# A strategy-based proof of the existence of the value in zero-sum differential games 

Pablo Maldonado and Miquel Oliu-Barton *

November 19, 2012


#### Abstract

The value of a zero-sum differential games is known to exist, under Isaacs condition, as the unique viscosity solution of a Hamilton-Jacobi-Bellman equation. In this note we provide a new proof via the construction of $\varepsilon$-optimal strategies, which is inspired in the "extremal aiming" method from [3].


## 1 Introduction

Let $U$ and $V$ be compact subsets of some euclidean space, let $\|\cdot\|$ be the euclidean norm in $\mathbb{R}^{n}$, and let $f:[0,1] \times \mathbb{R}^{n} \times U \times V \rightarrow \mathbb{R}^{n}$.

## Assumption 1:

1a. $f$ is uniformly bounded, i.e. $\|f\|:=\sup _{(t, x, u, v)}\|f(t, x, u, v)\|<+\infty$,
1b. $\exists c \geq 0$ such that $\forall(u, v) \in U \times V, \forall s, t \in[0,1], \forall x, y \in \mathbb{R}^{n}$ :

$$
\|f(t, x, u, v)-f(s, y, u, v)\| \leq c(|t-s|+\|x-y\|)
$$

The directional game For any $(t, x) \in[0,1] \times \mathbb{R}^{n}$ and any $\xi \in \mathbb{R}^{n}$, consider the one-shot game $\Gamma(t, x, \xi)$, with actions sets $U$ and $V$ and payoff function:

$$
(u, v) \mapsto\langle\xi, f(t, x, u, v)\rangle .
$$

Let $H^{-}(t, x, \xi)$ and $H^{+}(t, x, \xi)$ be its maxmin and minmax respectively:

$$
\begin{aligned}
H^{-}(t, x, \xi) & :=\max _{u \in U} \min _{v \in V}\langle\xi, f(t, x, u, v)\rangle, \\
H^{+}(t, x, \xi) & :=\min _{v \in V} \max _{u \in U}\langle\xi, f(t, x, u, v)\rangle .
\end{aligned}
$$

[^0]These functions satisfy $H^{-} \leq H^{+}$. If the equality $H^{+}(t, x, \xi)=H^{-}(t, x, \xi)$ holds, the game $\Gamma(t, x, \xi)$ has a value.

Assumption 2: $\forall(t, x, \xi) \in[0,1] \times \mathbb{R}^{n} \times \mathbb{R}^{n}$, the game $\Gamma(t, x, \xi)$ has a value $H(t, x, \xi)$.

### 1.1 An important Lemma

Introduce the sets of controls:

$$
\mathcal{U}=\{\mathbf{u}:[0,1] \rightarrow U, \text { measurable }\}, \quad \mathcal{V}=\{\mathbf{v}:[0,1] \rightarrow V, \text { measurable }\}
$$

Let $(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}, t_{0} \in[0,1],\left(x_{0}, w_{0}\right) \in\left(\mathbb{R}^{n}\right)^{2}$ and let $\left(u^{*}, v^{*}\right)$ be a couple of optimal actions in $\Gamma\left(t_{0}, x_{0}, x_{0}-w_{0}\right)$. Define two continuous trajectories in $\mathbb{R}^{n}, \mathbf{x}:\left[t_{0}, 1\right] \rightarrow \mathbb{R}^{n}$ and $\mathbf{w}:\left[t_{0}, 1\right] \rightarrow \mathbb{R}^{n}$, by:

$$
\begin{aligned}
\mathbf{x}\left(t_{0}\right)=x_{0}, & \text { and } \quad \dot{\mathbf{x}}(t)=f\left(t, \mathbf{x}(t), \mathbf{u}(t), v^{*}\right), \text {, a.e. } \\
\mathbf{w}\left(t_{0}\right)=w_{0}, & \text { and } \quad \dot{\mathbf{w}}(t)=f\left(t, \mathbf{w}(t), u^{*}, \mathbf{v}(t)\right), \text { a.e. }
\end{aligned}
$$

