the main point compared with the result of [32] concerns the fact that we can improve the range of  $\beta$  by assuming only  $\beta > 2$ . It would be possible also to improve the regularity condition on the initial data by working in Besov space invariant for the scaling of the system, but it is not the object of this paper.*

We are going to give your second result on Korteweg system with supercritical smallness condition on the initial data; before let us give the following definition:

**Definition 1.1** *We set  $q = \rho - 1$ ,  $m = \rho u$  and  $\rho = h$ .*

In the following we are dealing with the euclidian space  $\mathbb{R}^N$  with  $N \geq 2$ . Let us give our main result on the Korteweg system where we prove the existence of global strong solution with large initial data on the irrotational part.

**Theorem 1.2** *Suppose that we are under the conditions (1.3). Assume that  $m_0 \in B_{2,1}^{\frac{N}{2}-1}$  and  $q_0 \in B_{2,1}^{\frac{N}{2}}$  with  $h_0 \geq c > 0$ . Then there exists a constant  $\varepsilon_0$  depending on  $\frac{1}{h_0}$  such that if:*

$$\|m_0\|_{B_{2,1}^{\frac{N}{2}-1}} \leq \varepsilon_0,$$

*then there exists a unique global solution  $(q, m)$  for system (1.5) with  $h$  bounded away from zero and,*

$$h \in \tilde{C}(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}}) \cap L^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}+2}) \quad \text{and} \quad m \in \tilde{C}(\mathbb{R}^+; B_{2,1}^{\frac{N}{2}-1}) \cap L^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}-1} \cap B_{2,1}^{\frac{N}{2}+1}).$$

Let us give the plane of this paper, we shall remind in section 2 some auxiliary results of Gagliardo-Nirenberg's inequality and in section 3 the Littlewood-Paley theory. In section 4 and section 5, we will prove different a priori estimates on the density and the velocity which show the theorem 1.1. We will conclude in section 6 by the proof of theorem 1.2.

## Notation

In all the paper,  $C$  will stand for a harmless constant, and we will sometimes use the notation  $A \lesssim B$  equivalently to  $A \leq CB$ .

## 2 Auxiliary Assertions

We are going to recall some lemma which are also present in [32] and that we prefer to state for the sake of the completeness.

**Lemma 1** *Let  $\Omega \in \mathbb{R}^N$  be an arbitrary bounded domain satisfying the cone condition. Then the following inequality is valid for every function  $u \in W^{1,m}(\Omega)$ ,  $\int_{\Omega} u dx = 0$*

$$\|u\|_{L^q(\Omega)} \leq C_1 \|\nabla u\|_{L^m(\Omega)}^\alpha \|u\|_{L^r(\Omega)}^{1-\alpha}, \quad (2.7)$$

where  $\alpha = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{r} - \frac{1}{m} + \frac{1}{n}}$ , moreover if  $m < n$  then  $q \in [r, \frac{mn}{n-m}]$  for  $r \leq \frac{mn}{n-m}$  and  $q \in [\frac{mn}{n-m}, r]$  for  $r > \frac{mn}{n-m}$ . If  $m \geq n$  then  $q \in [r, +\infty)$  is arbitrary; moreover if  $m > n$  then equality (2.7) is also valid for  $q = +\infty$ .

Inequality (2.7) is a particular case of the more general inequalities proven in [9, 21, 10]. Let us mention that an inequality of the form (2.7) is valid for the function of class  $W^{1,m}(\Omega)$  when  $M = \frac{1}{|\Omega|} \int_{\Omega} u \, dx$  is not null. It suffices to consider  $v = u - M$  and apply inequality (2.7) to the function  $v$ . We obtain then the inequality

$$\|u\|_{L^q(\Omega)} \leq C_2(\|\nabla u\|_{L^m(\Omega)}^\alpha \|u\|_{L^r(\Omega)}^{1-\alpha} + \|u\|_{L^1(\Omega)}), \quad (2.8)$$

**Lemma 2** *Let  $\Omega \in \mathbb{R}^2$  be an arbitrary bounded domain satisfying the cone condition. Then every function  $u \in W^{1,m}(\Omega)$  with  $\int_{\Omega} u \, dx = 0$  satisfies the inequality*

$$\|u\|_{L^{\frac{2m}{2-m}}(\Omega)} \leq C_3(2-m)^{-\frac{1}{2}} \|\nabla u\|_{L^m(\Omega)}, \quad 1 \leq m < 2, \quad (2.9)$$

where  $C_3$  is a constant independent of  $m$  and the function  $u$ .

For a proof of this inequality see [33, 30]. The exact constant in inequality (2.9) is obtained in the article [30].

**Lemma 3** *Let  $\Omega \in \mathbb{R}^2$  be an arbitrary bounded domain satisfying the cone condition. Then for an arbitrary number  $\varepsilon$ ,  $1 \geq 2\varepsilon \geq 0$ , every function  $h \in W^{1, \frac{2m}{m+\delta}}(\Omega)$ ,  $m \geq 2$ ,  $1 \geq \delta \geq 0$ , satisfies the inequality*

$$\|h\|_{L^{2m}(\Omega)} \leq C_4(\|h\|_{L^1(\Omega)} + m^{\frac{1}{2}} \|\nabla h\|_{L^{\frac{2m}{m+\delta}}(\Omega)}^{1-s} \|h\|_{L^{2(1-\varepsilon)}(\Omega)}^s), \quad (2.10)$$

where  $s = (1-\varepsilon) \frac{1-\delta}{m-\delta(1-\varepsilon)}$  and  $C_4$  is a positive constant independent of  $m, \varepsilon, \delta$  and the function  $h$ .

### 3 Littlewood-Paley theory and Besov spaces

Throughout the paper,  $C$  stands for a constant whose exact meaning depends on the context. The notation  $A \lesssim B$  means that  $A \leq CB$ . For all Banach space  $X$ , we denote by  $C([0, T], X)$  the set of continuous functions on  $[0, T]$  with values in  $X$ . For  $p \in [1, +\infty]$ , the notation  $L^p(0, T, X)$  or  $L_T^p(X)$  stands for the set of measurable functions on  $(0, T)$  with values in  $X$  such that  $t \rightarrow \|f(t)\|_X$  belongs to  $L^p(0, T)$ . Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. We can use for instance any  $\varphi \in C^\infty(\mathbb{R}^N)$ , supported in  $\mathcal{C} = \{\xi \in \mathbb{R}^N / \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$  such that:

$$\sum_{l \in \mathbb{Z}} \varphi(2^{-l}\xi) = 1 \quad \text{if } \xi \neq 0.$$

Denoting  $h = \mathcal{F}^{-1}\varphi$ , we then define the dyadic blocks by:

$$\Delta_l u = \varphi(2^{-l}D)u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y) u(x-y) dy \quad \text{and} \quad S_l u = \sum_{k \leq l-1} \Delta_k u.$$

Formally, one can write that:

$$u = \sum_{k \in \mathbb{Z}} \Delta_k u.$$

This decomposition is called homogeneous Littlewood-Paley decomposition. Let us observe that the above formal equality does not hold in  $\mathcal{S}'(\mathbb{R}^N)$  for two reasons:

1. The right hand-side does not necessarily converge in  $\mathcal{S}'(\mathbb{R}^N)$ .
2. Even if it does, the equality is not always true in  $\mathcal{S}'(\mathbb{R}^N)$  (consider the case of the polynomials).

### 3.1 Homogeneous Besov spaces and first properties

**Definition 3.2** For  $s \in \mathbb{R}$ ,  $p \in [1, +\infty]$ ,  $q \in [1, +\infty]$ , and  $u \in \mathcal{S}'(\mathbb{R}^N)$  we set:

$$\|u\|_{B_{p,q}^s} = \left( \sum_{l \in \mathbb{Z}} (2^{ls} \|\Delta_l u\|_{L^p})^q \right)^{\frac{1}{q}}.$$

The Besov space  $B_{p,q}^s$  is the set of temperate distribution  $u$  such that  $\|u\|_{B_{p,q}^s} < +\infty$ .

**Remark 3** The above definition is a natural generalization of the nonhomogeneous Sobolev and Hölder spaces: one can show that  $B_{\infty,\infty}^s$  is the nonhomogeneous Hölder space  $C^s$  and that  $B_{2,2}^s$  is the nonhomogeneous space  $H^s$ .

**Proposition 3.1** The following properties holds:

1. there exists a constant universal  $C$  such that:  

$$C^{-1} \|u\|_{B_{p,r}^s} \leq \|\nabla u\|_{B_{p,r}^{s-1}} \leq C \|u\|_{B_{p,r}^s}.$$
2. If  $p_1 < p_2$  and  $r_1 \leq r_2$  then  $B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-N(1/p_1-1/p_2)}$ .
3.  $B_{p,r_1}^{s'} \hookrightarrow B_{p,r}^s$  if  $s' > s$  or if  $s = s'$  and  $r_1 \leq r$ .

Let now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J-M. Bony (see [?]) and rewrite on a generalized form in [?].

**Proposition 3.2** We have the following laws of product:

- For all  $s \in \mathbb{R}$ ,  $(p, r) \in [1, +\infty]^2$  we have:

$$\|uv\|_{B_{p,r}^s} \leq C (\|u\|_{L^\infty} \|v\|_{B_{p,r}^s} + \|v\|_{L^\infty} \|u\|_{B_{p,r}^s}). \quad (3.11)$$

- Let  $(p, p_1, p_2, r, \lambda_1, \lambda_2) \in [1, +\infty]^2$  such that:  $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$ ,  $p_1 \leq \lambda_2$ ,  $p_2 \leq \lambda_1$ ,  $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{\lambda_1}$  and  $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda_2}$ . We have then the following inequalities:  
if  $s_1 + s_2 + N \inf(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}) > 0$ ,  $s_1 + \frac{N}{\lambda_2} < \frac{N}{p_1}$  and  $s_2 + \frac{N}{\lambda_1} < \frac{N}{p_2}$  then:

$$\|uv\|_{B_{p,r}^{s_1+s_2-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}} \lesssim \|u\|_{B_{p_1,r}^{s_1}} \|v\|_{B_{p_2,\infty}^{s_2}}, \quad (3.12)$$

when  $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$  (resp  $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$ ) we replace  $\|u\|_{B_{p_1,r}^{s_1}}$  (resp  $\|v\|_{B_{p_2,\infty}^{s_2}}$ ) by  $\|u\|_{B_{p_1,1}^{s_1}}$  (resp  $\|v\|_{B_{p_2,\infty}^{s_2} \cap L^\infty}$ ), if  $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$  and  $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$  we take  $r = 1$ .

