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A strategy-based proof of the existence of the value in zero-sum differential games

Pablo Maldonado and Miquel Oliu-Barton *

November 19, 2012

Abstract

The value of a zero-sum differential games is known to exist, under Isaacs condition, as the unique viscosity solution of a Hamilton-Jacobi-Bellman equation. In this note we provide a new proof via the construction of \( \varepsilon \)-optimal strategies, which is inspired in the “extremal aiming” method from [3].

1 Introduction

Let \( U \) and \( V \) be compact subsets of some euclidean space, let \( \| \cdot \| \) be the euclidean norm in \( \mathbb{R}^n \), and let \( f : [0,1] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n \).

Assumption 1:

1a. \( f \) is uniformly bounded, i.e. \( \| f \| := \sup_{(t,x,u,v)} \| f(t,x,u,v) \| < +\infty \),

1b. \( \exists c \geq 0 \) such that \( \forall (u,v) \in U \times V, \forall s,t \in [0,1], \forall x,y \in \mathbb{R}^n : \)

\[
\| f(t,x,u,v) - f(s,y,u,v) \| \leq c(|t-s| + \|x-y\|),
\]

The directional game For any \( (t,x) \in [0,1] \times \mathbb{R}^n \) and any \( \xi \in \mathbb{R}^n \), consider the one-shot game \( \Gamma(t,x,\xi) \), with actions sets \( U \) and \( V \) and payoff function:

\[
(u,v) \mapsto \langle \xi, f(t,x,u,v) \rangle.
\]

Let \( H^-(t,x,\xi) \) and \( H^+(t,x,\xi) \) be its maxmin and minmax respectively:

\[
H^-(t,x,\xi) := \max_{u \in U} \min_{v \in V} \langle \xi, f(t,x,u,v) \rangle,
\]

\[
H^+(t,x,\xi) := \min_{v \in V} \max_{u \in U} \langle \xi, f(t,x,u,v) \rangle.
\]

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These functions satisfy $H^- \leq H^+$. If the equality $H^+(t, x, \xi) = H^-(t, x, \xi)$ holds, the game $\Gamma(t, x, \xi)$ has a value.

**Assumption 2:** $\forall (t, x, \xi) \in [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n$, the game $\Gamma(t, x, \xi)$ has a value $H(t, x, \xi)$.

### 1.1 An important Lemma

Introduce the sets of controls:

$$
\mathcal{U} = \{u : [0, 1] \rightarrow U, \text{ measurable}\}, \quad \mathcal{V} = \{v : [0, 1] \rightarrow V, \text{ measurable}\}.
$$

Let $(u, v) \in \mathcal{U} \times \mathcal{V}$, $t_0 \in [0, 1]$, $(x_0, w_0) \in (\mathbb{R}^n)^2$ and let $(u^*, v^*)$ be a couple of optimal actions in $\Gamma(t_0, x_0, x_0 - w_0)$. Define two continuous trajectories in $\mathbb{R}^n, x : [t_0, 1] \rightarrow \mathbb{R}^n$ and $w : [t_0, 1] \rightarrow \mathbb{R}^n$, by:

$$
x(t_0) = x_0, \quad \text{and} \quad \dot{x}(t) = f(t, x(t), u(t), v^*), \quad \text{a.e.}
$$

$$
w(t_0) = w_0, \quad \text{and} \quad \dot{w}(t) = f(t, w(t), u^*, v(t)), \quad \text{a.e.}
$$

The following lemma is inspired by Lemma 2.3.1 in [3].

**Lemma 1.** *Under Assumptions 1 and 2, there exists $A, B \geq 0$ such that $\forall t \in [t_0, 1]$:

$$
\|x(t) - w(t)\|^2 \leq (1 + (t - t_0)A)\|x_0 - w_0\|^2 + B(t - t_0)^2.
$$

*Proof.* Notation: let $d_0 := \|x_0 - w_0\|$ and $d(t) := \|x(t) - w(t)\|$. Then:

$$
d^2(t) = \|(x_0 - w_0) + \int_{t_0}^{t} f(s, x(s), u(s), v^*) - f(s, x(s), u^*, v(s)) ds\|^2. \quad (1.1)
$$