The following lemma is inspired by Lemma 2.3.1 in [3].
Lemma 1. Under Assumptions 1 and 2 , there exists $A, B \geq 0$ such that $\forall t \in\left[t_{0}, 1\right]$ :

$$
\|\mathbf{x}(t)-\mathbf{w}(t)\|^{2} \leq\left(1+\left(t-t_{0}\right) A\right)\left\|x_{0}-w_{0}\right\|^{2}+B\left(t-t_{0}\right)^{2}
$$

Proof. Notation: let $d_{0}:=\left\|x_{0}-w_{0}\right\|$ and $\mathbf{d}(t):=\|\mathbf{x}(t)-\mathbf{w}(t)\|$. Then:

$$
\begin{equation*}
\mathbf{d}^{2}(t)=\left\|\left(x_{0}-w_{0}\right)+\int_{t_{0}}^{t} f\left(s, \mathbf{x}(s), \mathbf{u}(s), v^{*}\right)-f\left(s, \mathbf{x}(s), u^{*}, \mathbf{v}(s)\right) d s\right\|^{2} \tag{1.1}
\end{equation*}
$$

The boundedness of $f$ implies that

$$
\begin{equation*}
\left\|\int_{t_{0}}^{t} f\left(s, \mathbf{x}(s), \mathbf{u}(s), v^{*}\right)-f\left(s, \mathbf{w}(s), u^{*}, \mathbf{v}(s)\right) d s\right\|^{2} \leq 4\|f\|^{2}\left(t-t_{0}\right)^{2} \tag{1.2}
\end{equation*}
$$

Claim: For all $s \in\left[t_{0}, 1\right]$, and for all $(u, v) \in U \times V$ :

$$
\begin{equation*}
\left\langle x_{0}-w_{0}, f\left(s, \mathbf{x}(s), u, v^{*}\right)-f\left(s, \mathbf{w}(s), u^{*}, v\right) d s\right\rangle \leq 2 C(s) d_{0}+c d_{0}^{2} \tag{1.3}
\end{equation*}
$$

where $C(s):=c(1+\|f\|)\left(s-t_{0}\right)$.
Let us prove this claim. Assumption 1 implies $\left\|\mathbf{x}(s)-x_{0}\right\| \leq\left(s-t_{0}\right)\|f\|$, and then:

$$
\left\|f\left(s, \mathbf{x}(s), u, v^{*}\right)-f\left(t_{0}, x_{0}, u, v^{*}\right)\right\| \leq c\left(\left(s-t_{0}\right)+\|f\|\left(s-t_{0}\right)\right)=C(s)
$$

Then, using Cauchy-Schwartz inequality, and the optimality of $v^{*}$ :

$$
\begin{aligned}
\left\langle x_{0}-w_{0}, f\left(s, \mathbf{x}(s), u, v^{*}\right)\right\rangle & \leq\left\langle x_{0}-w_{0}, f\left(t_{0}, x_{0}, u, v^{*}\right)\right\rangle+C(s) d_{0}, \\
& \leq H^{+}\left(t_{0}, x_{0}, x_{0}-w_{0}\right)+C(s) d_{0} .
\end{aligned}
$$

Similarly, Assumption 1 implies $\left\|\mathbf{w}(s)-x_{0}\right\| \leq d_{0}+\left(s-t_{0}\right)\|f\|$, and then:

$$
\left\|f\left(s, \mathbf{w}(s), u^{*}, v\right)-f\left(t_{0}, x_{0}, u^{*}, v\right)\right\| \leq C(s)+c d_{0}
$$

Using Cauchy-Schwartz inequality, and the optimality of $u^{*}$ :

$$
\begin{aligned}
\left\langle x_{0}-w_{0}, f\left(s, \mathbf{x}(s), u^{*}, v\right)\right\rangle & \geq\left\langle x_{0}-w_{0}, f\left(t_{0}, x_{0}, u^{*}, v\right)\right\rangle-\left(C(s)+c d_{0}\right) d_{0}, \\
& \geq H^{-}\left(t_{0}, x_{0}, x_{0}-w_{0}\right)-C(s) d_{0}-c d_{0}^{2} .
\end{aligned}
$$