If  $s_1 + s_2 = 0$ ,  $s_1 \in (\frac{N}{\lambda_1} - \frac{N}{p_2}, \frac{N}{p_1} - \frac{N}{\lambda_2}]$  and  $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$  then:

$$\|uv\|_{B_{p,\infty}^{-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}} \lesssim \|u\|_{B_{p_1,1}^{s_1}} \|v\|_{B_{p_2,\infty}^{s_2}}. \quad (3.13)$$

If  $|s| < \frac{N}{p}$  for  $p \geq 2$  and  $-\frac{N}{p'} < s < \frac{N}{p}$  else, we have:

$$\|uv\|_{B_{p,r}^s} \leq C \|u\|_{B_{p,r}^s} \|v\|_{B_{p,\infty}^{\frac{N}{p}} \cap L^\infty}. \quad (3.14)$$

**Remark 4** In the sequel  $p$  will be either  $p_1$  or  $p_2$  and in this case  $\frac{1}{\lambda} = \frac{1}{p_1} - \frac{1}{p_2}$  if  $p_1 \leq p_2$ , resp  $\frac{1}{\lambda} = \frac{1}{p_2} - \frac{1}{p_1}$  if  $p_2 \leq p_1$ .

**Corollary 1** Let  $r \in [1, +\infty]$ ,  $1 \leq p \leq p_1 \leq +\infty$  and  $s$  such that:

- $s \in (-\frac{N}{p_1}, \frac{N}{p_1})$  if  $\frac{1}{p} + \frac{1}{p_1} \leq 1$ ,
- $s \in (-\frac{N}{p_1} + N(\frac{1}{p} + \frac{1}{p_1} - 1), \frac{N}{p_1})$  if  $\frac{1}{p} + \frac{1}{p_1} > 1$ ,

then we have if  $u \in B_{p,r}^s$  and  $v \in B_{p_1,\infty}^{\frac{N}{p_1}} \cap L^\infty$ :

$$\|uv\|_{B_{p,r}^s} \leq C \|u\|_{B_{p,r}^s} \|v\|_{B_{p_1,\infty}^{\frac{N}{p_1}} \cap L^\infty}.$$

The study of non stationary PDE's requires space of type  $L^\rho(0, T, X)$  for appropriate Banach spaces  $X$ . In our case, we expect  $X$  to be a Besov space, so that it is natural to localize the equation through Littlewood-Paley decomposition. But, in doing so, we obtain bounds in spaces which are not type  $L^\rho(0, T, X)$  (except if  $r = p$ ). We are now going to define the spaces of Chemin-Lerner in which we will work, which are a refinement of the spaces  $L_T^\rho(B_{p,r}^s)$ .

**Definition 3.3** Let  $\rho \in [1, +\infty]$ ,  $T \in [1, +\infty]$  and  $s_1 \in \mathbb{R}$ . We set:

$$\|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1})} = \left( \sum_{l \in \mathbb{Z}} 2^{lrs_1} \|\Delta_l u(t)\|_{L^\rho(L^p)}^r \right)^{\frac{1}{r}}.$$

We then define the space  $\tilde{L}_T^\rho(B_{p,r}^{s_1})$  as the set of temperate distribution  $u$  over  $(0, T) \times \mathbb{R}^N$  such that  $\|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1})} < +\infty$ .

We set  $\tilde{C}_T(\tilde{B}_{p,r}^{s_1}) = \tilde{L}_T^\infty(\tilde{B}_{p,r}^{s_1}) \cap \mathcal{C}([0, T], B_{p,r}^{s_1})$ . Let us emphasize that, according to Minkowski inequality, we have:

$$\|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1})} \leq \|u\|_{L_T^\rho(B_{p,r}^{s_1})} \text{ if } r \geq \rho, \quad \|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1})} \geq \|u\|_{L_T^\rho(B_{p,r}^{s_1})} \text{ if } r \leq \rho.$$

**Remark 5** It is easy to generalize proposition 3.2, to  $\tilde{L}_T^\rho(B_{p,r}^{s_1})$  spaces. The indices  $s_1$ ,  $p$ ,  $r$  behave just as in the stationary case whereas the time exponent  $\rho$  behaves according to Hölder inequality.

In the sequel we will need of composition lemma in  $\tilde{L}_T^\rho(B_{p,r}^s)$  spaces.

**Lemma 4** Let  $s > 0$ ,  $(p, r) \in [1, +\infty]$  and  $u \in \tilde{L}_T^\rho(B_{p,r}^s) \cap L_T^\infty(L^\infty)$ .

1. Let  $F \in W_{loc}^{[s]+2,\infty}(\mathbb{R}^N)$  such that  $F(0) = 0$ . Then  $F(u) \in \tilde{L}_T^\rho(B_{p,r}^s)$ . More precisely there exists a function  $C$  depending only on  $s$ ,  $p$ ,  $r$ ,  $N$  and  $F$  such that:

$$\|F(u)\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \leq C(\|u\|_{L_T^\infty(L^\infty)}) \|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)}.$$

2. If  $v, u \in \tilde{L}_T^\rho(B_{p,r}^s) \cap L_T^\infty(L^\infty)$  and  $G \in W_{loc}^{[s]+3,\infty}(\mathbb{R}^N)$  then  $G(u) - G(v)$  belongs to  $\tilde{L}_T^\rho(B_p^s)$  and there exists a constant  $C$  depending only of  $s, p, N$  and  $G$  such that:

$$\begin{aligned} \|G(u) - G(v)\|_{\tilde{L}_T^\rho(B_{p,r}^s)} &\leq C(\|u\|_{L_T^\infty(L^\infty)}, \|v\|_{L_T^\infty(L^\infty)}) (\|v - u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} (1 + \|u\|_{L_T^\infty(L^\infty)} \\ &\quad + \|v\|_{L_T^\infty(L^\infty)}) + \|v - u\|_{L_T^\infty(L^\infty)} (\|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} + \|v\|_{\tilde{L}_T^\rho(B_{p,r}^s)})). \end{aligned}$$

Now we give some result on the behavior of the Besov spaces via some pseudodifferential operator (see [?]).

**Definition 3.4** Let  $m \in \mathbb{R}$ . A smooth function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  is said to be a  $\mathcal{S}^m$  multiplier if for all multi-index  $\alpha$ , there exists a constant  $C_\alpha$  such that:

$$\forall \xi \in \mathbb{R}^N, \quad |\partial^\alpha f(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}.$$

**Proposition 3.3** Let  $m \in \mathbb{R}$  and  $f$  be a  $\mathcal{S}^m$  multiplier. Then for all  $s \in \mathbb{R}$  and  $1 \leq p, r \leq +\infty$  the operator  $f(D)$  is continuous from  $B_{p,r}^s$  to  $B_{p,r}^{s-m}$ .

Let us now give some estimates for the heat equation:

**Proposition 3.4** Let  $s \in \mathbb{R}$ ,  $(p, r) \in [1, +\infty]^2$  and  $1 \leq \rho_2 \leq \rho_1 \leq +\infty$ . Assume that  $u_0 \in B_{p,r}^s$  and  $f \in \tilde{L}_T^{\rho_2}(B_{p,r}^{s-2+2/\rho_2})$ . Let  $u$  be a solution of:

$$\begin{cases} \partial_t u - \mu \Delta u = f \\ u_{t=0} = u_0. \end{cases}$$

Then there exists  $C > 0$  depending only on  $N, \mu, \rho_1$  and  $\rho_2$  such that:

$$\|u\|_{\tilde{L}_T^{\rho_1}(\tilde{B}_{p,r}^{s+2/\rho_1})} \leq C (\|u_0\|_{B_{p,r}^s} + \mu^{\frac{1}{\rho_2}-1} \|f\|_{\tilde{L}_T^{\rho_2}(B_{p,r}^{s-2+2/\rho_2})}).$$

If in addition  $r$  is finite then  $u$  belongs to  $C([0, T], B_{p,r}^s)$ .

## 4 Proof of theorem 1.1

In the sequel we shall work on the torus  $\Omega = \mathbb{T}^2$ . Let us start with recalling the energy estimate, when we multiply the momentum equation we get:

$$\begin{aligned} &\int_{\Omega} (\rho|u|^2(t, x) + \pi(\rho)(t, x)) dx + \int_0^t \int_{\Omega} |\nabla u|^2(s, x) ds dx \\ &+ \int_0^t \int_{\Omega} (1 + \lambda(\rho)(s, x)) (\operatorname{div} u)^2(s, x) ds dx \leq \int_{\Omega} (\rho_0(x)|u_0(x)|^2 + \Pi(\rho_0)(x)) dx \end{aligned} \quad (4.15)$$

with  $\pi$  defined as follows:

$$\pi(\rho) = a \left( \frac{1}{\gamma-1} (\rho^\gamma - \rho) - \rho + 1 \right) \text{ for } \gamma > 1.$$

Let us recall that  $P'(\rho) = \rho\pi''(\rho)$  what implies by convexity that  $\pi(\rho) \geq 0$ . Finally as we assume that:

$$C_1 = \int_{\Omega} \left( \frac{1}{2} \rho_0(x) |u_0(x)|^2 + \pi(\rho_0)(x) + \rho_0(x) \right) dx$$

is finite, we obtain at least formally (if  $\rho$  and  $u$  are enough regular for performing integration by parts) by energy estimate (4.15) and via the transport equation that:

$$\begin{aligned} \int_{\Omega} (\rho|u|^2(t, x) + \pi(\rho)(t, x) + \rho(t, x)) dx + \int_0^t \int_{\Omega} |\nabla u|^2(s, x) ds dx \\ + \int_0^t \int_{\Omega} (1 + \lambda(\rho)(s, x)) (\operatorname{div} u)^2(s, x) ds dx \leq C_1 \end{aligned} \quad (4.16)$$

Let us now explain how to get  $L^2((0, T) \times \Omega)$  estimates on  $u$ , we are going to follow Lions in [24] p4. Indeed by the momentum equation we have:

$$\left| \int_{\Omega} \rho u(t, x) dx \right| = \left| \int_{\Omega} \rho_0 u_0(x) dx \right| \leq \|\rho_0 u_0\|_{L^1(\Omega)}.$$

Next we use the Poincaré-Wirtinger inequality and we have:

$$\left| \int_{\Omega} \rho(t, x) [u(t, x) - \int_{\Omega} u(t, y) dy] dx \right| \leq C \|\rho(t, \cdot)\|_{L^\gamma} \|\nabla u\|_{L^2(\Omega)}.$$

Hence for all  $t \geq 0$ :

$$\left| \int_{\Omega} u(t, x) dx \right| \leq \frac{1}{\left( \int_{\Omega} \rho_0 dx \right)} (\|\rho_0 u_0\|_{L^1(\Omega)} + C \|\rho(t, \cdot)\|_{L^\gamma} \|\nabla u\|_{L^2(\Omega)}).$$

We conclude by Poincaré-Wirtinger inequality which implies that  $\left| \int_{\Omega} u(t, x) dx \right| + \|\nabla u\|_{L^2(\Omega)}$  is an equivalent norm to the usual one in  $H^1(\Omega)$ .  $\square$

Now we are just going to explain where in the proof we can slightly improve the range of the coefficient  $\beta$  in [32]. One of the main point of the proof in [32] consists in getting a priori estimates on the density in  $L^\infty(L^p(\mathbb{T}^2))$  for any  $p > 1$ . This is possible due to the viscosity coefficient  $\lambda(\rho) = \rho^\beta$  which provide such estimate at least if  $\beta$  is large enough. Let us follow the arguments of the proof of [32] and explain where by commutators estimates we can weaken the hypothesis  $\beta > 3$ .

