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Optimal Pricing of a Conspicuous Product During a Recession that Freezes Capital Markets[☆]

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Abstract

This paper considers the problem of how to price a conspicuous product when the economy is in a recession that disrupts capital markets. A conspicuous product in this context is a luxury good for which demand is increasing in brand image. Brand image here means the ability of a consumer to impress observers by conspicuously displaying consumption of the good. Brand image is built up when the good is priced high enough to make it exclusive, and eroded if the good is discounted.

Recession is modeled as having two effects: it reduces demand and it freezes capital markets so borrowing is not possible. In pricing the conspicuous product the firm faces the following trade-off. Reducing price helps maintain sales volume and cash flow in the face of reduced demand, but it also damages brand image and thus long term demand.

The paper analyzes the firm's pricing policy facing scenarios of mild, intermediate and severe recessions, while taking the threat of bankruptcy into account. For an intermediate recession the optimal solution is history-dependent. The results have implications for policy

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interventions in capital markets and for timing of mergers and acquisitions.

Key words: pricing, recession, conspicuous product, optimal control, Skiba point

JEL: C61, M31

1. Introduction

Standard recessions are a routine part of the business cycle. From a firm's perspective, they are periods of reduced demand. The 2008/2009 recession was non-standard inasmuch as it also involved widespread dislocation in capital markets; even firms with healthy fundamentals had difficulty borrowing. Firms were forced to maintain positive cash flow even if doing so required sacrifices to long-run profitability that might otherwise have been avoided, such as laying off employees with firm-specific capital, slashing R&D, or not investing in plant maintenance. All of these responses can be seen as drawing down a capital stock - be it human capital, technological know-how, production capacity, or brand image - more sharply than might have been optimal if borrowing were possible under terms available during normal times or even during standard recessions. This paper explores the example of firms sacrificing long-run brand reputation by lowering prices to improve short-term cash flow.

In particular, we consider the problem of how a firm should price a conspicuous product during a recession in which there is limited ability to borrow today based on the value the product's brand reputation will generate after the recession is over. We use the term "conspicuous product" to mean one for which demand is increasing in brand reputation and brand reputation in turn is increasing in price. This might be the case, for example, if the product's reputation for being expensive allows consumers to signal their wealth to observers and thereby enhance the reputation of the consumer. Examples of conspicuous products are fashion goods, luxury hotel rooms, and luxury cars. Indeed, our curiosity about this problem was piqued by articles in the New York Times describing the conundrum faces by luxury hotels (October 28, 2008; June 28, 2009).

Such so-called Veblen effects have fascinated economists for more than a hundred years (Bagwell and Bernheim, 1996), and a variety of models have been developed to explain why

it may be advantageous for consumers to behave this way (e.g., Bikhchandani et al., 1992; Coelho and McClure, 1993; Bagwell and Bernheim, 1996; Frijters, 1998; Corneo and Jeanne, 1999; Bianchi, 2002).

More recently there has been growing interest in looking at the phenomenon from the perspective of the firm that produces the good, rather than the consumers who buy them. Pricing conspicuous products is challenging even during normal times, so it has generated a modest literature (Amaldoss and Jain, 2005a,b; Kort et al., 2006; Caulkins et al., 2007). The brand image of a product is typically built up over time, and for a conspicuous product charging a high price raises the brand image. Setting the price optimally is thus a non-trivial issue since price has both a short-term direct effect on the quantity demanded today as well as a long term indirect effect on the entire demand curve in the future via its influence on the dynamic's of brand image.

To be specific, we imagine that for every price point there is a corresponding level of exclusivity. Actual brand image has inertia, like any capital stock. If the current price is raised to imply a level of exclusivity that is higher than the current brand image (which has been built up by past prices), then over time the brand's reputation will move up to reflect this more exclusive pricing. Conversely, discounting prices below those associated with the current brand image will erode that image. In other words, brand image is a stock or state variable that follows adjustment dynamics, always moving toward the level of exclusivity implied by the current price.

The present paper extends past research by considering how to price a conspicuous good during an economic recession like the one that started in 2008. In addition it broaches the general question of how firms might wish to draw down some capital stock - in this case a reputational stock - when the recession-induced reduction in demand is exacerbated by capital markets that do not function, so firms need to self-finance their operations.

Such recessions are uncommon, but by no means unprecedented. There were many panics in the 19th century, and the Asian financial crisis of 1997 is a more recent example, albeit on a regional not a global scale.

The self-financing constraint requires the firm to generate enough sales revenue to cover current operating costs. During a recession demand declines, particularly at the high end of the market (New York Times, October 28, 2008), creating pressure on sellers of conspicuous products to reduce price to improve cash flow. However, for exclusive brands such as the Four Seasons hotel, the first priority may be preserving the sanctity of the brand, so it would not want to reduce prices (New York Times, June 28, 2009).