The boundedness of $f$ implies that

$$
\|\int_{t_0}^{t} f(s, x(s), u(s), v^*) - f(s, w(s), u^*, v(s)) ds\|^2 \leq 4\|f\|^2(t - t_0)^2. \quad (1.2)
$$

**Claim:** For all $s \in [t_0, 1]$, and for all $(u, v) \in U \times V$:

$$
\langle x_0 - w_0, f(s, x(s), u, v^*) - f(s, w(s), u^*, v) ds \rangle \leq 2\|f\|d_0 + cd_0^2, \quad (1.3)
$$

where $C(s) := c(1 + \|f\|)(s - t_0)$.

Let us prove this claim. Assumption 1 implies $\|x(s) - x_0\| \leq (s - t_0)\|f\|$, and then:

$$
\|f(s, x(s), u, v^*) - f(t_0, x_0, u, v^*)\| \leq c((s - t_0) + \|f\|(s - t_0)) = C(s).
$$

Then, using Cauchy-Schwartz inequality, and the optimality of $v^*$:

$$
\langle x_0 - w_0, f(s, x(s), u, v^*) \rangle \leq \langle x_0 - w_0, f(t_0, x_0, u, v^*) \rangle + C(s)d_0,
$$

$$
\leq H^+(t_0, x_0, x_0 - w_0) + C(s)d_0.
$$

Similarly, Assumption 1 implies $\|w(s) - x_0\| \leq d_0 + (s - t_0)\|f\|$, and then:

$$
\|f(s, w(s), u^*, v) - f(t_0, x_0, u^*, v)\| \leq C(s) + cd_0.
$$
Finally, use the relations unique absolutely continuous solution in $x$. That is, for any $d$.

In this section, we give three direct consequences of Lemma 1. Let

1.2 Consequences

The claim follows from Assumption 1. Consider some sequence of times $\Pi = \{t_0 < t_1 < \cdots < t_N\}$ in $[0, 1]$, and let $\|\Pi\| := \max\{t_m - t_{m-1}, m = 1, \ldots, N\}$. Let $(u, v) \in \mathcal{U} \times \mathcal{V}$ be a fixed pair of controls. Define the trajectories $x$ and $w$ on $[t_0, t_N]$ inductively. Let $x(t_0) = x_0$, $w(t_0) = w_0$ and suppose that $x(t)$ and $w(t)$ are already defined on $[t_0, t_m]$. Let $(u_m^*, v_m^*) \in U \times V$ be a couple of optimal actions in $\Gamma(t_m, x(t_m), \mathbf{x}(t_m) - w(t_m))$. Then, on $[t_m, t_{m+1}]$, let $x$ and $w$ be the unique absolutely continuous solutions of:

$$
\frac{d}{dt}x(t) = f(t, x(t), u(t), v(t)),
\frac{d}{dt}w(t) = f(t, w(t), u^*_m, v(t)).
$$

Then, using the estimates (1.2) and (1.4), we have proved:

$$
\int_{t_0}^{t} \langle x(t) - w(t), f(s, x(s), u^*, v) - f(s, w(s), u^*, v) \rangle ds \leq (t - t_0)(C(t)d_0 + cd_0^2).
$$

Finally, use the properties for sets $W \subset [0, 1] \times \mathbb{R}^n$.

1.2 Consequences

In this section, we give three direct consequences of Lemma 1. Let $d : \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}$ denote the usual distance to a set in $\mathbb{R}^n$.

1. Consider some sequence of times $\Pi = \{t_0 < t_1 < \cdots < t_N\}$ in $[0, 1]$, and let $\|\Pi\| := \max\{t_m - t_{m-1}, m = 1, \ldots, N\}$. Let $(u, v) \in \mathcal{U} \times \mathcal{V}$ be a fixed pair of controls. Define the trajectories $x$ and $w$ on $[t_0, t_N]$ inductively. Let $x(t_0) = x_0$, $w(t_0) = w_0$ and suppose that $x(t)$ and $w(t)$ are already defined on $[t_0, t_m]$. Let $(u_m^*, v_m^*) \in U \times V$ be a couple of optimal actions in $\Gamma(t_m, x(t_m), \mathbf{x}(t_m) - w(t_m))$. Then, on $[t_m, t_{m+1}]$, let $x$ and $w$ be the unique absolutely continuous solutions of:

$$
\frac{d}{dt}x(t) = f(t, x(t), u(t), v(t)),
\frac{d}{dt}w(t) = f(t, w(t), u^*_m, v(t)).
$$