The claim follows from Assumption 2. In particular, it holds for $(u, v)=(\mathbf{u}(s), \mathbf{v}(s))$. Note that $\int_{t_{0}}^{t} 2 C(s) d s=\left(t-t_{0}\right) C(t)$. Thus, integrating (1.3) over $\left[t_{0}, t\right]$ yields:

$$
\begin{equation*}
\int_{t_{0}}^{t}\left\langle x_{0}-w_{0}, f\left(s, \mathbf{x}(s), \mathbf{u}(s), v^{*}\right)-f\left(s, \mathbf{w}(s), u^{*}, \mathbf{v}(s)\right) d s\right\rangle \leq\left(t-t_{0}\right)\left(C(t) d_{0}+c d_{0}^{2}\right) . \tag{1.4}
\end{equation*}
$$

Go back to (1.1) using the estimates (1.2) and (1.4). We have proved:

$$
\mathbf{d}^{2}(t) \leq d_{0}^{2}+4\|f\|^{2}\left(t-t_{0}\right)^{2}+2\left(t-t_{0}\right) C(t) d_{0}+2 c\left(t-t_{0}\right) d_{0}^{2} .
$$

Finally, use the relations $d_{0} \leq 1+d_{0}^{2}, C(t) \leq c(1+\|f\|)$ and $\left(t-t_{0}\right) C(t)=c(1+$ $\|f\|)\left(t-t_{0}\right)^{2}$ to obtain the result, with $A=3 c+2\|f\|$ and $B=4\|f\|^{2}+2 c(1+\|f\|)$.

### 1.2 Consequences

In this section, we give three direct consequences of Lemma 1 . Let $d: \mathbb{R}^{n} \times \mathcal{P}\left(\mathbb{R}^{n}\right) \rightarrow$ $\mathbb{R}$ denote the usual distance to a set in $\mathbb{R}^{n}$.

1. Consider some sequence of times $\Pi=\left\{t_{0}<t_{1}<\cdots<t_{N}\right\}$ in [0, 1], and let $\|\Pi\|:=\max \left\{t_{m}-t_{m-1}, m=1, \ldots, N\right\}$. Let $(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$ be a fixed pair of controls. Define the trajectories $\mathbf{x}$ and $\mathbf{w}$ on $\left[t_{0}, t_{N}\right]$ inductively. Let $\mathbf{x}\left(t_{0}\right)=x_{0}$, $\mathbf{w}\left(t_{0}\right)=w_{0}$ and suppose that $\mathbf{x}(t)$ and $\mathbf{w}(t)$ are already defined on $\left[t_{0}, t_{m}\right]$. Let $\left(u_{m}^{*}, v_{m}^{*}\right) \in U \times V$ be a couple of optimal actions in $\Gamma\left(t_{m}, \mathbf{x}\left(t_{m}\right), \mathbf{x}\left(t_{m}\right)-\mathbf{w}\left(t_{m}\right)\right)$. Then, on $\left[t_{m}, t_{m+1}\right]$, let $\mathbf{x}$ and $\mathbf{w}$ be the unique absolutely continuous solutions of:

$$
\begin{aligned}
\dot{\mathbf{x}}(t) & =f\left(t, \mathbf{x}(t), \mathbf{u}(t), v_{m}^{*}\right) \\
\dot{\mathbf{w}}(t) & =f\left(t, \mathbf{w}(t), u_{m}^{*}, \mathbf{v}(t)\right)
\end{aligned}
$$

Corollary 1.1. Under Assumptions 1 and 2:

$$
\left\|\mathbf{x}\left(t_{N}\right)-\mathbf{w}\left(t_{N}\right)\right\|^{2} \leq e^{A}\left(\left\|x_{0}-w_{0}\right\|^{2}+B\|\Pi\|\right)
$$

Proof. For any $0 \leq m \leq N$, let $d_{m}:=\left\|\mathbf{x}\left(t_{m}\right)-\mathbf{w}\left(t_{m}\right)\right\|$. Lemma 1 yields:

$$
d_{m}^{2} \leq\left(1+\left(t_{m}-t_{m-1}\right) A\right) d_{m-1}^{2}+B\left(t_{m}-t_{m-1}\right)^{2} .
$$

