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# An extension of the Crouzeix–Raviart space to general meshes with application to quasi-incompressible linear elasticity and Stokes flow

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## Abstract

In this work we introduce a discrete functional space on general polygonal or polyhedral meshes which mimics two important properties of the standard Crouzeix–Raviart space, namely the continuity of mean values at interfaces and the existence of an interpolator which preserves the mean value of the gradient inside each element. The construction borrows ideas from both Cell Centered Galerkin and Hybrid Finite Volume methods. More specifically, the discrete function space is defined from cell and face unknowns by introducing a suitable piecewise affine reconstruction on a pyramidal subdivision of the original mesh. This subdivision is fictitious in the sense that the original mesh is the only one that needs to be manipulated by the end-user. Two applications are considered in which the discrete space plays an important role, namely (i) the design of a locking-free primal (as opposed to mixed) method for quasi-incompressible linear elasticity on general polygonal meshes; (ii) the design of an inf-sup stable method for the Stokes equations on general polygonal or polyhedral meshes for which the velocity approximation is unaffected by the presence of large irrotational body forces. In both cases, the relation between the proposed method and classical finite volume as well as finite element methods on standard meshes is investigated. Finally, it is shown how similar ideas can be exploited to mimic key properties of the lowest-order Raviart–Thomas space on general polygonal or polyhedral meshes.

**Mathematics Subject Classification.** 65N08, 65N30, 74B05, 76D07

## 1 Introduction

In the context of industrial simulators, lowest-order methods for the discretization of diffusive problems on general polygonal or polyhedral meshes have received an increasing attention over the last few years. The reasons are multifold. On the one hand, allowing general polyhedral elements may ease the discretization of complex domains, while supporting nonmatching interfaces is a key ingredient for nonconforming  $h$ -adaptivity. On the other hand, disposing of discretization methods applicable to general meshes is mandatory whenever the user cannot adapt the mesh to the needs of the numerical scheme. This is the case, e.g., in the context

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of computational geosciences, where the discretization of the subsoil is developed in a separate stage, and is focused on integrating physical and geometrical data resulting from the seismic analysis. Fairly general meshes can thus be encountered, featuring, e.g., nonmatching interfaces corresponding to geological faults or general polyhedral elements resulting from the degeneration of hexahedral cells in eroded layers. Polyhedral elements may also be present in near wellbore regions, where the use of radial meshes can be prompted by (qualitative) a priori knowledge of the solution. Nonconforming  $h$ -refinement can also appear at specific locations where the resolution needs to be increased or when moving fronts are present; cf., e.g., Chainais-Hillairet, Enchéry, and Mamaghani [13]. In this context, the use of lowest-order methods is justified both by the inherent uncertainty associated to physical data and by the need to keep computational costs within affordable bounds.

Among the methods that have appeared in recent years, the most directly related to the present work are the Mimetic Finite Difference (MFD) method of Brezzi, Lipnikov et al. [11, 9, 10], the Hybrid Finite Volume (HFV) method of Eymard, Gallouët and Herbin [24, 25], and the Mixed Finite Volume (MFV) method of Droniou and Eymard [21]. The close relation between these methods has recently been investigated in [22]. The main result of the present work is the construction of a discrete space of piecewise affine functions which extends two important properties of the classical Crouzeix–Raviart space [15] to general meshes, namely,

- (CR1) the continuity of mean values at interfaces. Since we deal with piecewise affine functions, this property is equivalent to the continuity at the faces barycenters;
- (CR2) the existence of an interpolator which preserves the mean value of the gradient inside each element. In the context of quasi-incompressible linear elasticity, this turns out to be a key property to accurately approximate nontrivial solenoidal fields.

Let  $\mathcal{K}_h$  denote a general polygonal or polyhedral mesh of the bounded domain  $\Omega$  matching the regularity requirements discussed in Section 2. In the spirit of Cell Centered Galerkin (ccG) methods [16], the discrete space is constructed in three steps:

- (i) we fix the vector space  $\mathbb{V}_h$  of face- and cell-centered degrees of freedom (DOFs) on  $\mathcal{K}_h$ ;
- (ii) we define a discrete gradient reconstruction operator  $\mathfrak{G}_h$  acting on  $\mathbb{V}_h$ . The reconstructed gradient is piecewise constant on a fictitious pyramidal submesh  $\mathcal{P}_h$  obtained by subdividing each element using one interior point (the cell center), and it results from the sum of two terms: a consistent part depending on face unknowns only plus a subgrid correction involving both face- and cell-centered DOFs. The continuity of mean values at interfaces is ensured by finely tuning the latter contribution;
- (iii) we define an affine reconstruction operator  $\mathfrak{R}_h$  acting on  $\mathbb{V}_h$  which maps every vector of DOFs on a broken affine function on  $\mathcal{P}_h$ . This function is obtained by perturbing the (unique) face unknown associated to each pyramid with a linear correction based on the discrete gradient  $\mathfrak{G}_h$ . The discrete space is then defined as

$$\mathfrak{CR}(\mathcal{K}_h) := \mathfrak{R}_h(\mathbb{V}_h) \subset \mathbb{P}_d^1(\mathcal{P}_h),$$

with  $\mathbb{P}_d^1(\mathcal{P}_h)$  space of broken affine functions on  $\mathcal{P}_h$ , cf. (4).

The pyramidal submesh is fictitious in the sense that all the relevant geometric information can be computed on the primal mesh, which is therefore the only one that needs to be described and manipulated by the end-user. In Appendix A it is shown how similar ideas can be used to construct a  $\mathbf{H}(\text{div}; \Omega)$ -conforming discrete space on general meshes which mimics two key

properties of the lowest-order Raviart–Thomas space on matching simplicial meshes, namely the (full) continuity of normal values at interfaces and the approximation of vector-valued fields.

The first application we consider is linear elasticity, which was also the original motivation for this work. Our main result is the design of a primal (as opposed to mixed) discretization method based on the  $\mathfrak{CA}(\mathcal{K}_h)$  space which is locking-free in the quasi-incompressible limit in two space dimensions (the required regularity estimate for the continuous solution is still an open problem in higher space dimensions). To the best of our knowledge, our method is the first to achieve this result on general polygonal meshes without resorting to a (possibly more expensive) mixed formulation.

It has been long known that the accuracy of  $H^1$ -conforming lowest-order finite element approximations deteriorates when quasi-incompressible materials are considered, i.e., for large values of the first Lamé parameter  $\lambda$ ; cf., e.g., [7, Section 11.3]. This is a consequence of the fact that the convergence estimates are not uniform in  $\lambda$ , which, in turn, reflects the inability of the discretization space to accurately represent nontrivial solenoidal fields. One classical way of circumventing this problem is the use of a mixed formulation, where the solution is characterized as the saddle point of a Lagrangian functional involving both discrete displacements and stresses. The resulting methods converge uniformly in  $\lambda$ , but they are often computationally more expensive than primal methods where the displacement is the sole unknown. In this context, we recall, e.g., the PEERS method by Arnold, Brezzi and Douglas [4], the  $p$ -version method of Vogelius [33] (which does not directly approximate the stresses, but whose study relies on a stability condition for a saddle point problem), the mixed method of Stenberg [31], the nonconforming methods of Falk [26], and the mixed methods of Chavan, Lamichhane and Wohlmuth [14], and Lamichhane and Stephan [30]. All these methods require matching triangular or quadrilateral meshes. General meshes matching regularity assumptions that are similar to the ones proposed in this work have been recently considered by Beirão Da Veiga [5], who introduces a mixed MFD method which does not lock in the quasi-incompressible limit.

The problem of locking has also been addressed without resorting to mixed formulations, and several methods can be found in the literature. In this work we take inspiration, in particular, from the classical paper of Brenner and Sung [8], where the authors propose a locking-free method on matching triangular meshes based on the nonconforming Crouzeix–Raviart element [15]. Here, the properties (CR1) and (CR2) play an important role in deriving an error estimate uniform in  $\lambda$ . Another source of inspiration is the work of Hansbo and Larson [28, 29], where a locking-free Discontinuous Galerkin (dG) method on matching triangular meshes is analyzed; see also Di Pietro and Nicaise [20] for an extension to composite materials. In this case, the coercivity of the method is ensured by a least-square penalization of interface jumps. In our method, a similar device is required in the case of mixed boundary conditions to invoke Korn’s inequality on broken polynomial spaces [6] in the stability proof. For the pure displacement problem, the more convenient and naturally coercive Navier–Cauchy formulation allows to circumvent the use of Korn’s inequality.

An interesting property of our method that deserves to be mentioned is that, up to a suitable treatment of the right-hand side, it is possible to recover a local conservation property similar to those encountered in finite volume methods. In this case, the face unknowns can be interpreted as Lagrange multipliers of the flux continuity constraint. Yet another treatment of the right-hand side allows to recover the Crouzeix–Raviart method of Brenner and Sung [8] on matching simplicial meshes.

The second application is inspired by the recent work of Galvin, Linke, Rebholz and Wilson [27], where the authors study the effect of large irrotational body forces on the numerical solution of the Stokes equations. For the purposes of the present work, we consider the case when a Helmholtz decomposition of the source term is available. At the continuous level, it is

known that the velocity is unaffected by the irrotational component of body forces. A key point to mimic this property at the discrete level is to have a tight control of the discrete divergence of the velocity approximation as well as to carefully design the discretization of body forces. This requires, in particular, an appropriate choice of the approximation spaces for the velocity and the pressure. When working with matching simplicial meshes, these requirements can be met using the classical Crouzeix–Raviart element to approximate the velocity components together with a piecewise constant approximation of the pressure. The method proposed in this work extends this formulation to general meshes by approximating the velocity components in the space  $\mathfrak{CA}(\mathcal{K}_h)$ . The main properties of the resulting scheme are inf-sup stability and robustness with respect to irrotational body forces on general polygonal or polyhedral meshes. Also in this case, suitable modifications of the right-hand side allow to establish a link with finite volume and standard finite element methods on matching simplicial meshes.

The paper is organized as follows. In Section 2 we introduce the notation and define the concept of admissible mesh sequence and pyramidal submesh inspired by [17, 16]. The main novelty of this section is the proof that the pyramidal submesh inherits the shape- and contact-regularity properties of the original mesh provided a set of cell centers can be defined for the latter. Section 3 details the construction of the space  $\mathfrak{CA}(\mathcal{K}_h)$  as well as the proofs of the properties (CR1) and (CR2). In Section 4 we present two applications of the space  $\mathfrak{CA}(\mathcal{K}_h)$ , namely the approximation of the linear elasticity problem for quasi-incompressible materials and the discretization of the Stokes equations with large irrotational body forces. Finally, in Appendix A we show how the ideas of Section 3 can be used to construct a discrete space which mimics the properties of the lowest-order Raviart–Thomas space on general polygonal or polyhedral meshes.

## 2 Admissible mesh sequences

Following [17, Chapter 1] and [16, Section 1], we introduce in this section the concept of admissible mesh sequence of a bounded connected polygonal or polyhedral domain  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ . For the sake of brevity, we only give the proofs of the new results, and refer to [17, 16] for further details.

### 2.1 Shape- and contact-regularity

Let  $\mathcal{H} \subset \mathbb{R}_*^+$  denote a countable set having 0 as its unique accumulation point. We consider mesh sequences  $\mathcal{K}_{\mathcal{H}} := (\mathcal{K}_h)_{h \in \mathcal{H}}$  where, for all  $h \in \mathcal{H}$ ,  $\mathcal{K}_h$  denotes a finite collection of nonempty disjoint open polyhedra  $\mathcal{K}_h = \{K\}$  such that  $\bar{\Omega} = \bigcup_{K \in \mathcal{K}_h} \bar{K}$  and  $h = \max_{K \in \mathcal{K}_h} h_K$  ( $h_K$  denotes here the diameter of the element  $K \in \mathcal{K}_h$ ). We say that a hyperplanar closed connected subset  $F$  of  $\bar{\Omega}$  is a mesh face if it has positive  $(d-1)$ -dimensional measure and if either there exist  $K_1, K_2 \in \mathcal{K}_h$  such that  $F \subset \partial K_1 \cap \partial K_2$  (and  $F$  is called an interface) or there exists  $K \in \mathcal{K}_h$  such that  $F \subset \partial K \cap \partial \Omega$  (and  $F$  is called a boundary face). Interfaces are collected in the set  $\mathcal{F}_{\mathcal{K}_h}^i$ , boundary faces in  $\mathcal{F}_{\mathcal{K}_h}^b$  and we let  $\mathcal{F}_{\mathcal{K}_h} := \mathcal{F}_{\mathcal{K}_h}^i \cup \mathcal{F}_{\mathcal{K}_h}^b$ . The diameter of a face  $F \in \mathcal{F}_{\mathcal{K}_h}$ , is denoted by  $h_F$ . Moreover, we set, for all  $K \in \mathcal{K}_h$ ,  $\mathcal{F}_K := \{F \in \mathcal{F}_{\mathcal{K}_h} \mid F \subset \partial K\}$ . According to the context, the notation  $|\cdot|$  is used for the  $d$ - or the  $(d-1)$ -dimensional Lebesgue measure. In the rest of this section, we discuss some fairly general regularity conditions on the mesh sequence  $\mathcal{K}_{\mathcal{H}}$  that allow to prove basic results such as trace and inverse inequalities and polynomial approximation properties.

**Definition 1** (Shape- and contact-regularity). *The mesh sequence  $\mathcal{K}_{\mathcal{H}}$  is shape- and contact-regular if for all  $h \in \mathcal{H}$ ,  $\mathcal{K}_h$  admits a matching simplicial submesh  $\mathcal{T}_h$  such that*

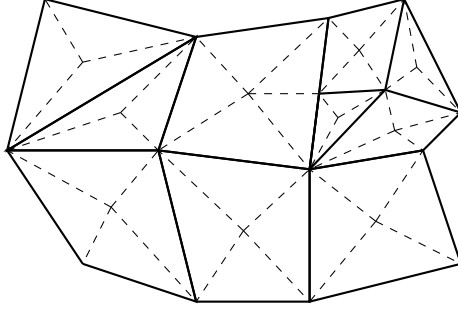


Figure 1: Mesh  $\mathcal{K}_h$  (solid lines) and pyramidal submesh  $\mathcal{P}_h$  (dashed lines)

- (i) Shape-regularity. *There exists a real  $\varrho_1 > 0$  independent of  $h$  such that, for all  $h \in \mathcal{H}$  and all simplex  $T \in \mathcal{T}_h$  of diameter  $h_T$  and inradius  $r_T$ , there holds  $\varrho_1 h_T \leq r_T$ ;*
- (ii) Contact-regularity. *There exists a real  $\varrho_2 > 0$  independent of  $h$  such that, for all  $h \in \mathcal{H}$ , all  $K \in \mathcal{K}_h$ , and all  $T \in \mathcal{T}_K := \{T \in \mathcal{T}_h \mid T \subset K\}$ , there holds  $\varrho_2 h_K \leq h_T$ .*

## 2.2 Cell centers and pyramidal submesh

The discrete space introduced in this work requires to identify a set of points which play a pivotal role in the construction.

**Definition 2** (Cell centers). *The mesh sequence  $\mathcal{K}_{\mathcal{H}}$  admits a set of cell centers if*

- (i) *for all  $h \in \mathcal{H}$  and all  $K \in \mathcal{K}_h$ , there exists a point  $\mathbf{x}_K$  such that  $K$  is star-shaped with respect to  $\mathbf{x}_K$  (the cell center);*
- (ii) *there exists a real  $\varrho_3 > 0$  independent of  $h$  such that for all  $h \in \mathcal{H}$ , all  $K \in \mathcal{K}_h$ , and all  $F \in \mathcal{F}_K$ , denoting by  $d_{K,F}$  the orthogonal distance between  $\mathbf{x}_K$  and  $F$ ,*

$$d_{K,F} \geq \varrho_3 h_K. \quad (1)$$

Let  $\mathcal{K}_{\mathcal{H}}$  denote a mesh sequence which admits a set of cell centers. We define for all  $h \in \mathcal{H}$  the pyramidal submesh

$$\mathcal{P}_h = \{K_F\}_{K \in \mathcal{K}_h, F \in \mathcal{F}_K},$$

where, for all  $K \in \mathcal{K}_h$  and all  $F \in \mathcal{F}_K$ ,  $K_F$  denotes the open pyramid of apex  $\mathbf{x}_K$  and base  $F$ . An example of mesh  $\mathcal{K}_h$  and associated pyramidal submesh  $\mathcal{P}_h$  is provided in Figure 1. Each element of  $\mathcal{P}_h$  is associated to a unique element  $K \in \mathcal{K}_h$  and a unique face  $F \in \mathcal{F}_K$ . When this link is irrelevant, the generic element of  $\mathcal{P}_h$  is noted  $P$  instead of  $K_F$ . The pyramids  $\{K_F\}_{K \in \mathcal{K}_h, F \in \mathcal{F}_K}$  are nondegenerated owing to assumption (1). In the two-dimensional case,  $\mathcal{P}_h$  is a matching simplicial mesh while, in higher dimension, it is in general not simplicial. Owing to the planarity of faces, there holds for all  $K \in \mathcal{K}_h$  and all  $F \in \mathcal{F}_K$ ,

$$|K_F| = \frac{|F| d_{K,F}}{d}. \quad (2)$$

The set of faces of  $\mathcal{P}_h$  (including the mesh faces in  $\mathcal{F}_{\mathcal{K}_h}$  as well as the lateral faces of the pyramids) is denoted by  $\mathcal{F}_{\mathcal{P}_h}$  and we let  $\mathcal{F}_{\mathcal{P}_h}^i := \mathcal{F}_{\mathcal{P}_h} \setminus \mathcal{F}_{\mathcal{K}_h}^b$  and  $\mathcal{F}_{\mathcal{P}_h}^b := \mathcal{F}_{\mathcal{K}_h}^b$ . Additionally, for all  $P \in \mathcal{P}_h$ , we introduce the set  $\mathcal{F}_P := \{F \in \mathcal{F}_{\mathcal{P}_h} \mid F \subset \partial P\}$ . The following result shows that the pyramidal mesh sequence  $\mathcal{P}_{\mathcal{H}} := (\mathcal{P}_h)_{h \in \mathcal{H}}$  inherits the shape- and contact-regularity properties of the original mesh sequence  $\mathcal{K}_{\mathcal{H}}$ .