## 5 A priori estimates on the density and the velocity

First as in [32], we are going to recall some estimates for solutions to the following two Neumann problems:

$$\Delta \xi = \operatorname{div}(\rho u), \quad \int_{\Omega} \xi dx = 0, \quad \partial_{x_1} \xi|_{x_1=0, x_1=1} = \partial_{x_2} \xi|_{x_2=0, x_2=1} = 0. \quad (5.17)$$

$$\Delta \eta = \operatorname{div}(\operatorname{div}(\rho u \otimes u)), \quad \int_{\Omega} \eta dx = 0, \quad \partial_{x_1} \eta|_{x_1=0, x_1=1} = \partial_{x_2} \eta|_{x_2=0, x_2=1} = 0. \quad (5.18)$$

Therefore by [23] we have solution to the problems (5.17) and (5.18), whereas the estimates for singular integrals in [28] provide the following inequalities:

$$\begin{aligned}\|\nabla(\Delta)^{-1}\operatorname{div}(\rho u)\|_{L^{2m}} &\lesssim m\|\rho u\|_{L^{2m}}, \quad 1 \leq m < +\infty, \\ \|\nabla(\Delta)^{-1}\operatorname{div}(\rho u)\|_{L^{2-r}} &\lesssim \|\rho u\|_{L^{2-r}}, \quad 1 \geq 2r \geq 0, \\ \|R_{i,j}(\rho u_i u_j)\|_{L^{2m}} &\lesssim m\|\rho u \otimes u\|_{L^{2m}}, \quad 1 \leq m < +\infty,\end{aligned}$$

Here we have roughly written  $\xi = (\Delta)^{-1}\operatorname{div}(\rho u)$  and  $\eta = R_{i,j}(\rho u_i u_j)$  (with the summation notation).

By Hölder's inequalities we obtain:

$$\begin{aligned}\|\nabla(\Delta)^{-1}\operatorname{div}(\rho u)\|_{L^{2m}} &\lesssim m\|\rho\|_{L^{\frac{2mk}{k-1}}}\|u\|_{L^{2mk}}, \\ \|\nabla(\Delta)^{-1}\operatorname{div}(\rho u)\|_{L^{2-r}} &\lesssim \|\rho\|_{L^{\frac{2-r}{r}}}\|\sqrt{\rho}u\|_{L^2}^{\frac{1}{2}}, \\ \|R_{i,j}(\rho u_i u_j)\|_{L^{2m}} &\lesssim m\|\rho\|_{L^{\frac{2mk}{k-1}}}\|u\|_{L^{4mk}}^2,\end{aligned}\tag{5.19}$$

where  $k > 1$ ,  $m \geq 1$  and  $r \geq 1$ ,  $1 \geq 2r \geq 0$ .

From the estimate of lemma 1-3, we obtain:

$$\begin{aligned}\|u\|_{L^{2m}} &\lesssim m^{\frac{1}{2}}\|\nabla u\|_{L^2}, \quad m > 2, \\ \|(\Delta)^{-1}\operatorname{div}(\rho u)\|_{L^{2m}} &\lesssim m^{\frac{1}{2}}\|\nabla(\Delta)^{-1}\operatorname{div}(\rho u)\|_{L^{\frac{2m}{m+1}}}, \quad m > 2,\end{aligned}\tag{5.20}$$

We set now:

$$\varphi(t) = \int_{\Omega} (\operatorname{curl}u^2(t, x) + (2 + \lambda(\rho))\operatorname{div}u^2(t, x))dx,$$

we obtain then by using (5.20), (5.19) with  $r = \frac{2}{m+1}$  and the energy inequality (4.16):

$$\|(\Delta)^{-1}\operatorname{div}(\rho u)\|_{L^{2m}} \lesssim m^{\frac{1}{2}}\|\rho\|_{L^m}^{\frac{1}{2}}, \quad m > 2,\tag{5.21}$$

Similarly we have:

$$\begin{aligned}\|\nabla(\Delta)^{-1}\operatorname{div}(\rho u)\|_{L^{2m}} &\lesssim m^{\frac{3}{2}}k^{\frac{1}{2}}(\varphi(t))^{\frac{1}{2}}\|\rho\|_{L^{\frac{2mk}{k-1}}}, \quad m > 2, \quad k > 1, \\ \|R_{i,j}(\rho u_i u_j)\|_{L^{2m}} &\lesssim m^2k\varphi(t)\|\rho\|_{L^{\frac{2mk}{k-1}}}, \quad m > 2, \quad k > 1.\end{aligned}\tag{5.22}$$

## 5.1 Gain of integrability for the density

Following [32] the plan of the proof of [32], we are interested in getting a gain of integrability on the density. We follow here the method of Lions in [24] to get a gain of integrability on the pressure and the argument developed in [32]. Apply the operator  $(\Delta)^{-1}\operatorname{div}$  to the momentum equation, we obtain:

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta)^{-1}\operatorname{div}(\rho u) + [R_{ij}, u_j](\rho u_i) - (2 + \lambda(\rho))\operatorname{div}u \\ + P(\rho) - \frac{1}{|\Omega|} \int_{\Omega} (P(\rho)(t, x) - (2 + \lambda(\rho))\operatorname{div}u)dx = 0.\end{aligned}\tag{5.23}$$

We will set in the sequel:

$$B = (2 + \lambda(\rho))\operatorname{div}u - P(\rho). \quad (5.24)$$

Next if we renormalize the mass equation we have:

$$\partial_t \theta(\rho) + u \cdot \nabla \theta(\rho) + \rho \theta'(\rho) \operatorname{div}u = 0.$$

where we have set:

$$\theta(\rho) = \int_1^\rho \frac{1}{s} (2 + \lambda(s)) ds = 2 \ln \rho + \frac{1}{\beta} (\rho^\beta - 1).$$

Finally we get the following transport equation:

$$\begin{aligned} \frac{\partial}{\partial t} [(\Delta)^{-1} \operatorname{div}(\rho u) + \theta(\rho)] + u \cdot \nabla [(\Delta)^{-1} \operatorname{div}(\rho u) + \theta(\rho)] + [R_{ij}, u_j](\rho u_i) \\ + P(\rho) - \frac{1}{|\Omega|} \int_\Omega [P(\rho)(t, x) - (2 + \lambda(\rho)) \operatorname{div}u] dx = 0, \end{aligned} \quad (5.25)$$

Denote by  $f$  the function:

$$f(t, x) = \max(0, (\Delta)^{-1} \operatorname{div}(\rho u) + \theta(\rho))$$

and multiply the equation (5.25) by the function  $\rho f^{2m-1}$  with  $m \in \mathbb{N}$  and  $m \geq 4$  and integrate over  $\Omega$ , we obtain:

$$\begin{aligned} \frac{1}{2m} \frac{d}{dt} \int_\Omega \rho f^{2m} dx + \int_\Omega \rho P(\rho) f^{2m-1} dx + \int_\Omega [R_{ij}, u_j](\rho u_i) \rho f^{2m-1} dx \\ + \int_\Omega B dx \int_\Omega \rho f^{2m-1} dx = 0. \end{aligned} \quad (5.26)$$

As in [32] we set:

$$Z(t) = \left( \int_\Omega \rho f^{2m}(t, x) \right)^{\frac{1}{2m}} \quad (5.27)$$

Using Hölder's inequality ( $\frac{2m-1}{2m} + \frac{1}{2m} = 1$ ,  $\frac{\beta}{2m\beta+1} + \frac{1}{2m(2m\beta+1)} = \frac{1}{2m}$ ), we begin with estimating the term  $|\int_\Omega [R_{ij}, u_j](\rho u_i) \rho f^{2m-1} dx|$  in (5.26) as follows:

$$\begin{aligned} \left| \int_\Omega [R_{ij}, u_j](\rho u_i) \rho f^{2m-1} dx \right| &\leq \int_\Omega |[R_{ij}, u_j](\rho u_i)| \rho^{\frac{1}{2m}} \rho^{\frac{2m-1}{2m}} f^{2m-1} dx \\ &\leq \| |[R_{ij}, u_j](\rho u_i)| \rho^{\frac{1}{2m}} \|_{L^{2m}(\Omega)} Z^{2m-1}(t) \\ &\leq \|\rho\|_{L^{2m\beta+1}(\Omega)}^{\frac{1}{2m}} \| [R_{ij}, u_j](\rho u_i) \|_{L^{2m+\frac{1}{\beta}}(\Omega)} Z^{2m-1}(t) \end{aligned} \quad (5.28)$$

Next we recall a result of R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes in [6], which says that the following map:

$$\begin{aligned} W^{1,r_1}(\mathbb{T}^N)^N \times L^{r_2}(\mathbb{T}^N)^N &\rightarrow W^{1,r_3}(\mathbb{T}^N)^N \\ (a, b) &\rightarrow [a_j, R_{i,j}] b_i \end{aligned}$$

is continuous for any  $N \geq 2$  as soon as  $\frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2}$ . Hence we obtain that  $[R_{ij}, u_j](\rho u_i)$  belongs in  $W^{1,p}$  (where  $\frac{1}{p} = \frac{1}{2} + \frac{1}{2(m+\frac{1}{2\beta})k} + \frac{k-1}{2(m+\frac{1}{2\beta})k}$  with  $k > 1$  and  $p = 2 - \frac{2}{m+1+\frac{1}{2\beta}} < 2$ ) with the following inequality:

$$\begin{aligned} \|[R_{ij}, u_j](\rho u_i)\|_{W^{1,p}(\Omega)} &\leq C \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^{2(m+\frac{1}{2\beta})k}(\Omega)} \|\rho\|_{L^{\frac{2(m+\frac{1}{2\beta})k}{k-1}}(\Omega)}, \\ &\leq C \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^{2(m+\frac{1}{2\beta})k}(\Omega)} \|\rho\|_{L^{2m\beta+1}(\Omega)}, \end{aligned} \quad (5.29)$$

where we have choose  $k$  such that  $\frac{2(m+\frac{1}{2\beta})k}{k-1} = 2m\beta + 1$ , let  $k = \frac{2m\beta+1}{2m(\beta-1)+1-\frac{1}{\beta}}$ . We verifies that  $\frac{1}{q} = \frac{1}{p} - \frac{1}{2} = \frac{1}{2m+\frac{1}{\beta}}$ . Next by using lemma 2 and (5.29) we get:

$$\begin{aligned} \|[R_{ij}, u_j](\rho u_i)(t, \cdot)\|_{L^{2m+\frac{1}{\beta}}(\Omega)} &\lesssim (m^{\frac{1}{2}} \|\nabla u(t, \cdot)\|_{L^2(\Omega)} \|u(t, \cdot)\|_{L^{2(m+\frac{1}{2\beta})k}(\Omega)} \|\rho(t, \cdot)\|_{L^{2m\beta+1}(\Omega)} \\ &\quad + |\int_{\Omega} [R_{ij}, u_j](\rho u_i)(t, x) dx|). \end{aligned}$$

We can easily bound the last term on the right hand side by using the continuity of the Riez transform in  $L^p(\omega)$  with  $1 < p < +\infty$ :

$$|\int_{\Omega} [R_{ij}, u_j](\rho u_i)(t, x) dx| \lesssim \|\rho(t, \cdot)\|_{L^\gamma(\Omega)} \|u(t, \cdot)\|_{H^1(\Omega)}^2.$$