Then, by induction: $d_{N}^{2} \leq \exp \left(A \sum_{m=1}^{N} t_{m}-t_{m-1}\right)\left(d_{0}^{2}+B \sum_{m=1}^{N}\left(t_{m}-t_{m-1}\right)^{2}\right)$. The result follows, since $t_{N}-t_{0} \leq 1$ and $\sum_{m=1}^{N}\left(t_{m}-t_{m-1}\right)^{2} \leq\|\Pi\|$.
2. For any $\left(t_{0}, x_{0}\right) \in[0,1] \times \mathbb{R}^{n}$ and $(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$, let $\mathbf{x}=\mathbf{x}\left[t_{0}, x_{0}, \mathbf{u}, \mathbf{v}\right]$ be the unique absolutely continuous solution in $\left[t_{0}, 1\right]$ of:

$$
\mathbf{x}\left(t_{0}\right)=x_{0}, \quad \text { and } \quad \dot{\mathbf{x}}(t)=f(t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{v}(t)) \text {, a.e. }
$$

That is, $\mathbf{x}\left[t_{0}, x_{0}, \mathbf{u}, \mathbf{v}\right]$ is the trajectory induced by the initial position $\left(t_{0}, x_{0}\right)$ and the controls $(\mathbf{u}, \mathbf{v})$. For any $u \in U$, let $\mathbf{x}\left[t_{0}, x_{0}, u, \mathbf{v}\right]$ be the trajectory induced by $\left(t_{0}, x_{0}, \mathbf{v}\right)$ and the constant control $\mathbf{u} \equiv u$.

Define two properties for sets $\mathcal{W} \subset\left[t_{0}, 1\right] \times \mathbb{R}^{n}$.

- P1: For any $t \in\left[t_{0}, 1\right], \mathcal{W}(t):=\left\{x \in \mathbb{R}^{n} \mid(t, x) \in \mathcal{W}\right\}$ is closed and nonempty.
- P2: For any $(t, x) \in \mathcal{W}$ and any $t_{1} \in[t, 1]$ :

$$
\sup _{u \in U} \inf _{\mathbf{v} \in \mathcal{V}} d\left(\mathbf{x}[t, x, u, \mathbf{v}]\left(t_{1}\right), \mathcal{W}\left(t_{1}\right)\right)=0,
$$

where $d$ is the usual distance in $\mathbb{R}^{n}$.
Corollary 1.2. Let $\mathcal{W} \subset\left[t_{0}, 1\right] \times \mathbb{R}^{n}$ satisfy $\mathbf{P} 1$ and $\mathbf{P} 2$. Under Assumptions 1 and 2 , there exists $v^{*} \in V$ such that, $\forall t \in\left[t_{0}, 1\right], \forall \mathbf{u} \in \mathcal{U}$ :

$$
d^{2}\left(\mathbf{x}\left[t_{0}, x_{0}, \mathbf{u}, v^{*}\right](t), \mathcal{W}(t)\right) \leq\left(1+\left(t-t_{0}\right) A\right) d^{2}\left(x_{0}, \mathcal{W}\left(t_{0}\right)\right)+B\left(t-t_{0}\right)^{2} .
$$

Proof. Let $w_{0} \in \operatorname{argmin}_{w \in \mathcal{W}\left(t_{0}\right)}\left\|x_{0}-w\right\|$ be some closest point (which exists by $\mathbf{P 1})$. Let $\left(u^{*}, v^{*}\right)$ be optimal in $\Gamma\left(t_{0}, x_{0}, x_{0}-w_{0}\right)$. By $\mathbf{P} 2, \forall \varepsilon>0, \exists \mathbf{v}_{\varepsilon}$ such that $\mathbf{w}_{\varepsilon}(t):=\mathbf{x}\left[t_{0}, w_{0}, u^{*}, \mathbf{v}_{\varepsilon}\right](t)$ satisfies $d\left(\mathbf{w}_{\varepsilon}(t), \mathcal{W}(t)\right) \leq \varepsilon$. The triangular equality implies $d(\mathbf{x}(t), \mathcal{W}(t)) \leq\left\|\mathbf{x}(t)-\mathbf{w}_{\varepsilon}(t)\right\|+\varepsilon$. Taking the limit, as $\varepsilon \rightarrow 0$ :

$$
d^{2}(\mathbf{x}(t), \mathcal{W}(t)) \leq \lim _{\varepsilon \rightarrow 0}\left\|\mathbf{x}(t)-\mathbf{w}_{\varepsilon}(t)\right\|^{2}
$$

where $\left\|\mathbf{x}(t)-\mathbf{w}_{\varepsilon}(t)\right\|^{2} \leq\left(1+\left(t-t_{0}\right) A\right)\left\|x_{0}-w_{0}\right\|^{2}+B\left(t-t_{0}\right)^{2}$ for any $\varepsilon>0$, by Lemma 1 , and where $\left\|x_{0}-w_{0}\right\|=d\left(x_{0}, \mathcal{W}\left(t_{0}\right)\right)$ by definition.
3. Putting Corollaries 1.1 and 1.2 together, one obtains the following result.