We describe this problem by setting up a two-stage dynamic model. During Stage 1 (the recession), the firm has to price so that its operations are self-financing despite a recession-induced reduction in demand. The firm takes into account that the recession's duration is unknown and beyond the firm's influence because the crisis is too big for any one firm to bring it to an early end. In Stage 2 the recession is over, demand returns to its normal level, and there are normal (perfect) capital markets, implying that the firm can borrow and lend as much as it wants at a fixed interest rate.

The results, in brief, are as follows. When the recession is mild, price is positively related to brand image and over time the brand image converges to a constant value. However, the moment the recession is over, the optimal price jumps upwards.

When the recession is severe, the firm has to lower its price to keep demand at a sufficiently high level. This erodes brand image, which in turn leads to reduced demand so that the price needs to be reduced even further. This process cannot continue indefinitely. At some point brand image is so low that no price exists such that the resulting revenue covers the firm's operating costs. At that point the firm must go bankrupt since, by assumption, it cannot borrow during the recession. Whether bankruptcy actually occurs depends on what happens first: the end of the recession or reaching the level of brand image at which maximized current net revenue falls below operating costs.

A special case occurs when the recession is so severe or the initial brand image is so low that insolvency happens immediately. Otherwise, whether the firm goes bankrupt is in part a matter of luck, even if the firm is managed optimally.

In case of an intermediate recession the firm's optimal behavior is history dependent.

Where the firm ends up in the long term depends on the initial brand image. If initial brand image is low, the resulting price path is qualitatively similar to the severe recession case, price decreases over time, and the firm will go bankrupt if the recession lasts too long. If the initial brand image is high enough, the price path is qualitatively the same as in the mild recession case; there is no bankruptcy, price is increasing with brand image, and both price and brand image will approach stable levels as the recession continues.

The paper is organized as follows. Section 2 presents the model, while Section 3 contains the analysis and the results. Section 4 contrasts these results with the outcome if the duration of the recession is known and then if capital markets continue to function during the recession, leading to some interesting managerial implications for policy makers and owners. Section 5 concludes.

2. The Model

The continuous time model has two stages, a recession stage of unknown length followed by a second stage of normal business conditions that persists indefinitely. The state variable $A(t)$ denotes the firm's brand image at time t , while the control variable $p(t)$ is the price the firm charges. Demand is influenced by both price and brand image, $Q(p, A)$, and we assume the following specification in the normal period:

$$Q = m - p/A^\beta,$$

where m , the potential market size, and β are positive constants. We restrict $\beta < 1$ because the more fashionable the product, the more specialized is its market niche, so further increases in reputation will influence a smaller customer base. As usual quantity is decreasing in price. Furthermore, demand is increasing in brand image, but at a decreasing rate. Also, when brand image is high, a price increase will have less effect on demand, which is sensible for a conspicuous good.

We assume the recession shifts the demand curve downwards by a given amount, $\alpha > 0$,

for any given price and reputation, i.e.

$$Q = m - p/A^\beta - \alpha. \quad (1)$$

Hence, the constant α denotes the severity of the recession, and the proportionate reduction in demand is greatest for relatively high priced goods, which is sensible inasmuch as recessions make consumers more value focused.

Charging a high price contributes positively to brand image. This is captured in the following formulation:

$$\dot{A} = \kappa (\gamma p - A),$$

where γ and κ are positive constants. The parameter γ is a scaling parameter reflecting the relationship between brand image and price. The level of exclusivity associated with price p is γp , and if price is held at p indefinitely, then brand image will approach that level. Parameter κ governs how fast brand image adjusts when the current image is out of step with the current price. The smaller κ is the longer the firm can milk a strong brand image without undermining its reputation. On the other hand, if κ is high, the firm is in a good position to raise prices, since it only has to weather reduced demand for a relatively short time before brand image rises.

We assume the firm incurs a fixed cost per unit time, C , as long as it remains in business. These cover all the usual sources of fixed costs, from marketing budgets to retirees' health care costs, and are only expressed as a rate per unit time because this is a dynamic model.

Stage 2 models the period after the recession is over when demand has returned to normal ($\alpha = 0$), the capital market operates (perfectly), and there is no self-financing constraint. So the firm maximizes its discounted cash flow stream, by solving the following model:

$$\begin{aligned} S(A_0) &= \max_{p(\cdot)} \int_0^\infty e^{-rt} (p(m - p/A^\beta) - C) dt, \\ \dot{A} &= \kappa (\gamma p - A), \quad A(0) = A_0, \\ Q &= m - p/A^\beta \geq 0, \quad p \geq 0. \end{aligned}$$

Note: p and Q should obviously never be negative. It will turn out that p will always be strictly positive, but sometimes it will be optimal to let Q fall to zero, which we interpret as selling at the minimum level sufficient only for the price p to be observed and so influence image, but not high enough to generate meaningful revenues. The value function $S(A)$ of the Stage 2 problem will appear in the salvage value function of the Stage 1 problem.