By (5.20) and the previous inequalities we obtain finally:

$$\|[R_{ij}, u_j](\rho u_i)(t, \cdot)\|_{L^{2m+\frac{1}{\beta}}(\Omega)} \leq Cm \|\nabla u\|_{L^2(\Omega)}^2 \|\rho\|_{L^{2m\beta+1}(\Omega)} + \|\rho(t, \cdot)\|_{L^\gamma(\Omega)} \|u(t, \cdot)\|_{H^1(\Omega)}^2. \quad (5.30)$$

We have then from (5.28) and (5.30):

$$|\int_{\Omega} [R_{ij}, u_j](\rho u_i) \rho f^{2m-1} dx| \lesssim (m \|\rho\|_{L^{2m\beta+1}(\Omega)}^{1+\frac{1}{2m}} \varphi(t) + \|\rho(t, \cdot)\|_{L^\gamma(\Omega)} \|u(t, \cdot)\|_{H^1(\Omega)}^2) Z^{2m-1}(t).$$

Next as in [32] we get:

$$\begin{aligned} |\int_{\Omega} B dx \int_{\Omega} \rho f^{2m-1} dx| &\lesssim Z^{2m-1}(t) \|\rho\|_{L^1}^{\frac{1}{2m}} \int_{\Omega} ((2+\lambda)|\operatorname{div} u| + P) dx \\ &\lesssim Z^{2m-1}(t) (1 + (\varphi(t))^{\frac{1}{2}} (\int_{\Omega} (2+\lambda(\rho)) dx)^{\frac{1}{2}}) \\ &\lesssim Z^{2m-1}(t) (1 + \varphi(t)^{\frac{1}{2}} + \|\rho\|_{L^{2m\beta+1}(\Omega)}^{\frac{\beta}{2}} \varphi(t)^{\frac{1}{2}}). \end{aligned}$$

Collecting all the above inequalities, we obtain:

$$\begin{aligned} Z(t) &\lesssim 1 + \int_0^t \|\rho\|_{L^\gamma(\Omega)}(\tau) \|u\|_{H^1(\Omega)}^2(\tau) d\tau + \int_0^t m \varphi(\tau) \|\rho\|_{L^{2m\beta+1}(\Omega)}^{1+\frac{1}{2m}}(\tau) d\tau \\ &\quad + \int_0^t \varphi(\tau)^{\frac{1}{2}} \|\rho\|_{L^{2m\beta+1}(\Omega)}^{\frac{\beta}{2}}(\tau) d\tau. \end{aligned} \quad (5.31)$$

As we have seen that  $u$  belongs in  $L^2((0, t), H^1(\Omega))$  we have:

$$\int_0^t \|\rho\|_{L^\gamma(\Omega)}(\tau) \|u\|_{H^1(\Omega)}^2(\tau) d\tau \lesssim 1.$$

We obtain then:

$$Z(t) \lesssim 1 + \int_0^t m\varphi(\tau) \|\rho\|_{L^{2m\beta+1}(\Omega)}^{1+\frac{1}{2m}}(\tau) d\tau + \int_0^t \varphi(\tau)^{\frac{1}{2}} \|\rho\|_{L^{2m\beta+1}(\Omega)}^{\frac{\beta}{2}}(\tau) d\tau. \quad (5.32)$$

Next following [32] we introduce the measurable sets:

$$\Omega_1(t) = \{x \in \Omega / \rho \geq 2m'\} \quad \text{and} \quad \Omega_2(t) = \{x \in \Omega_1(t) / \theta(\rho) + (\Delta)^{-1} \operatorname{div}(\rho u) > 0\}.$$

We then have:

$$\|\rho\|_{L^{2m\beta+1}(\Omega)}^\beta \lesssim \left( \int_{\Omega_1(t)} \rho^{2m\beta+1} dx \right)^{\frac{\beta}{2m\beta+1}} + 1, \quad (5.33)$$

Moreover by the definition of the function  $\theta(\rho)$ , we have:

$$\left( \int_{\Omega_1(t)} \rho^{2m\beta+1} dx \right)^{\frac{\beta}{2m\beta+1}} \lesssim \left( \int_{\Omega_1(t)} \rho \theta(\rho)^{2m} dx \right)^{\frac{\beta}{2m\beta+1}} \quad (5.34)$$

Using the fact that on  $\Omega_1(t) \setminus \Omega_2(t)$  we have  $0 \leq \theta(\rho) \leq |(\Delta)^{-1} \operatorname{div}(\rho u)|$ , we derive the following estimate:

$$\begin{aligned} \int_{\Omega_1(t)} \rho \theta(\rho)^{2m} dx &= \int_{\Omega_2(t)} \rho (\theta(\rho) + (\Delta)^{-1} \operatorname{div}(\rho u) - (\Delta)^{-1} \operatorname{div}(\rho u))^{2m} dx \\ &\quad + \int_{\Omega_1(t) \setminus \Omega_2(t)} \rho \theta(\rho)^{2m} dx \\ &\leq 2^{2m-1} \left( \int_{\Omega_2(t)} \rho f(\rho)^{2m} dx + \int_{\Omega_2(t)} \rho |(\Delta)^{-1} \operatorname{div}(\rho u)|^{2m} dx \right) \\ &\quad + \int_{\Omega_1(t) \setminus \Omega_2(t)} \rho |(\Delta)^{-1} \operatorname{div}(\rho u)|^{2m} dx \\ &\leq 2^{2m} (Z^{2m}(t) + \int_{\Omega} \rho |(\Delta)^{-1} \operatorname{div}(\rho u)|^{2m} dx) \end{aligned}$$

From estimates (5.33) and (5.34) we deduce:

$$\|\rho\|_{L^{2m\beta+1}(\Omega)}^\beta \leq C((m')^\beta + Z(t)^{\frac{2m\beta}{2m\beta+1}}(t) + \left( \int_{\Omega} \rho |(\Delta)^{-1} \operatorname{div}(\rho u)|^{2m} dx \right)^{\frac{\beta}{2m\beta+1}})$$

In view of estimate (5.21), (5.22), we have:

$$\begin{aligned} \int_{\Omega} \rho |(\Delta)^{-1} \operatorname{div}(\rho u)|^{2m} dx &\leq \|\rho\|_{L^{2m\beta+1}(\Omega)} \|(\Delta)^{-1} \operatorname{div}(\rho u)\|_{L^{2m+\frac{1}{\beta}}(\Omega)}^{2m} \\ &\leq \|\rho\|_{L^{2m\beta+1}(\Omega)} \left( \left(m + \frac{1}{2\beta}\right)^{\frac{1}{2}} \|\rho\|_{L^{m+\frac{1}{\beta}}(\Omega)}^{\frac{1}{2}} \right)^{2m} \\ &\leq C^m m^m \|\rho\|_{L^{2m\beta+1}(\Omega)}^{m+1}. \end{aligned}$$

Finally:

$$\|\rho\|_{L^{2m\beta+1}(\Omega)}^\beta \lesssim \left( Z(t)^{\frac{2m\beta}{2m\beta+1}}(t) + m^{\frac{1}{2}} \|\rho\|_{L^{2m\beta+1}(\Omega)}^{\frac{\beta(m+1)}{2m\beta+1}} \right). \quad (5.35)$$

By Young's inequality (with  $q = \frac{2m\beta+1}{m+1}$  and  $p = \frac{2\beta}{2\beta-1} + \frac{1}{m(2\beta-1)}$ ), we obtain:

$$\|\rho\|_{L^{2m\beta+1}(\Omega)}^\beta \lesssim \left( Z(t) + \frac{1}{\varepsilon} m^{\frac{\beta}{2\beta-1} + \frac{1}{2m(2\beta-1)}} + \varepsilon \|\rho\|_{L^{2m\beta+1}(\Omega)}^\beta \right).$$

By bootstrap, we get:

$$\|\rho\|_{L^{2m\beta+1}(\Omega)}^\beta \lesssim Z(t) + m^{\frac{\beta}{2\beta-1}}. \quad (5.36)$$

Therefore (5.31) and (5.36) give the following inequality:

$$\|\rho\|_{L^{2m\beta+1}(\Omega)}^\beta \lesssim \left( m^{\frac{\beta}{2\beta-1}} + \int_0^t m\varphi(\tau) \|\rho\|_{L^{2m\beta+1}(\Omega)}^{1+\frac{1}{2m}}(\tau) d\tau + \int_0^t \varphi(\tau)^{\frac{1}{2}} \|\rho\|_{L^{2m\beta+1}(\Omega)}^{\frac{\beta}{2}}(\tau) d\tau \right)$$

Next by Young's inequality, we have:

$$\|\rho\|_{L^{2m\beta+1}(\Omega)}^\beta \lesssim \left( 1 + m^{\frac{\beta}{2\beta-1}} + \int_0^t m\varphi(\tau) \|\rho\|_{L^{2m\beta+1}(\Omega)}^{1+\frac{1}{2m}}(\tau) d\tau + \int_0^t \|\rho\|_{L^{2m\beta+1}(\Omega)}^\beta(\tau) d\tau \right)$$

Using the fact that  $\varphi(t) \in L^1(0, T)$  and applying Grönwall's inequality, we have that:

$$\|\rho\|_{L^{2m\beta+1}(\Omega)}^\beta \leq C \left( 1 + m^{\frac{\beta}{2\beta-1}} + \int_0^t m\varphi(\tau) \|\rho\|_{L^{2m\beta+1}(\Omega)}^{1+\frac{1}{2m}}(\tau) d\tau, \right)$$

where  $C$  depends on  $t$ . Denote:

$$y(t) = m^{-\frac{1}{\beta-1}} \|\rho\|_{L^{2m\beta+1}(\Omega)}, \quad t \in [0, T]. \quad (5.37)$$

Then:

$$y^\beta(t) m^{\frac{\beta}{\beta-1}} \leq C \left( 1 + m^{\frac{\beta}{2\beta-1}} + m m^{(1+\frac{1}{2m})\frac{1}{\beta-1}} \int_0^t \varphi(\tau) y^{1+\frac{1}{2m}}(\tau) d\tau \right)$$

and we have:

$$m m^{(1+\frac{1}{2m})\frac{1}{\beta-1}} = m^{\frac{\beta}{\beta-1} + \frac{1}{2m(\beta-1)}}.$$

We have then:

$$y^\beta(t) \leq C \left( 1 + m^{\frac{-\beta^2}{(\beta-1)(2\beta-1)}} + m^{\frac{1}{2m(\beta-1)}} \int_0^t \varphi(\tau) y^{1+\frac{1}{2m}}(\tau) d\tau \right)$$

where  $\frac{\beta}{2\beta-1} - \frac{\beta}{\beta-1} = \frac{-\beta^2}{(\beta-1)(2\beta-1)} < 0$ .