Corollary 1.3. Let $\mathcal{W} \subset\left[t_{0}, 1\right] \times \mathbb{R}^{n}$ satisfy $\mathbf{P 1}$ and $\mathbf{P} 2$, let $\Pi=\left\{t_{0}<\cdots<t_{N}\right\}$ be a sequence of times, and let $x_{0} \in \mathcal{W}\left(t_{0}\right)$. Under Assumptions 1 and 2 , there exist $v_{0}^{*}, \ldots, v_{N-1}^{*} \in V$ such that, for $\mathbf{v} \equiv v_{m}^{*}$, on $\left[t_{m}, t_{m+1}\right]$, and for all $\mathbf{u} \in \mathcal{U}$ :

$$
d^{2}\left(\mathbf{x}\left[t_{0}, x_{0}, \mathbf{u}, \mathbf{v}\right]\left(t_{N}\right), \mathcal{W}\left(t_{N}\right)\right) \leq e^{A} B\|\Pi\| .
$$

## 2 Differential Games

For any $\left(t_{0}, x_{0}\right) \in[0,1] \times \mathbb{R}^{n}$, consider now the zero-sum differential with the following two-controlled dynamic

$$
\mathbf{x}\left(t_{0}\right)=x_{0}, \quad \text { and } \dot{\mathbf{x}}(t)=f(t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{v}(t)), \text { a.e. on }\left[t_{0}, 1\right] .
$$

Definition 2.1. A strategy for player 2 is a map $\beta: \mathcal{U} \rightarrow \mathcal{V}$ such that, for some finite partition $t_{0}<t_{1}<\cdots<t_{N}=1$ of $\left[t_{0}, 1\right], \forall \mathbf{u}_{1}, \mathbf{u}_{2} \in \mathcal{U}$ :

$$
\mathbf{u}_{1} \equiv \mathbf{u}_{2} \text { a.e. on }\left[t_{0}, t_{m}\right] \Longrightarrow \beta\left(\mathbf{u}_{1}\right) \equiv \beta\left(\mathbf{u}_{2}\right) \text { a.e. on }\left[t_{0}, t_{m+1} \wedge 1\right] .
$$

These strategies are called nonanticipative strategies with delay (NAD) in [1], in contrast to the classical nonanticipative strategies. The strategies for player 1 are defined in a dual manner. Let $\mathcal{B}$ (resp. $\mathcal{A}$ ) the set of strategies for Player 2 (resp. $1)$. For any pair of strategies $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$, [1] establishes the following crucial result: there exists a unique pair $(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$ such that $\alpha(\mathbf{v})=\mathbf{u}$, and $\beta(\mathbf{u})=\mathbf{v}$. Denote by $\mathbf{x}\left[t_{0}, x_{0}, \alpha, \beta\right]$ the trajectory induced by the pair $(\mathbf{u}, \mathbf{v})$.

Let $g: \mathbb{R}^{n} \rightarrow \mathbb{R}$ some function. The differential game with initial time $t_{0}$, initial state $x_{0}$, and terminal payoff $g$ is denoted by $\mathcal{G}\left(t_{0}, x_{0}\right)$. Introduce the upper and lower value functions:

$$
\begin{aligned}
V^{-}\left(t_{0}, x_{0}\right) & :=\sup _{\alpha \in \mathcal{A}} \inf _{\beta \in \mathcal{B}} g\left(\mathbf{x}\left[t_{0}, x_{0}, \alpha, \beta\right](1)\right) \\
V^{+}\left(t_{0}, x_{0}\right) & :=\inf _{\beta \in \mathcal{B}} \sup _{\alpha \in \mathcal{A}} g\left(\mathbf{x}\left[t_{0}, x_{0}, \alpha, \beta\right](1)\right)
\end{aligned}
$$