**Lemma 3** (Shape- and contact-regularity of the pyramidal submesh). *Let  $\mathcal{K}_{\mathcal{H}}$  admit a set of cell centers. Then, if  $\mathcal{K}_{\mathcal{H}}$  is shape- and contact-regular, the same holds for  $\mathcal{P}_{\mathcal{H}}$ .*

*Proof.* Let  $h \in \mathcal{H}$ . By assumption,  $\mathcal{K}_h$  admits a matching simplicial submesh  $\mathcal{T}_h$ . A matching simplicial submesh  $\mathfrak{T}_h$  of the pyramidal submesh  $\mathcal{P}_h$  can be constructed as follows: For all  $K \in \mathcal{K}_h$  and all  $F \in \mathcal{F}_K$  (i) a  $(d-1)$ -simplicial mesh  $\mathfrak{S}_F$  of  $F$  is obtained taking the trace of  $\mathcal{T}_h$  on  $F$ ; (ii) a  $d$ -simplicial mesh  $\mathfrak{T}_{K_F}$  of the pyramid  $K_F$  is then obtained connecting the (hyperplanar) elements in  $\mathfrak{S}_F$  to the cell center. A matching simplicial submesh of  $\mathcal{P}_h$  is obtained by setting

$$\mathfrak{T}_h := \bigcup_{K \in \mathcal{K}_h, F \in \mathcal{F}_K} \mathfrak{T}_{K_F}.$$

(i) *Shape-regularity.* We have to prove that there exists a real  $\varrho'_1$  independent of  $h$  such that  $\varrho'_1 h_T \leq r_T$  for all  $T \in \mathfrak{T}_h$ . Let  $K_F \in \mathcal{P}_h$  and  $T \in \mathfrak{T}_{K_F}$  be given. Denoting by  $r_T$  the inradius of  $T$ , letting  $\mathcal{A}_T := |\partial T|$  and  $\sigma := \partial T \cap F$ , there holds  $d|T| = r_T \mathcal{A}_T = |\sigma| d_{K,F}$ , hence

$$r_T = \frac{|\sigma| d_{K,F}}{\mathcal{A}_T}. \quad (3)$$

Since the  $(d-1)$ -dimensional measure of each face of  $T$  is bounded by  $h_K^{d-1}$  and  $T$  has  $(d+1)$  faces, there holds  $\mathcal{A}_T \leq (d+1)h_K^{d-1}$ . Let now  $S \in \mathcal{T}_h$  be the unique simplex such that  $\partial S \cap F = \sigma$  and  $S \subset K$ . Denoting by  $r_\sigma$  the inradius of  $\sigma$  and observing that  $r_\sigma \geq r_S$  by a simple argument based on the Pythagorean theorem, it is inferred  $|\sigma| \geq |\mathfrak{B}_{d-1}| r_\sigma^{d-1} \geq |\mathfrak{B}_{d-1}| r_S^{d-1} \geq |\mathfrak{B}_{d-1}| (\varrho_1 \varrho_2)^{d-1} h_K^{d-1}$  owing to the shape- and contact-regularity of  $\mathcal{K}_h$  ( $\mathfrak{B}_{d-1}$  denotes here the  $(d-1)$ -dimensional unit ball). Plugging these inequalities into (3), it is inferred

$$r_T \geq \frac{|\mathfrak{B}_{d-1}| (\varrho_1 \varrho_2)^{d-1}}{d+1} d_{K,F} \geq \varrho_3 \frac{|\mathfrak{B}_{d-1}| (\varrho_1 \varrho_2)^{d-1}}{d+1} h_T,$$

and the conclusion follows with  $\varrho'_1 = \varrho_3 |\mathfrak{B}_{d-1}| (\varrho_1 \varrho_2)^{d-1} / (d+1)$ .

(ii) *Contact-regularity.* We have to prove that there exists a real  $\varrho'_2$  independent of  $h$  such that, for all  $K_F \in \mathcal{P}_h$  and all  $T \in \mathfrak{T}_{K_F}$ ,  $\varrho'_2 h_{K_F} \leq h_T$ . To this end, we invoke (1) to infer, for all  $K_F \in \mathcal{P}_h$  and all  $T \in \mathfrak{T}_{K_F}$ ,  $h_T \geq d_{K,F} \geq \varrho_3 h_K \geq \varrho_3 h_{K_F}$ , where  $h_{K_F}$  denotes the diameter of  $K_F$ . The conclusion follows with  $\varrho'_2 = \varrho_3$ .  $\square$

### 2.3 Broken function spaces

For  $\mathcal{S}_h \in \{\mathcal{K}_h, \mathcal{P}_h\}$  and an integer  $k \geq 0$ , we introduce the broken polynomial space

$$\mathbb{P}_d^k(\mathcal{S}_h) := \{v \in L^2(\Omega) \mid \forall S \in \mathcal{S}_h, v|_S \in \mathbb{P}_d^k(S)\}, \quad (4)$$

where  $\mathbb{P}_d^k$  denotes the space of polynomial functions of total degree at most  $k$ . Broken polynomial spaces are a special instance of broken Sobolev spaces: For an integer  $l \geq 1$ ,

$$H^l(\mathcal{S}_h) := \{v \in L^2(\Omega) \mid \forall S \in \mathcal{S}_h, v|_S \in H^l(S)\}.$$

For further use, we define the broken gradient denoted by  $\nabla_h$  and acting on functions  $v \in H^1(\mathcal{S}_h)$  such that  $(\nabla_h v)|_S := \nabla(v|_S)$ , for all  $S \in \mathcal{S}_h$ . We also define the broken divergence of a field  $\mathbf{v} \in H^1(\mathcal{S}_h)^d$  denoted by  $\nabla_h \cdot \mathbf{v}$  and the broken symmetric tensor-gradient  $\underline{\varepsilon}_h(\mathbf{v})$  respectively as the trace and as the symmetric part of the broken tensor-gradient  $\nabla_h \mathbf{v}$ . The shape- and contact-regularity of the mesh sequences  $\mathcal{K}_{\mathcal{H}}$  and  $\mathcal{P}_{\mathcal{H}}$  are instrumental to prove the following result, see [17, Lemmata 1.46 and 1.49].

**Lemma 4** (Trace inequalities). *Let  $\mathcal{K}_{\mathcal{H}}$  be a shape- and contact-regular mesh sequence admitting a set of cell centers and denote by  $\mathcal{P}_{\mathcal{H}}$  the corresponding sequence of pyramidal submeshes. Then, there exist two reals  $C_{\text{tr}}$  and  $C_{\text{tr},c}$  independent of  $h$  (but depending on the polynomial degree) such that, for all  $h \in \mathcal{H}$  with  $\mathcal{S}_h \in \{\mathcal{K}_h, \mathcal{P}_h\}$ ,*

$$\forall v_h \in \mathbb{P}_d^k(\mathcal{P}_h), \forall P \in \mathcal{P}_h, \forall F \in \mathcal{F}_P, \quad \|v_h\|_{L^2(F)} \leq C_{\text{tr}} h_F^{-1/2} \|v_h\|_{L^2(P)}, \quad (5)$$

$$\forall v \in H^1(\mathcal{S}_h), \forall S \in \mathcal{S}_h, \forall F \in \mathcal{F}_S, \quad \|v\|_{L^2(F)} \leq C_{\text{tr},c} \left( h_S^{-1} \|v\|_{L^2(S)}^2 + h_S |v|_{H^1(S)}^2 \right)^{1/2}. \quad (6)$$

We next define some trace operators commonly used in the context of nonconforming finite element methods. For every interface  $F \in \mathcal{F}_{\mathcal{S}_h}^i$ ,  $\mathcal{S}_h \in \{\mathcal{K}_h, \mathcal{P}_h\}$ , we introduce an arbitrary but fixed ordering of the elements  $S_1$  and  $S_2$  such that  $F \subset \partial S_1 \cap \partial S_2$  and let  $\mathbf{n}_F := \mathbf{n}_{S_1, F} = -\mathbf{n}_{S_2, F}$ , where  $\mathbf{n}_{S_i, F}$ ,  $i \in \{1, 2\}$ , denotes the unit normal to  $F$  pointing out of  $S_i$ . The orientation of the normal remains coherent when  $F \in \mathcal{F}_{\mathcal{K}_h}^i$  is regarded as an element of  $\mathcal{F}_{\mathcal{P}_h}^i$ . For all  $S \in \mathcal{S}_h$ , we also introduce the symbol  $\mathbf{n}_S$  to denote the vector-valued field such that  $\mathbf{n}_{S|F} = \mathbf{n}_{S, F}$  for all  $F \in \mathcal{F}_S$ . On boundary faces  $F \in \mathcal{F}_{\mathcal{P}_h}^b$ ,  $\mathbf{n}_F$  denotes the unit normal pointing out of  $\Omega$ .

**Definition 5** (Jumps and averages). *Let  $v$  denote a scalar-valued function defined on  $\Omega$  smooth enough to admit on all  $F \in \mathcal{F}_{\mathcal{P}_h}$  a possibly two-valued trace. Then, if  $F \in \mathcal{F}_{\mathcal{P}_h}^i$  with  $F \subset \partial P_1 \cap \partial P_2$ , the jump and average of  $v$  at  $F$  are defined for a.e.  $\mathbf{x} \in F$  as*

$$\llbracket v \rrbracket_F(\mathbf{x}) := v|_{P_1}(\mathbf{x}) - v|_{P_2}(\mathbf{x}), \quad \{v\}_F(\mathbf{x}) := \frac{1}{2} (v|_{P_1}(\mathbf{x}) + v|_{P_2}(\mathbf{x})).$$

If  $F \in \mathcal{F}_{\mathcal{P}_h}^b$  with  $F = \partial P \cap \partial \Omega$ , we conventionally set  $\llbracket v \rrbracket_F(\mathbf{x}) = \{v\}_F(\mathbf{x}) := v|_P(\mathbf{x})$ .

When applied to vector-valued functions, both the jump and average operators act component-wise. Whenever no confusion can arise, we omit the subscript  $F$  and the variable  $\mathbf{x}$  and simply write  $\llbracket v \rrbracket$ ,  $\{v\}$ .

## 2.4 Polynomial approximation

The last requirement on the mesh sequence is that regular functions can be optimally approximated by broken polynomial functions.

**Definition 6** (Optimal polynomial approximation). *The mesh sequence  $\mathcal{K}_{\mathcal{H}}$  has optimal polynomial approximation properties if for all  $h \in \mathcal{H}$ , all  $K \in \mathcal{K}_h$ , all polynomial degree  $k \geq 0$ , all  $s \in \{0, \dots, k+1\}$  and all  $v \in H^s(K)$ , there holds*

$$|v - \Pi_h^k v|_{H^m(K)} \leq C_{\text{app}} h_K^{s-m} |v|_{H^s(K)} \quad \forall m \in \{0, \dots, s\}, \quad (7)$$

where  $C_{\text{app}}$  is independent of both  $K$  and  $h$ , and  $\Pi_h^k$  denotes the  $L^2$ -orthogonal projector on  $\mathbb{P}_d^k(\mathcal{K}_h)$ .

A sufficient condition for a shape- and contact-regular mesh sequence  $\mathcal{K}_{\mathcal{H}}$  to have optimal approximation properties is that every element  $K \in \mathcal{K}_h$  is star-shaped with respect to a ball whose diameter is uniformly comparable to  $h_K$ , cf. [7, Chapter 4]. Note that under this assumption, the mesh clearly admits a set of cell centers. In what follows, we only need polynomial approximation for polynomial degrees 0 and 1.

**Definition 7** (Admissible mesh sequence). *We say that the mesh sequence  $\mathcal{K}_{\mathcal{H}}$  is admissible if it is shape- and contact-regular, it admits a set of cell centers, and it possesses optimal polynomial approximation properties.*



The regularity of an admissible mesh sequence is described by the reals  $\varrho_1$ ,  $\varrho_2$ , and  $\varrho_3$ , which are therefore collectively referred to as mesh regularity parameters. Admissible mesh sequences include general polyhedral discretizations with possibly nonmatching interfaces, see Figure 1 for an example.

**Proposition 8** (Approximation on mesh faces). *For an admissible mesh sequence  $\mathcal{K}_{\mathcal{H}}$  there holds for all  $h \in \mathcal{H}$ , all  $K \in \mathcal{K}_h$ , all  $F \in \mathcal{F}_K$ , all polynomial degree  $k \geq 0$ , all  $s \in \{0, \dots, k+1\}$ , and all  $v \in H^s(K)$ ,*

$$\|v - \Pi_h^k v\|_{L^2(F)} \leq C h_K^{s-1/2} |v|_{H^s(K)},$$

where  $C = C_{\text{tr,c}} C_{\text{app}}$  with  $C_{\text{tr,c}}$  defined as in (6) and  $C_{\text{app}}$  as in (7).

*Proof.* The result is an immediate consequence of the trace inequality (6) with  $\mathcal{S}_h = \mathcal{K}_h$  and of the approximation properties of the  $L^2$ -orthogonal projector.  $\square$

### 3 Mimicking the properties of the lowest-order Crouzeix–Raviart space on general meshes

In this section we present a piecewise affine functional space obtained in the spirit of [16] that extends some important properties of the Crouzeix–Raviart space [15] to general polygonal or polyhedral meshes.

#### 3.1 Construction

Our goal is to extend two key properties of the classical Crouzeix–Raviart space to admissible polygonal or polyhedral meshes, namely (i) the continuity of the mean value (or, equivalently, the barycentric value) of discrete functions at interfaces; (ii) the existence of an interpolator that preserves the mean value of the gradient inside each element. In the spirit of ccG methods, the discrete space is obtained in three steps by prescribing (i) the vector space of degrees of freedom  $\mathbb{V}_h$ ; (ii) a gradient reconstruction  $\mathfrak{G}_h$ , i.e., a linear operator which maps every vector of DOFs onto a vector field piecewise constant on  $\mathcal{P}_h$ ; (iii) a piecewise affine reconstruction  $\mathfrak{A}_h$  defined from a point value and a linear perturbation based on the gradient reconstruction which maps every vector of DOFs onto a piecewise affine function on the pyramidal submesh  $\mathcal{P}_h$ . In the spirit of HFV methods [25], the vector space of DOFs contains cell and face unknowns and is defined by

$$\mathbb{V}_h := \left\{ \mathbb{v}_h = ((v_K \in \mathbb{R})_{K \in \mathcal{K}_h}, (v_F \in \mathbb{R})_{F \in \mathcal{F}_{\mathcal{K}_h}}) \in \mathbb{R}^{\mathcal{K}_h} \times \mathbb{R}^{\mathcal{F}_{\mathcal{K}_h}} \right\}.$$

The gradient operator generalizes the one introduced in [24], and is composed of a consistent contribution piecewise constant on the primal mesh  $\mathcal{K}_h$  plus a subgrid correction piecewise constant on the pyramidal submesh  $\mathcal{P}_h$ . More precisely,  $\mathfrak{G}_h : \mathbb{V}_h \rightarrow \mathbb{P}_d^0(\mathcal{P}_h)^d$  realizes the mapping  $\mathbb{V}_h \ni \mathbb{v}_h \mapsto \mathfrak{G}_h(\mathbb{v}_h)$  with

$$\mathfrak{G}_h(\mathbb{v}_h)|_{K_F} = \mathbf{G}_{K_F}(\mathbb{v}_h) := \mathbf{G}_K(\mathbb{v}_h) + \mathbf{R}_{K_F}(\mathbb{v}_h), \quad \forall K \in \mathcal{K}_h, F \in \mathcal{F}_K, \quad (8)$$

where

$$\mathbf{G}_K(\mathbb{v}_h) := \frac{1}{|K|} \sum_{F \in \mathcal{F}_K} |F| v_F \mathbf{n}_{K,F}, \quad \mathbf{R}_{K_F}(\mathbb{v}_h) := \frac{\eta}{d_{K,F}} (v_F - v_K - \mathbf{G}_K(\mathbb{v}_h) \cdot (\bar{\mathbf{x}}_F - \mathbf{x}_K)) \mathbf{n}_{K,F}, \quad (9)$$

and  $\eta > 0$  is a user-dependent parameter. With a slight abuse in notation, the symbols  $\mathbf{G}_{K_F}(\mathbb{v}_h)$ ,  $\mathbf{G}_K(\mathbb{v}_h)$ , and  $\mathbf{R}_{K_F}(\mathbb{v}_h)$  will also be used to denote the corresponding constant fields on  $K_F$ ,  $K$ ,

and  $K_F$  respectively. The choice  $\eta = d^{1/2}$  is proposed in [25] to recover the classical two-point finite volume method on superadmissible meshes; cf. [25, Lemma 2.1]. In our case, the choice  $\eta = d$  is a key ingredient to prove the continuity of mean values at interfaces; cf. Lemma 10. The reconstruction operator  $\mathfrak{R}_h$  realizes the mapping  $\mathbb{V}_h \ni v_h \mapsto \mathfrak{R}_h(v_h) \in \mathbb{P}_d^1(\mathcal{P}_h)$  with

$$\mathfrak{R}_h(v_h)|_{K_F}(\mathbf{x}) = v_F + \mathfrak{G}_h(v_h)|_{K_F} \cdot (\mathbf{x} - \bar{\mathbf{x}}_F), \quad \forall K_F \in \mathcal{P}_h, \forall \mathbf{x} \in K_F, \quad (10)$$

where  $\bar{\mathbf{x}}_F := \langle \mathbf{x} \rangle_F$  denotes the barycenter of  $F \in \mathcal{F}_{\mathcal{P}_h}$  and, for a function  $\varphi$  integrable on  $F$ , we have introduced the notation  $\langle \varphi \rangle_F := \int_F \varphi / |F|$ . By construction, there holds  $\nabla_h \mathfrak{R}_h = \mathfrak{G}_h$ . We are now ready to introduce the discrete space

$$\mathfrak{C}\mathfrak{R}(\mathcal{K}_h) := \mathfrak{R}_h(\mathbb{V}_h). \quad (11)$$

**Remark 9** (Choice of the starting point in (10)). *The affine reconstruction in  $K_F$  is obtained by perturbing the face unknown  $v_F$ , unlike [16], where the cell unknown  $v_K$  is used instead. As a consequence, letting  $v_h := \mathfrak{R}_h(v_h)$ , there holds, for all  $K \in \mathcal{K}_h$  and all  $F \in \mathcal{F}_K$ ,  $v_h|_{K_F}(\bar{\mathbf{x}}_F) = v_F$ . Since the reconstruction is piecewise affine, this implies the continuity of mean values for all  $F \in \mathcal{F}_{\mathcal{K}_h}^i$ .*

### 3.2 Continuity of mean values at interfaces

In this section we extend Remark 9 by proving that the choice  $\eta = d$  in (9) yields the continuity of the mean values (or, equivalently, the barycentric values) of discrete functions across all the interfaces in  $\mathcal{F}_{\mathcal{P}_h}^i$  (including lateral pyramidal faces).