Recalling that  $\beta > 1$  and  $\varphi(t) \in L^1(0, T)$  we find that for  $m$  big enough:

$$y^\beta(t) \leq C \left( C_1 + \int_0^t \varphi(\tau) y^\beta(\tau) d\tau \right)$$

whence by Grönwall inequality:

$$y(t) \leq C, \quad t \in [0, T],$$

where  $C$  depends on  $t$ . We thus have:

$$\|\rho\|_{L^{2m\beta+1}} \leq Cm^{\frac{1}{\beta-1}}, \quad t \in [0, T].$$

Hence the inequality:

$$\|\rho\|_{L^k(\Omega)}(t) \leq Ck^{\frac{1}{\beta-1}}, \quad t \in [0, T]. \quad (5.38)$$

is valid for every  $k \geq 1$ , with  $C$  a positive constant independent of  $k \geq 1$  but depending of the time.  $\square$

**Remark 6** *Let us point out that the estimate (5.38) is the key point in order to improve the range on  $\beta$ . Indeed this last one is a refinement of the corresponding one in [32]. In particular we will be able to obtain the energy estimates (5.66) only with assuming  $\beta > 2$ .*

## 5.2 Second a priori estimate for the velocity

In this section, we are going to furnish estimates on the velocity by using the gain of integrability on the density proved in the previous section. We are going essentially to follow the proof of [32] and to emphasize on the key point where we will only need the hypothesis  $\beta > 2$ . We begin with recalling some equation on the effective pressure defined in [24] and the rotational *curl*. We set:

$$\begin{aligned} A &= \operatorname{curl} u \quad \text{and} \quad B = (2 + \lambda(\rho))\operatorname{div} u - P(\rho), \\ L &= \frac{1}{\rho}(\partial_y A + \partial_x B) \quad \text{and} \quad H = \frac{1}{\rho}(-\partial_x A + \partial_y B). \end{aligned}$$

We now want to obtain some estimates on the unknowns  $A$  and  $B$ , let us start with rewriting the momentum equation under the following eulerian form:

$$\partial_t u + u \cdot \nabla u - \frac{1}{\rho} \Delta u - \frac{1}{\rho} \nabla((\mu + \lambda(\rho))\operatorname{div} u) + \nabla\left(\frac{P(\rho)}{\gamma\rho}\right) = 0 \quad (5.39)$$

Next if we apply the operator *curl*, we get:

$$\partial_t A + u \cdot \nabla A + A \operatorname{div} u = \partial_y L - \partial_x H. \quad (5.40)$$

Next we apply the operator *div* to the momentum equation (5.39):

$$\partial_t \operatorname{div} u + u \cdot \nabla u - \frac{1}{\rho} \Delta u - \frac{1}{\rho} \nabla((\mu + \lambda(\rho))\operatorname{div} u) + \nabla\left(\frac{P(\rho)}{\gamma\rho}\right) = 0, \quad (5.41)$$

and via the mass equation we have:

$$\begin{aligned} \partial_t B + U \cdot \nabla B - \rho(2 + \lambda)\left(B\left(\frac{1}{2 + \lambda}\right)' + \left(\frac{P}{2 + \lambda}\right)'\right) \operatorname{div} u \\ + (2 + \lambda)(U_x^2 + 2U_y V_x + V_y^2) = (2 + \lambda)(L_x + H_y). \end{aligned} \quad (5.42)$$

As in [32] multiplying the equation(5.40) by  $A$  and integrate over  $\Omega$  we obtain:

$$\int_{\Omega} \frac{1}{2} \frac{d}{dt} [A^2] dx + \frac{1}{2} \int_{\Omega} \operatorname{div} u A^2 dx + \int_{\Omega} (L \partial_y A - H \partial_x A) dx = 0. \quad (5.43)$$

Similarly multiplying the equation(5.42) by  $\frac{1}{2+\lambda}B$  and integrate over  $\Omega$  we have:

$$\begin{aligned} & \int_{\Omega} \frac{1}{2+\lambda} \frac{d}{dt} (\frac{1}{2}B^2) dx - \frac{1}{2} \int_{\Omega} \operatorname{div} u \frac{B^2}{2+\lambda} dx - \frac{1}{2} \int_{\Omega} u \cdot \nabla (\frac{1}{2+\lambda}) B^2 dx \\ & - \int_{\Omega} \rho B \operatorname{div} u (B(\frac{1}{2+\lambda})' + (\frac{P}{2+\lambda})') dx + \int_{\Omega} B(U_x^2 + 2U_y V_x + V_y^2) dx \\ & + \int_{\Omega} (L\partial_x B + H\partial_y B) dx = 0. \end{aligned} \quad (5.44)$$

We recall now that:

$$\partial_t (\frac{1}{2+\lambda}) + (\frac{1}{2+\lambda})' \rho \operatorname{div} u + \nabla (\frac{1}{2+\lambda}) \cdot u = 0.$$

By combining the previous equality and (5.44) we get:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \frac{d}{dt} (\frac{1}{2+\lambda} B^2) dx - \frac{1}{2} \int_{\Omega} \operatorname{div} u B^2 (\frac{1}{2+\lambda} - \rho(\frac{1}{2+\lambda})') dx \\ & - \int_{\Omega} \rho B \operatorname{div} u (B(\frac{1}{2+\lambda})' + (\frac{P}{2+\lambda})') dx + \int_{\Omega} B(\operatorname{div} u)^2 dx \\ & + 2 \int_{\Omega} B(\partial_y U \partial_x V - \partial_x U \partial_y V) dx + \int_{\Omega} (L\partial_x B + H\partial_y B) dx = 0. \end{aligned} \quad (5.45)$$

Summing (5.43) and (5.45) we have:

$$\begin{aligned} & \int_{\Omega} \frac{1}{2} \frac{d}{dt} [(A^2 + \frac{B^2}{2+\lambda})] dx + \int_{\Omega} \frac{1}{2} \operatorname{div} u A^2 dx + \int_{\Omega} \frac{(A_y + B_x)^2 + (-A_x + B_y)^2}{\rho} dx \\ & - \frac{1}{2} \int_{\Omega} B^2 \operatorname{div} u (\frac{1}{2+\lambda} - \rho(\frac{1}{2+\lambda})') dx + 2 \int_{\Omega} B(U_y V_x - U_x V_y) dx \\ & - \int_{\Omega} \rho B \operatorname{div} u (B(\frac{1}{2+\lambda})' + (\frac{P}{2+\lambda})') dx + \int_{\Omega} B \operatorname{div} u^2 dx = 0. \end{aligned}$$

As:

$$\operatorname{div} u^2 = \operatorname{div} u (\frac{B}{2+\lambda} + \frac{P}{2+\lambda}),$$

we deduce:

$$\begin{aligned} & \int_{\Omega} \frac{1}{2} \frac{d}{dt} [(A^2 + \frac{B^2}{2+\lambda})] dx + \int_{\Omega} \frac{1}{2} \operatorname{div} u A^2 dx + \int_{\Omega} \frac{(A_y + B_x)^2 + (-A_x + B_y)^2}{\rho} dx \\ & + \int_{\Omega} \frac{1}{2} B^2 \operatorname{div} u (\frac{1}{2+\lambda} - \rho(\frac{1}{2+\lambda})') dx + \int_{\Omega} B \operatorname{div} u (\frac{P}{2+\lambda} - \rho(\frac{P}{2+\lambda})') dx \\ & + 2 \int_{\Omega} B(U_y V_x - U_x V_y) dx - \int_{\Omega} \rho B \operatorname{div} u (B(\frac{1}{2+\lambda})' + (\frac{P}{2+\lambda})') dx = 0. \end{aligned} \quad (5.46)$$

Let us set:

$$\begin{aligned} Z(t) &= \left( \int_{\Omega} (A^2 + \frac{B^2}{2+\lambda}) dx \right)^{\frac{1}{2}} \\ a(t) &= \left( \int_{\Omega} \frac{(A_y + B_x)^2 + (-A_x + B_y)^2}{\rho} dx \right)^{\frac{1}{2}}, \quad t \in [0, T]. \end{aligned} \quad (5.47)$$

Next we have:

$$\int_{\Omega} ((A_y + B_x)^2 + (-A_x + B_y)^2) dx = \int_{\Omega} (A_x^2 + A_y^2 + B_x^2 + B_y^2) dx.$$

Let us observe that for every  $r$ ,  $1 \geq 4r > 0$ , from the result on elliptic system and by Hölder inequalities we get as in [32]:

$$\|\nabla A\|_{L^{2(1-r)}(\Omega)} + \|\nabla B\|_{L^{2(1-r)}(\Omega)} \leq C \left( \int_{\Omega} \frac{(A_y + B_x)^2 + (-A_x + B_y)^2}{\rho} dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \rho^{\frac{1-r}{r}} dx \right)^{\frac{r}{2(1-r)}}.$$

From (5.38), we have:

$$(\|\nabla A\|_{L^{2(1-r)}(\Omega)} + \|\nabla B\|_{L^{2(1-r)}(\Omega)}) \leq C \left( \frac{1}{r} \right)^{\frac{1}{2(\beta-1)}} a(t) \quad (5.48)$$

**Remark 7** *Let us point out that the estimate (5.48) is better than the corresponding one in [32] due to the better estimate (5.38).*

Moreover via (5.38) we also obtain the following inequality:

$$(\|\nabla u\|_{L^2(\Omega)} + \|A\|_{L^2(\Omega)} + \|\sqrt{2 + \lambda} \operatorname{div} u\|_{L^2(\Omega)}) \leq C(1 + Z(t)), \quad t \in [0, T]. \quad (5.49)$$

Now, are interested in providing other estimates for the non positive terms of the equality (5.46).

### Estimates for the terms of (5.46)

Following [32], using (??), the lemma 1 (with  $\alpha = \frac{1-\varepsilon}{2(1-2\varepsilon)}$ ) and Young's inequality (with  $p = \frac{2(1-2\varepsilon)}{1-\varepsilon}$ ,  $q = \frac{2(1-2\varepsilon)}{1-3\varepsilon}$  and  $p_1 = \frac{(1-2\varepsilon)(2+\varepsilon)}{1-\varepsilon}$ ,  $q_1 = \frac{(1-2\varepsilon)(2+\varepsilon)}{1-2\varepsilon-2\varepsilon^2}$ ) the , we obtain:

$$\begin{aligned} \left| \frac{1}{2} \int_{\Omega} \operatorname{div} u A^2 \right| &\leq \frac{1}{2} \|\operatorname{div} u(t)\|_{L^2} \|A(t)\|_{L^4}^2, \\ &\leq C \|\operatorname{div} u\|_{L^2} \left( \frac{1}{2} \|A\|_{L^2} \right)^{\frac{1-3\varepsilon}{1-2\varepsilon}} \left( \frac{1}{2} \|\nabla A\|_{L^{2(1-\varepsilon)}} \right)^{\frac{1-\varepsilon}{1-2\varepsilon}} \\ &\leq C(1 + Z(t)) Z(t)^{\frac{1-3\varepsilon}{1-2\varepsilon}} \left( \left( \frac{1}{\varepsilon} \right)^{\frac{1}{2(\beta-1)}} a(t) \right)^{\frac{1-\varepsilon}{1-2\varepsilon}} \\ &\leq \delta a^2(t) + C(\delta)(1 + Z(t))^{\frac{2(1-2\varepsilon)}{1-3\varepsilon}} Z(t)^2 \left( \frac{1}{\varepsilon} \right)^{\frac{1}{\beta-1}} \\ &\leq \delta a^2(t) + C(\delta)(1 + Z(t)^2)^{2+\frac{\varepsilon}{1-3\varepsilon}} \left( \frac{1}{\varepsilon} \right)^{\frac{1}{\beta-1}}. \end{aligned}$$