In Stage 1 the recession reduces demand by the constant α . Also, since banks do not lend and stock prices are so low that issuing new equity is not a sensible alternative, at each moment the firm's revenues must cover its current operating cost. So in Stage 1 we impose the self-financing constraint

$$p(m - p/A^\beta - \alpha) - C \geq 0. \quad (2)$$

A more complicated two-state model would allow the firm to generate and retain excess cash early in the recession to spend down later by tracking cash on hand in a second state variable. By omitting that variable we are implicitly assuming the firm continues to pay out dividends to shareholders when it generates excess cash.

The self-financing constraint implies that there is a price, such that the firm generates only just enough net revenue to cover its operating costs, C . Using (2), this price can be determined as

$$p = \frac{1}{2}A^\beta(m - \alpha) \pm \sqrt{(\alpha - m)^2 - 4CA^{-\beta}}, \quad (3)$$

where the positive root is always preferred because it has a more beneficial effect on reputation. The minimum viable reputation is that which keeps the radical nonnegative in the price equation above, i.e.,

$$A_{\min} = \left(\frac{4C}{(m - \alpha)^2} \right)^{\frac{1}{\beta}}. \quad (4)$$

If at any time during Stage 1 the firm's reputation A falls below A_{\min} , then the firm is forced into bankruptcy. The minimum viable reputation can also be explained by considering the price ($p_s = (m - \alpha)A^\beta/2$) that maximizes immediate cash flow without taking into account

effects on future brand image and setting that maximized cash flow to zero.

We assume that the firm does not know when the recession will end, and that the firm cannot influence the timing of the end of the recession. In particular, we assume that the duration of the recession is exponentially distributed with mean $1/\mu$.

Assuming that the firm maximizes its expected discounted cash flow stream over time (T), the objective function consists of the expected gains during the recession period, which is $\int_0^s e^{-rt}(pQ - C) dt$, plus gains during the non-recession period, given by $S(A(s))$, both weighted by the exponential distribution's density function $\mu e^{-\mu t}$. Thus, employing integration by parts, we find that

$$\begin{aligned} & \int_0^T \left(\int_0^s e^{-rt} (p(m - p/A^\beta - \alpha) - C) dt + S(A(s)) \right) \mu e^{-\mu s} ds \\ = & \int_0^T e^{-(r+\mu)s} [p(m - p/A^\beta - \alpha) - C + \mu S(A)] ds \\ & - e^{-\mu T} \int_0^T e^{-rt} [p(m - p/A^\beta - \alpha) - C] dt. \end{aligned}$$

Note that the first term describes the profit during a recession period at time t multiplied with the probability that the recession endures that long plus the profits from a subsequent normal period that begins at t multiplied with the probability that the recession ends at time t . The last term takes into account that the gain of the firm has to be reduced, since there is a positive probability for the recession to endure. This probability is exactly $e^{-\mu T}$ and is balanced in a way, that if the recession would never end, i.e. $\mu = 0$, the total gain is zero, independent of end time T .

Taking into account that the recession may take so long that the firm faces bankruptcy,

we assume a free end time problem yielding:

$$\begin{aligned}
V &= \max_{p(\cdot), T} \left\{ \int_0^T e^{-(r+\mu)t} [p(m - p/A^\beta - \alpha) - C + \mu S(A)] dt \right. \\
&\quad \left. - e^{-\mu T} \int_0^T e^{-rt} [p(m - p/A^\beta - \alpha) - C] dt \right\}, \\
\dot{A} &= \kappa(\gamma p - A), \quad A(0) = A_0, \\
p(m - p/A^\beta - \alpha) - C &\geq 0, \\
p &\geq 0, \quad A(T) \geq A_{\min}.
\end{aligned}$$

In principle the firm has basically two possibilities. Either the pricing policy creates a *positive bankruptcy probability*. Then denoting by T the possible bankruptcy time, the firm will go bankrupt if the recession lasts more than T years, which happens with probability $e^{-\mu T}$. The value of T is chosen optimally, and thus endogenously, such that $\frac{dV}{dT} = 0$. The firm goes bankrupt at the moment that no price exists such that the current revenue matches the fixed cost C .