**Lemma 10** (Continuity of mean values at interfaces). *Let  $\mathcal{K}_h$  belong to an admissible mesh sequence and, if  $\mathcal{K}_h$  is not matching simplicial, assume  $\eta = d$  in (9). Then, there holds for all  $v_h \in \mathfrak{C}\mathfrak{R}(\mathcal{K}_h)$ ,*

$$\forall F \in \mathcal{F}_{\mathcal{P}_h}^i, \quad \langle \llbracket v_h \rrbracket \rangle_F = 0.$$

*Proof.* Let  $F \in \mathcal{F}_{\mathcal{P}_h}^i$ ,  $v_h \in \mathbb{V}_h$ , and let  $v_h := \mathfrak{R}_h(v_h) \in \mathfrak{C}\mathfrak{R}(\mathcal{K}_h)$ . We distinguish two cases. (i) If  $F \in \mathcal{F}_{\mathcal{K}_h}^i$  is a face of the primal mesh  $\mathcal{K}_h$ , the fact that  $\langle \llbracket v_h \rrbracket \rangle_F = 0$  is an immediate consequence of Remark 9; (ii) if  $F \in \mathcal{F}_{\mathcal{P}_h}^i \setminus \mathcal{F}_{\mathcal{K}_h}^i$  is a lateral pyramidal face, there exist a unique element  $K \in \mathcal{K}_h$  and two faces  $F_1, F_2 \in \mathcal{F}_K$  such that  $F \subset \partial K_{F_1} \cap \partial K_{F_2}$  (cf. Figure 2a). There holds for  $i \in \{1, 2\}$  (cf. Figure 2b),

$$(\bar{\mathbf{x}}_F - \bar{\mathbf{x}}_{F_i}) \cdot \mathbf{n}_{K, F_i} = (\bar{\mathbf{x}}_F - \mathbf{x}_K) \cdot \mathbf{n}_{K, F_i} + (\mathbf{x}_K - \bar{\mathbf{x}}_{F_i}) \cdot \mathbf{n}_{K, F_i} = \left( \frac{d-1}{d} - 1 \right) d_{K, F_i} = -\frac{d_{K, F_i}}{d},$$

where we have used the fact that  $\bar{\mathbf{x}}_F$  is the barycenter of  $F$  to treat the term  $(\bar{\mathbf{x}}_F - \mathbf{x}_K) \cdot \mathbf{n}_{K, F_i}$ . Using the above result together with (9) it is inferred for  $i \in \{1, 2\}$ ,

$$\alpha_i := \mathbf{R}_{K, F_i}(v_h) \cdot (\bar{\mathbf{x}}_F - \bar{\mathbf{x}}_{F_i}) = -\frac{\eta}{d} (v_{F_i} - v_K - \mathbf{G}_K(v_h) \cdot (\bar{\mathbf{x}}_{F_i} - \mathbf{x}_K)).$$

Using the definition of the jump operator and substituting the expression (10) for the barycentric values  $v_h|_{K_{F_i}}(\bar{\mathbf{x}}_F)$ ,  $i \in \{1, 2\}$ , we obtain

$$\begin{aligned} \langle \llbracket v_h \rrbracket \rangle_F &= v_h|_{K_{F_1}}(\bar{\mathbf{x}}_F) - v_h|_{K_{F_2}}(\bar{\mathbf{x}}_F) \\ &= v_{F_1} - v_{F_2} - \mathbf{G}_K(v_h) \cdot (\bar{\mathbf{x}}_{F_1} - \bar{\mathbf{x}}_{F_2}) + \alpha_1 - \alpha_2 \\ &= \left( 1 - \frac{\eta}{d} \right) (v_{F_1} - v_{F_2} - \mathbf{G}_K(v_h) \cdot (\bar{\mathbf{x}}_{F_1} - \bar{\mathbf{x}}_{F_2})). \end{aligned} \quad (12)$$

If  $\mathcal{K}_h$  is a matching simplicial mesh,  $\mathbf{G}_K(v_h)$  coincides by definition with the standard Crouzeix–Raviart gradient (cf. also Proposition 11), hence the second factor in (12) vanishes, otherwise the assumption  $\eta = d$  is needed to conclude that  $\langle \llbracket v_h \rrbracket \rangle_F = 0$ . This concludes the proof.  $\square$

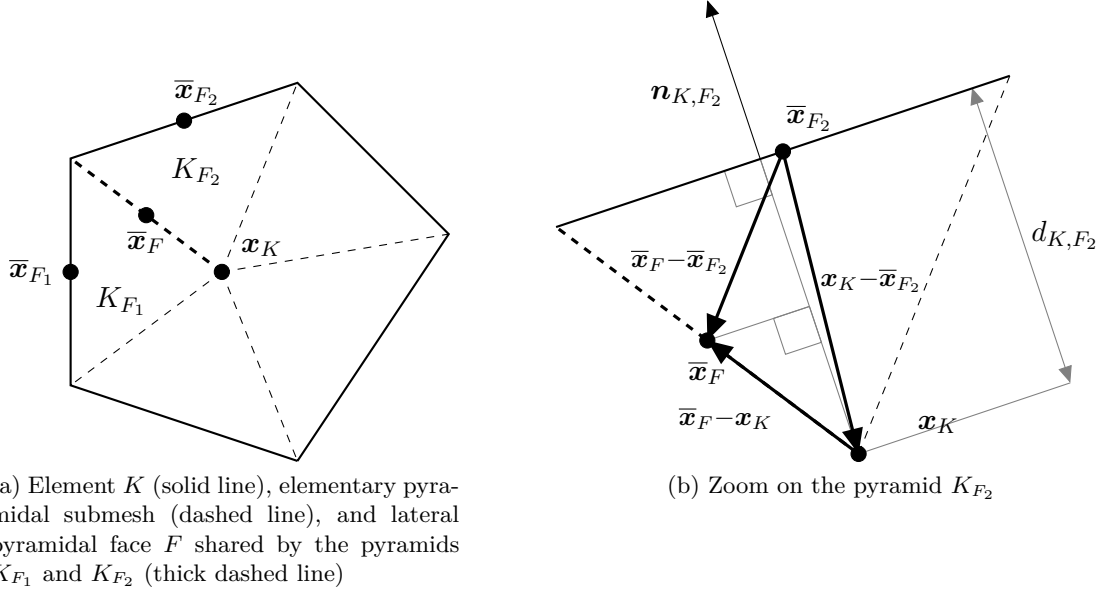


Figure 2: Notation for the proof of Lemma 10

### 3.3 The matching simplicial case

To gain further insight, we examine the case when  $\mathcal{K}_h$  is matching simplicial and prove a result that links the space  $\mathfrak{CR}(\mathcal{K}_h)$  with the classical Crouzeix–Raviart space  $\mathbb{CR}(\mathcal{K}_h)$ . Let

$$W(\mathcal{P}_h) := \{v \in H^1(\mathcal{P}_h) \mid \langle \llbracket v \rrbracket \rangle_F = 0 \text{ for all } F \in \mathcal{F}_{\mathcal{P}_h}^i\},$$

and denote by  $\mathcal{I}_h^{\mathbb{CR}} : W(\mathcal{P}_h) \rightarrow \mathbb{CR}(\mathcal{K}_h)$  the interpolator that maps a function  $v \in W(\mathcal{P}_h)$  on the function  $v_h := \mathcal{I}_h^{\mathbb{CR}}(v) \in \mathbb{CR}(\mathcal{K}_h)$  such that  $v_h(\bar{\mathbf{x}}_F) = \langle v \rangle_F$  for all  $F \in \mathcal{F}_{\mathcal{K}_h}$ . Defining  $\mathcal{I}_h^{\mathbb{CR}}$  for functions in  $W(\mathcal{P}_h)$  is necessary for the discussion in Sections 4.2.4 and 4.3.3.

**Proposition 11** (The matching simplicial case). *Assume that  $\mathcal{K}_h$  is a matching simplicial mesh. Then, for all  $\eta > 0$  in (9) there holds*

$$\mathbb{CR}(\mathcal{K}_h) \subset \mathfrak{CR}(\mathcal{K}_h). \quad (13)$$

*Proof.* Let  $v_h \in \mathbb{CR}(\mathcal{K}_h)$  and set  $\mathfrak{v}_h := ((v_h(\mathbf{x}_K))_{K \in \mathcal{K}_h}, (v_h(\bar{\mathbf{x}}_F))_{F \in \mathcal{F}_{\mathcal{K}_h}})$ . By definition there holds (cf. (9))  $\mathbf{G}_K(\mathfrak{v}_h) = (\nabla_h v_h)|_K$  for all  $K \in \mathcal{K}_h$ . Using the linearity of  $v_h$  inside each element it is inferred  $\mathbf{R}_{K_F}(\mathfrak{v}_h) = \mathbf{0}$ , hence  $\mathbf{G}_{K_F}(\mathfrak{v}_h) = \mathbf{G}_K(\mathfrak{v}_h) = (\nabla_h v_h)|_K$  for all  $F \in \mathcal{F}_K$ . As a consequence, we conclude that  $v_h = \mathfrak{R}_h(\mathfrak{v}_h) \in \mathfrak{CR}(\mathcal{K}_h)$ .  $\square$

### 3.4 Approximation

In this section we introduce a suitable interpolator on the  $\mathfrak{CR}(\mathcal{K}_h)$  space and study its approximation properties. Let  $\mathcal{I}_h^{\mathfrak{CR}} : H^1(\Omega) \rightarrow \mathfrak{CR}(\mathcal{K}_h)$  be such that, for all  $v \in H^1(\Omega)$ ,  $\mathcal{I}_h^{\mathfrak{CR}}(v) = \mathfrak{R}_h(\mathfrak{v}_h)$  with

$$\mathfrak{v}_h \ni \mathfrak{v}_h = ((\Pi_h^1 v(\mathbf{x}_K))_{K \in \mathcal{K}_h}, (\langle v \rangle_F)_{F \in \mathcal{F}_{\mathcal{K}_h}}). \quad (14)$$

When applied to vector-valued fields,  $\mathcal{I}_h^{\mathfrak{CR}}$  acts component-wise.

**Lemma 12** (Approximation in  $\mathfrak{CR}(\mathcal{K}_h)$ ). For all  $\eta > 0$  in (9) and all  $v \in H^1(\Omega)$  there holds with  $v_h := \mathcal{I}_h^{\mathfrak{CR}}(v) \in \mathfrak{CR}(\mathcal{K}_h)$ ,

$$\Pi_h^0(\nabla_h v_h) = \Pi_h^0(\nabla v), \quad (15)$$

where  $\Pi_h^0$  denotes the  $L^2$ -orthogonal projector on  $\mathbb{P}_d^0(\mathcal{K}_h)^d$ . Moreover, there exists a real  $C > 0$  independent of the meshsize such that, for all  $h \in \mathcal{H}$ , all  $K \in \mathcal{K}_h$ , all  $v \in H^1(\Omega) \cap H^{l+1}(\mathcal{K}_h)$ ,  $l \in \{0, 1\}$ , there holds with  $v_h := \mathcal{I}_h^{\mathfrak{CR}}(v)$

$$\|v - v_h\|_{L^2(K)} + h_K \|\nabla v - \nabla_h v_h\|_{L^2(K)^d} \leq Ch_K^{l+1} |v|_{H^{l+1}(K)}. \quad (16)$$

*Proof.* To avoid naming generic constants, we use the notation  $a \lesssim b$  for the inequality  $a \leq Cb$  with  $C > 0$  independent of the meshsize.

(i) *Equality* (15). For a given  $v \in H^1(\Omega)$ , let  $\mathfrak{v}_h$  be defined as in (14). We start by noting the following orthogonality relation (cf. [25, eq. (27)]) valid for all  $\mathfrak{w}_h \in \mathbb{V}_h$  and all  $K \in \mathcal{K}_h$ :

$$\sum_{F \in \mathcal{F}_K} |K_F| \mathbf{R}_{K_F}(\mathfrak{w}_h) = \mathbf{0}. \quad (17)$$

As a consequence, for all  $K \in \mathcal{K}_h$  there holds

$$\langle \nabla_h v_h \rangle_K = \mathbf{G}_K(\mathfrak{v}_h) = \frac{1}{|K|} \sum_{F \in \mathcal{F}_K} |F| \langle v \rangle_F \mathbf{n}_{K,F} = \frac{1}{|K|} \int_{\partial K} v \mathbf{n}_K = \langle \nabla v \rangle_K,$$

where we have used the planarity of faces and Green's formula. Relation (15) follows.

(ii) *Inequality* (16). Let  $K \in \mathcal{K}_h$  and define  $\mathfrak{v}_h$  as in (14). We first estimate  $\|\nabla v - \nabla_h v_h\|_{L^2(K)^d}$ . Using (8) together with the previous point and the triangular inequality it is inferred

$$\|\nabla v - \nabla_h v_h\|_{L^2(K)^d} \leq \|\nabla v - \Pi_h^0(\nabla v)\|_{L^2(K)^d} + \left( \sum_{F \in \mathcal{F}_K} |K_F| |\mathbf{R}_{K_F}(\mathfrak{v}_h)|^2 \right)^{\frac{1}{2}} := \mathfrak{T}_1 + \mathfrak{T}_2.$$

Using the approximation properties of the  $L^2$ -orthogonal projector it is readily inferred  $\mathfrak{T}_1 \lesssim h_K^l |v|_{H^{l+1}(K)}$ .

To estimate the second term, we preliminarily observe that there holds for all  $F \in \mathcal{F}_K$  with  $w_h := \Pi_h^1 v$  ( $\Pi_h^1$  denotes here the  $L^2$ -orthogonal projector on  $\mathbb{P}_d^1(\mathcal{K}_h)$ ),

$$\mathbf{R}_{K_F}(\mathfrak{v}_h) = \frac{\eta}{d_{K,F}} (\langle v \rangle_F - w_h(\mathbf{x}_K) - \mathbf{G}_K(\mathfrak{v}_h) \cdot (\bar{\mathbf{x}}_F - \mathbf{x}_K)) \mathbf{n}_{K,F} = \frac{\eta}{d_{K,F}} (\alpha_{K,F} + \beta_{K,F}) \mathbf{n}_{K,F}, \quad (18)$$

where  $\alpha_{K,F} := \langle v \rangle_F - \langle w_h|_K \rangle_F$ ,  $\beta_{K,F} := (\nabla w_h|_K - \mathbf{G}_K(\mathfrak{v}_h)) \cdot (\bar{\mathbf{x}}_F - \mathbf{x}_K)$ , and we have used the fact that  $w_h|_K$  is affine in  $K$  to infer  $w_h(\mathbf{x}_K) = \langle w_h|_K \rangle_F - \nabla w_h|_K \cdot (\bar{\mathbf{x}}_F - \mathbf{x}_K)$ . There follows from equation (18)

$$\mathfrak{T}_2^2 \lesssim \sum_{F \in \mathcal{F}_K} \frac{|K_F|}{d_{K,F}^2} |\alpha_{K,F}|^2 + \sum_{F \in \mathcal{F}_K} \frac{|K_F|}{d_{K,F}^2} |\beta_{K,F}|^2 := \mathfrak{T}_{2,1} + \mathfrak{T}_{2,2}.$$

Using (2), the Cauchy–Schwarz inequality, the mesh regularity assumption (1), the fact that  $\text{card}(\mathcal{F}_K)$  is bounded uniformly in  $h$  (cf. [17, Lemma 1.41]), and Proposition 8 it is inferred,

$$\mathfrak{T}_{2,1} = \frac{1}{d} \sum_{F \in \mathcal{F}_K} \frac{1}{d_{K,F} |F|} \left( \int_F v - w_h \right)^2 \leq \frac{1}{d \varrho_3} \sum_{F \in \mathcal{F}_K} \frac{1}{h_K} \|v - w_h\|_{L^2(F)}^2 \lesssim h_K^{2l} |v|_{H^{l+1}(K)}^2.$$

On the other hand, since  $|\bar{\mathbf{x}}_F - \mathbf{x}_K| \leq h_K$  and both  $\nabla w_h|_K$  and  $\mathbf{G}_K(\mathfrak{v}_h)$  are constant on  $K$ , there holds

$$\mathfrak{T}_{2,2} \leq \sum_{F \in \mathcal{F}_K} |K_F| \frac{h_K^2}{d_{K,F}^2} |\nabla w_h|_K - \mathbf{G}_K(\mathfrak{v}_h)|^2 \leq \frac{1}{\varrho_3^2} \|\nabla w_h|_K - \Pi_h^0(\nabla v)\|_{L^2(K)^d}^2 \lesssim h_K^{2l} |v|_{H^{l+1}(K)}^2,$$

where we have used the mesh regularity assumption (1) together with (15), and concluded using the approximation properties of the  $L^2$ -orthogonal projector. Gathering up the bounds on  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$  it is inferred

$$\|\nabla v - \nabla_h v_h\|_{L^2(K)^d} \lesssim h_K^l |v|_{H^{l+1}(K)}. \quad (19)$$

To complete the proof of inequality (16) it only remains to estimate  $\|v - v_h\|_{L^2(K)}$ . To this end, letting again  $w_h := \Pi_h^1 v$ , we apply the triangular inequality to infer

$$\|v - v_h\|_{L^2(K)} \leq \|v - w_h\|_{L^2(K)} + \|w_h - v_h\|_{L^2(K)} := \mathfrak{T}_1 + \mathfrak{T}_2.$$

The approximation properties of the  $L^2$ -orthogonal projector readily yield  $\mathfrak{T}_1 \lesssim h_K^{l+1} |v|_{H^{l+1}(K)}$ . For the second term, we notice that for all  $F \in \mathcal{F}_K$  and all  $\mathbf{x} \in K_F$ , the linearity of both  $w_h|_K$  and  $v_h|_{K_F}$  yields

$$w_h|_K(\mathbf{x}) = \langle w_h|_K \rangle_F + \nabla w_h|_K \cdot (\mathbf{x} - \bar{\mathbf{x}}_F), \quad v_h|_{K_F}(\mathbf{x}) = \langle v \rangle_F + \nabla v_h|_{K_F} \cdot (\mathbf{x} - \bar{\mathbf{x}}_F).$$

As a consequence,

$$\|w_h - v_h\|_{L^2(K_F)}^2 \lesssim \int_{K_F} (\langle w_h|_K - v \rangle_F)^2 + \int_{K_F} [(\nabla w_h|_K - \nabla v_h|_{K_F}) \cdot (\mathbf{x} - \bar{\mathbf{x}}_F)]^2 := \mathfrak{T}_{2,1} + \mathfrak{T}_{2,2}.$$

Using (2), the Cauchy–Schwarz inequality, and Proposition 8 it is inferred

$$\mathfrak{T}_{2,1} = \frac{|F|}{d} \frac{d_{K,F}}{d} (\langle w_h|_K - v \rangle_F)^2 \leq \frac{d_{K,F}}{d} \|w_h|_K - v\|_{L^2(F)}^2 \lesssim h_K^{2(l+1)} |v|_{H^{l+1}(K)}^2.$$

Since  $|\mathbf{x} - \bar{\mathbf{x}}_F| \leq h_K$  for all  $\mathbf{x} \in K_F$  and both  $\nabla w_h|_K$  and  $\nabla v_h|_{K_F}$  are constant on  $K_F$ , the estimate (19) yields

$$\mathfrak{T}_{2,2} \leq h_K^2 \|\nabla w_h|_K - \nabla v_h|_{K_F}\|_{L^2(K_F)^d}^2 \lesssim h_K^{2(l+1)} |v|_{H^{l+1}(K)}^2.$$

Summing over  $F \in \mathcal{F}_K$ , using the bounds for  $\mathfrak{T}_{2,1}$  and  $\mathfrak{T}_{2,2}$  together with the fact that  $\text{card}(\mathcal{F}_K)$  is bounded uniformly in  $h$ , it is inferred  $\mathfrak{T}_2 \lesssim h_K^{l+1} |v|_{H^{l+1}(K)}$ , thereby yielding  $\|v - v_h\|_{L^2(K)} \lesssim h_K^{l+1} |v|_{H^{l+1}(K)}$  and therefore concluding the proof.  $\square$

**Remark 13** (The matching simplicial case). *When  $\mathcal{K}_h$  is matching simplicial, the proof of Lemma 12 can be simplified exploiting the result of Proposition 11 to infer*

$$\inf_{v_h \in \mathfrak{CR}(\mathcal{K}_h)} \|v - v_h\|_{H^1(K)} \leq \inf_{v_h \in \mathbb{CR}(\mathcal{K}_h)} \|v - v_h\|_{H^1(K)},$$

and conclude using the approximation properties of the standard Crouzeix–Raviart space.

We next examine the approximation of the divergence of a vector-valued field. For all  $\mathbf{w} \in \mathbf{H}(\text{div}; \mathcal{K}_h) := \{\mathbf{v} \in L^2(\Omega)^d \mid \forall K \in \mathcal{K}_h, \nabla \cdot (\mathbf{v}|_K) \in L^2(K)\}$  we introduce the following discrete divergence piecewise constant on  $\mathcal{K}_h$ :

$$D_h(\mathbf{w}) := \Pi_h^0(\nabla_h \cdot \mathbf{w}). \quad (20)$$

An immediate consequence of the the first point in Lemma 12 is that the discrete vector space  $\mathfrak{CR}(\mathcal{K}_h)^d$  possesses the following approximation property.

**Corollary 14** (Divergence approximation). *Let  $\mathbf{v} \in H^1(\Omega)^d$  and  $\mathbf{v}_h := \mathcal{I}_h^{\mathfrak{CR}}(\mathbf{v})$ . There holds*

$$D_h(\mathbf{v}_h) = \Pi_h^0(\nabla \cdot \mathbf{v}).$$

Moreover, there exists a real  $C > 0$  independent of the meshsize such that, for all  $h \in \mathcal{H}$ , all  $K \in \mathcal{K}_h$ , and all  $\mathbf{v} \in H^1(\Omega)^d \cap \mathbf{H}^1(\text{div}; \mathcal{K}_h)$  with  $\mathbf{H}^1(\text{div}; \mathcal{K}_h) := \{\mathbf{v} \in \mathbf{H}(\text{div}; \Omega) \mid \nabla_h \cdot \mathbf{v} \in H^1(\mathcal{K}_h)\}$  and  $\mathbf{v}_h := \mathcal{I}_h^{\mathfrak{CR}}(\mathbf{v})$ ,

$$\|\nabla \cdot \mathbf{v} - D_h(\mathbf{v}_h)\|_{L^2(K)} + h_K |\nabla \cdot \mathbf{v} - D_h(\mathbf{v}_h)|_{H^1(K)} \leq Ch_K |\nabla \cdot \mathbf{v}|_{H^1(K)}.$$

## 4 Applications

In this section we present two examples of applications where the properties of the extended Crouzeix–Raviart space (11) are instrumental in designing a suitable numerical approximation, namely linear elasticity for quasi-incompressible materials and the Stokes equations with large irrotational sources.