The inequality $V^{-} \leq V^{+}$holds everywhere. If $V^{-}\left(t_{0}, x_{0}\right)=V^{+}\left(t_{0}, x_{0}\right)$, the game $\mathcal{G}\left(t_{0}, x_{0}\right)$ has a value. Notice that its lower and upper Hamiltonian of are precisely the maxmin and the minmax of the directional games defined in Section 1. Consequently, Assumption 2 is precisely Isaacs' condition.
Assumption 3: $g$ is $c$-Lipschitz continuous, i.e. $|g(x)-g(y)| \leq c\|x-y\|, \forall x, y \in \mathbb{R}^{n}$.

### 2.1 Existence and characterization of the value

Let $\phi:\left[t_{0}, 1\right] \times \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a real function satisfying the following properties:
(i) $x \mapsto \phi(t, x)$ is lower semicontinuous, $\forall t \in\left[t_{0}, 1\right]$,
(ii) $\forall(t, x) \in\left[t_{0}, 1\right] \times \mathbb{R}^{n}, \forall t_{1} \in[t, 1]:$

$$
\phi(t, x) \geq \sup _{u \in U} \inf _{\mathbf{v} \in \mathcal{V}} \phi\left(t_{1}, \mathbf{x}[t, x, u, \mathbf{v}]\left(t_{1}\right)\right)
$$

(iii) $\phi(1, x) \geq g(x), \forall x \in \mathbb{R}^{n}$.

For any $\ell \in \mathbb{R}$, define the $\ell$-level set of $\phi$ by:

$$
\begin{equation*}
\mathcal{W}_{\ell}^{\phi}=\left\{(t, x) \in\left[t_{0}, 1\right] \times \mathbb{R}^{n} \mid \phi(t, x) \leq \ell\right\} \tag{2.1}
\end{equation*}
$$

Lemma 2. For any $\ell \geq \phi\left(t_{0}, x_{0}\right)$, the $\ell$-level set of $\phi$ satisfies $\mathbf{P} 1$ and P2.
Proof. Note that $\mathcal{W}_{\ell}^{\phi}\left(t_{0}\right)$ is nonempty, since $x_{0} \in \mathcal{W}_{\ell}^{\phi}\left(t_{0}\right)$. (i) implies that $\mathcal{W}_{\ell}^{\phi}(t)$ is a closed set, $\forall t \in[0,1]$. On the other hand, by $(i i)$ for all $(t, x) \in\left[t_{0}, 1\right] \times \mathbb{R}^{n}$, $t_{1} \in[t, 1], u \in U$, and $n \in \mathbb{N}^{*}$, there exists $\mathbf{v}_{n} \in \mathcal{V}$ such that:

$$
\begin{equation*}
\phi(t, x) \geq \phi\left(t_{1}, \mathbf{x}\left[t, x, u, \mathbf{v}_{n}\right]\left(t_{1}\right)\right)-\frac{1}{n} \tag{2.2}
\end{equation*}
$$

The boundedness of $f$ implies that $x_{n}:=\mathbf{x}\left[t, x, u, \mathbf{v}_{n}\right]\left(t_{1}\right)$ belongs to some compact set. Consider a subsequence $\left(x_{n}\right)_{n}$ such that $\lim _{n \rightarrow \infty} \phi\left(t_{1}, x_{n}\right)=\liminf _{n \rightarrow \infty} \phi\left(t_{1}, x_{n}\right)$, and such that $\left(x_{n}\right)_{n}$ converges to some $\bar{x} \in \mathbb{R}^{n}$. Then, taking the limit, as $n \rightarrow \infty$, in (2.2) implies, using $(i)$ and $\ell \geq \phi(t, x)$ :

$$
\phi\left(t_{1}, \bar{x}\right) \leq \lim _{n \rightarrow \infty} \phi\left(t_{1}, x_{n}\right) \leq \phi(t, x) \leq \ell
$$

Hence $\bar{x} \in \mathcal{W}_{\ell}^{\phi}\left(t_{1}\right)$, and $\inf _{n \in \mathbb{N}^{*}} d\left(\mathbf{x}\left[t, x, u, \mathbf{v}_{n}\right]\left(t_{1}\right), \mathcal{W}_{\ell}^{\phi}\right)=0$. In particular, $\mathcal{W}_{\ell}^{\phi}\left(t_{1}\right)$ is nonempty, and $\mathbf{P 1}$ and $\mathbf{P} 2$ hold.