Or if the firm applies a pricing policy such that the *probability of bankruptcy is zero*, it will live on forever, i.e. $T = \infty$, and the Stage 1 problem then becomes:

$$\begin{aligned}
V &= \max_{p(\cdot)} \int_0^\infty e^{-(r+\mu)t} [p(m - p/A^\beta - \alpha) - C + \mu S(A)] dt, \\
\text{s.t.} \quad \dot{A} &= \kappa(\gamma p - A), \quad A(0) = A_0, \\
p(m - p/A^\beta - \alpha) - C &\geq 0, \quad p \geq 0
\end{aligned}$$

It is important to realize that, depending on $A(0)$ and the parameter values, there can be three different scenarios. First, the situation can be so good that under the optimal policy the probability of bankruptcy is zero. Then the infinite time model automatically applies. Second, the situation can be so bad that a policy leading to a positive bankruptcy probability is unavoidable. Then the free end time model prevails. Third, the firm can be in a situation such that it can choose between a policy associated with zero bankruptcy probability, or a policy leading to bankruptcy with positive probability. In this situation both models must

be analyzed, and the case yielding highest objective value is the policy the firm will choose.

3. Results

3.1. Parameter Values for Numerical Calculations

Since not all results can be derived analytically, some numerical calculations have to be executed. This implies that we are restricted to studying certain scenarios. To clarify which situations we cover, we motivate our parameter value choices below. Moreover, we analyze the model for different gradations of recession severity by considering different values of the key parameter α . The following parameter values were used:

$$\begin{array}{cccccccc} r & \gamma & \beta & C & \kappa & m & \mu & \alpha \\ \hline 0.04 & 5 & 0.5 & 7.5 & 2 & 3 & 0.5 & [0, 1] \end{array} \quad (5)$$

The values of r and μ are understandable given that we measure time in years. The expected length of the recession is $1/\mu$. So the value $\mu = 0.5$ implies the expected duration of the recession is two years.

To get some grip on the values of m , β , and α , consider the long run steady state when there is no recession. There it holds that $A = \gamma p$. Substituting this equality into the demand equation and maximizing profit gives

$$\tilde{p} = \left(\frac{m\gamma^\beta}{2-\beta} \right)^{\frac{1}{1-\beta}}, \quad \tilde{Q} = m \frac{1-\beta}{2-\beta}.$$

For analytical convenience we take $\beta = 0.5$ and then choose $m = 3$ to normalize the value of \tilde{Q} to 1. That normalization lets us define the recession-induced decline in potential market, α , as the percentage reduction in demand felt the moment the recession starts, assuming that the firm had been in this long run profit maximizing no-recession equilibrium.

Strictly speaking, parameter γ is not necessary. It could be set to 1 without loss of generality. But that would force reputation and price to have the same values in steady state, which can make interpretation confusing. So we arbitrarily set $\gamma = 5$.

Since the Stage 1 problem contains the objective value of Stage 2, we start by analyzing Stage 2.

3.2. Analysis of Stage 2

The Hamiltonian is

$$H = p(m - p/A^\beta) - C + \lambda\kappa(\gamma p - A),$$

where λ is the co-state variable.¹ The control variable p is restricted by $m - p/A^\beta \geq 0$. Thus, the Lagrangian is

$$L = H + \pi(m - p/A^\beta).$$

The Legendre-Clebsch condition is always fulfilled as $L_{pp} = -2A^{-\beta} < 0$. The Lagrange multiplier is denoted by π and has to be non-negative.

The co-state equation is

$$\dot{\lambda} = r\lambda - H_A = (r + \kappa)\lambda - \frac{(p + \pi)\beta p}{A^{\beta+1}}. \quad (6)$$

If the restrictions are not binding, π is zero and the optimal dynamic price satisfies

$$p = \frac{1}{2}A^\beta(m + \kappa\lambda\gamma). \quad (7)$$

It can be shown that the sufficiency conditions concerning concavity of the Hamiltonian are fulfilled if no constraint is violated.

Differentiating (7) w.r.t. time, and using (6) and (7) itself, gives

$$\dot{p} = p(r + \kappa - \kappa\beta) - \frac{1}{2}A^\beta m(r + \kappa) + \frac{1}{2A}p^2\kappa\beta\gamma. \quad (8)$$

¹Note: omitting the factor λ_0 in the Hamiltonian is justified since the problem is normal. This can be seen by the fact that for $\lambda_0 = 0$ the Hamiltonian becomes linear in p and the co-state dynamics independent of A . Therefore the Hamiltonian maximizing condition yields $p = mA^\beta$, which is obviously inferior to a solution with $p = p_s$ for example.

The steady state is

$$\hat{p} = \left(\frac{\kappa + r - \frac{1}{2}\kappa\beta}{\frac{1}{2}\gamma^\beta m (r + \kappa)} \right)^{\frac{1}{\beta-1}}, \quad \hat{A} = \gamma \left(\frac{\kappa + r - \frac{1}{2}\kappa\beta}{\frac{1}{2}\gamma^\beta m (r + \kappa)} \right)^{\frac{1}{\beta-1}}. \quad (9)$$

Hence, an equilibrium in the first quadrant is only possible if $\beta < 2\left(1 + \frac{r}{\kappa}\right)$. It is easily checked that the equilibrium (9) lies in the region where $Q = m - p/A^\beta > 0$ iff $\beta \leq 1 + \frac{r}{\kappa}$.