### 4.1 Discrete $H_0^1(\Omega)$ -norm

For problems naturally set in  $H_0^1(\Omega)$ , boundary conditions can be accounted for in a strong manner by introducing the discrete space

$$\mathfrak{CA}_0(\mathcal{K}_h) := \mathfrak{A}_h(\mathbb{V}_{h,0}), \quad \mathbb{V}_{h,0} = \{v_h \in \mathbb{V}_h \mid v_F = 0, \forall F \in \mathcal{F}_{\mathcal{K}_h}^b\}. \quad (21)$$

In the following proposition we show that the  $L^2$ -norm of the broken gradient is a norm on  $\mathfrak{CA}_0(\mathcal{K}_h)$  by proving uniform discrete equivalence with the usual dG norm, cf. [17, Section 5.1]:

$$\|v_h\|_{\text{dG}}^2 := \|\nabla_h v_h\|_{L^2(\Omega)^d}^2 + |v_h|_{\mathbb{J}}^2, \quad |v_h|_{\mathbb{J}}^2 := \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} \frac{1}{h_F} \|[[v_h]]\|_{L^2(F)}^2. \quad (22)$$

**Proposition 15** (Discrete norm). *Assume  $\eta = d$  in (9). Then, there exists a real  $C > 0$  independent of the meshsize such that, for all  $v_h \in \mathfrak{CA}_0(\mathcal{K}_h)$ ,*

$$\|\nabla_h v_h\|_{L^2(\Omega)^d} \leq \|v_h\|_{\text{dG}} \leq C \|\nabla_h v_h\|_{L^2(\Omega)^d}.$$

*Proof.* The notation  $a \lesssim b$  stands for  $a \leq Cb$  with  $C > 0$  independent of the meshsize. Clearly,  $\|\nabla_h v_h\|_{L^2(\Omega)^d} \leq \|v_h\|_{\text{dG}}$  for all  $v_h \in \mathfrak{CA}_0(\mathcal{K}_h)$ . To prove that  $\|v_h\|_{\text{dG}} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d}$  for all  $v_h \in \mathfrak{CA}_0(\mathcal{K}_h)$ , it suffices to show that  $|v_h|_{\mathbb{J}} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d}$ . Let  $P \in \mathcal{P}_h$  and  $F \in \mathcal{F}_P$ . Owing to the linearity of  $v_h$  inside  $P$  there holds for all  $\mathbf{x} \in P$ ,  $v_{h|P}(\mathbf{x}) = \langle v_{h|P} \rangle_F + \nabla v_{h|P} \cdot (\mathbf{x} - \bar{\mathbf{x}}_F)$ . This together with Lemma 10 and the discrete trace inequality (5) yields

$$\|[[v_h]]\|_{L^2(F)} = \|[[v_h]] - \langle [[v_h]] \rangle_F\|_{L^2(F)} \leq h_F \|\nabla_h v_h\|_{L^2(F)^d} \lesssim h_F^{1/2} \sum_{P \in \mathcal{P}_F} \|\nabla v_{h|P}\|_{L^2(P)^d}, \quad (23)$$

where we have set  $\mathcal{P}_F := \{P \in \mathcal{P}_h \mid F \subset \partial P\}$ . Using (22) together with (23) and the Cauchy–Schwarz inequality it is inferred

$$|v_h|_{\mathbb{J}}^2 = \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} \frac{1}{h_F} \|[[v_h]] - \langle [[v_h]] \rangle_F\|_{L^2(F)}^2 \lesssim \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} \sum_{P \in \mathcal{P}_F} \|\nabla v_{h|P}\|_{L^2(P)^d}^2 \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d}^2, \quad (24)$$

where the last bound is a consequence of the fact that the maximum number of faces of a pyramid is bounded uniformly in  $h$  since  $\mathcal{P}_h$  is shape- and contact-regular, cf. [17, Lemma 1.41].  $\square$

### 4.2 Quasi-incompressible linear elasticity

We tackle here the question of the accurate approximation of the linear elasticity equations in the quasi-incompressible limit on general polygonal or polyhedral meshes. The main sources of inspiration are the classical works of Brenner and Sung [8], and Hansbo and Larson [28, 29].

### 4.2.1 Setting

We consider a homogeneous elastic material occupying the polygonal/polyhedral domain  $\Omega \subset \mathbb{R}^d$ , and whose mechanical properties are described by the real Lamé parameters  $\lambda$  and  $\mu$  with  $0 < \mu < +\infty$  and  $0 < \lambda \leq +\infty$ . The mechanical behavior of the material is governed by the linear elasticity equations,

$$\begin{aligned} -\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) &= \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} & \text{on } \partial\Omega, \end{aligned} \quad (25)$$

where  $\mathbf{u}$  denotes the vector-valued displacement field,  $\mathbf{f} \in L^2(\Omega)^d$  the forcing term, and the Cauchy stress tensor is defined for all  $\mathbf{v} \in H^1(\Omega)^d$  by Hooke's law,

$$\underline{\underline{\sigma}}(\mathbf{v}) := 2\mu \underline{\underline{\varepsilon}}(\mathbf{v}) + \lambda \nabla \cdot \mathbf{v} \underline{\underline{I}}_d, \quad \underline{\underline{\varepsilon}}(\mathbf{v}) := \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T).$$

Setting  $\mathbf{U} := H_0^1(\Omega)^d$ , the weak formulation of problem (25) reads: Find  $\mathbf{u} \in \mathbf{U}$  such that

$$\tilde{a}(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{L^2(\Omega)^d} \quad \forall \mathbf{v} \in \mathbf{U}, \quad (26)$$

where  $\tilde{a}(\mathbf{u}, \mathbf{v}) := (\underline{\underline{\sigma}}(\mathbf{u}), \underline{\underline{\varepsilon}}(\mathbf{v}))_{L^2(\Omega)^{d,d}}$ . The well-posedness of the weak formulation (26) relies on Korn's inequality in  $\mathbf{U}$  (cf., e.g., [1, Lemma 5.3.2]),

$$\|\nabla \mathbf{v}\|_{L^2(\Omega)^{d,d}} \leq \sqrt{2} \|\underline{\underline{\varepsilon}}(\mathbf{v})\|_{L^2(\Omega)^{d,d}} \quad \forall \mathbf{v} \in \mathbf{U}. \quad (27)$$

The use of Korn's inequality can be circumvented for the pure displacement problem using the following alternative Navier–Cauchy formulation: Find  $\mathbf{u} \in \mathbf{U}$  such that

$$a(\mathbf{u}, \mathbf{v}) := (\mu \nabla \mathbf{u}, \nabla \mathbf{v})_{L^2(\Omega)^{d,d}} + ((\mu + \lambda) \nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_{L^2(\Omega)} = (\mathbf{f}, \mathbf{v})_{L^2(\Omega)^d} \quad \forall \mathbf{v} \in \mathbf{U}. \quad (28)$$

This naturally coercive formulation is equivalent to (26) for homogeneous materials and pure Dirichlet boundary conditions. Throughout the rest of this section we focus on this alternative form. The treatment of mixed Dirichlet/Neumann boundary conditions is briefly addressed in Section 4.2.5. We recall the following regularity result for problem (25) in  $d = 2$ ; cf., e.g., [8, Lemma 2.2].

**Lemma 16** (Regularity). *Let  $d = 2$  and assume that  $\Omega$  is convex. Then, problem (25) has a unique solution  $\mathbf{u} \in \mathbf{U}_* := \mathbf{U} \cap H^2(\Omega)^d$ . Moreover, there exists a real  $C_\mu > 0$  only depending on  $\Omega$  and  $\mu$  but not on  $\lambda$ , such that, for  $\lambda$  large enough,*

$$\mathcal{N}_{\text{el}}(\mathbf{u}) := \|\mathbf{u}\|_{H^2(\Omega)^d} + |\lambda \nabla \cdot \mathbf{u}|_{H^1(\Omega)} \leq C_\mu \|\mathbf{f}\|_{L^2(\Omega)^d}. \quad (29)$$

This a priori estimate implies that, if  $\lambda \rightarrow +\infty$ , the divergence of the displacement field approaches zero, corresponding to a quasi-incompressible material. A generalization of this result to composite materials with piecewise constant mechanical properties is proved in [20], where a locking-free dG discretization on matching triangular meshes is introduced. Note that the regularity estimate (29) is stronger than a simple bound on  $|\lambda^{1/2} \nabla \cdot \mathbf{u}|_{H^1(\Omega)}$ .

### 4.2.2 Discretization

Let  $\mathcal{K}_h$  belong to an admissible mesh sequence in the sense of Definition 7. We consider an approximation of the displacement field in the space

$$\mathbf{U}_h := \mathfrak{CR}_0(\mathcal{K}_h)^d,$$

with  $\mathfrak{N}_0(\mathcal{K}_h)$  defined by (21). Henceforth we assume the choice  $\eta = d$  in (9), so that the continuity of mean values stated in Lemma 10 holds. Lowest-order, locking-free discretizations of (25) satisfy an estimate of the form

$$\|\mathbf{u} - \mathbf{u}_h\|_{\text{el}} \leq Ch \|\mathbf{f}\|_{L^2(\Omega)^d}, \quad (30)$$

where  $\|\cdot\|_{\text{el}}$  is a (discrete) energy norm and  $C > 0$  is a constant possibly depending on  $\mu$  and on the mesh regularity parameters but independent of  $h$ ,  $\lambda$ , and  $\mathbf{u}$ . The key point is here that the multiplicative constant in the right-hand side of (30) does not blow up in the limit  $\lambda \rightarrow +\infty$ , i.e., the method converges uniformly with respect to  $\lambda$ . To obtain (30) we prove that in general space dimension there holds with  $\mathcal{N}_{\text{el}}(\mathbf{u})$  defined by (29),

$$\|\mathbf{u} - \mathbf{u}_h\|_{\text{el}} \leq C_{\text{el}} h \mathcal{N}_{\text{el}}(\mathbf{u}),$$

where  $C_{\text{el}} > 0$  has the same dependencies as  $C$ , and then invoke the regularity estimate in Lemma 16 to conclude in  $d = 2$ . The discrete problem reads: Find  $\mathbf{u}_h \in \mathbf{U}_h$  such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)^d} \quad \forall \mathbf{v}_h \in \mathbf{U}_h, \quad (31)$$

with discrete bilinear form  $a_h$  such that

$$a_h(\mathbf{w}, \mathbf{v}) := (\mu \nabla_h \mathbf{w}, \nabla_h \mathbf{v})_{L^2(\Omega)^{d,d}} + (\mu \nabla_h \cdot \mathbf{w}, \nabla_h \cdot \mathbf{v})_{L^2(\Omega)} + (\lambda D_h(\mathbf{w}), D_h(\mathbf{v}))_{L^2(\Omega)}. \quad (32)$$

The last term in the right-hand side is treated using the discrete divergence  $D_h$  defined in Corollary 14. The approximation properties of  $D_h$  are instrumental to ensure that  $\lambda$  only appears in terms of the form  $|\lambda \nabla \cdot \mathbf{u}|_{H^1(\Omega)}$  in the right-hand side of the error estimate (cf., in particular, the bound for the consistency term in the proof of Theorem 20).

**Remark 17** (Implementation). *In the practical implementation of (31) cell-centered unknowns can be eliminated from the global linear system by solving an inexpensive local problem inside each element in the spirit of [3]. This is the case since, for each element, the cell unknown is only linked with the face unknowns located on the boundary of the element.*

The energy norm associated to the bilinear form  $a_h$  is

$$\|\mathbf{v}\|_{\text{el}}^2 := a_h(\mathbf{v}, \mathbf{v}) = \|\mu^{1/2} \nabla_h \mathbf{v}\|_{L^2(\Omega)^{d,d}}^2 + \|\mu^{1/2} \nabla_h \cdot \mathbf{v}\|_{L^2(\Omega)}^2 + \|\lambda^{1/2} D_h(\mathbf{v})\|_{L^2(\Omega)}^2. \quad (33)$$

**Lemma 18** (Coercivity). *There holds for all  $\mathbf{v}_h \in \mathbf{U}_h$ ,*

$$a_h(\mathbf{v}_h, \mathbf{v}_h) = \|\mathbf{v}_h\|_{\text{el}}^2 \geq \mu \|\nabla_h \mathbf{v}_h\|_{L^2(\Omega)^{d,d}}^2.$$

The well-posedness of the discrete problem (31) then follows from the Lax–Milgram Lemma and Proposition 15. Let now  $\mathbf{U}_{*h} := \mathbf{U}_* + \mathbf{U}_h$  with  $\mathbf{U}_*$  defined in Lemma 16, and extend the bilinear form  $a_h$  to  $\mathbf{U}_{*h} \times \mathbf{U}_{*h}$ . Assuming that the exact solution belongs to the augmented space  $\mathbf{U}_*$  ensures that the boundary terms in the expression of the consistency error are well-defined, cf. Lemma 19. This additional regularity is verified, e.g., under the assumptions of Lemma 16.

**Lemma 19** (Weak consistency). *Let  $\mathbf{u} \in \mathbf{U}$  denote the solution to (28) and additionally assume that  $\mathbf{u} \in \mathbf{U}_*$ . Then, there holds for all  $\mathbf{v}_h \in \mathbf{U}_h$ ,*

$$a_h(\mathbf{u}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)^d} + \mathcal{E}_h(\mathbf{v}_h),$$

where, letting  $\underline{\tau}(\mathbf{u}) := \mu \nabla \mathbf{u} + (\mu + \lambda)(\nabla \cdot \mathbf{u}) \underline{I}_d$ ,

$$\mathcal{E}_h(\mathbf{v}_h) := \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} (\underline{\tau}(\mathbf{u}) \mathbf{n}_F, \llbracket \mathbf{v}_h \rrbracket)_{L^2(F)^d} + (\lambda(D_h(\mathbf{u}) - \nabla \cdot \mathbf{u}), \nabla_h \cdot \mathbf{v}_h)_{L^2(\Omega)}. \quad (34)$$



*Proof.* Observing that  $(\lambda D_h(\mathbf{u}), D_h(\mathbf{v}_h))_{L^2(\Omega)} = (\lambda D_h(\mathbf{u}), \nabla_h \cdot \mathbf{v}_h)_{L^2(\Omega)}$ , summing and subtracting  $(\lambda \nabla \cdot \mathbf{u}, \nabla_h \cdot \mathbf{v}_h)_{L^2(\Omega)}$  from the right-hand side of (32) with  $(\mathbf{w}, \mathbf{v}) = (\mathbf{u}, \mathbf{v}_h)$  there holds

$$a_h(\mathbf{u}, \mathbf{v}_h) = (\underline{\mathcal{T}}(\mathbf{u}), \nabla_h \mathbf{v}_h)_{L^2(\Omega)^{d,d}} + (\lambda(D_h(\mathbf{u}) - \nabla \cdot \mathbf{u}), \nabla_h \cdot \mathbf{v}_h)_{L^2(\Omega)}.$$

Integrating by parts the first term element-wise, rearranging the boundary contributions, and using the fact that  $\llbracket \underline{\mathcal{T}}(\mathbf{u}) \rrbracket_F \mathbf{n}_F = \mathbf{0}$  and  $\{\underline{\mathcal{T}}(\mathbf{u})\}_F \mathbf{n}_F = \underline{\mathcal{T}}(\mathbf{u}) \mathbf{n}_F$  for all  $F \in \mathcal{F}_{\mathcal{P}_h}^i$ , it is inferred

$$\begin{aligned} (\underline{\mathcal{T}}(\mathbf{u}), \nabla_h \mathbf{v}_h)_{L^2(\Omega)^{d,d}} &= -(\nabla \cdot \underline{\mathcal{T}}(\mathbf{u}), \mathbf{v}_h)_{L^2(\Omega)^d} + \sum_{P \in \mathcal{P}_h} (\underline{\mathcal{T}}(\mathbf{u}) \mathbf{n}_P, \mathbf{v}_h|_P)_{L^2(\partial P)^d} \\ &= (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)^d} + \sum_{F \in \mathcal{F}_{\mathcal{P}_h}^i} (\llbracket \underline{\mathcal{T}}(\mathbf{u}) \rrbracket_F \mathbf{n}_F, \{\mathbf{v}_h\})_{L^2(F)^d} + \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} (\{\underline{\mathcal{T}}(\mathbf{u})\}_F \mathbf{n}_F, \llbracket \mathbf{v}_h \rrbracket)_{L^2(F)^d} \\ &= (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)^d} + \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} (\underline{\mathcal{T}}(\mathbf{u}) \mathbf{n}_F, \llbracket \mathbf{v}_h \rrbracket)_{L^2(F)^d}, \end{aligned}$$

where we have used the fact that  $-\nabla \cdot \underline{\mathcal{T}}(\mathbf{u}) = \mathbf{f}$  almost everywhere in  $\Omega$  as a consequence of (28). This concludes the proof.  $\square$

In the following theorem the continuity of mean values at interfaces plays an important role in estimating the boundary contribution in the consistency error.