We now are interested in estimating the term in (5.46) corresponding to:

$$\begin{aligned} I_1 &= \left| \frac{1}{2} \int_{\Omega} B^2 \operatorname{div} u \left( \frac{1}{2+\lambda} - \rho \left( \frac{1}{2+\lambda} \right)' \right) dx \right| \\ &= \left| \frac{1}{2} \int_{\Omega} B^2 \left( \frac{B}{2+\lambda} + \frac{P}{2+\lambda} \right) \left( \frac{1}{2+\lambda} - \rho \left( \frac{1}{2+\lambda} \right)' \right) dx \right| \end{aligned}$$

Easily there exist a positive constant  $C > 0$  such that:

$$\left| \frac{1}{2+\lambda} - \rho\left(\frac{1}{2+\lambda}\right)' \right| \leq C,$$

for all  $\rho \in [0, +\infty)$ . We deduce that:

$$I_1 \leq C(m')^\beta \left( \int_{\Omega} \frac{|B|^3}{2+\lambda} dx + \int_{\Omega} \frac{|B|^2}{2+\lambda} |P| dx \right).$$

By Young's inequality we have:

$$\begin{aligned} \left| \int_{\Omega} B \operatorname{div} u \left( \frac{P}{2+\lambda} - \rho\left(\frac{P}{2+\lambda}\right)' \right) dx \right| &= \left| \int_{\Omega} B \left( \frac{B}{2+\lambda} + \frac{P}{2+\lambda} \right) \left( \frac{P}{2+\lambda} - \rho\left(\frac{P}{2+\lambda}\right)' \right) dx \right| \\ &\leq C \left( 1 + \int_{\Omega} \frac{|B|^3}{2+\lambda} dx \right). \end{aligned}$$

Now, the last term in (5.46) can be treated as follows:

$$\left| 2 \int_{\Omega} B (U_y V_x - U_x V_y) dx \right| \leq \int_{\Omega} |B| (U_x^2 + U_y^2 + V_x^2 + V_y^2) dx$$

Via the previous estimate, the notations (5.47), and using the equality (5.46), we get:

$$\begin{aligned} \frac{1}{2} (Z^2(t)) + a^2(t) &\leq \delta a^2(t) + C(\delta) (1 + Z(t)^2)^{1 + \frac{\varepsilon}{1-3\varepsilon}} \left( \frac{1}{\varepsilon} \right)^{\frac{2}{\beta-1}} \\ &+ C \left( 1 + \int_{\Omega} \frac{|B|^3}{2+\lambda} dx \right) + \int_{\Omega} |B| (U_x^2 + U_y^2 + V_x^2 + V_y^2) dx. \end{aligned} \quad (5.50)$$

It remains to estimate the two last terms on the right hand side of (5.50). In this goal, from (2.10) we have:

$$\|B\|_{L^{2m}(\Omega)} \leq C \left( \|B\|_{L^1(\Omega)} + m^{\frac{1}{2}} \|\nabla B\|_{L^{\frac{2m}{m+\varepsilon}}(\Omega)}^{1-s} \|B\|_{L^{2(1-\varepsilon)}(\Omega)}^s \right) \quad (5.51)$$

where:  $s = \frac{(1-\varepsilon)^2}{m-\varepsilon(1-\varepsilon)}$  and  $C > 0$  is a positive constant independent of  $m > 2$ .

Now in inequalities (5.50) and (5.51) we fix  $\varepsilon = 2^{-m}$  with  $m > 2$ . Using estimate (5.38) for the density, we derive the inequalities:

$$\begin{aligned} \|B\|_{L^1(\Omega)} &= \int_{\Omega} |B| dx = \int_{\Omega} \left( \frac{1}{2+\lambda} \right)^{\frac{1}{2}} |B| (2+\lambda)^{\frac{1}{2}} dx \leq \|(2+\lambda)^{\frac{1}{2}}\|_{L^2(\Omega)} Z(t) \\ &\leq CZ(t). \end{aligned}$$

$$\begin{aligned} \|B\|_{L^{2(1-\varepsilon)}(\Omega)}^s &= \left( \int_{\Omega} \left( \frac{1}{2+\lambda} \right)^{1-\varepsilon} |B|^{2(1-\varepsilon)} (2+\lambda)^{1-\varepsilon} dx \right)^{\frac{s}{2(1-\varepsilon)}} \\ &\leq Z(t)^s \|2+\lambda\|_{L^{\frac{1-\varepsilon}{2}}(\Omega)}^{\frac{s}{2}} \leq C \left( \frac{1}{\varepsilon} \right)^{\frac{\beta s}{2(\beta-1)}} Z^s(t) \leq C 2^{2sm} Z^s(t), \\ &\leq CZ^s(t). \end{aligned}$$

From inequalities (5.48) and (5.51) we finally obtain:

$$\begin{aligned} \|B\|_{L^{2m}(\Omega)} &\leq C(Z(t) + m^{\frac{1}{2}}(\frac{m}{\varepsilon})^{\frac{1-s}{2(\beta-1)}}(a(t))^{1-s}Z^s(t)), \\ &\leq C(Z(t) + m^{\frac{1}{2}}(\frac{m}{\varepsilon})^{\frac{1-s}{2(\beta-1)}}(a(t))^{1-s}Z^s(t)). \end{aligned} \quad (5.52)$$

Now dealing with the integral with  $|B|^3$ , we have:

$$\begin{aligned} \int_{\Omega} \frac{|B|^3}{2+\lambda} dx &= \int_{\Omega} \frac{|B|^{2-\frac{1}{m-1}}}{(2+\lambda)^{1-\frac{1}{2(m-1)}}} (\frac{1}{2+\lambda})^{\frac{1}{2(m-1)}} |B|^{1+\frac{1}{m-1}} dx \\ &\leq \int_{\Omega} \frac{|B|^{2-\frac{1}{m-1}}}{(2+\lambda)^{1-\frac{1}{2(m-1)}}} |B|^{\frac{m}{m-1}} dx \leq Z(t)^{2-\frac{1}{m-1}} \|B\|_{L^{2m}(\Omega)}^{\frac{m}{m-1}}, \\ &\leq (\int_{\Omega} \frac{|B|^2}{2+\lambda} dx)^{1-\frac{1}{2(m-1)}} (\int_{\Omega} |B|^{2m} dx)^{\frac{1}{2(m-1)}} \leq^{-1+\frac{1}{2(m-1)}} Z(t)^{2-\frac{1}{m-1}} \|B\|_{L^{2m}(\Omega)}^{\frac{m}{m-1}} \\ &\leq CZ^{2-\frac{1}{m-1}}(t) (Z(t)^{\frac{m}{m-1}} + m^{\frac{m}{2(m-1)}} (\frac{m}{\varepsilon})^{\frac{m(1-s)}{2(\beta-1)(m-1)}} (a(t))^{\frac{m(1-s)}{m-1}} Z^{\frac{ms}{m-1}}(t)), \\ &\leq C(Z(t)^3 + m^{\frac{m}{2(m-1)}} (\frac{m}{\varepsilon})^{\frac{m(1-s)}{2(\beta-1)(m-1)}} (a(t))^{\frac{m(1-s)}{m-1}} Z^{2+\frac{ms-1}{m-1}}(t)), \end{aligned}$$

where  $C > 0$  is a positive constant independent of  $m > 2$  and  $\varepsilon = 2^{-m}$ . Finally applying applying Young's inequality with  $p = \frac{2(m-1)}{m(1-s)}$  and  $q = \frac{2(m-1)}{m(s+1)-2}$  we have:

$$\begin{aligned} \int_{\Omega} \frac{|B|^3}{2+\lambda} dx &\leq C(Z^3(t) + m^{\frac{1}{2}}(\frac{m}{\varepsilon})^{\frac{1}{2(\beta-1)}} a^{\frac{m(1-s)}{m-1}}(t) Z^{2+\frac{ms-1}{m-1}}(t)), \\ &\leq \delta a^2(t) + C(\delta)(Z^3(t) + m^{\frac{m-1}{m(s+1)-2}} (\frac{m}{\varepsilon})^{\frac{m-1}{(\beta-1)(m(s+1)-2)}} Z^{4+\frac{2(1-ms)}{m(s+1)-2}}(t)) \end{aligned} \quad (5.53)$$

From (5.51), we verify that:

$$1 - ms = 1 - \frac{m(1-\varepsilon)^2}{m-\varepsilon(1-\varepsilon)} = \varepsilon \frac{m(2-\varepsilon) + \varepsilon - 1}{m-\varepsilon(1-\varepsilon)}, \quad (5.54)$$

and then:

$$\lim_{m \rightarrow +\infty} (2^m(1-ms)) = 2,$$

hence via (5.53) and (5.54) we get:

$$\int_{\Omega} \frac{|B|^3}{2+\lambda} dx \leq \delta a^2(t) + C(\delta)((1+Z^2(t))^2 + m(\frac{m}{\varepsilon})^{\frac{1}{\beta-1}}(1+Z^2(t))^{2+\frac{1-ms}{m(s+1)-2}}). \quad (5.55)$$

Now, consider the last term in (5.50):

$$I_2 = \int_{\Omega} |B|(U_x^2 + U_y^2 + V_x + V_y^2) \leq \|B\|_{L^{2m}(\Omega)} (\int_{\Omega} (|\nabla U|^2 + |\nabla V|^2)^{\frac{2m}{2m-1}} dx)^{1-\frac{1}{2m}}.$$

Recalling the relation  $3 > \frac{4m}{2m-1} > 2$ ,  $m > 2$ , from the properties of elliptic system [?] we derive the inequality:

$$(\int_{\Omega} (|\nabla U|^2 + |\nabla V|^2)^{\frac{2m}{2m-1}} dx)^{1-\frac{1}{2m}} \leq C(\|\operatorname{div} u\|_{L^{\frac{4m}{2m-1}}(\Omega)}^2 + \|A\|_{L^{\frac{4m}{2m-1}}(\Omega)}^2).$$

Thus the previous inequality furnish the estimate:

$$I_2 \leq C \|B\|_{L^{2m}(\Omega)} (\|\operatorname{div} u\|_{L^{\frac{4m}{2m-1}}(\Omega)}^2 + \|A\|_{L^{\frac{4m}{2m-1}}(\Omega)}^2). \quad (5.56)$$

Next we have as  $A$  vanishes on the boundary of the domain  $\Omega$ , we apply the Gagliardo-Nirenberg inequality:

$$\begin{aligned} \|A\|_{L^{\frac{4m}{2m-1}}(\Omega)}^2 &\leq C \|A\|_{L^2(\Omega)}^{2-\frac{1-\varepsilon}{m(1-2\varepsilon)}} \|\nabla A\|_{L^{2(1-\varepsilon)}(\Omega)}^{\frac{1-\varepsilon}{m(1-2\varepsilon)}}, \\ &\leq CZ^{2-\frac{1-\varepsilon}{m(1-2\varepsilon)}}(t) \left(\frac{1}{\varepsilon}\right)^{\frac{1}{2(\beta-1)}} a(t)^{\frac{1-\varepsilon}{m(1-2\varepsilon)}} \leq CZ^{2-\frac{1-\varepsilon}{m(1-2\varepsilon)}}(t) (a(t))^{\frac{1-\varepsilon}{m(1-2\varepsilon)}}. \end{aligned} \quad (5.57)$$