From determining the Jacobian in the equilibrium we straightforwardly obtain that $\det J < 0 \iff \beta < 1$. We have proved the following proposition.

Proposition 1. *There exists a unique interior steady state (9) for $\beta \leq 1 + \frac{r}{\kappa}$. If furthermore $\beta < 1$ then the steady state is a saddle point.*

While for $\beta < 1$ the equilibrium is a saddle point that lies in the region with $Q > 0$, a positive sales volume may still not pertain for the complete saddle point path.

If the constraint concerning the non-negativity of the demand is binding the control variable is given by $p = mA^\beta$. The Lagrange multiplier can then be determined as $\pi = A^\beta(m + \kappa\gamma\lambda) - 2p = A^\beta(\kappa\gamma\lambda - m)$. It can be shown that there is no admissible steady state with active control constraints.

The phase diagram in Figure 1 makes clear that $Q = 0$ for A sufficiently small, implying that current revenue is zero there. However, note that p is at its upper bound for $Q = 0$, and the advantage of a large p is that brand image A is increasing quickly, which raises future demand. If the reputation is larger than its steady state value, a decision maker would, however, accept a loss of reputation to strengthen the immediate demand.

PUT FIGURE 1 HERE

For the numerical calculation of $S'(A)$ in the Stage 1 problem we note that $S'(A) = \lambda(A)$ with $\lambda(A)$ the value of the co-state of the non-recession corresponding to the value of A , which is justified since the problem is autonomous.

3.3. Analysis of Stage 1

The Hamiltonian and Lagrangian are

$$\begin{aligned} H &= (1 - e^{\mu(t-T)}) (p(m - p/A^\beta - \alpha) - C) + \mu S(A) + \lambda \kappa (\gamma p - A), \\ L &= H + \pi (m - p/A^\beta - \alpha) \end{aligned}$$

where again λ is the co-state variable and π the Lagrange multiplier. The control variable p is restricted by $p(m - p/A^\beta - \alpha) - C \geq 0$ and $p \geq 0$. As long as these constraints are not binding, the optimal price satisfies

$$p = \frac{1}{2} A^\beta \left(m - \alpha + \frac{\kappa \lambda \gamma}{1 - e^{\mu(t-T)}} \right).$$

The cash flow constraint can also be written as

$$\begin{aligned} p &\in [p_{\min}, p_{\max}], \\ p_{\min} &= \frac{1}{2} A^\beta (m - \alpha) - \frac{1}{2} \sqrt{A^\beta (A^\beta (m - \alpha)^2 - 4C)}, \\ p_{\max} &= \frac{1}{2} A^\beta (m - \alpha) + \frac{1}{2} \sqrt{A^\beta (A^\beta (m - \alpha)^2 - 4C)}. \end{aligned}$$

Charging a price $p < p_s$ is never optimal. Hence, the region of possible prices that can occur on an optimal trajectory in Stage 1 is

$$p \in [p_s, p_{\max}]. \quad (10)$$

The dynamic co-state equation is

$$\dot{\lambda} = (r + \mu) \lambda - H_A = (r + \mu + \kappa) \lambda - \frac{(1 - e^{\mu(t-T)} + \pi) \beta p^2}{A^{\beta+1}} - \mu S'(A).$$

If the self-financing constraint is active the Lagrange multiplier is determined as $\pi = -\frac{\lambda \kappa \gamma}{m - 2pA^{-\beta} - \alpha} - 1 + e^{\mu(t-T)}$. The Legendre-Clebsch condition is fulfilled as $L_{pp} = -(1 - e^{\mu(t-T)} + \pi) 2A^{-\beta} \leq 0$.

In the infinite time horizon case the transversality condition $\lim_{t \rightarrow \infty} e^{-rt} \lambda(t) \rightarrow 0$ has to hold.

Employing a strategy with a finite time horizon raises the question of when it is optimal to cease operation. Let us assume that a firm can either try to survive as long as possible and go bankrupt at $A(T) = A_{\min}$ with $\lambda(T)$ free or stop earlier. Then it must find a reputation greater than A_{\min} where the transversality condition $\lambda(T) = 0$ has to hold. In order to determine T optimally it must hold that $\frac{dV}{dT} = 0$, thus, in both cases

$$H(T) = -\mu \int_0^T e^{-rt} (p(m - p/A^\beta - \alpha) - C) dt \quad (11)$$

must be fulfilled. If $\lambda(T) = 0$ this condition cannot hold if $S(A) > 0$, which is the case for the parameter values used here. Thus, we can exclude $A(T) > A_{\min}$, meaning that a decision maker has no incentive to liquidate if there is still at least some chance of survival.

Ideally, we would like to proceed as in Stage 2, by deriving \dot{p} to obtain the dynamic system for A and p . However, we are not able to determine analytically the value of the firm $S(A)$ in Stage 2. This implies that no analytical expression is available for $S'(A)$ and thus for $\dot{\lambda}$. Therefore, we proceed with a numerical analysis.