**Theorem 20** (Error estimate for (31)). *Let  $\mathbf{u} \in \mathbf{U}$  denote the solution to (28) and additionally assume that  $\mathbf{u} \in \mathbf{U}_*$ . Then, there exists  $C_{\text{el}} > 0$  independent of the meshsize, of  $\lambda$ , and of  $\mathbf{u}$  such that, denoting by  $\mathbf{u}_h \in \mathbf{U}_h$  the solution to (31), there holds*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\text{el}} \leq C_{\text{el}} h \mathcal{N}_{\text{el}}(\mathbf{u}). \quad (35)$$

*Proof.* We use the notation  $a \lesssim b$  for the inequality  $a \leq Cb$  where  $C > 0$  has the same dependence as the constant  $C_{\text{el}}$  in (35). The Cauchy–Schwarz inequality yields boundedness in the form  $a_h(\mathbf{w}, \mathbf{v}) \leq \|\mathbf{w}\|_{\text{el}} \|\mathbf{v}\|_{\text{el}}$  for all  $(\mathbf{w}, \mathbf{v}) \in \mathbf{U}_{*h} \times \mathbf{U}_{*h}$ . This together with Lemmata 18 and 19 and the Second Strang Lemma [32] (cf. also [23, Lemma 2.25]) yields:

$$\|\mathbf{u} - \mathbf{u}_h\|_{\text{el}} \lesssim \inf_{\mathbf{v}_h \in \mathbf{U}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\text{el}} + \sup_{\mathbf{v}_h \in \mathbf{U}_h \setminus \{\mathbf{0}\}} \frac{|\mathcal{E}_h(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{\text{el}}} := \mathfrak{T}_1 + \mathfrak{T}_2. \quad (36)$$

The first term in the right-hand side depends on the approximation properties of the discrete space in the  $\|\cdot\|_{\text{el}}$ -norm, whereas the second is linked to the consistency error. Let  $\mathbf{w}_h := \mathcal{I}_h^{\text{CR}}(\mathbf{u}) \in \mathbf{U}_h$ . Using Lemma 12 twice and Corollary 14 respectively to treat the three terms in the right-hand side of (33) with  $\mathbf{v} = \mathbf{u} - \mathbf{w}_h$  there holds,

$$\mathfrak{T}_1 \leq \|\mathbf{u} - \mathbf{w}_h\|_{\text{el}} \lesssim h \|\mathbf{u}\|_{H^2(\Omega)^d} + h |\lambda^{1/2} \nabla \cdot \mathbf{u}|_{H^1(\Omega)}. \quad (37)$$

To treat the consistency error, denote by  $\mathfrak{T}_{2,1}$  and  $\mathfrak{T}_{2,2}$  the two terms in the right-hand side of (34). Let

$$\underline{\phi}_\mu := \mu(\nabla \mathbf{u} - \Pi_h^0(\nabla \mathbf{u})), \quad \psi_s := s(\nabla \cdot \mathbf{u} - \Pi_h^0(\nabla \cdot \mathbf{u})), \quad \text{with } s \in \{\mu, \lambda\}.$$

Clearly,  $\psi_\mu = \text{tr}(\underline{\phi}_\mu)$ . Using the continuity of mean values at interfaces together with the fact that both  $\{\mu \Pi_h^0(\nabla \mathbf{u})\}_F$  and  $\{s \Pi_h^0(\nabla \cdot \mathbf{u})\}_F$ ,  $s \in \{\mu, \lambda\}$ , are constant on every  $F \in \mathcal{F}_{\mathcal{P}_h}$ , it is inferred

$$\begin{aligned} \mathfrak{T}_{2,1} &= \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} (\{\underline{\mathcal{T}}(\mathbf{u}) - \Pi_h^0(\underline{\mathcal{T}}(\mathbf{u}))\}_F \mathbf{n}_F, \llbracket \mathbf{v}_h \rrbracket - \langle \llbracket \mathbf{v}_h \rrbracket \rangle_F)_{L^2(F)^d} \\ &= \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} (\{\underline{\phi}_\mu + \psi_\mu \underline{I}_d\}_F \mathbf{n}_F, \llbracket \mathbf{v}_h \rrbracket - \langle \llbracket \mathbf{v}_h \rrbracket \rangle_F)_{L^2(F)^d} + \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} (\{\psi_\lambda\}_F \mathbf{n}_F, \llbracket \mathbf{v}_h \rrbracket - \langle \llbracket \mathbf{v}_h \rrbracket \rangle_F)_{L^2(F)^d}. \end{aligned}$$

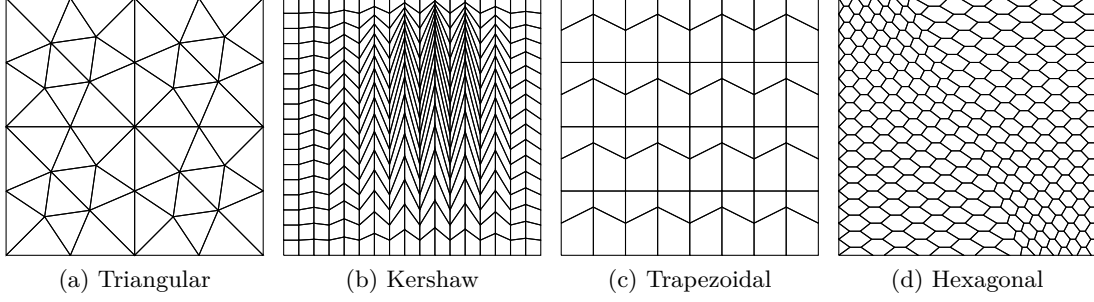


Figure 3: Members of the mesh families for the numerical test of Section 4.2.3.

The Cauchy–Schwarz inequality followed by the trace inequality (5), the fact that the maximum number of faces of a pyramid is bounded uniformly in  $h$ , the approximation properties of the  $L^2$ -orthogonal projection, and (24) yield

$$\begin{aligned} \mathfrak{T}_{2,1} &\lesssim \left\{ \sum_{P \in \mathcal{P}_h} h \left( \|\underline{\phi}_\mu|_P\|_{L^2(\partial P)^{d,d}}^2 + \|\psi_\lambda|_P\|_{L^2(\partial P)}^2 \right) \right\}^{\frac{1}{2}} \times \left\{ \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} h_F^{-1} \|[[\mathbf{v}_h]] - \langle [[\mathbf{v}_h]] \rangle_F\|_{L^2(F)^d}^2 \right\}^{\frac{1}{2}} \\ &\lesssim h \mathcal{N}_{\text{el}}(\mathbf{u}) \|\mathbf{v}_h\|_{\text{el}}, \end{aligned} \quad (38)$$

where the bound  $\|\nabla_h \mathbf{v}_h\|_{L^2(\Omega)^{d,d}} \lesssim \|\mathbf{v}_h\|_{\text{el}}$  is a consequence of Lemma 18. Finally, using the Cauchy–Schwarz inequality together with the approximation properties of the  $L^2$ -orthogonal projection and Lemma 18 it is inferred,

$$\mathfrak{T}_{2,2} \leq \|\lambda(\Pi_h^0(\nabla \cdot \mathbf{u}) - \nabla \cdot \mathbf{u})\|_{L^2(\Omega)} \|\nabla_h \cdot \mathbf{v}_h\|_{L^2(\Omega)} \lesssim h |\lambda \nabla \cdot \mathbf{u}|_{H^1(\Omega)} \|\mathbf{v}_h\|_{\text{el}}. \quad (39)$$

Using inequalities (37), (38), and (39) to bound the right-hand side of (36) the conclusion follows.  $\square$

**Corollary 21** (Uniform convergence with respect to  $\lambda$  in  $d = 2$ ). *If  $d = 2$  the locking-free error estimate (30) holds with  $C = C_{\text{el}} C_\mu$ .*

**Remark 22** (Use of Lemma 16). *In the proof of Theorem 20, the a priori bound on  $|\lambda \nabla \cdot \mathbf{u}|_{H^1(\Omega)}$  is only required to bound  $\mathfrak{T}_2$ . For  $\mathfrak{T}_1$ , the weaker regularity estimate  $\|\mathbf{u}\|_{H^2(\Omega)^d} + |\lambda^{1/2} \nabla \cdot \mathbf{u}|_{H^1(\Omega)} \lesssim \|\mathbf{f}\|_{L^2(\Omega)^d}$  is sufficient.*

**Remark 23** ( $L^2$ -error estimate). *Optimal error estimates for the  $L^2$ -error on the displacement can be derived using the Aubin–Nitsche trick based on the symmetry of the method.*

### 4.2.3 A numerical example: the closed cavity problem

To assess the robustness of the method (31) in the quasi-incompressible limit we consider the closed cavity problem of Hansbo and Larson [29]. The implementation relies on the general framework recently introduced in [18], to which we refer for further details. We let  $\Omega := (0, 1)^2$ ,  $\mathbf{f} \equiv \mathbf{0}$ , and prescribe a horizontal displacement  $\mathbf{u} = (1, 0)$  on the upper side and  $\mathbf{u} = (0, 0)$  on the remaining three. The elastic modulus and Poisson’s ratio are chosen as  $E = 1000$  and  $\nu = 0.4999$  respectively. The Lamé parameters can be derived from the following relations:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

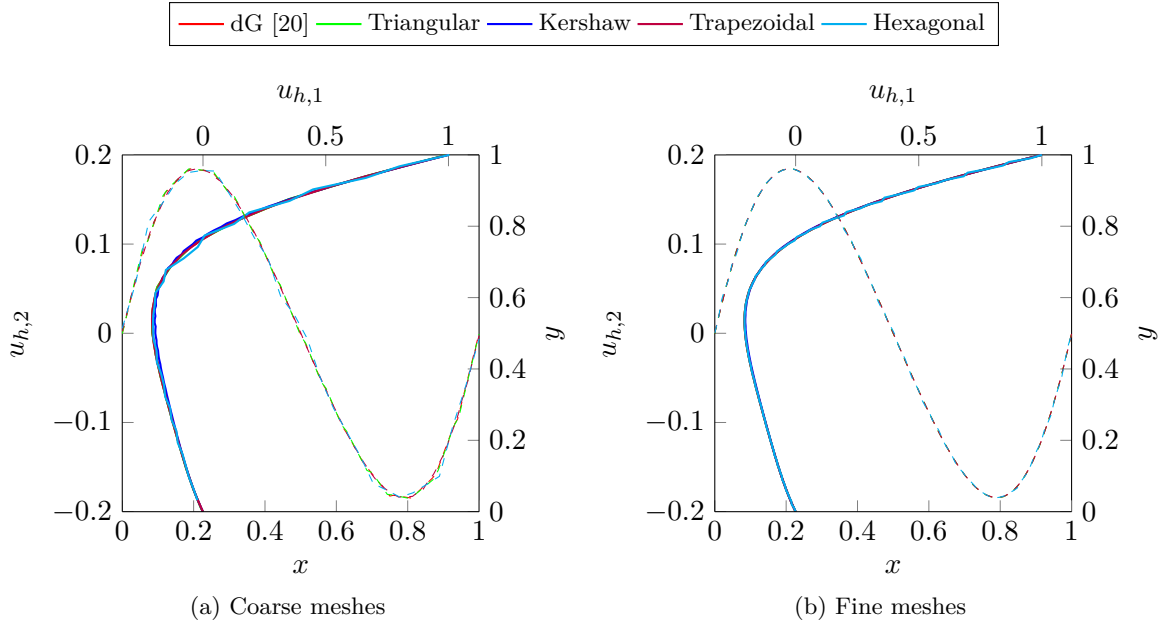


Figure 4: Results for the closed cavity problem on a coarse and a fine mesh extracted from the mesh families of Figure 3. *Solid lines*: horizontal displacement  $u_{h,1}$  along the vertical centerline. *Dashed lines*: vertical displacement  $u_{h,2}$  along the horizontal centerline.

From the first relation it is apparent that  $\lambda \rightarrow +\infty$  as  $\nu \rightarrow 1/2$ . Although this problem does not exhibit the regularity required by Theorem 20, it is included as it is one of the simplest benchmarks for numerical locking. The discrete problem (31) is solved on the four mesh sequences depicted in Figure 3. From each mesh family, a coarse and a fine meshes are selected featuring roughly the same number of elements. Figure 4 depicts the values of the horizontal displacement  $u_{h,1}$  along the vertical centerline  $x = 1/2$  as well as the values of the vertical displacement  $u_{h,2}$  along the horizontal centerline  $y = 1/2$ . The results of the (more expensive) dG method of [20] on the matching triangular mesh are also included for comparison. As predicted, no sign of numerical locking is observed.

#### 4.2.4 Variations and links with finite volume and nonconforming finite element methods

**Flux formulation and local conservation** Following [25, Section 2.4] it is possible to reformulate the discrete bilinear form (32) in terms of numerical fluxes. More specifically, let  $\mathbf{w}_h, \mathbf{v}_h \in \mathbf{U}_h$  be two discrete functions and denote by  $\mathbf{w}_h = (\mathbf{w}_{h,i})_{1 \leq i \leq d} \in \mathbb{V}_{h,0}^d$  and  $\mathbf{v}_h = (\mathbf{v}_{h,i})_{1 \leq i \leq d} \in \mathbb{V}_{h,0}^d$  the corresponding vectors of DOFs, where, for all  $i \in \{1, \dots, d\}$ ,  $\mathbf{w}_{h,i}$  and  $\mathbf{v}_{h,i}$  are the vectors of DOFs associated to the  $i$ th components of  $\mathbf{w}_h$  and  $\mathbf{v}_h$  respectively. Our goal is to show that there exists a family of numerical fluxes  $(\Phi_{K,F}(\mathbf{w}_h))_{K \in \mathcal{K}_h, F \in \mathcal{F}_K}$  with  $\Phi_{K,F}(\mathbf{w}_h) = (\Phi_{K,F,i}(\mathbf{w}_h))_{1 \leq i \leq d}$  such that

$$a_h(\mathbf{w}_h, \mathbf{v}_h) = \sum_{i=1}^d \sum_{K \in \mathcal{K}_h} \sum_{F \in \mathcal{F}_K} \Phi_{K,F,i}(\mathbf{w}_h)(v_{F,i} - v_{K,i}), \quad (40)$$

with  $a_h$  defined by (32). The main interest of this alternative formulation is that it allows to prove a local conservation property similar to those encountered in standard finite volume methods. Recalling the expression (40) for the bilinear form  $a_h$  and using the cell center as a

quadrature node to approximate the right-hand side in each element, the discrete problem (31) in algebraic form reads: Find  $\mathbf{u}_h \in \mathbb{V}_{h,0}^d$  such that for all  $\mathbf{v}_h \in \mathbb{V}_{h,0}^d$  there holds,

$$\sum_{i=1}^d \sum_{K \in \mathcal{K}_h} \sum_{F \in \mathcal{F}_K} \Phi_{K,F,i}(\mathbf{u}_h)(v_{F,i} - v_{K,i}) = \sum_{i=1}^d \sum_{K \in \mathcal{K}_h} |K| f_i(\mathbf{x}_K) v_{K,i}, \quad (41)$$

where  $\mathbb{V}_{h,0}$  is defined by (21). Consider now an interface  $F \in \mathcal{F}_{\mathcal{K}_h}^i$  such that  $F \subset \partial K_1 \cap \partial K_2$ , and let  $\mathbf{v}_{h,i}$  be such that  $v_{F,i} = 1$ ,  $v_{F',i} = 0$  for all  $F' \in \mathcal{F}_{\mathcal{K}_h} \setminus \{F\}$ , and  $v_{K,i} = 0$  for all  $K \in \mathcal{K}_h$ . There follows from (41),

$$\Phi_{K_1,F,i}(\mathbf{u}_h) = -\Phi_{K_2,F,i}(\mathbf{u}_h), \quad (42)$$

i.e., the method is locally conservative. An important remark is that the loading term does not appear in (42) since its approximation in (41) only involves cell DOFs.

**Proposition 24** (Flux formulation). *For all  $\mathbf{w}_h, \mathbf{v}_h \in \mathbf{U}_h$ , the flux formulation (40) is obtained setting for all  $K \in \mathcal{K}_h$ ,  $F \in \mathcal{F}_K$ , and  $i \in \{1, \dots, d\}$ ,*

$$\Phi_{K,F,i}(\mathbf{w}_h) := \sum_{F' \in \mathcal{F}_K} |K_{F'}| \left[ \mu \mathbf{G}_{K_{F'}}(\mathbf{w}_{h,i}) \cdot \mathbf{y}_{F',F}^K + \left( \sum_{j=1}^d \mu \mathbf{G}_{K_{F'}}(\mathbf{w}_{h,j}) + \lambda \mathbf{G}_K(\mathbf{w}_{h,j}) \right) \cdot \mathbf{e}_j (\mathbf{y}_{F',F}^K \cdot \mathbf{e}_i) \right],$$

where  $\mathbf{w}_h, \mathbf{v}_h \in \mathbb{V}_{h,0}^d$  are the vectors of DOFs associated to  $\mathbf{w}_h$  and  $\mathbf{v}_h$  respectively,  $(\mathbf{e}_i)_{1 \leq i \leq d}$  denotes the canonical basis of  $\mathbb{R}^d$ , and

$$\mathbf{y}_{F',F}^K := \begin{cases} \frac{|F|}{|K|} \mathbf{n}_{K,F} + \frac{\eta}{d_{K,F}} \left( 1 - \frac{|F|}{|K|} \mathbf{n}_{K,F} \cdot (\bar{\mathbf{x}}_F - \mathbf{x}_K) \right) \mathbf{n}_{K,F} & \text{if } F = F', \\ \frac{|F|}{|K|} \mathbf{n}_{K,F} - \frac{\eta}{d_{K,F}|K|} |F| \mathbf{n}_{K,F} \cdot (\bar{\mathbf{x}}_{F'} - \mathbf{x}_K) \mathbf{n}_{K,F'} & \text{otherwise.} \end{cases} \quad (43)$$

*Proof.* For all  $\mathbf{v}_h \in \mathbb{V}_h$ , all  $K \in \mathcal{K}_h$ , and all  $F' \in \mathcal{F}_K$  there holds with  $\mathbf{G}_{K_{F'}}(\mathbf{v}_h)$  defined by (8) (cf. [25, eq. (26) et seq.]),

$$\mathbf{G}_{K_{F'}}(\mathbf{v}_h) = \sum_{F \in \mathcal{F}_K} \mathbf{y}_{F',F}^K (v_F - v_K). \quad (44)$$

Using (8) and (10), and observing that  $(\lambda D_h(\mathbf{w}_h), D_h(\mathbf{v}_h))_{L^2(\Omega)} = (\lambda D_h(\mathbf{w}_h), \nabla_h \cdot \mathbf{v}_h)_{L^2(\Omega)}$  owing to (20) together with the properties of the  $L^2$ -orthogonal projector, it is inferred

$$a_h(\mathbf{w}_h, \mathbf{v}_h) = \sum_{i=1}^d \sum_{K \in \mathcal{K}_h} \sum_{F' \in \mathcal{F}_K} |K_{F'}| \left[ \mu \mathbf{G}_{K_{F'}}(\mathbf{w}_{h,i}) + \left( \sum_{j=1}^d \mu \mathbf{G}_{K_{F'}}(\mathbf{w}_{h,j}) \cdot \mathbf{e}_j + \lambda \mathbf{G}_K(\mathbf{w}_{h,j}) \cdot \mathbf{e}_j \right) \mathbf{e}_i \right] \cdot \mathbf{G}_{K_{F'}}(\mathbf{v}_{h,i}).$$

The conclusion follows using the expression (44) for  $\mathbf{G}_{K_{F'}}(\mathbf{v}_{h,i})$  and exchanging the sums of indices  $F$  and  $F'$ .  $\square$

**Link with the Crouzeix–Raviart solution on matching simplicial meshes** Assume that  $\mathcal{K}_h$  is a matching simplicial mesh and let

$$\mathbb{C}\mathbb{R}_0(\mathcal{K}_h) := \{v_h \in \mathbb{C}\mathbb{R}(\mathcal{K}_h) \mid v_h(\bar{\mathbf{x}}_F) = 0, \forall F \in \mathcal{F}_{\mathcal{K}_h}^b\}.$$

We show in this section that a suitable modification of the right-hand side allows to recover the Crouzeix–Raviart solution  $\hat{\mathbf{u}}_h \in \mathbb{C}\mathbb{R}_0(\mathcal{K}_h)^d$  such that (cf. [8]),

$$a_h(\hat{\mathbf{u}}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)^d} \quad \forall \mathbf{v}_h \in \mathbb{C}\mathbb{R}_0(\mathcal{K}_h)^d. \quad (45)$$

We consider the following variation of (31): Find  $\mathbf{u}_h \in \mathbf{U}_h$  such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathcal{I}_h^{\text{CR}}(\mathbf{v}_h))_{L^2(\Omega)^d} \quad \forall \mathbf{v}_h \in \mathbf{U}_h, \quad (46)$$

where the sole difference with respect to (31) lies in the treatment of the right-hand side.