**Corollary 1.1.** Under Assumptions 1 and 2:

$$
\|x(t_N) - w(t_N)\|^2 \leq e^{A\|\Pi\|}(\|x_0 - w_0\|^2 + B\|\Pi\|).
$$

**Proof.** For any $0 \leq m \leq N$, let $d_m := \|x(t_m) - w(t_m)\|$. Lemma 1 yields:

$$
d_m^2 \leq (1 + (t_m - t_{m-1})A)d_{m-1}^2 + B(t_m - t_{m-1})^2.
$$

Then, by induction: $d_N^2 \leq \exp(A \sum_{m=1}^{N} (t_m - t_{m-1})A)d_0^2 + B \sum_{m=1}^{N} (t_m - t_{m-1})^2$. The result follows, since $t_N - t_0 \leq 1$ and $\sum_{m=1}^{N} (t_m - t_{m-1})^2 \leq \|\Pi\|$. \qed

2. For any $(t_0, x_0) \in [0, 1] \times \mathbb{R}^n$ and $(u, v) \in \mathcal{U} \times \mathcal{V}$, let $x = x[t_0, x_0, u, v]$ be the unique absolutely continuous solution in $[t_0, 1]$ of:

$$
x(t_0) = x_0, \quad \text{and} \quad \frac{d}{dt}x(t) = f(t, x(t), u(t), v(t)), \ \text{a.e.}
$$

That is, $x[t_0, x_0, u, v]$ is the trajectory induced by the initial position $(t_0, x_0)$ and the controls $(u, v)$. For any $u \in U$, let $x[t_0, x_0, u, v]$ be the trajectory induced by $(t_0, x_0, v)$ and the constant control $u \equiv u$.

Define two properties for sets $W \subset [t_0, 1] \times \mathbb{R}^n$. 

\begin{itemize}
\item $W$ is the set of trajectories induced by the initial position $(t_0, x_0)$ and the controls $(u, v)$.
\item $W$ is the set of trajectories induced by $(t_0, x_0, v)$ and the constant control $u \equiv u$.
\end{itemize}
• **P1:** For any $t \in [t_0, 1]$, $W(t) := \{x \in \mathbb{R}^n \mid (t, x) \in W\}$ is closed and nonempty.

• **P2:** For any $(t, x) \in W$ and any $t_1 \in [t, 1]$:

$$
\sup_{u \in U} \inf_{v \in V} d(x[t, x, u, v](t_1), W(t_1)) = 0,
$$

where $d$ is the usual distance in $\mathbb{R}^n$.

**Corollary 1.2.** Let $W \subset [t_0, 1] \times \mathbb{R}^n$ satisfy **P1** and **P2**. Under Assumptions 1 and 2, there exists $v^* \in V$ such that, $\forall t \in [t_0, 1], \forall u \in U$:

$$
d^2(x[t_0, x_0, u, v^*](t), W(t)) \leq (1 + (t - t_0)A)d^2(x_0, W(t_0)) + B(t - t_0)^2.
$$

**Proof.** Let $w_0 \in \arg\min_{w \in W(t_0)} ||x_0 - w||$ be some closest point (which exists by **P1**). Let $(u^*, v^*)$ be optimal in $\Gamma(t_0, x_0, x_0 - w_0)$. By **P2**, $\forall \varepsilon > 0, \exists v_\varepsilon$ such that $w_\varepsilon(t) := x[t_0, x_0, u^*, v_\varepsilon](t) \text{satisfies } d(w_\varepsilon(t), W(t)) \leq \varepsilon$. The triangular equality implies $d(x(t), W(t)) \leq ||x(t) - w_\varepsilon(t)|| + \varepsilon$. Taking the limit, as $\varepsilon \to 0$:

$$
d^2(x(t), W(t)) \leq \lim_{\varepsilon \to 0} ||x(t) - w_\varepsilon(t)||^2,
$$

where $||x(t) - w_\varepsilon(t)||^2 \leq (1 + (t - t_0)A)||x_0 - w_0||^2 + B(t - t_0)^2$ for any $\varepsilon > 0$, by Lemma 1, and where $||x_0 - w_0|| = d(x_0, W(t_0))$ by definition.