Since  $B = (2 + \lambda)\operatorname{div} u - P$ , estimate (5.38) provides:

$$\begin{aligned} \|\operatorname{div} u\|_{L^{\frac{4m}{2m-1}}(\Omega)}^2 &= \left\| \frac{B}{2 + \lambda} + \frac{P}{2 + \lambda} \right\|_{L^{\frac{4m}{2m-1}}(\Omega)}^2, \\ &\leq C \left( \left\| \frac{B}{2 + \lambda} \right\|_{L^{\frac{4m}{2m-1}}(\Omega)}^2 + 1 \right). \end{aligned} \quad (5.58)$$

We can now deal with the right-hand side of (5.58) as follows:

$$\begin{aligned} \left\| \frac{B}{2 + \lambda} + \frac{P}{2 + \lambda} \right\|_{L^{\frac{4m}{2m-1}}(\Omega)}^2 &\leq \left( \int_{\Omega} \frac{|B|^{\frac{2m(2m-3)}{(m-1)(2m-1)}}}{2 + \lambda} |B|^{\frac{2m}{(m-1)(2m-1)}} dx \right)^{1-\frac{1}{2m}}, \\ &\leq \|B\|_{L^{2m}(\Omega)}^{\frac{1}{m-1}} \left( \int_{\Omega} \frac{|B|^2}{(2 + \lambda)^{\frac{(m-1)(2m-1)}{m(2m-3)}}} dx \right)^{\frac{2m-3}{2m-2}} \leq \|B\|_{L^{2m}(\Omega)}^{\frac{1}{m-1}} \left( \int_{\Omega} \frac{|B|^2}{(2 + \lambda)} dx \right)^{\frac{2m-3}{2m-2}}, \\ &\leq \|B\|_{L^{2m}(\Omega)}^{\frac{1}{m-1}} (Z(t))^{\frac{2m-3}{2m-2}} = \|B\|_{L^{2m}(\Omega)}^{\frac{1}{m-1}} (Z(t))^{2-\frac{1}{m-1}}. \end{aligned}$$

Thus,

$$\|\operatorname{div} u\|_{L^{\frac{4m}{2m-1}}(\Omega)}^2 \leq C(1 + (Z(t))^{2-\frac{1}{m-1}} \|B\|_{L^{2m}(\Omega)}^{\frac{1}{m-1}}). \quad (5.59)$$

Using estimates (5.57) and (5.59), from (5.56) we have:

$$I_2 \leq C \|B\|_{L^{2m}(\Omega)} (1 + Z^{2-\frac{1-\varepsilon}{m(1-2\varepsilon)}}(t) (a(t))^{\frac{1-\varepsilon}{m(1-2\varepsilon)}} + Z^{2-\frac{1}{m-1}}(t) \|B\|_{L^{2m}(\Omega)}^{\frac{1}{m-1}}).$$

Using estimate (5.52) for  $\|B\|_{L^{2m}(\Omega)}$ , we finally get:

$$\begin{aligned} I_2 &\leq C \left( (Z(t))^{3-\frac{1-\varepsilon}{m(1-2\varepsilon)}} (a(t))^{\frac{1-\varepsilon}{m(1-2\varepsilon)}} + Z(t) + Z^3(t) \right. \\ &\quad \left. + m^{\frac{1}{2}} \left(\frac{m}{\varepsilon}\right)^{\frac{1-s}{2(\beta-1)}} Z^{2+s-\frac{1-\varepsilon}{m(1-2\varepsilon)}}(t) (a(t))^{1-s+\frac{1-\varepsilon}{m(1-2\varepsilon)}} \right. \\ &\quad \left. + m^{\frac{1}{2}} \left(\frac{m}{\varepsilon}\right)^{\frac{m(1-s)}{2(\beta-1)(m-1)}} Z^{2+\frac{ms-1}{m-1}}(t) (a(t))^{\frac{m(1-s)}{m-1}} + m^{\frac{1}{2}} \left(\frac{m}{\varepsilon}\right)^{\frac{1-s}{2(\beta-1)}} Z^s(t) (a(t))^{1-s} \right). \end{aligned} \quad (5.60)$$

Using Young's inequality, we treat the summand in (5.60) as follows:

$$\begin{aligned}
C(Z(t))^{3-\frac{1-\varepsilon}{m(1-2\varepsilon)}}(a(t))^{\frac{1-\varepsilon}{m(1-2\varepsilon)}} &\leq \delta a^2(t) + C(1 + Z^2(t))^2, \\
C(Z(t) + Z^3(t)) &\leq C(1 + Z^2(t))^2, \\
Cm^{\frac{1}{2}}\left(\frac{m}{\varepsilon}\right)^{\frac{1-s}{2(\beta-1)}}Z^{2+s-\frac{1-\varepsilon}{m(1-2\varepsilon)}}(t)(a(t))^{1-s+\frac{1-\varepsilon}{m(1-2\varepsilon)}} &\leq \\
\delta a^2(t) + Cm\left(\frac{m}{\varepsilon}\right)^{\frac{1}{\beta-1}}(1 + Z^2(t))^{2+\frac{1-ms+(2ms-1)\varepsilon}{(1+s)m(1-2\varepsilon)-1+\varepsilon}}, & \\
Cm^{\frac{1}{2}}\left(\frac{m}{\varepsilon}\right)^{\frac{1-s}{2(\beta-1)}}Z^s(t)(a(t))^{1-s} &\leq \delta a^2(t) + Cm\left(\frac{m}{\varepsilon}\right)^{\frac{1}{\beta-1}}(1 + Z^2(t)), \\
m^{\frac{1}{2}}\left(\frac{m}{\varepsilon}\right)^{\frac{m(1-s)}{2(\beta-1)(m-1)}}Z^{2+\frac{ms-1}{m-1}}(t)(a(t))^{\frac{m(1-s)}{m-1}} &\leq \\
\delta a^2(t) + Cm\left(\frac{m}{\varepsilon}\right)^{\frac{1}{\beta-1}}(1 + Z^2(t))^{2+\frac{1-ms}{(1+s)m-2}} &
\end{aligned}$$

Here  $\delta$  is a small positive constant to be mentioned below. From inequality (5.60) we derive that:

$$\begin{aligned}
I_2 &\leq \delta a^2(t) + C\left(m\left(\frac{m}{\varepsilon}\right)^{\frac{1}{\beta-1}}(1 + Z^2(t))^{2+\frac{1-ms+(2ms-1)\varepsilon}{(1+s)m(1-2\varepsilon)-1+\varepsilon}}\right. \\
&\quad \left.+ (1 + Z^2(t))^2 + m\left(\frac{m}{\varepsilon}\right)^{\frac{1}{\beta-1}}(1 + Z^2(t)) + m\left(\frac{m}{\varepsilon}\right)^{\frac{1}{\beta-1}}(1 + Z^2(t))^{2+\frac{1-ms}{(1+s)m-2}}\right) \quad (5.61)
\end{aligned}$$

From (5.55) and (5.61), and inequality (5.50) we have:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(Z^2(t) + a^2(t)) &\leq \delta a^2(t) + C(\delta)(1 + Z^2(t))^{2+\frac{\varepsilon}{3-\varepsilon}}\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\beta-1}} \\
&\quad + C(1 + \delta a^2(t)) + CC(\delta)(1 + Z^2(t))^2 + CC(\delta)\left(m\left(\frac{m}{\varepsilon}\right)^{\frac{1}{\beta-1}}(1 + Z^2(t))^{2+\frac{1-ms}{m(s+1)-2}}\right. \\
&\quad \left.+ 4\delta a^2(t) + C(((1 + Z^2(t))^2 + m\left(\frac{m}{\varepsilon}\right)^{\frac{1}{\beta-1}}(1 + Z^2(t))^{2+\frac{1-ms+(2ms-1)\varepsilon}{(1+s)m(1-2\varepsilon)-1+\varepsilon}}\right. \\
&\quad \left.+ m\left(\frac{m}{\varepsilon}\right)^{\frac{1}{\beta-1}}(1 + Z^2(t))^2 + m\left(\frac{m}{\varepsilon}\right)^{\frac{1}{\beta-1}}(1 + Z^2(t))^{2+\frac{1-ms}{m(s+1)-2}})\right). \quad (5.62)
\end{aligned}$$

Choose  $\delta > 0$  such that:

$$5\delta + \delta C = \frac{1}{2}.$$

Since  $s = \frac{(1-\varepsilon)^2}{m-\varepsilon(1-\varepsilon)}$  and  $\varepsilon = 2^{-m}$ ,  $m > 2$ , we have:

$$\frac{1-ms}{m(s+1)-2} \leq 4\varepsilon, \quad \frac{1-ms+(2ms-1)\varepsilon}{(1+s)m(1-2\varepsilon)-1+\varepsilon} \leq 4\varepsilon \quad \text{and} \quad \frac{\varepsilon}{1-3\varepsilon} \leq 4\varepsilon.$$

Then by (5.62) and the fact that  $Z^2(t) \in L^1(0, T)$ , we obtain the inequality with  $0 < \bar{T} < \frac{T}{2}$ :

$$\frac{1}{2}\frac{d}{dt}(1 + Z^2(t)) + a^2(t) \leq m\left(\frac{m}{\varepsilon}\right)^{\frac{1}{\beta-1}}(1 + Z^2(t))^{2+4\varepsilon}, \quad (5.63)$$

From (5.63) we have for  $0 \leq t < T$ :

$$\frac{1}{(1 + Z^2(t))^{4\varepsilon}} - \frac{1}{(1 + Z^2(\bar{T}))^{4\varepsilon}} + Cm\varepsilon\left(\frac{m}{\varepsilon}\right)^{\frac{1}{\beta-1}} \geq 0.$$

**Remark 8** *Let us point out that the last inequality is better than in [32] and allows us to assume only  $\beta > 2$ .*

Now, take  $N > 2$  such that:

$$1 - CN\varepsilon\left(\frac{N}{\varepsilon}\right)^{\frac{1}{\beta-1}}(1 + Z^2(\bar{0}))^{4\varepsilon} \geq \frac{1}{2}, \quad \varepsilon = 2^{-N}.$$

Here the fact that  $\beta > 2$  allows to conclude and by this fact improve the results of [32]. We get finally that for  $0 \leq t < T$ :

$$Z^2(t) \leq 2^{2N-2}(1 + Z^2(0)) - 1, \quad t \in [0, T]. \quad (5.64)$$

Now, from inequality (5.63) we get moreover that:

$$\int_0^T a^2(t)dt \leq C. \quad (5.65)$$

Now by estimate (5.38) for the density, there exists a positive constant  $C$  depending continuously on the data of the problem and such that:

$$\begin{aligned} \sup_{0 < t < T} \int_{\Omega} \left( (\operatorname{curl} u)^2 + \frac{1}{2 + \lambda(\rho)} \left( (2 + \lambda(\rho)) \operatorname{div} u - P(\rho) \right)^2 \right) (t, x) dx &\leq C, \\ \sup_{0 < t < T} \int_{\Omega} \left( (\operatorname{curl} u)^2 + (2 + \lambda(\rho)) (\operatorname{div} u)^2 \right) (t, x) dx &\leq C, \\ \int_0^T \int_{\Omega} \frac{(A_y + B_x)^2 + (-A_x + B_y)^2}{\rho} dx dy &\leq C. \end{aligned} \quad (5.66)$$

The rest of the proof follows exactly the same lines than in [32] and then we refer to [32].