### 2.1.1 Extremal strategies in $\mathcal{G}\left(t_{0}, x_{0}\right)$

Let $\Pi=\left\{t_{0}<\cdots<t_{N}=1\right\}$ be partition of $\left[t_{0}, 1\right]$, let $\|\Pi\|=\max \left\{t_{m}-t_{m-1}, m=\right.$ $1, \ldots, N\}$, and let $\mathcal{W}^{\phi} \subset\left[t_{0}, 1\right] \times \mathbb{R}^{n}$ be the $\phi\left(t_{0}, x_{0}\right)$-level set of $\phi$.

Definition 2.2. An extremal strategy $\beta=\beta(\phi, \Pi)$ is defined inductively: suppose $\beta$ is already defined on $\left[t_{0}, t_{m}\right]$ and let $x_{m}=\mathbf{x}\left[t_{0}, x_{0}, \mathbf{u}, \beta\right]\left(t_{m}\right)$. Then, $\forall \mathbf{u} \in \mathcal{U}$ :

- If $x_{m} \in \mathcal{W}^{\phi}\left(t_{m}\right)$, set $\beta(\mathbf{u})(s)=v$, for any $v \in V, \forall s \in\left[t_{m}, t_{m+1}\right)$.
- If $x_{m} \notin \mathcal{W}^{\phi}\left(t_{m}\right)$, let $w_{m} \in \operatorname{argmin}_{w \in \mathcal{W}^{\phi}\left(t_{m}\right)}\left\|x_{m}-w_{m}\right\|$ be some closest point, and let $v_{m}^{*}$ be some optimal action in the directional game $\Gamma\left(t_{m}, x_{m}, x_{m}-w_{m}\right)$. Set $\beta(\mathbf{u})(s)=v_{m}^{*}, \forall s \in\left[t_{m}, t_{m+1}\right)$.

These strategies are inspired by the extremal aiming method of Krasovskii and Subbotin (see Section 2.4 in [3]). Notice that $\beta$ is defined up to some selection rule since $V$, the set of closest points and the set of minimizers may have more than one element.

Proposition 2.1. Under Assumptions 1, 2 and $3, \exists C \geq 0$ such that:

$$
g\left(\mathbf{x}\left[t_{0}, x_{0}, \mathbf{u}, \beta(\mathbf{u})\right](1)\right) \leq \phi\left(t_{0}, x_{0}\right)+C \sqrt{\|\Pi\|}, \quad \forall \mathbf{u} \in \mathcal{U}
$$

for any extremal strategy $\beta=\beta(\phi, \Pi)$.
Proof. $\mathcal{W}^{\phi}$ satsifies P1 and P2 by Lemma 2. Applying Corollary 1.3:

$$
d^{2}\left(x_{N}, \mathcal{W}^{\phi}\left(t_{N}\right)\right) \leq e^{A} B\|\Pi\| .
$$

Now, by (iii), and since $t_{N}=1$ :

$$
\mathcal{W}^{\phi}\left(t_{N}\right)=\left\{x \in \mathbb{R}^{n} \mid \phi(1, x) \leq \phi\left(t_{0}, x_{0}\right)\right\} \subset\left\{x \in \mathbb{R}^{n} \mid g(x) \leq \phi\left(t_{0}, x_{0}\right)\right\} .
$$

Let $w_{N} \in \operatorname{argmin}_{w \in \mathcal{W}^{\phi}(1)}\left\|x_{N}-w\right\|$ be some closest point. By Assumption 3:

$$
g\left(x_{N}\right) \leq g\left(w_{N}\right)+c\left\|x_{N}-w_{N}\right\| \leq \phi\left(t_{0}, x_{0}\right)+c d\left(x_{N}, \mathcal{W}^{\phi}\left(t_{N}\right)\right) .
$$

The result follows, recalling that $x_{N}=\mathbf{x}\left[t_{0}, x_{0}, \mathbf{u}, \beta(\mathbf{u})\right](1)$. Explicitly, $C=c \sqrt{e^{A} B}$.