3. Putting Corollaries 1.1 and 1.2 together, one obtains the following result.

**Corollary 1.3.** Let $W \subset [t_0, 1] \times \mathbb{R}^n$ satisfy **P1** and **P2**, let $\Pi = \{t_0 < \cdots < t_N\}$ be a sequence of times, and let $x_0 \in W(t_0)$. Under Assumptions 1 and 2, there exist $v_0^*, \ldots, v_{N-1}^* \in V$ such that, for $v \equiv v_m^*$, on $[t_m, t_{m+1}]$, and for all $u \in U$:

$$
d^2(x[t_0, x_0, u, v](t_N), W(t_N)) \leq e^4B||\Pi||.
$$

2 **Differential Games**

For any $(t_0, x_0) \in [0, 1] \times \mathbb{R}^n$, consider now the zero-sum differential with the following two-controlled dynamic

$$
x(t_0) = x_0, \quad \dot{x}(t) = f(t, x(t), u(t), v(t)), \text{ a.e. on } [t_0, 1].
$$

**Definition 2.1.** A strategy for player 2 is a map $\beta : U \to V$ such that, for some finite partition $t_0 < t_1 < \cdots < t_N = 1$ of $[t_0, 1], \forall u_1, u_2 \in U$:

$$
u_1 \equiv u_2 \text{ a.e. on } [t_0, t_m] \implies \beta(u_1) \equiv \beta(u_2) \text{ a.e. on } [t_0, t_{m+1} \wedge 1].
$$

These strategies are called nonanticipative strategies with delay (NAD) in [1], in contrast to the classical nonanticipative strategies. The strategies for player 1 are defined in a dual manner. Let $B$ (resp. $A$) the set of strategies for Player 2 (resp. 1). For any pair of strategies $(\alpha, \beta) \in A \times B$, [1] establishes the following crucial result: there exists a unique pair $(u, v) \in U \times V$ such that $\alpha(v) = u$, and $\beta(u) = v$. Denote by $x[t_0, x_0, \alpha, \beta]$ the trajectory induced by the pair $(u, v)$.
Let \( g : \mathbb{R}^n \to \mathbb{R} \) some function. The differential game with initial time \( t_0 \), initial state \( x_0 \), and terminal payoff \( g \) is denoted by \( G(t_0, x_0) \). Introduce the upper and lower value functions:

\[
V^-(t_0, x_0) := \sup_{\alpha \in A} \inf_{\beta \in B} g(x[t_0, x_0, \alpha, \beta](1)),
V^+(t_0, x_0) := \inf_{\beta \in B} \sup_{\alpha \in A} g(x[t_0, x_0, \alpha, \beta](1)).
\]

The inequality \( V^- \leq V^+ \) holds everywhere. If \( V^-(t_0, x_0) = V^+(t_0, x_0) \), the game \( G(t_0, x_0) \) has a value. Notice that its lower and upper Hamiltonian of are precisely the maxmin and the minmax of the directional games defined in Section 1. Consequently, Assumption 2 is precisely Isaacs’ condition.

**Assumption 3:** \( g \) is \( \alpha \)-Lipschitz continuous, i.e. \( |g(x) - g(y)| \leq c\|x - y\|, \forall x, y \in \mathbb{R}^n \).

### 2.1 Existence and characterization of the value

Let \( \phi : [t_0, 1] \times \mathbb{R}^n \to \mathbb{R} \) be a real function satisfying the following properties:

(i) \( x \mapsto \phi(t, x) \) is lower semicontinuous, \( \forall t \in [t_0, 1] \),

(ii) \( \forall (t, x) \in [t_0, 1] \times \mathbb{R}^n, \forall t_1 \in [t, 1] \):

\[
\phi(t, x) \geq \sup_{u \in U} \inf_{v \in V} \phi(t_1, x(t, u, v)(t_1)),
\]

(iii) \( \phi(1, x) \geq g(x), \forall x \in \mathbb{R}^n \).