**Lemma 25** (Relation between (45) and (46)). *There holds  $\mathbf{u}_h = \hat{\mathbf{u}}_h$ .*

*Proof.* Let  $\mathbb{V}_{h,0}^d \ni \hat{\mathbf{u}}_h = (\hat{\mathbf{u}}_{h,i})_{1 \leq i \leq d}$  be such that, for  $i \in \{1, \dots, d\}$ ,

$$\hat{\mathbf{u}}_{h,i} = ((\hat{\mathbf{u}}_{h,i}(\mathbf{x}_K))_{K \in \mathcal{K}_h}, (\hat{\mathbf{u}}_{h,i}(\bar{\mathbf{x}}_F))_{F \in \mathcal{F}_{\mathcal{K}_h}}) \in \mathbb{V}_{h,0}.$$

Clearly, for all  $K \in \mathcal{K}_h$ , all  $F \in \mathcal{F}_K$ , and all  $i \in \{1, \dots, d\}$ ,  $\mathbf{R}_{K_F}(\hat{\mathbf{u}}_{h,i}) = \mathbf{0}$ , hence  $\mathbf{G}_{K_F}(\hat{\mathbf{u}}_{h,i}) = \mathbf{G}_K(\hat{\mathbf{u}}_{h,i}) = (\nabla_h \hat{\mathbf{u}}_{h,i})|_K$ . As a consequence,  $\hat{\mathbf{u}}_h = \mathfrak{R}_h(\hat{\mathbf{u}}_h)$ . Accounting for this fact, there holds for all  $\mathbf{v}_h \in \mathbf{U}_h$  such that  $\mathbf{v}_h = \mathfrak{R}_h(\mathbb{v}_h)$  with  $\mathbb{v}_h \in \mathbb{V}_{h,0}^d$ ,

$$\begin{aligned} a_h(\hat{\mathbf{u}}_h, \mathbf{v}_h) &= \sum_{i=1}^d \sum_{K \in \mathcal{K}_h} \sum_{F \in \mathcal{F}_K} |K_F| \left\{ \mu \mathbf{G}_K(\hat{\mathbf{u}}_{h,i}) \cdot \mathbf{G}_{K_F}(\mathbb{v}_{h,i}) + \mu \mathbf{G}_K(\hat{\mathbf{u}}_{h,i}) \cdot \mathbf{e}_i D_{K_F}(\mathbb{v}_h) \right. \\ &\quad \left. + \lambda \mathbf{G}_K(\hat{\mathbf{u}}_{h,i}) \cdot \mathbf{e}_i D_K(\mathbb{v}_h) \right\} \\ &= \sum_{i=1}^d \sum_{K \in \mathcal{K}_h} |K| \left\{ \mu \mathbf{G}_K(\hat{\mathbf{u}}_{h,i}) \cdot \mathbf{G}_K(\mathbb{v}_{h,i}) + (\mu + \lambda) \mathbf{G}_K(\hat{\mathbf{u}}_{h,i}) \cdot \mathbf{e}_i D_K(\mathbb{v}_h) \right\} \\ &= a_h(\hat{\mathbf{u}}_h, \mathcal{I}_h^{\text{CR}}(\mathbf{v}_h)) = (\mathbf{f}, \mathcal{I}_h^{\text{CR}}(\mathbf{v}_h))_{L^2(\Omega)^d}, \end{aligned}$$

where the first passage is a consequence of (17) and we have let, for the sake of conciseness  $D_K(\mathbb{v}_h) := \sum_{j=1}^d \mathbf{G}_K(\mathbb{v}_{h,j}) \cdot \mathbf{e}_j$  and  $D_{K_F}(\mathbb{v}_h) := \sum_{j=1}^d \mathbf{G}_{K_F}(\mathbb{v}_{h,j}) \cdot \mathbf{e}_j$ . Owing to the coercivity of  $a_h$ , problem (46) admits a unique solution and we therefore conclude that  $\hat{\mathbf{u}}_h = \mathbf{u}_h$ .  $\square$

#### 4.2.5 Mixed boundary conditions

For the sake of completeness, we briefly address here the case of mixed boundary conditions for the linear elasticity problem. For simplicity of exposition, we focus on the homogeneous case, the generalization to the nonhomogeneous one being straightforward. The continuous problem reads

$$\begin{aligned} -\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega^{\text{D}}, \\ \underline{\underline{\sigma}}(\mathbf{u})\mathbf{n} &= \mathbf{0} && \text{on } \partial\Omega^{\text{N}}, \end{aligned} \quad (47)$$

where  $\partial\Omega^{\text{D}}$  and  $\partial\Omega^{\text{N}}$  are such that  $\partial\Omega^{\text{D}} \neq \emptyset$ ,  $\partial\Omega^{\text{D}} \cap \partial\Omega^{\text{N}} = \emptyset$  and  $\partial\Omega^{\text{D}} \cup \partial\Omega^{\text{N}} = \partial\Omega$ . Let  $\mathbf{U} := H_{\text{D}}^1(\Omega)^d$  with  $H_{\text{D}}^1(\Omega) := \{v \in H^1(\Omega) \mid v|_{\partial\Omega^{\text{D}}} = 0\}$ . When dealing with mixed Dirichlet/Neumann boundary conditions, the weak formulation (28) is no longer equivalent to the continuous problem (47), and (26) must be used instead: Find  $\mathbf{u} \in \mathbf{U}$  such that

$$\tilde{a}(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{L^2(\Omega)^d} \quad \forall \mathbf{v} \in \mathbf{U}. \quad (48)$$

The well-posedness of the weak formulation (48) relies on Korn's inequality (27), which is still valid on the space  $H_{\text{D}}^1(\Omega)^d$ . Likewise, the regularity estimate of Lemma 16 holds for problem (47) in the case of a convex polygonal domain, with the above definition of  $\mathbf{U}$  and  $\mathbf{U}_* := \mathbf{U} \cap H^2(\Omega)^d$ . We introduce the following  $H_{\text{D}}^1(\Omega)$ -like discrete space to approximate the displacement field:

$$\mathfrak{R}_{\text{D}}(\mathcal{K}_h) := \mathfrak{R}_h(\mathbb{V}_{h,\text{D}}), \quad \mathbb{V}_{h,\text{D}} = \{\mathbb{v}_h \in \mathbb{V}_h \mid v_F = 0, \forall F \in \mathcal{F}_{\mathcal{K}_h}^{\text{b,D}}\},$$

where  $\mathcal{F}_{\mathcal{K}_h}^{\text{b},\text{D}} := \{F \in \mathcal{F}_{\mathcal{K}_h}^{\text{b}} \mid F \subset \partial\Omega^{\text{D}}\}$ . We also introduce the set  $\mathcal{F}_{\mathcal{K}_h}^{\text{b},\text{N}} := \mathcal{F}_{\mathcal{K}_h}^{\text{b}} \setminus \mathcal{F}_{\mathcal{K}_h}^{\text{b},\text{D}}$ . Note that  $\|\nabla_h v_h\|_{L^2(\Omega)^d}$  is still a norm on  $\mathfrak{R}_D(\mathcal{K}_h)$ . With  $\mathbf{U}_h := \mathfrak{R}_D(\mathcal{K}_h)^d$  the discrete problem reads: Find  $\mathbf{u}_h \in \mathbf{U}_h$  such that

$$\tilde{a}_h(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)^d} \quad \forall \mathbf{v}_h \in \mathbf{U}_h, \quad (49)$$

with discrete bilinear form  $\tilde{a}_h$  such that

$$\tilde{a}_h(\mathbf{w}, \mathbf{v}) := (2\mu \underline{\underline{\varepsilon}}_h(\mathbf{w}), \underline{\underline{\varepsilon}}_h(\mathbf{v}))_{L^2(\Omega)^{d,d}} + (\lambda D_h(\mathbf{w}), D_h(\mathbf{v}))_{L^2(\Omega)} + \sum_{F \in \mathcal{F}_{\mathcal{P}_h} \setminus \mathcal{F}_{\mathcal{K}_h}^{\text{b},\text{N}}} h_F^{-1} (2\mu \llbracket \mathbf{w} \rrbracket, \llbracket \mathbf{v} \rrbracket)_{L^2(F)^d}. \quad (50)$$

The least-square stabilization of the jumps, inspired from [29], is necessary to ensure the coercivity of the bilinear form  $\tilde{a}_h$  based on the following discrete Korn's inequality, which is a variant of [6, eq. (1.12)].

**Lemma 26** (Discrete Korn's inequality). *There exists a real  $C_K > 0$ , independent of the mesh-size but depending on the mesh regularity parameters and on  $\Omega$ , such that there holds for all  $\mathbf{v} = (v_i)_{1 \leq i \leq d} \in H^1(\mathcal{P}_h)^d$ ,*

$$\|\nabla_h \mathbf{v}\|_{L^2(\Omega)^{d,d}} \leq C_K \left( \|\underline{\underline{\varepsilon}}_h(\mathbf{v})\|_{L^2(\Omega)^{d,d}}^2 + |\mathbf{v}|_{\text{J},\text{D}}^2 \right)^{\frac{1}{2}},$$

where  $|\mathbf{v}|_{\text{J},\text{D}}^2 := \sum_{i=1}^d |v_i|_{\text{J},\text{D}}^2$  and  $|\mathbf{v}|_{\text{J},\text{D}}^2 := \sum_{F \in \mathcal{F}_{\mathcal{P}_h} \setminus \mathcal{F}_{\mathcal{K}_h}^{\text{b},\text{N}}} h_F^{-1} \|\llbracket \mathbf{v} \rrbracket\|_{L^2(F)}^2$ .

**Remark 27** (Penalty term). *We stress that in this case it is not possible to integrate the penalty term in (50) using the face barycenter as a quadrature point, since this would yield a vanishing contribution. A quadrature rule exact for polynomials of degree at least 2 must be used instead.*

The energy norm associated to the bilinear form  $\tilde{a}_h$  is

$$\|\mathbf{v}\|_{\text{el}}^2 := \tilde{a}_h(\mathbf{v}, \mathbf{v}) = \|(2\mu)^{1/2} \underline{\underline{\varepsilon}}_h(\mathbf{v})\|_{L^2(\Omega)^{d,d}}^2 + \|\lambda^{1/2} D_h(\mathbf{v})\|_{L^2(\Omega)}^2 + (2\mu)^{1/2} |\mathbf{v}|_{\text{J},\text{D}}^2. \quad (51)$$

**Lemma 28** (Coercivity). *There holds for all  $\mathbf{v}_h \in \mathbf{U}_h$ ,*

$$\tilde{a}_h(\mathbf{v}_h, \mathbf{v}_h) = \|\mathbf{v}_h\|_{\text{el}}^2 \geq 2\mu C_K^{-2} \|\nabla_h \mathbf{v}_h\|_{L^2(\Omega)^{d,d}}^2.$$

The well-posedness of problem (49) is now straightforward. Defining  $\mathbf{U}_{*h} := \mathbf{U}_* + \mathbf{U}_h$  and extending  $\tilde{a}_h$  to  $\mathbf{U}_{*h} \times \mathbf{U}_{*h}$ , the convergence analysis can be carried out in a similar way as in Section 4.2.2. Assuming  $\mathbf{u} \in \mathbf{U}_*$ , the consistency error becomes, for all  $\mathbf{v}_h \in \mathbf{U}_h$ ,

$$\mathcal{E}_h(\mathbf{v}_h) := \sum_{F \in \mathcal{F}_{\mathcal{P}_h} \setminus \mathcal{F}_{\mathcal{K}_h}^{\text{b},\text{N}}} (\underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n}_F, \llbracket \mathbf{v}_h \rrbracket)_{L^2(F)^d} + (\lambda(D_h(\mathbf{u}) - \nabla \cdot \mathbf{u}), \nabla_h \cdot \mathbf{v}_h)_{L^2(\Omega)}.$$

Using the same notation as in the proof of Theorem 20, with  $\mathbf{w}_h := \mathcal{I}_h^{\text{er}}(\mathbf{u}) \in \mathbf{U}_h$ , the term  $\mathfrak{T}_1$  can be handled as

$$\mathfrak{T}_1 \leq \|\mathbf{u} - \mathbf{w}_h\|_{\text{el}} \lesssim h \|\mathbf{u}\|_{H^2(\Omega)^d} + h |\lambda^{1/2} \nabla \cdot \mathbf{u}|_{H^1(\Omega)},$$

using Lemma 12, Corollary 14, and the trace inequality (6) with  $\mathcal{S}_h = \mathcal{P}_h$  combined with Lemma 12, respectively to treat the three terms in the right-hand side of (51) with  $\mathbf{v} = \mathbf{u} - \mathbf{w}_h$ . The terms  $\mathfrak{T}_{2,1}$  and  $\mathfrak{T}_{2,2}$  can be treated by using exactly the same arguments as in the proof of Theorem 20. Finally, if  $d = 2$ , we again obtain a locking-free error estimate of the form (30).

### 4.3 Stokes: inf-sup stability and large irrotational body forces

We address in this section the questions of the discrete inf-sup stability and the numerical treatment of large irrotational body forces in the approximation of the Stokes problem on general polygonal or polyhedral meshes. This section also contains the proofs of some results announced but not proved by the authors in [19].

#### 4.3.1 Setting

We consider the viscous flow of a Newtonian fluid of constant viscosity taken equal to 1 for the sake of simplicity. The Stokes equations governing the motion of the fluid read

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} & \text{on } \partial\Omega, \\ \int_{\Omega} p &= 0, \end{aligned} \tag{52}$$

where  $\mathbf{f} \in L^2(\Omega)^d$  represents the volumic body force acting on the fluid. Let  $\mathbf{U} := H_0^1(\Omega)^d$ ,  $P := L_0^2(\Omega)$  with  $L_0^2(\Omega) := \{q \in L^2(\Omega) \mid \int_{\Omega} q = 0\}$ , and set  $\mathbf{X} := \mathbf{U} \times P$ . The weak formulation of problem (52) reads: Find  $(\mathbf{u}, p) \in \mathbf{X}$  such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - b(\mathbf{u}, q) = (\mathbf{f}, \mathbf{v})_{L^2(\Omega)^d} \quad \forall (\mathbf{v}, q) \in \mathbf{X}, \tag{53}$$

where

$$a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v})_{L^2(\Omega)^{d,d}}, \quad b(\mathbf{v}, p) = -(\nabla \cdot \mathbf{v}, p)_{L^2(\Omega)}. \tag{54}$$

#### 4.3.2 Discretization

Let  $\mathcal{K}_h$  belong to an admissible mesh sequence. We consider an approximation based on the following spaces for the velocity and pressure respectively:

$$\mathbf{U}_h := \mathfrak{C}\mathfrak{R}_0(\mathcal{K}_h)^d, \quad P_h := \mathbb{P}_d^0(\mathcal{K}_h) \cap L_0^2(\Omega).$$

We assume the choice  $\eta = d$  in (9), so that the continuity of mean values stated in Lemma 10 holds. As for the discretization of the linear elasticity problem discussed in Section 4.2, this property is exploited to estimate the consistency error; cf. the proof of Theorem 33. We equip  $\mathbf{U}_h$  with the norm  $\|\mathbf{v}\|_{\mathbf{U}} := \|\nabla_h \mathbf{v}\|_{L^2(\Omega)^{d,d}}$ , see Proposition 15, and  $P_h$  with the norm  $\|q\|_P := \|q\|_{L^2(\Omega)}$ . Finally, we introduce the product space  $\mathbf{X}_h := \mathbf{U}_h \times P_h$  equipped with the norm

$$\|(\mathbf{v}, q)\|_{\text{sto}}^2 := \|\mathbf{v}\|_{\mathbf{U}}^2 + \|q\|_P^2. \tag{55}$$

We consider the following discrete problem: Find  $(\mathbf{u}_h, p_h) \in \mathbf{X}_h$  such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) - b_h(\mathbf{u}_h, q_h) = (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)^d} \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h, \tag{56}$$

where

$$a_h(\mathbf{w}, \mathbf{v}) = (\nabla_h \mathbf{w}, \nabla_h \mathbf{v})_{L^2(\Omega)^{d,d}}, \quad b_h(\mathbf{v}, q) = -(\nabla_h \cdot \mathbf{v}, q)_{L^2(\Omega)}. \tag{57}$$

Alternatively, problem (56) can be formulated as: Find  $(\mathbf{u}_h, p_h) \in \mathbf{X}_h$  such that

$$c_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)^d} \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h,$$

with bilinear form  $c_h((\mathbf{w}, r), (\mathbf{v}, q)) = a_h(\mathbf{w}, \mathbf{v}) + b_h(\mathbf{v}, r) - b_h(\mathbf{w}, q)$ . The stability of the discrete problem (56) relies on the coercivity of the bilinear form  $a_h$  (readily following from Proposition 15) and on the inf-sup stability of the bilinear form  $b_h$ .

**Lemma 29** (inf-sup stability for  $b_h$ ). *There exists  $\beta > 0$  independent of the meshsize, such that, for all  $q_h \in P_h$ ,*

$$\beta \|q_h\|_P \leq \sup_{\mathbf{w}_h \in \mathbf{U}_h \setminus \{0\}} \frac{b_h(\mathbf{w}_h, q_h)}{\|\mathbf{w}_h\|_{\mathbf{U}}}. \quad (58)$$

*Proof.* The notation  $a \lesssim b$  stands for  $a \leq Cb$  with  $C > 0$  independent of the meshsize. Let  $q_h \in P_h \subset P$ . We invoke the surjectivity of the divergence operator from  $\mathbf{U}$  to  $P$  to infer the existence of  $\mathbf{v} \in \mathbf{U}$  and a real  $C_\Omega > 0$  solely depending on  $\Omega$  such that  $\nabla \cdot \mathbf{v} = q_h$  and  $\|\mathbf{v}\|_{H^1(\Omega)^d} \leq C_\Omega \|q_h\|_{L^2(\Omega)}$ . Let  $\mathbf{v}_h := \mathcal{I}_h^{\text{opt}}(\mathbf{v}) \in \mathbf{U}_h$ . Using the triangular inequality followed by (16) with  $l = 0$ , it is inferred

$$\|\mathbf{v}_h\|_{\mathbf{U}} \leq \|\nabla \mathbf{v} - \nabla_h \mathbf{v}_h\|_{L^2(\Omega)^{d,d}} + \|\nabla \mathbf{v}\|_{L^2(\Omega)^{d,d}} \lesssim \|\mathbf{v}\|_{H^1(\Omega)^d},$$

which can be interpreted as a  $H^1$ -stability property for the  $\mathcal{I}_h^{\text{opt}}$  interpolator. Moreover, using the fact that  $q_h$  is piecewise constant, there follows from the definition of the  $L^2$ -orthogonal projector together with Corollary 14,

$$b_h(\mathbf{v}_h, q_h) = -(\nabla_h \cdot \mathbf{v}_h, q_h)_{L^2(\Omega)} = -(\Pi_h^0(\nabla_h \cdot \mathbf{v}_h), q_h)_{L^2(\Omega)} = -(\Pi_h^0(\nabla \cdot \mathbf{v}), q_h)_{L^2(\Omega)} = b_h(\mathbf{v}, q_h).$$

Finally, denoting by  $\$$  the supremum in (58) and using the previous intermediate results,

$$\|q_h\|_{L^2(\Omega)}^2 = (\nabla \cdot \mathbf{v}, q_h)_{L^2(\Omega)} = -b_h(\mathbf{v}, q_h) = -b_h(\mathbf{v}_h, q_h) \leq \$\|\mathbf{v}_h\|_{\mathbf{U}} \lesssim \$\|\mathbf{v}\|_{H^1(\Omega)^d} \lesssim \$\|q_h\|_{L^2(\Omega)},$$

which concludes the proof.  $\square$

The following inf-sup condition for  $c_h$  is a classical consequence of Lemma 29 together with the coercivity of  $a_h$ .