We consider how the firm's optimal strategy depends on the severity of the recession, as governed by parameter α . Figure 2 and Table 1 reveal that three regions can be identified, corresponding to a mild, intermediate and a severe recession, respectively. Besides the level of the minimum possible reputation value A_{\min} , Figure 2 shows that in Region I there exists one steady state \hat{A}_I , in Region II we have two steady states, \hat{A}_I and \hat{A}_{II} , while no steady states exist in Region III.

PUT TABLE 1 HERE

PUT FIG. 2 HERE

3.3.1. Mild recession

The recession being mild implies that the value of α is relatively low. Figure 3 depicts the optimal trajectories for $\alpha = 0.7$. Recall that if brand image is ever below A_{\min} (see (4)) no price exists such that revenue counterbalances the cost C , and the firm must go bankrupt immediately.

PUT FIG. 3 HERE

As long as brand image exceeds A_{\min} , the trajectory converges to the steady state (\hat{A}_I, \hat{p}_I) . This happens in a monotonic way so if $A_0 \geq A_{\min}$ then A never falls below A_{\min} and the firm never goes bankrupt. Thus if $A_0 < A_{\min}$ the firm always shuts down as soon as the recession starts; otherwise there is zero probability of going bankrupt. For no initial condition is there an intermediate probability of going bankrupt. A qualitatively similar solution pertains for any $\alpha < 0.7106$. For this reason we classify recessions falling into this category as *mild*.

If the initial brand image is between A_{\min} and the steady state \hat{A}_I , it is optimal for the firm to raise the price over time. For A_0 at the lower end of this range, the firm should initially choose the highest possible price, meaning a price such that the self-financing constraint is binding. This increases brand reputation as fast as is possible. Later, once A is greater than 47.42, the optimal trajectory leaves the cash flow constraint, and the firm begins to generate positive net operating revenues as it approaches the steady state.

Figure 3 confirms (10) that the optimal dynamic price is always at least as great as p_s , i.e., that in setting price the firm considers the positive effect of a high price on the future values of brand image and thus on future demand levels. It also shows that A_{\min} is the point of intersection between the cash flow constraint and the short-term profit maximizing prize.

3.3.2. Severe recession

If there is a severe recession then the firm can never avoid facing a positive risk of bankruptcy. More precisely, there exists a finite time T such that if the recession is not over yet, cash flow will become negative.

A severe recession occurs for $\alpha \geq 0.8366$. Figure 4 depicts the case $\alpha = 0.85$. For larger values of the initial brand image, price has to be below A/γ to generate demand. Consequently, brand image decreases and the optimal trajectory always approaches A_{\min} at which point bankruptcy results.

If A_0 is greater than 80.79, the optimal price is below the maximum price possible under the cash flow constraint. This erodes reputational capital faster than is strictly necessary, but allows the firm to produce a positive net cash flow. Once A has fallen below 80, price is kept as high as the cash flow constraint will allow in order to reduce the speed at which brand image declines. The closer A is to A_{\min} , the higher the bankruptcy probability, but bankruptcy is never inevitable. If the firm is lucky, the recession ends before A_{\min} is reached.

It might seem odd that even when $A > 80$ the firm would risk using a lower price and hence a faster erosion of brand reputation. However, companies are in business to maximize expected profit, not to maximize the probability of survival. Taking risks is a normal part of business, and in this case the firm accepts a greater risk of subsequent bankruptcy in order to generate current income for the owners.

As it turns out, with these parameters the firm is not being terribly daring. It starts charging the maximum possible price at a point such that it can still survive 9.63 more years of recession and the probability the recession lasts that much longer is only $e^{-0.5 \cdot 9.63} = 0.8\%$.

This low bankruptcy risk is a direct consequence of the parameter choice. For instance, when we increase the discount rate to $r = 0.1$, the bankruptcy probability becomes 1.35%. If we subsequently raise the adjustment speed of brand image by increasing κ from 2 to 4, the bankruptcy probability further increases to 7.13%. This probability can be made even larger, even beyond 50%, by further adjusting these parameters and/or the expected duration of the recession.

PUT FIG. 4 HERE

3.3.3. *Intermediate recession*

Next, consider the α -region $(0.7106, 0.8366)$, which is the interval between the above cases and so is an *intermediate recession*. Compared to the mild recession case, an additional

steady state $(\hat{A}_{II}, \hat{p}_{II})$ arises, which results in history dependent behavior, as illustrated in Figure 5. In particular, $(\hat{A}_{II}, \hat{p}_{II})$ acts as a weak Skiba point (see, e.g., Grass et al., 2008; Caulkins et al., 2007; Haunschmied et al., 2003), below which reputation and hence demand is so low that the firm acts as if the recession is severe, and above which reputation and hence demand is so high that the firm acts as if the recession is mild. If the brand image exceeds \hat{A}_{II} , the optimal trajectories always converge to the steady state \hat{A}_I in a monotonic way, so the firm lives on forever and bankruptcy never occurs.