Proposition 2.1 applies to any function satisfying (i), (ii) and (iii). Consequently, under Assumptions 1, 2 and 3:

$$
\begin{equation*}
V^{+}\left(t_{0}, x_{0}\right) \leq \inf \left\{\phi\left(t_{0}, x_{0}\right) \mid \phi:\left[t_{0}, 1\right] \times \mathbb{R}^{n} \rightarrow \mathbb{R} \text { satisfying }(i),(i i),(i i i)\right\} \tag{2.3}
\end{equation*}
$$

Theorem 2.3. Under Assumptions 1, 2 and 3, the differential game $\mathcal{G}\left(t_{0}, x_{0}\right)$ has a value, characterized as:

$$
\begin{equation*}
\mathbf{V}\left(t_{0}, x_{0}\right)=\min _{\substack{\phi \text { satisfying } \\(i),(i i),(i i i)}} \phi\left(t_{0}, x_{0}\right) . \tag{2.4}
\end{equation*}
$$

The strategies $\beta(\mathbf{V}, \Pi)$ are asymptotically optimal for player 2 , as $\|\Pi\| \rightarrow 0$.

Proof. By (2.3), it is enough to prove that $V^{-}$satisfies (i), (ii) and (iii), where (iii) is immediate. Assumption 1, and Gronwall's lemma imply that $\forall t \in\left[t_{0}, 1\right]$, $\forall(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$, and $\forall x, y \in \mathbb{R}^{n}:$

$$
\left.\left.\| \mathbf{x}\left[t_{0}, x, \mathbf{u}, \mathbf{v}\right](t)\right)-\mathbf{x}\left[t_{0}, y, \mathbf{u}, \mathbf{v}\right](t)\right) \mid \leq e^{c\left(t-t_{0}\right)}\|x-y\| .
$$

Assumption 2 gives then, $\forall(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$, and $\forall x, y \in \mathbb{R}^{n}$ :

$$
\left|g\left(\mathbf{x}\left[t_{0}, x, \mathbf{u}, \mathbf{v}\right](1)\right)-g\left(\mathbf{x}\left[t_{0}, y, \mathbf{u}, \mathbf{v}\right](1)\right)\right| \leq c e^{c\left(1-t_{0}\right)}\|x-y\| .
$$

Thus, by standard arguments, $x \mapsto V^{-}(t, x)$ is $c e^{c}$-Lipschitz continuous $\forall t \in\left[t_{0}, 1\right]$ and, in particular, $V^{-}$satisfies $(i)$. On the other hand, $(i i)$ is a weak version of the classical dynamic programming principle (see [2], for nonanticipative strategies, and [1] for NAD strategies, defined above): $\forall(t, x) \in\left[t_{0}, 1\right] \times \mathbb{R}^{n}, \forall t_{1} \in[t, 1]$ :

$$
V^{-}(t, x)=\sup _{\alpha \in \mathcal{A}} \inf _{\mathbf{v} \in \mathcal{V}} V^{-}\left(t_{1}, \mathbf{x}[t, x, \alpha(\mathbf{v}), \mathbf{v}]\left(t_{1}\right)\right) .
$$

Finally, let $\beta(\mathbf{V}, \Pi)$ be an extremal strategy. By Corollary 2.1:

$$
g\left(\mathbf{x}\left[t_{0}, x_{0}, \mathbf{u}, \beta(\mathbf{V}, \Pi)(\mathbf{u})\right](1)\right) \leq \mathbf{V}\left(t_{0}, x_{0}\right)+C \sqrt{\|\Pi\|}, \quad \forall \mathbf{u} \in \mathcal{U}
$$

Consequenly, for any $\varepsilon>0, \beta(\mathbf{V}, \Pi)$ is $\varepsilon$-optimal for sufficiently small $\|\Pi\|$.

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[^0]:    *The authors are particularly indebted with Pierre Cardaliaguet, Marc Quincampoix and Sylvain Sorin for their careful reading and comments on earlier drafts. This work was partially supported by the Commission of the European Communities under the 7th Framework Programme Marie Curie Initial Training Network (FP7-PEOPLE-2010-ITN), project SADCO, contract number 264735.