## 6 Proof of theorem 1.2

The existence part of the theorem is proved by an iterative method. We define a sequence  $(q^n, m^n)$  such that:

$$\begin{cases} \partial_t q^0 - \mu \Delta q^0 + \operatorname{div} m^0 = 0, \\ \partial_t m^0 - \mu \Delta m^0 + r m^0 = 0, \\ (q^0, m^0) = (q_0, m_0). \end{cases}$$

Assuming that  $(q^n, m^n)$  is in  $E_T$  with:

$$E_T = (\tilde{C}_T(B_{2,1}^{\frac{N}{2}}) \cap L_T^1(B_{2,1}^{\frac{N}{2}+2})) \times (\tilde{C}_T(B_{2,1}^{\frac{N}{2}-1}) \cap L_T^1(B_{2,1}^{\frac{N}{2}+1} \cap B_{2,1}^{\frac{N}{2}-1}))^N,$$

we define then  $q^{n+1} = q^0 + \bar{q}^{n+1}$ ,  $m^{n+1} = m^0 + \bar{m}^{n+1}$  such that  $(\bar{q}_{n+1}, \bar{m}_{n+1})$  be the solution of the following system:

$$\begin{cases} \partial_t \bar{q}^{n+1} - \mu \Delta \bar{q}^{n+1} + \operatorname{div} \bar{m}^{n+1} = 0, \\ \partial_t \bar{m}^{n+1} - \mu \Delta \bar{m}^{n+1} + r \bar{m}^{n+1} = G^n, \\ (\bar{q}^{n+1}, \bar{m}^{n+1})|_{t=0} = (0, 0), \end{cases}$$

with:

$$G^n = -\operatorname{div} \left( \frac{m^n}{h^n} \otimes m^n \right)$$

We also set:  $h^n = q^n + 1$ .

**1) First step, uniform bounds:**

Let  $\varepsilon$  be a small parameter and by proposition 3.4, we have for any  $T > 0$ :

$$\begin{aligned} \|q^0\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}}) \cap L_T^1(B_{2,1}^{\frac{N}{2}+2})} &\leq C \|q_0\|_{B_{2,1}^{\frac{N}{2}}}, \\ \|m^0\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}-1}) \cap L_T^1(B_{2,1}^{\frac{N}{2}-1} \cap B_{2,1}^{\frac{N}{2}+1})} &\leq C \|m_0\|_{B_{2,1}^{\frac{N}{2}-1}}. \end{aligned} \quad (6.67)$$

We are going to show by induction that for  $\varepsilon > 0$  small enough:

$$(\mathcal{P}_n) \quad \|(\bar{q}^n, \bar{m}^n)\|_{F_T} \leq \varepsilon.$$

As  $(\bar{q}_0, \bar{m}_0) = (0, 0)$  the result is true for  $n = 0$ . We suppose now  $(\mathcal{P}_n)$  true and we are going to show  $(\mathcal{P}_{n+1})$ .

To begin with we are going to show that  $1 + q^n$  is positive. Indeed we have:  $h^0 = h_1^0 + h_2^0$  such that:

$$\begin{aligned} \partial_t h_1^0 - \mu \Delta h_1^0 &= 0, \\ (h_1^0)_{/t=0} &= h_0. \end{aligned}$$

and:

$$\begin{aligned} \partial_t h_2^0 - \mu \Delta h_2^0 &= -\operatorname{div} m^0, \\ (h_2^0)_{/t=0} &= h_0. \end{aligned}$$

By proposition (3.4) and (6.67) we have for any  $T > 0$ :

$$\|h_2^0\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq C \|m_0\|_{B_{2,1}^{\frac{N}{2}-1}}. \quad (6.68)$$

By maximum principle, we have for any  $t > 0$ :

$$h_1^0(t, x) \geq \min_{x \in \mathbb{R}^N} h_0(x) \geq c > 0.$$

We deduce that for  $\eta = \|m_0\|_{B_{2,1}^{\frac{N}{2}-1}}$  (at least inferior to  $\frac{c}{4C}$  with the  $C$  of (6.68)) small enough and any  $t > 0$ :

$$h^0(t, x) / g e q \frac{3c}{4} > 0,$$

and

$$q^0(t, x) \geq \frac{3c}{4} - 1.$$

and by definition of  $q^n$  and the assumption  $\cap P_n$  that:

$$q^n(t, x) \geq \frac{3c}{4} - 1 - \varepsilon.$$

In particular for  $\varepsilon$  small enough at least  $\varepsilon \leq \frac{c}{4}$ , we deduce that:

$$h^n = 1 + q^n \geq \frac{c}{2} > 0. \quad (6.69)$$

In order to bound  $(\bar{q}^n, \bar{m}^n)$  in  $E_T$ , we shall use proposition 3.4 and in particular estimating  $G^n$  in  $L_T^1(B_{2,1}^{\frac{N}{2}-1})$ . By using proposition 3.2, (6.69) and lemma4, we obtain:

$$\begin{aligned}
& \|\operatorname{div}\left(\frac{m^n}{h^n} \otimes m^n\right)\|_{L_T^1(B_{2,1}^{\frac{N}{2}-1})} \leq \left\|\frac{m^n}{h^n} \otimes m^n\right\|_{L_T^1(B_{2,1}^{\frac{N}{2}-1})}, \\
& \leq C\|m^n\|_{L_T^2(B_{2,1}^{\frac{N}{2}})}^2 \left(\frac{1}{1+q^n} - 1\right)_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} + 1, \\
& \leq C(\|m^0\|_{L_T^2(B_{2,1}^{\frac{N}{2}})}^2 + \|\bar{m}^n\|_{L_T^2(B_{2,1}^{\frac{N}{2}})})^2 (1 + C(\|\frac{1}{h^n}\|_{L_T^\infty})) (\|q^n\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} + 1), \\
& \leq C(\|m^0\|_{L_T^2(B_{2,1}^{\frac{N}{2}})}^2 + \|\bar{m}^n\|_{L_T^2(B_{2,1}^{\frac{N}{2}})})^2 (\|\bar{q}^n\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} + \|q^0\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} + 1),
\end{aligned} \tag{6.70}$$

Therefore by using (6.70), the proposition 3.4 and  $(\mathcal{P}_n)$  we obtain for any  $T > 0$ :

$$\begin{aligned}
\|(\bar{q}^{n+1}, \bar{m}^{n+1})\|_{F_T} & \leq C(\|m^0\|_{L_T^2(B_{2,1}^{\frac{N}{2}})}^2 + \varepsilon)^2 (\varepsilon + \|q^0\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} + 1), \\
& \leq C(\eta + \varepsilon)^2 (2 + \|q^0\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})})
\end{aligned} \tag{6.71}$$

By choosing  $\eta = \varepsilon$  and  $\varepsilon \leq \frac{1}{2C(2+\|q^0\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})})}$ , this implies  $(\mathcal{P})_{n+1}$ . We have shown by induction that  $(q^n, m^n)$  is uniformly bounded in  $F_T$  for any  $T > 0$ .

### Second Step: Convergence of the sequence

We shall prove that  $(q^n, m^n)$  is a Cauchy sequence in the Banach space  $F_T$ , hence converges to some  $(q, m) \in F_T$ .

Let:

$$\delta q^n = q^{n+1} - q^n \quad \text{and} \quad \delta m^n = m^{n+1} - m^n.$$

The system verified by  $(\delta q^n, \delta m^n)$  reads:

$$\begin{cases} \partial_t \delta q^n - \mu \Delta \delta q^n + \operatorname{div} \delta m^n = 0, \\ \partial_t \delta m^n - \mu \Delta \delta m^n + r \delta m^n = G^n - G_{n-1}, \\ \delta q^n(0) = 0, \quad \delta m^n(0) = 0. \end{cases}$$

Applying propositions 3.4 and using  $(\mathcal{P}_n)$ , we get for any  $T > 0$ :

$$\begin{aligned}
\|(\delta q^n, \delta m^n)\|_{F_T} & \leq C\|G^n - G_{n-1}\|_{L_T^1(B_{2,1}^{N/2-1})}, \\
& \leq C\left(\left\|\frac{\delta m^n}{h^n} \otimes m^n\right\|_{L_T^1(B_{2,1}^{N/2})} + \left\|\frac{\delta m^n}{h^n} \otimes m^{n-1}\right\|_{L_T^1(B_{2,1}^{N/2})} + \|m^n \otimes m^{n-1}\left(\frac{1}{h^n} - \frac{1}{h^{n-1}}\right)\|_{L_T^1(B_{2,1}^{N/2})}\right).
\end{aligned}$$

By using proposition 3.2 and lemma 4, we get:

$$\|(\delta q^n, \delta m^n)\|_{F_T} \leq C\varepsilon\|(\delta q^{n-1}, \delta m^{n-1})\|_{F_T}.$$

So by taking  $\varepsilon$  enough small we have proved that  $(q^n, m^n)$  is a Cauchy sequence in  $F_T$  which is a Banach space. It implies that  $(q^n, m^n)$  converge to a limit  $(q, m)$  in  $F_T$ . It is easy to verify that  $(q, m)$  is a solution of the system (1.5).

### 3) Uniqueness of the solution:

The proof is similar to the proof of contraction, indeed we need the same type of estimates. Let us consider two solutions in  $E_T$ :  $(q_1, m_1)$  and  $(q_2, m_2)$  of the system (1.5) with the same initial data. With no loss of generality, one can assume that  $(q_1, m_1)$  is the solution found in the previous section. We thus have:

$$(\mathcal{H}) \quad q_1(t, x) \geq -\frac{1}{2}.$$

We note:

$$\delta q = q_2 - q_1, \quad \delta m = m_2 - m_1,$$

which verifies the system:

$$\begin{cases} \partial_t \delta q - \mu \Delta \delta q + \operatorname{div} \delta m = 0, \\ \partial_t \delta m - \mu \Delta \delta m + r \delta m = -\operatorname{div} \left( \frac{m_1}{h_1} \otimes m_1 \right) + \operatorname{div} \left( \frac{m_1}{h_1} \otimes m_1 \right) \end{cases}$$

By using proposition 3.2, 3.4 and lemma 4 on  $[0, T_1]$  with  $0 < T$  we have:

$$\|(\delta q, \delta m)\|_{E_T} \leq A(T) \|(\delta q, \delta m)\|_{E_T},$$

such that for  $T$  small enough  $A(T) \leq \frac{1}{2}$ . We thus obtain:  $\delta q = 0, \delta m = 0$  on  $[0, T]$ . And we repeat the argument in order to prove that:  $\delta q = 0, \delta m = 0$  on  $\mathbb{R}^+$ . This concludes the proof of theorem 1.1.  $\square$

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