**Lemma 30** (inf-sup stability for  $c_h$ ). *There exists a real  $\gamma > 0$  independent of the meshsize such that, for all  $(\mathbf{v}_h, q_h) \in \mathbf{X}_h$ ,*

$$\gamma \|(\mathbf{v}_h, q_h)\|_{\text{sto}} \leq \sup_{(\mathbf{w}_h, r_h) \in \mathbf{X}_h \setminus \{0\}} \frac{c_h((\mathbf{v}_h, q_h), (\mathbf{w}_h, r_h))}{\|(\mathbf{w}_h, r_h)\|_{\text{sto}}}.$$

The well-posedness of the discrete problem (56) follows. Let now  $\mathbf{U}_* := \mathbf{U} \cap H^2(\Omega)^d$ ,  $P_* := P \cap H^1(\Omega)$ ,  $\mathbf{X}_* := \mathbf{U}_* \times P_*$ , and define the augmented spaces

$$\mathbf{U}_{*h} := \mathbf{U}_* + \mathbf{U}_h, \quad P_{*h} := P_* + P_h, \quad \mathbf{X}_{*h} := \mathbf{U}_{*h} \times P_{*h}.$$

The bilinear forms  $a_h$ ,  $b_h$ , and  $c_h$  are extended to  $\mathbf{U}_{*h} \times \mathbf{U}_{*h}$ ,  $\mathbf{U}_{*h} \times P_{*h}$ , and  $\mathbf{X}_{*h} \times \mathbf{X}_{*h}$  respectively. As in Lemma 19, we assume additional regularity for the exact solution to ensure that the boundary terms in the expression of the consistency error are well-defined. Sufficient conditions for the regularity  $(\mathbf{u}, p) \in \mathbf{X}_*$  to hold are studied, e.g., by Cattabriga [12] and Amrouche and Girault [2].

**Lemma 31** (Weak consistency). *Let  $(\mathbf{u}, p) \in \mathbf{X}$  denote the solution of (53), and additionally assume  $(\mathbf{u}, p) \in \mathbf{X}_*$ . Then, there holds, for all  $(\mathbf{v}_h, q_h) \in \mathbf{X}_h$ ,*

$$a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p) - b_h(\mathbf{u}, q_h) = (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)^d} + \mathcal{E}_h(\mathbf{v}_h),$$

with consistency error

$$\mathcal{E}_h(\mathbf{v}_h) := \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} (\nabla \mathbf{u} \mathbf{n}_F - p \mathbf{n}_F, \llbracket \mathbf{v}_h \rrbracket)_{L^2(F)^d}.$$



*Proof.* Element-wise integration by parts of the terms in the discrete momentum equation together with the fact that  $b_h(\mathbf{u}, q_h) = -(\nabla \cdot \mathbf{u}, q_h)_{L^2(\Omega)} = 0$  yields

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p) - b_h(\mathbf{u}, q_h) &= (-\Delta \mathbf{u} + \nabla p, \mathbf{v}_h)_{L^2(\Omega)^d} \\ &+ \sum_{F \in \mathcal{F}_{\mathcal{P}_h}^i} (\llbracket \nabla \mathbf{u} - p \underline{I}_d \rrbracket \mathbf{n}_F, \{\mathbf{v}_h\})_{L^2(F)^d} + \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} (\{\nabla \mathbf{u} - p \underline{I}_d\} \mathbf{n}_F, \llbracket \mathbf{v}_h \rrbracket)_{L^2(F)^d}. \end{aligned}$$

The conclusion follows using the continuity of the flux for the continuous problem to infer  $\llbracket \nabla \mathbf{u} - p \underline{I}_d \rrbracket_F \mathbf{n}_F = \mathbf{0}$  and  $\{\nabla \mathbf{u} - p \underline{I}_d\}_F \mathbf{n}_F = (\nabla \mathbf{u} - p \underline{I}_d) \mathbf{n}_F$  for all  $F \in \mathcal{F}_{\mathcal{P}_h}^i$ .  $\square$

The following result is obtained applying several times the Cauchy–Schwarz inequality and recalling (55).

**Lemma 32** (Boundedness). *There exists a real  $C_{\text{bnd}} > 0$  independent of the meshsize such that, for all  $(\mathbf{v}, q) \in \mathbf{X}_{*h}$  and all  $(\mathbf{w}_h, r_h) \in \mathbf{X}_h$ ,*

$$c_h((\mathbf{v}, q), (\mathbf{w}_h, r_h)) \leq C_{\text{bnd}} \|(\mathbf{v}, q)\|_{\text{sto}} \|(\mathbf{w}_h, r_h)\|_{\text{sto}}.$$

**Theorem 33** (Error estimate for (56)). *Under the assumptions of Lemma 31, there holds with  $C > 0$  independent of the meshsize, of  $\mathbf{u}$ , and of  $p$ , and  $\mathcal{N}_{\text{sto}}(\mathbf{u}, p) := \|\mathbf{u}\|_{H^2(\Omega)^d} + \|p\|_{H^1(\Omega)}$ ,*

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\text{sto}} \leq Ch \mathcal{N}_{\text{sto}}(\mathbf{u}, p).$$

*Proof.* The notation  $a \lesssim b$  stands for  $a \leq Cb$  with  $C > 0$  independent of the meshsize. Lemmata 30, 31, and 32 together with the Second Strang Lemma yield the following estimate:

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\text{sto}} \lesssim \inf_{(\mathbf{v}_h, q_h) \in \mathbf{X}_h} \|(\mathbf{u} - \mathbf{v}_h, p - q_h)\|_{\text{sto}} + \sup_{\mathbf{v}_h \in \mathbf{U}_h \setminus \{\mathbf{0}\}} \frac{|\mathcal{E}_h(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{\mathbf{U}}} := \mathfrak{T}_1 + \mathfrak{T}_2.$$

The first term in the right-hand side of the error estimate is related to the approximation properties of  $\mathbf{X}_h$  with respect to the  $\|\cdot\|_{\text{sto}}$ -norm, whereas the second term is linked to the consistency error. Letting  $(\mathbf{w}_h, r_h) := (\mathcal{I}_h^{\text{cor}}(\mathbf{u}), \Pi_h^0 p) \in \mathbf{X}_h$  (the fact that  $r_h$  has vanishing mean value on  $\Omega$  results from the properties of the  $L^2$ -orthogonal projector), there holds

$$\mathfrak{T}_1 \leq \|(\mathbf{u} - \mathbf{w}_h, p - r_h)\|_{\text{sto}} \leq \|\mathbf{u} - \mathbf{w}_h\|_{\mathbf{U}} + \|p - r_h\|_P \lesssim h \mathcal{N}_{\text{sto}}(\mathbf{u}, p),$$

where the last inequality results from (16) together with the approximation properties of the  $L^2$ -orthogonal projector. To treat the second term we use a similar argument as in Theorem 20. Let

$$\underline{\phi} := \nabla \mathbf{u} - \Pi_h^0(\nabla \mathbf{u}), \quad \psi := p - \Pi_h^0 p.$$

Invoking the continuity of mean values at interfaces together with the fact that both  $\{\Pi_h^0(\nabla \mathbf{u})\}_F$  and  $\{\Pi_h^0 p\}_F$  are constant on every  $F \in \mathcal{F}_{\mathcal{P}_h}$  yields for all  $\mathbf{v}_h \in \mathbf{U}_h$ ,

$$\mathcal{E}_h(\mathbf{v}_h) = \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} (\{\underline{\phi}\} \mathbf{n}_F, \llbracket \mathbf{v}_h \rrbracket - \langle \llbracket \mathbf{v}_h \rrbracket \rangle_F)_{L^2(F)^d} - \sum_{F \in \mathcal{F}_{\mathcal{P}_h}} (\{\psi\} \mathbf{n}_F, \llbracket \mathbf{v}_h \rrbracket - \langle \llbracket \mathbf{v}_h \rrbracket \rangle_F)_{L^2(F)^d}.$$

Proceeding as in the proof of Theorem 20 gives  $\mathfrak{T}_2 \lesssim h \mathcal{N}_{\text{sto}}(\mathbf{u}, p)$ , thereby concluding the proof.  $\square$

**Remark 34** ( $L^2$ -error estimate for the velocity). *Optimal error estimates for the  $L^2$ -error on the velocity can be derived proceeding as in [17, Section 6.1.3.3].*

### 4.3.3 Variations and links with finite volume and nonconforming finite element methods

**Flux formulation and local conservation** In this section we reformulate the discrete problem (56) in terms of numerical fluxes and identify a local conservation property. Let  $(\mathbf{w}_h, r_h), (\mathbf{v}_h, q_h) \in \mathbf{X}_h$  be two discrete functions, and denote by  $(\mathfrak{w}_h, \mathfrak{r}_h) \in \mathbb{X}_h$  and  $(\mathfrak{v}_h, \mathfrak{q}_h) \in \mathbb{X}_h$  the corresponding vectors of DOFs, where we have set

$$\mathbb{P}_h := \{\mathfrak{q}_h \in \mathbb{R}^{\mathcal{K}_h} \mid \sum_{K \in \mathcal{K}_h} |K| q_K = 0\}, \quad \mathbb{X}_h := \mathbb{V}_{h,0}^d \times \mathbb{P}_h.$$

The vector of DOFs for each component of the velocity is denoted by a subscript as in Section 4.2.4. Our goal is to show that there exist two families of numerical fluxes  $(\Phi_{K,F}(\mathfrak{w}_h, \mathfrak{r}_h))_{K \in \mathcal{K}_h, F \in \mathcal{F}_K}$  with  $\Phi_{K,F}(\mathfrak{w}_h, \mathfrak{r}_h) = (\Phi_{K,F,i}(\mathfrak{w}_h, \mathfrak{r}_h))_{1 \leq i \leq d}$  and  $(\phi_F(\mathfrak{w}_h))_{F \in \mathcal{F}_{\mathcal{K}_h}}$  such that

$$\begin{aligned} a_h(\mathbf{w}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, r_h) &= \sum_{i=1}^d \sum_{K \in \mathcal{K}_h} \sum_{F \in \mathcal{F}_K} \Phi_{K,F,i}(\mathfrak{w}_h, \mathfrak{r}_h) (v_{F,i} - v_{K,i}), \\ -b_h(\mathbf{w}_h, q_h) &= \sum_{F \in \mathcal{F}_{\mathcal{K}_h}} \phi_F(\mathfrak{w}_h) \llbracket \mathfrak{q}_h \rrbracket_F, \end{aligned} \quad (59)$$

where  $a_h$  and  $b_h$  are defined by (57) and, with a slight abuse in notation, we have set for all  $F \in \mathcal{F}_{\mathcal{K}_h}$   $\llbracket \mathfrak{q}_h \rrbracket_F := \llbracket q_h \rrbracket_F$ . Using the cell center as a quadrature node in each element to approximate the right-hand side, the discrete problem (56) in algebraic form reads: Find  $(\mathfrak{u}_h, \mathfrak{p}_h) \in \mathbb{X}_h$  such that for all  $(\mathfrak{v}_h, \mathfrak{q}_h) \in \mathbb{X}_h$  there holds,

$$\sum_{i=1}^d \sum_{K \in \mathcal{K}_h} \sum_{F \in \mathcal{F}_K} \Phi_{K,F,i}(\mathfrak{u}_h, \mathfrak{p}_h) (v_{F,i} - v_{K,i}) + \sum_{F \in \mathcal{F}_{\mathcal{K}_h}} \phi_F(\mathfrak{u}_h) \llbracket \mathfrak{q}_h \rrbracket_F = \sum_{i=1}^d \sum_{K \in \mathcal{K}_h} |K| f_i(\mathbf{x}_K) v_{K,i}.$$

Proceeding as in Section 4.2.4, we can prove that for every interface  $F \in \mathcal{F}_{\mathcal{K}_h}^i$  such that  $F \subset \partial K_1 \cap \partial K_2$  there holds,

$$\Phi_{K_1,F}(\mathfrak{u}_h, \mathfrak{p}_h) = -\Phi_{K_2,F}(\mathfrak{u}_h, \mathfrak{p}_h).$$

Moreover, the mass flux  $\phi_F(\mathfrak{u}_h)$  is single-valued, and therefore conservative.

**Proposition 35** (Flux formulation). *For all  $(\mathbf{w}_h, r_h), (\mathbf{v}_h, q_h) \in \mathbf{X}_h$ , the flux formulation (59) is obtained setting for all  $K \in \mathcal{K}_h$  and all  $F \in \mathcal{F}_K$ ,*

$$\Phi_{K,F,i}(\mathfrak{w}_h, \mathfrak{r}_h) := \sum_{F' \in \mathcal{F}_K} |K_{F'}| (\mathbf{G}_{K_{F'}}(\mathfrak{w}_{h,i}) + r_K \mathbf{e}_i) \cdot \mathbf{y}_{F',F}^K \quad \forall i \in \{1, \dots, d\}, \quad (60a)$$

$$\phi_F(\mathfrak{w}_h) := |F| \sum_{i=1}^d w_{F,i} n_{F,i} \quad \forall F \in \mathcal{F}_{\mathcal{K}_h}, \quad (60b)$$

where  $(\mathfrak{w}_h, \mathfrak{r}_h), (\mathfrak{v}_h, \mathfrak{q}_h) \in \mathbb{X}_h$  are the vectors of DOFs associated to  $(\mathbf{w}_h, r_h)$  and  $(\mathbf{v}_h, q_h)$  respectively,  $(\mathbf{e}_i)_{1 \leq i \leq d}$  denotes the canonical basis of  $\mathbb{R}^d$ , and the family of vectors  $(\mathbf{y}_{F',F}^K)_{K \in \mathcal{K}_h, F, F' \in \mathcal{F}_K}$  is defined by (43).

*Proof.* The proof of (60a) is similar to that of Proposition 24 and is not detailed for the sake of conciseness. To prove (60b) we observe that integration by parts yields

$$-b_h(\mathbf{w}_h, q_h) = -(\mathbf{w}_h, \nabla_h q_h)_{L^2(\Omega)^d} + \sum_{F \in \mathcal{F}_{\mathcal{K}_h}^i} (\llbracket \mathbf{w}_h \rrbracket \cdot \mathbf{n}_F, \{q_h\})_{L^2(F)} + \sum_{F \in \mathcal{F}_{\mathcal{K}_h}} (\{\mathbf{w}_h\} \cdot \mathbf{n}_F, \llbracket q_h \rrbracket)_{L^2(F)}.$$

The first and the second terms in the right-hand side vanish owing to the fact that  $(\nabla_h q_h)|_K = \mathbf{0}$  for all  $K \in \mathcal{K}_h$  and to Lemma 10 respectively. For the last term we use the fact that  $\llbracket q_h \rrbracket_F \in \mathbb{P}_d^0(F)$  for all  $F \in \mathcal{F}_{\mathcal{K}_h}$  to infer  $(\{\mathbf{w}_h\}_F \cdot \mathbf{n}_F, \llbracket q_h \rrbracket_F)_{L^2(F)} = |F| \langle \mathbf{w}_h \rangle_F \cdot \mathbf{n}_F \llbracket q_h \rrbracket_F = |F| \sum_{i=1}^d w_{F,i} n_{F,i} \llbracket \mathfrak{q}_h \rrbracket_F$ . This concludes the proof.  $\square$

**Link with the Crouzeix–Raviart solution on matching simplicial meshes** Assume that  $\mathcal{K}_h$  is a matching simplicial mesh. The classical Crouzeix–Raviart/ $\mathbb{P}_d^0$  method consists in finding  $(\hat{\mathbf{u}}_h, \hat{p}_h) \in \hat{\mathbf{X}}_h$  with  $\hat{\mathbf{X}}_h := \mathbb{C}\mathbb{R}_0(\mathcal{K}_h)^d \times P_h$  such that

$$a_h(\hat{\mathbf{u}}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, \hat{p}_h) - b_h(\hat{\mathbf{u}}_h, q_h) = (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)^d} \quad \forall (\mathbf{v}_h, q_h) \in \hat{\mathbf{X}}_h. \quad (61)$$

We consider the following variation of (56): Find  $(\mathbf{u}_h, p_h) \in \mathbf{X}_h$  such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) - b_h(\mathbf{u}_h, q_h) = (\mathbf{f}, \mathcal{I}_h^{\mathbb{C}\mathbb{R}}(\mathbf{v}_h))_{L^2(\Omega)^d} \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h, \quad (62)$$

where the sole difference lies in the treatment of the right-hand side. The proof of the following result is similar to that of Lemma 25 and is omitted for the sake of conciseness.

**Proposition 36** (Relation between (62) and (61)). *There holds  $\mathbf{u}_h = \hat{\mathbf{u}}_h$ .*

#### 4.3.4 Large irrotational body forces

The method of the previous section can be adapted to problems featuring volumic body forces with large irrotational components, as it is the case in natural convection with large Rayleigh numbers. As recently pointed out in Galvin, Linke, et al. [27], a careful design of the numerical method is required in this case to preserve an accurate approximation of the velocity field. We assume here the following Helmholtz decomposition of the volumic body forces in (52):

$$\mathbf{f} = \Psi - \nabla\varphi, \quad (63)$$

where  $\Psi \in \mathbf{H}_0(\text{div}; \Omega) := \{\mathbf{v} \in \mathbf{H}(\text{div}; \Omega) \mid \nabla \cdot \mathbf{v} = 0\}$  is a solenoidal vector field and  $\varphi \in H^1(\Omega) \cap L_0^2(\Omega)$  is a scalar potential. The weak formulation of problem (52) with right-hand side given by (63) reads: Find  $(\mathbf{u}, p) \in \mathbf{X}$  such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - b(\mathbf{u}, q) = l(\mathbf{v}) \quad \forall (\mathbf{v}, q) \in \mathbf{X}, \quad (64)$$

with bilinear forms  $a$  and  $b$  defined by (54) and  $l(\mathbf{v}) := (\Psi, \mathbf{v})_{L^2(\Omega)^d} - b(\mathbf{v}, \varphi)$ . Denoting by  $(\mathbf{u}_\Psi, p_\Psi)$  the solution to (64) with  $\varphi \equiv 0$  (no irrotational body forces), there holds

$$\mathbf{u} = \mathbf{u}_\Psi, \quad p = p_\Psi - \varphi. \quad (65)$$

As pointed out in [27, 19], mimicking or approaching property (65) at the discrete level is a key ingredient to design a discretization method yielding an accurate approximation of the velocity field for large values of  $\|\varphi\|_{H^1(\Omega)}$ . We consider the following approximation to (64): Find  $(\mathbf{u}_h, p_h) \in \mathbf{X}_h$  such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) - b_h(\mathbf{u}_h, q_h) = l_h(\mathbf{v}_h) \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h, \quad (66)$$

with bilinear forms  $a_h$  and  $b_h$  defined by (57) and  $l_h(\mathbf{v}_h) := (\Psi, \mathbf{v}_h)_{L^2(\Omega)^d} - b_h(\mathbf{v}_h, \Pi_h^0 \varphi)$ . The sole difference with respect to (56) lies in the treatment of the source term, which is designed so that the following property holds true.