For any \( \ell \in \mathbb{R} \), define the \( \ell \)-level set of \( \phi \) by:

\[
W^\phi_\ell = \{(t, x) \in [t_0, 1] \times \mathbb{R}^n | \phi(t, x) \leq \ell\}. \tag{2.1}
\]

**Lemma 2.** For any \( \ell \geq \phi(t_0, x_0) \), the \( \ell \)-level set of \( \phi \) satisfies \( P1 \) and \( P2 \).

**Proof.** Note that \( W^\phi_\ell(t_0) \) is nonempty, since \( x_0 \in W^\phi_\ell(t_0) \). (i) implies that \( W^\phi_\ell(t) \) is a closed set, \( \forall t \in [0, 1] \). On the other hand, by (ii) for all \( (t, x) \in [t_0, 1] \times \mathbb{R}^n \), \( t_1 \in [t, 1] \), \( u \in U \), and \( n \in \mathbb{N}^* \), there exists \( v_n \in V \) such that:

\[
\phi(t, x) \geq \phi(t_1, x(t, u, v_n)(t_1)) - \frac{1}{n}. \tag{2.2}
\]

The boundedness of \( f \) implies that \( x_n := x[t, x, u, v_n](t_1) \) belongs to some compact set. Consider a subsequence \( (x_n)_n \) such that \( \lim_{n \to \infty} \phi(t_1, x_n) = \lim_{n \to \infty} \phi(t_1, x_n) \), and such that \( (x_n)_n \) converges to some \( \bar{x} \in \mathbb{R}^n \). Then, taking the limit, as \( n \to \infty \), in (2.2) implies, using (i) and \( \ell \geq \phi(t, x) \):

\[
\phi(t_1, \bar{x}) \leq \lim_{n \to \infty} \phi(t_1, x_n) \leq \phi(t, x) \leq \ell.
\]

Hence \( \bar{x} \in W^\phi_\ell(t_1) \), and \( \inf_{n \in \mathbb{N}^*} d(x[t, x, u, v_n](t_1), W^\phi_\ell) = 0 \). In particular, \( W^\phi_\ell(t_1) \) is nonempty, and \( P1 \) and \( P2 \) hold.
2.1.1 Extremal strategies in \( G(t_0, x_0) \)

Let \( \Pi = \{ t_0 < \cdots < t_N = 1 \} \) be partition of \( [t_0, 1] \), let \( \| \Pi \| = \max \{ t_m - t_{m-1}, \, m = 1, \ldots, N \} \), and let \( \mathcal{W}^\phi \subset [t_0, 1] \times \mathbb{R}^n \) be the \( \phi(t_0, x_0) \)-level set of \( \phi \).

**Definition 2.2.** An extremal strategy \( \beta = \beta(\phi, \Pi) \) is defined inductively: suppose \( \beta \) is already defined on \( [t_0, t_m] \) and let \( \beta = \beta(t_0, x_0, u, \beta(t_m)) \). Then, \( \forall u \in U \):

- If \( x_m \in \mathcal{W}^\phi(t_m) \), set \( \beta(u)(s) = v \), for any \( v \in V \), \( \forall s \in [t_m, t_{m+1}] \).
- If \( x_m \notin \mathcal{W}^\phi(t_m) \), let \( w_m \in \arg\min_{w \in \mathcal{W}^\phi(t_m)} \| x_m - w_m \| \) be some closest point, and let \( v_m^* \) be some optimal action in the directional game \( \Gamma(t_m, x_m, x_m - w_m) \).

Set \( \beta(u)(s) = v_m^*, \forall s \in [t_m, t_{m+1}] \).

These strategies are inspired by the extremal aiming method of Krasovskii and Subbotin (see Section 2.4 in [3]). Notice that \( \beta \) is defined up to some selection rule since \( V \), the set of closest points and the set of minimizers may have more than one element.