PUT FIG. 5 HERE

However, if $A_0 \in (A_{\min}, \hat{A}_{II})$, the trajectory approaches A_{\min} . As in the severe recession case, the firm needs to reduce its price to improve sales. However, this leads to a decline in brand image. If brand image declines all the way to A_{\min} before the end of the recession then bankruptcy occurs. However, with positive probability the recession will be over before this occurs. Figure 6 shows the bankruptcy probability as a function of the initial brand image A_0 . In between A_{\min} and \hat{A}_{II} the bankruptcy probability is decreasing because the larger the initial brand image, the longer it takes to reach A_{\min} , and therefore, the greater the chance that in the mean time the recession will be over.

PUT FIG. 6 HERE

4. Robustness

Two essential attributes of our modeling of the recession are that its duration is unknown (specifically, the duration is exponentially distributed) and that it freezes capital markets. One way to enhance intuition about our system is to compare the results above with those obtained if one or the other of these two attributes is modified.

To assess the impact of uncertainty we considered a deterministic variant where the end of the recession is fixed at $\tau = 2$ (matching the expected length of the recession in the stochastic case). Figure 7 presents the solution, depicting both the Stage 1 (lines (a)-(c)) and Stage 2 (line (d)) prices for different initial levels of reputation.

PUT FIG. 7 HERE

If the initial brand image level lies in between A_{\min} and A_b , the optimal trajectory will reach A_{\min} before the recession is over, which implies that the firm will surely go bankrupt. For an initial brand image between A_b and A_{II} the recession will be over before A_{\min} is reached, so bankruptcy does not occur. Hence, the main difference is that in the deterministic case bankruptcy either occurs with certainty or surely does not occur. In the stochastic case the bankruptcy probability is a continuous function of brand image in the severe and in the intermediate recession case.

PUT FIG. 8 HERE

Turning to the second alternate recession model, where duration is unknown but there is no self-financing constraint, it can be shown that the optimal trajectory always converges to the steady state A_I (see Figure 8). No matter how indebted the firm is, and hence how poorly it has performed for past investors, the forward-looking prospects are strong enough to justify fresh infusions of capital. This is because with an exponentially distributed recession length, no matter how long the recession has already lasted, one does not become any more pessimistic (or optimistic) about how much longer it will last.

Hence, the history dependent behavior is caused by the self-financing constraint. Satisfying the constraint requires that revenue at least matches the operating cost. If the initial brand image is below A_{II} , the firm has to reduce its price to keep demand at a sufficiently high level. However, the low price further reduces brand image so that future demand is lowered. Absent the self-financing constraint, the firm would set a high price and initially make losses, but brand image and hence demand increases over time, allowing the firm to make profits later on. However, this can only happen when the firm is able to finance its losses by borrowing money.

PUT FIG. 9 HERE

Figure 9 plots the present value of the firm's expected future earnings as a function of initial brand reputation ($A(0)$) for three cases: no recession, a "normal recession" with

reduces demand but does not disrupt capital markets, and a recession that forces the firm to self-finance. The reduction in demand in both types of recessions is the same ($\alpha = 0.83$). Yet, the rather striking result is that while reducing demand reduces firm value (the decline from the dashed no recession line to the dotted and then solid normal recession lines), the destruction of value can be much greater if capital markets are also disrupted. Specifically, the destruction in value is much greater with these parameter values for firms whose initial brand reputation is not high; in contrast, for firms with a reputation strong enough to generate net income to cover operating expenses even during the recession, there is no extra effect of losing access to capital markets. This suggests two managerial implications. From the perspective of government or society more generally, policy interventions that stimulate demand may have value (depending on larger general equilibrium considerations that are beyond the scope of this model), but in some circumstances interventions to correct the market failure of non-functioning capital markets could be even more valuable. From the perspective of private enterprise, loss of capital markets may stimulate mergers if firms with positive cash flow can merge with, or acquire, firms for whom loss of financing opportunities is a mortal threat. Such a merger might create value if it allows the weaker firm to avoid liquidation. That is, when normal borrowing is not possible, those with positive cash flow might be able to snap up some bargain acquisitions.

5. Conclusion

This paper analyzes the problem of a firm selling a conspicuous product during a recession when cutting prices both improves short term cash flow and erodes brand reputation in the long-run. The paper was inspired by the luxury hotel business dealing with this trade off during the recession that started in 2008.

Dynamic effects are important, so we set up a two stage optimal control model. In the first stage the firm faces a recession that reduces demand and prevents access to outside financing. Once the recession is over, which happens at an unknown point in time, the firm enters the second stage during which demand is back to its normal level and it is possible for the firm to borrow and issue new equity shares.