**Proposition 37** (Discrete counterpart of property (65)). *Denote by  $(\mathbf{u}_{\Psi,h}, p_{\Psi,h})$  the solution to problem (66) with  $\varphi \equiv 0$ . There holds*

$$\mathbf{u}_h = \mathbf{u}_{\Psi,h}, \quad p_h = p_{\Psi,h} - \Pi_h^0 \varphi.$$

*Proof.* There holds

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h + \Pi_h^0 \varphi) - b_h(\mathbf{u}_h, q_h) = (\boldsymbol{\Psi}, \mathbf{v}_h)_{L^2(\Omega)^d}, \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h.$$

Hence, owing to the well-posedness of the discrete problem (56),  $(\mathbf{u}_{\boldsymbol{\Psi},h}, p_{\boldsymbol{\Psi},h}) = (\mathbf{u}_h, p_h + \Pi_h^0 \varphi)$ . The conclusion follows.  $\square$

**Theorem 38** (Error estimate for (66)). *Under the assumptions of Lemma 31 there holds with real numbers  $C_1 > 0$  and  $C_2 > 0$  independent of the meshsize, of  $\mathbf{u}$ , and of  $p$  but depending on the mesh regularity parameters and on  $\Omega$ ,*

$$\|\mathbf{u} - \mathbf{u}_h\|_U \leq C_1 h \mathcal{N}_{\text{sto}}(\mathbf{u}_{\boldsymbol{\Psi}}, p_{\boldsymbol{\Psi}}), \quad \|p - p_h\|_P \leq C_2 h (\mathcal{N}_{\text{sto}}(\mathbf{u}_{\boldsymbol{\Psi}}, p_{\boldsymbol{\Psi}}) + \|\varphi\|_{H^1(\Omega)}).$$

*Proof.* Using the estimate in Theorem 33 for the approximate solution to problem (64) with  $\varphi \equiv 0$  we infer

$$\|\mathbf{u}_{\boldsymbol{\Psi}} - \mathbf{u}_{\boldsymbol{\Psi},h}\|_U + \|p_{\boldsymbol{\Psi}} - p_{\boldsymbol{\Psi},h}\|_P \leq Ch \mathcal{N}_{\text{sto}}(\mathbf{u}_{\boldsymbol{\Psi}}, p_{\boldsymbol{\Psi}}), \quad (67)$$

where we have used the notation of Proposition 37 and  $C > 0$  has the same dependencies as  $C_1$  and  $C_2$ . The estimate for  $\|\mathbf{u} - \mathbf{u}_h\|_U$  is an immediate consequence of (65) and Proposition 37. To estimate  $\|p - p_h\|_P$  we invoke again (65) and Proposition 37 to infer

$$\|p - p_h\|_P \leq \|p_{\boldsymbol{\Psi}} - p_{\boldsymbol{\Psi},h}\|_P + \|\varphi - \Pi_h^0 \varphi\|_P,$$

and conclude using (67) and the approximation properties of the  $L^2$ -orthogonal projector.  $\square$

**Remark 39** (Robustness with respect to large irrotational body forces). *An important consequence of Theorem 38 is that the velocity approximation is not affected by the irrotational part of the source term, which yields the robustness of the method.*

## A Mimicking the properties of the lowest-order Raviart–Thomas space on general meshes

In the spirit of Section 3.1, it is possible to extend two classical properties of the lowest-order Raviart–Thomas space to general polygonal or polyhedral meshes, namely (i) the (full) continuity of normal components of discrete functions at interfaces and (ii) the existence of an interpolator which preserves the mean value of the divergence inside each element. Since the construction as well as the proofs are similar to the ones presented in Section 3, only the main points are detailed.

For a matching simplicial mesh, the standard lowest-order Raviart–Thomas space is defined as

$$\mathbb{RT}_d^0(\mathcal{K}_h) := \mathbb{P}_d^0(\mathcal{K}_h)^d + \mathbf{x} \mathbb{P}_d^0(\mathcal{K}_h), \quad (68)$$

i.e.,  $\mathbb{RT}_d^0(\mathcal{K}_h)$  is spanned by vector-valued functions obtained by a linear perturbation of a piecewise constant vector-valued field based on a piecewise constant, isotropic gradient. To perform a similar construction on general polygonal or polyhedral meshes, we consider the following space of DOFs, composed of vector cell unknowns and scalar face unknowns associated to the normal component of the discrete vector field:

$$\mathbb{V}_h := \{\mathbf{v}_h = ((\mathbf{v}_K \in \mathbb{R}^d)_{K \in \mathcal{K}_h}, (v_F^n \in \mathbb{R})_{F \in \mathcal{F}_{\mathcal{K}_h}})\}.$$

As it is the case for the extension of the Crouzeix–Raviart space discussed in Section 3, cell unknowns are used to define a piecewise constant subgrid correction on the pyramidal submesh.

The main difference with respect to the construction of Section 3 is that we now define an isotropic instead of a full gradient operator. More specifically, we introduce the operator  $\mathfrak{G}_h : \mathbb{V}_h \rightarrow \mathbb{P}_d^0(\mathcal{P}_h)$  which realizes the mapping  $\mathbb{V}_h \ni \mathbb{v}_h \mapsto \mathfrak{G}_h(\mathbb{v}_h) \in \mathbb{P}_d^0(\mathcal{P}_h)$  with

$$\mathfrak{G}_h(\mathbb{v}_h)|_{K_F} = G_K(\mathbb{v}_h) + R_{K_F}(\mathbb{v}_h), \quad \forall K \in \mathcal{K}_h, \forall F \in \mathcal{F}_K,$$

where

$$G_K(\mathbb{v}_h) := \frac{1}{d|K|} \sum_{F \in \mathcal{F}_K} v_F^n \mathbf{n}_F \cdot \mathbf{n}_{K,F}, \quad R_{K_F}(\mathbb{v}_h) := \frac{\eta}{d_{K,F}} (v_F^n \mathbf{n}_F - \mathbf{v}_K - G_K(\mathbb{v}_h)(\bar{\mathbf{x}}_F - \mathbf{x}_K)) \cdot \mathbf{n}_{K,F}, \quad (69)$$

and  $\eta > 0$  is a user-dependent parameter. Extending the definition (68) to general nonsimplicial meshes, we define the reconstruction operator  $\mathfrak{R}_h : \mathbb{V}_h \rightarrow \mathbb{RT}_d^0(\mathcal{P}_h)$  which realizes the mapping  $\mathbb{V}_h \ni \mathbb{v}_h \mapsto \mathfrak{R}_h(\mathbb{v}_h) \in \mathbb{RT}_d^0(\mathcal{P}_h)$  with

$$\mathfrak{R}_h(\mathbb{v}_h)|_{K_F}(\mathbf{x}) = \mathbf{v}_K + \mathfrak{G}_h(\mathbb{v}_h)|_{K_F}(\mathbf{x} - \mathbf{x}_K), \quad \forall K_F \in \mathcal{P}_h, \forall \mathbf{x} \in K_F. \quad (70)$$

Unlike (10), for all  $K \in \mathcal{K}_h$  and all  $F \in \mathcal{F}_K$ , there holds  $\mathbf{v}_K = \mathfrak{R}_h(\mathbb{v}_h)(\mathbf{x}_K)$ , i.e., the cell unknown can now be interpreted as the value of the reconstruction at the cell center. This is a consequence of selecting the cell center as a starting point in (70). We consider the discrete space

$$\mathfrak{RT}(\mathcal{K}_h) := \mathfrak{R}_h(\mathbb{V}_h).$$

**Lemma 40** ( $\mathbf{H}(\text{div}; \Omega)$ -conformity). *Assume  $\eta = 1$  in (69). Then, for all  $\mathbf{v}_h \in \mathfrak{RT}(\mathcal{K}_h)$  and all  $F \in \mathcal{F}_{\mathcal{P}_h}^i$ , there holds for all  $\mathbf{x} \in F$ ,*

$$\llbracket \mathbf{v}_h \rrbracket_F(\mathbf{x}) \cdot \mathbf{n}_F = 0.$$

*Proof.* Let  $\mathbf{v}_h \in \mathfrak{RT}(\mathcal{K}_h)$  with  $\mathbf{v}_h = \mathfrak{R}_h(\mathbb{v}_h)$ ,  $F \in \mathcal{F}_{\mathcal{P}_h}^i$ , and  $\mathbf{x} \in F$ . We distinguish two cases.

(i)  $F \in \mathcal{F}_{\mathcal{K}_h}^i$  is an interface of the primal mesh  $\mathcal{K}_h$ . Let  $K_1, K_2 \in \mathcal{K}_h$  be the elements such that  $F \subset \partial K_1 \cap \partial K_2$  and, for  $i \in \{1, 2\}$  let, for the sake of brevity,  $G_i := G_{K_i}(\mathbb{v}_h)$ ,  $R_i := R_{K_{iF}}(\mathbb{v}_h)$ ,  $d_i := d_{K_i, F}(\mathbf{n}_{K_i, F} \cdot \mathbf{n}_F)$ , and

$$\alpha_i := R_i(\mathbf{x} - \mathbf{x}_{K_i}) \cdot \mathbf{n}_F = R_i d_i = \eta (v_F^n \mathbf{n}_F - \mathbf{v}_{K_i} - G_i(\bar{\mathbf{x}}_F - \mathbf{x}_{K_i})) \cdot \mathbf{n}_F,$$

where we have used the fact that  $\mathbf{x} \in F$  to infer  $(\mathbf{x} - \mathbf{x}_{K_i}) \cdot \mathbf{n}_F = d_i$  and the fact that  $\mathbf{n}_F = \mathbf{n}_{K_1, F} = -\mathbf{n}_{K_2, F}$  to infer  $(\mathbf{n}_{K_1, F} \cdot \mathbf{n}_F) \mathbf{n}_{K_1, F} = \mathbf{n}_F$ . Algebraic manipulations yield

$$\alpha_1 - \alpha_2 = -\eta [(\mathbf{v}_{K_1} - \mathbf{v}_{K_2}) \cdot \mathbf{n}_F + G_1 d_1 - G_2 d_2].$$

Using the previous relation in the definition of the jump at  $\mathbf{x} \in F$  it is inferred,

$$\begin{aligned} \llbracket \mathbf{v}_h \rrbracket_F(\mathbf{x}) \cdot \mathbf{n}_F &= \mathbf{v}_h|_{K_{1F}}(\mathbf{x}) \cdot \mathbf{n}_F - \mathbf{v}_h|_{K_{2F}}(\mathbf{x}) \cdot \mathbf{n}_F \\ &= (\mathbf{v}_{K_1} - \mathbf{v}_{K_2}) \cdot \mathbf{n}_F + G_1 d_1 - G_2 d_2 + \alpha_1 - \alpha_2 \\ &= (1 - \eta) [(\mathbf{v}_{K_1} - \mathbf{v}_{K_2}) \cdot \mathbf{n}_F + G_1 d_1 - G_2 d_2]. \end{aligned}$$

As a consequence, the jump vanishes provided  $\eta = 1$ .

(ii)  $F \in \mathcal{F}_{\mathcal{P}_h}^i \setminus \mathcal{F}_{\mathcal{K}_h}^i$  is a lateral pyramidal face. In this case, there exist a unique element  $K \in \mathcal{K}_h$  and two faces  $F_1, F_2 \in \mathcal{F}_K$  such that  $F \subset \partial K_{F_1} \cap \partial K_{F_2}$  (cf. Figure 2a). There holds letting for the sake of brevity  $R_i := R_{K_{F_i}}(\mathbb{v}_h)$ ,  $i \in \{1, 2\}$ ,

$$\llbracket \mathbf{v}_h \rrbracket_F(\mathbf{x}) \cdot \mathbf{n}_F = \mathbf{v}_h|_{K_{F_1}}(\mathbf{x}) \cdot \mathbf{n}_F - \mathbf{v}_h|_{K_{F_2}}(\mathbf{x}) \cdot \mathbf{n}_F = (R_1 - R_2)(\mathbf{x} - \mathbf{x}_K) \cdot \mathbf{n}_F = 0,$$

since  $(\mathbf{x} - \mathbf{x}_K)$  and  $\mathbf{n}_F$  are orthogonal by definition. This concludes the proof.  $\square$

**Remark 41** (Role of  $\eta$  in the proof of Lemma 40). *Unlike Lemma 10, the subgrid correction parameter is here used to enforce the continuity of the normal component across interfaces belonging to the primal mesh rather than across lateral pyramidal faces. This is a consequence of choosing the cell center instead of the face barycenter as a starting point in (70).*

Let  $\mathcal{I}_h^{\mathfrak{RT}} : \mathbf{H}(\operatorname{div}; \Omega) \rightarrow \mathfrak{RT}(\mathcal{K}_h)$  be such that, for all  $\mathbf{v} \in \mathbf{H}(\operatorname{div}; \Omega)$ ,  $\mathcal{I}_h^{\mathfrak{RT}}(\mathbf{v}) := \mathfrak{R}_h(\mathbf{v}_h)$  with

$$\mathbb{V}_h \ni \mathbf{v}_h = ((\Pi_h^1 \mathbf{v}(\mathbf{x}_K))_{K \in \mathcal{K}_h}, (\langle \mathbf{v} \rangle_F \cdot \mathbf{n}_F)_{F \in \mathcal{F}_{\mathcal{K}_h}}).$$

The following result summarizes the most relevant approximation properties of  $\mathcal{I}_h^{\mathfrak{RT}}$ . The proof is omitted as it closely resembles that of Lemma 12.

**Lemma 42** (Approximation in  $\mathfrak{RT}(\mathcal{K}_h)$ ). *For all  $\eta > 0$  in (69) and all  $\mathbf{v} \in \mathbf{H}(\operatorname{div}; \Omega)$  there holds with  $\mathbf{v}_h := \mathcal{I}_h^{\mathfrak{RT}}(\mathbf{v})$ ,*

$$D_h(\mathbf{v}_h) = \Pi_h^0(\nabla \cdot \mathbf{v}).$$

*Moreover, there exists a real  $C > 0$  independent of the meshsize such that, for all  $h \in \mathcal{H}$ , all  $K \in \mathcal{K}_h$ , and all  $\mathbf{v} \in \mathbf{H}(\operatorname{div}; \Omega) \cap \mathbf{H}^1(\operatorname{div}; \mathcal{K}_h)$  with  $\mathbf{v}_h := \mathcal{I}_h^{\mathfrak{RT}}(\mathbf{v})$ , there holds*

$$\|\mathbf{v} - \mathbf{v}_h\|_{L^2(K)^d} + \|\nabla \cdot \mathbf{v} - D_h(\mathbf{v}_h)\|_{L^2(K)} \leq Ch_K \left( |\mathbf{v}|_{H^1(K)^d} + |\nabla \cdot \mathbf{v}|_{H^1(K)} \right).$$

**Remark 43** (The matching simplicial case). *When  $\mathcal{K}_h$  is matching simplicial, in the spirit of Proposition 11, we can prove that the standard Raviart–Thomas space is a subspace of  $\mathfrak{RT}(\mathcal{K}_h)$ . This can then be accounted for in the proof of Lemma 42 as in Remark 13. Note that, unlike Lemma 10, the assumption  $\eta = 1$  in Lemma 40 is here mandatory also in the matching simplicial case, since the cell unknown has been chosen as a starting point of the reconstruction, see Remark 41.*

## References

- [1] G. Allaire. *Analyse numérique et optimisation*. Les éditions de l’École Polytechnique, Palaiseau, 2009.
- [2] C. Amrouche and V. Girault. On the existence and regularity of the solution of Stokes problem in arbitrary dimension. *Proc. Japan. Acad.*, 67:171–175, 1991.
- [3] D. N. Arnold and F. Brezzi. Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates. *RAIRO Math. Model. Numer. Anal.*, 19(1):7–32, 1985.
- [4] D. N. Arnold, F. Brezzi, and J. Douglas. PEERS: A new mixed finite element for plane elasticity. *Japan Journal of Industrial and Applied Mathematics*, 1(2):347–367, 1984.
- [5] L. Beirão Da Veiga. A mimetic discretization method for linear elasticity. *M2AN Math. Model. Numer. Anal.*, 44(2):231–250, 2010.
- [6] S. C. Brenner. Korn’s inequalities for piecewise  $H^1$  vector fields. *Math. Comp.*, 73(247):1067–1087, 2004.
- [7] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.
- [8] S. C. Brenner and L.-Y. Sung. Linear finite element methods for planar linear elasticity. *Math. Comp.*, 59(200):321–338, 1992.

- [9] F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of the mimetic finite difference method for diffusion problems on polyhedral meshes. *SIAM J. Numer. Anal.*, 43(5):1872–1896, 2005.
- [10] F. Brezzi, K. Lipnikov, M. Shashkov, and V. Simoncini. A new discretization methodology for diffusion problems on generalized polyhedral meshes. *Comput. Methods Appl. Mech. Engrg.*, 196(37–40):3682–3692, 2007.
- [11] F. Brezzi, K. Lipnikov, and V. Simoncini. A family of mimetic finite difference methods on polygonal and polyhedral meshes. *Math. Mod. Meths. Appli. Sci. (M3AS)*, 15(10):1533–1551, 2005.
- [12] L. Cattabriga. Su un problema al contorno relativo al sistema di equazioni di Stokes. *Rend. Sem. Mat. Univ. Padova*, 31:308–340, 1961.
- [13] C. Chainais-Hillairet, G. Enchéry, and M. Mamaghani. Development of a refinement criterion for adaptive mesh refinement in steam-assisted gravity drainage simulation. *Comput. Geosc.*, 15(1):17–34, 2011.
- [14] K. S. Chavan, B. P. Lamichhane, and B. I. Wohlmuth. Locking-free finite element methods for linear and nonlinear elasticity in 2D and 3D. *Comput. Methods Appl. Mech. Engrg.*, 196:4075–4086, 2007.
- [15] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. *RAIRO Modél. Math. Anal. Num.*, 7(3):33–75, 1973.
- [16] D. A. Di Pietro. Cell centered Galerkin methods for diffusive problems. *M2AN Math. Model. Numer. Anal.*, 46(1):111–144, 2012.
- [17] D. A. Di Pietro and A. Ern. *Mathematical aspects of discontinuous Galerkin methods*, volume 69 of *Mathématiques & Applications*. Springer-Verlag, Berlin, 2011.
- [18] D. A. Di Pietro, J.-M. Gratien, and C. Prud’homme. A domain-specific embedded language in C++ for lowest-order discretizations of diffusive problems on general meshes. *BIT Numerical Mathematics*, 2012. Published online. DOI: 10.1007/s10543-012-0403-3.
- [19] D. A. Di Pietro and S. Lemaire. An inf-sup stable and robust discretization of the Stokes equations with large irrotational sources on general meshes. *C. R. Acad. Sci. Paris, Ser. I*, 2012. Submitted. Preprint available at <http://hal.archives-ouvertes.fr/hal-00708270>.
- [20] D. A. Di Pietro and S. Nicaise. A locking-free discontinuous Galerkin method for linear elasticity in locally nearly incompressible heterogeneous media. *App. Num. Math.*, 63:105–116, 2013.
- [21] J. Droniou and R. Eymard. A mixed finite volume scheme for anisotropic diffusion problems on any grid. *Numer. Math.*, 105(1):35–71, 2006.
- [22] J. Droniou, R. Eymard, T. Gallouët, and R. Herbin. A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods. *Math. Mod. Meths. Appli. Sci. (M3AS)*, 20(2):265–295, 2010.
- [23] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

- [24] R. Eymard, T. Gallouët, and R. Herbin. A new finite volume scheme for anisotropic diffusion problems on general grids: convergence analysis. *C. R. Acad. Sci. Paris, Ser. I*, 344:403–406, 2007.
- [25] R. Eymard, T. Gallouët, and R. Herbin. Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. SUSI: a scheme using stabilization and hybrid interfaces. *IMA J. Num. Anal.*, 30(4):1009–1043, 2010.
- [26] R. S. Falk. Nonconforming finite element methods for the equations of linear elasticity. *Math. Comp.*, 57(196):529–550, 1991.
- [27] K. J. Galvin, A. Linke, L. G. Rebholz, and N. E. Wilson. Stabilizing poor mass conservation in incompressible flow problems with large irrotational forcing and application to thermal convection. *Comput. Methods Appl. Mech. Engrg.*, 237–240:166–176, 2012.
- [28] P. Hansbo and M. G. Larson. Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche’s method. *Comput. Methods Appl. Mech. Engrg.*, 191(17–18):1895–1908, 2002.
- [29] P. Hansbo and M. G. Larson. Discontinuous Galerkin and the Crouzeix–Raviart element: application to elasticity. *M2AN Math. Model. Numer. Anal.*, 37(1):63–72, 2003.
- [30] B. P. Lamichhane and E. P. Stephan. A symmetric mixed finite element method for nearly incompressible elasticity based on biorthogonal systems. *Numer. Methods for Partial Differential Equations*, 28(4):1336–1353, 2012.
- [31] R. Stenberg. A family of mixed finite elements for the elasticity problem. *Numer. Math.*, 53:513–538, 1988.
- [32] G. Strang. Variational crimes in the finite element method. In A. Aziz, editor, *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*. Academic Press, New York, 1972.
- [33] M. Vogelius. An analysis of the  $p$ -version of the finite element method for nearly incompressible materials. Uniformly valid, optimal error estimates. *Numer. Math.*, 41:39–53, 1983.