**Proposition 2.1.** Under Assumptions 1, 2 and 3, \( \exists C \geq 0 \) such that:

\[
g(x[t_0, x_0, u, \beta(u)](1)) \leq \phi(t_0, x_0) + C\sqrt{\| \Pi \|}, \quad \forall u \in U,
\]

for any extremal strategy \( \beta = \beta(\phi, \Pi) \).

**Proof.** \( \mathcal{W}^\phi \) satisfies \( \textbf{P1} \) and \( \textbf{P2} \) by Lemma 2. Applying Corollary 1.3:

\[
d^2(x_N, \mathcal{W}^\phi(t_N)) \leq e^A B \| \Pi \|.
\]

Now, by \( (iii) \), and since \( t_N = 1 \):

\[
\mathcal{W}^\phi(t_N) = \{ x \in \mathbb{R}^n : \phi(1, x) \leq \phi(t_0, x_0) \} \subset \{ x \in \mathbb{R}^n : g(x) \leq \phi(t_0, x_0) \}.
\]

Let \( w_N \in \arg\min_{w \in \mathcal{W}^\phi(1)} \| x_N - w \| \) be some closest point. By Assumption 3:

\[
g(x_N) \leq g(w_N) + c\| x_N - w_N \| \leq \phi(t_0, x_0) + cd(x_N, \mathcal{W}^\phi(t_N)).
\]

The result follows, recalling that \( x_N = x[t_0, x_0, u, \beta(u)](1) \). Explicitly, \( C = c\sqrt{e^A B} \).

\( \square \)

Proposition 2.1 applies to any function satisfying \( (i) \), \( (ii) \) and \( (iii) \). Consequently, under Assumptions 1, 2 and 3:

\[
V^+(t_0, x_0) \leq \inf \{ \phi(t_0, x_0) \mid \phi : [t_0, 1] \times \mathbb{R}^n \to \mathbb{R} \text{ satisfying } (i), (ii), (iii) \}.
\]

(2.3)

**Theorem 2.3.** Under Assumptions 1, 2 and 3, the differential game \( G(t_0, x_0) \) has a value, characterized as:

\[
V(t_0, x_0) = \min_{\phi \text{ satisfying } (i), (ii), (iii)} \phi(t_0, x_0).
\]

(2.4)

The strategies \( \beta(V, \Pi) \) are asymptotically optimal for player 2, as \( \| \Pi \| \to 0 \).
Proof. By (2.3), it is enough to prove that $V^-$ satisfies (i), (ii) and (iii), where (iii) is immediate. Assumption 1, and Gronwall's lemma imply that $\forall t \in [t_0, 1], \forall (u, v) \in U \times V$, and $\forall x, y \in \mathbb{R}^n$:

$$\|x[t_0, x, u, v](t) - x[t_0, y, u, v](t)\| \leq e^{c(t-t_0)}\|x - y\|.$$  

Assumption 2 gives then, $\forall (u, v) \in U \times V$, and $\forall x, y \in \mathbb{R}^n$:

$$|g(x[t_0, x, u, v](1)) - g(x[t_0, y, u, v](1))| \leq c e^{c(1-t_0)}\|x - y\|.$$  

Thus, by standard arguments, $x \mapsto V^-(t, x)$ is $ce^c$-Lipschitz continuous $\forall t \in [t_0, 1]$ and, in particular, $V^-$ satisfies (i). On the other hand, (ii) is a weak version of the classical dynamic programming principle (see [2], for nonanticipative strategies, and [1] for NAD strategies, defined above): $\forall (t, x) \in [t_0, 1] \times \mathbb{R}^n, \forall t_1 \in [t, 1]$:

$$V^-(t, x) = \sup_{\alpha \in A} \inf_{v \in V} V^-(t_1, x[t, x, \alpha(v), v](t_1)).$$

Finally, let $\beta(V, \Pi)$ be an extremal strategy. By Corollary 2.1:

$$g(x[t_0, x_0, u, \beta(V, \Pi)(u)](1)) \leq V(t_0, x_0) + C \sqrt{\|\Pi\|}, \quad \forall u \in U.$$  

Consequently, for any $\varepsilon > 0$, $\beta(V, \Pi)$ is $\varepsilon$-optimal for sufficiently small $\|\Pi\|$.

References