We study firm behavior for recessions of different severity. With a mild recession, if the initial reputation is sufficient to operate at all, the firm can operate indefinitely without risk of bankruptcy. With a severe recession, the firm always has to price so low that reputation erodes and the firm will go bankrupt if the recession is not over in time. For an intermediate recession the firm's long run position is determined by the initial brand image. This history dependence is caused by the self financing constraint. If the brand image is initially low, the firm's limited financial means cause an increased probability of bankruptcy over time, but with a high enough initial brand image the firm survives the recession for sure.

Optimal solution values can be much lower with the capital constraint than with just the recession-induced reduction in demand. This implies that government intervention to kick-start capital markets may have a benefit (although presumably it also has costs, which are beyond the scope of this paper, so the conclusion is only that there can be benefits, not that the benefits justify the costs). They also show how people or firms with large cash reserves when capital markets freeze might be able to acquire or merge at bargain prices with firms that have sound long-run fundamentals but poor cash positions.

One interesting topic for future research may be the following. In our current model the shareholders collect all profits immediately as dividends. Consequently, they never let the firm retain profits as a cash cushion. To make the latter possible it would be interesting to add a cash inventory as a second state variable. Then the bankruptcy condition would change such that bankruptcy would occur at the moment that the firm runs out of cash.

Another extension would consider a three period problem that distinguishes the restoration of the capital market from the restoration of demand. That is, Stage 1 would be the same as in the current model. In Stage 2 the self-financing constraint is removed, but the reduction in demand persists. Stage 3 would, like Stage 2 in the present model, represent full restoration of normal business conditions.

Finally, yet another interesting topic for future research would be to reconsider the dynamic brand image equation. For example, Amaldoss and Jain (2008, 2010) divide consumers in two groups, "leaders" and "followers". The leaders' utility derived from purchasing the

good decreases in the number of followers who are also consuming the good, whereas the followers' utility increases in the number of leaders who are using it. Where Amaldoss and Jain use a two-period formulation (without budget constraint), our approach could extend this research to a fully dynamic setting.

References

- Amaldoss, W., Jain, S., 2005a. Conspicuous consumption and sophisticated thinking. *Management Science* 51 (10), 1449–1466.
- Amaldoss, W., Jain, S., 2005b. Pricing of conspicuous goods: a competitive analysis of social effects. *Journal of Marketing Research* 42 (1), 30–42.
- Amaldoss, W., Jain, S., 2008. Trading up: a strategic analysis of reference group effects. *Marketing Science* 27 (5), 932–942.
- Amaldoss, W., Jain, S., 2010. Reference groups and product line decisions: an experimental investigation of limited editions and product proliferation. *Management Science* 56 (4), 621–644.
- Bagwell, L. S., Bernheim, B. D., 1996. Veblen effects in a theory of conspicuous consumption. *The American Economic Review* 86 (3), 349–373.
- Bianchi, M., 2002. Novelty, preferences, and fashion: when goods are unsettling. *Journal of Economic Behavior & Organization* 47 (1), 1–18.
- Bikhchandani, S., Hirshleifer, D., Welsh, I., 1992. A theory of fads, fashion, custom, and cultural change as informational cascades. *Journal of Political Economy* 100 (5), 992–1026.
- Caulkins, J. P., Hartl, R. F., Kort, P. M., Feichtinger, G., 2007. Imitators chasing innovators in product space. *Journal of Economic Dynamics & Control* 31 (5), 1535–1556.

- Coelho, P. R. P., McClure, J. E., 1993. Toward an economic theory of fashion. *Economic Inquiry* 31 (4), 595–608.
- Corneo, G., Jeanne, O., 1999. Segmented communication and fashionable behavior. *Journal of Economic Behavior & Organization* 39 (4), 371–385.
- Frijters, P., 1998. A model of fashion and status. *Economic Modeling* 15 (4), 501–517.
- Grass, D., Caulkins, J. P., Feichtinger, G., Tragler, G., Behrens, D. A., 2008. *Optimal Control of Nonlinear Processes: With Applications in Drugs, Corruption and Terror*. Springer, Heidelberg.
- Haunschmied, J. L., Kort, P. M., Hartl, R. F., Feichtinger, G., 2003. A DNS-curve in a two-state capital accumulation model: a numerical analysis. *Journal of Economic Dynamics & Control* 27 (4), 701–716.
- Kort, P. M., Caulkins, J. P., Hartl, R. F., Feichtinger, G., 2006. Brand image and brand dilution in the fashion industry. *Automatica* 42 (8), 1363–1370.
- The New York Times, 2008. Dim days for luxury hotels. By J. Sharkey, October 28.
- The New York Times, 2009. Pillow fights at the Four Seasons. By D. Segal, June 28.

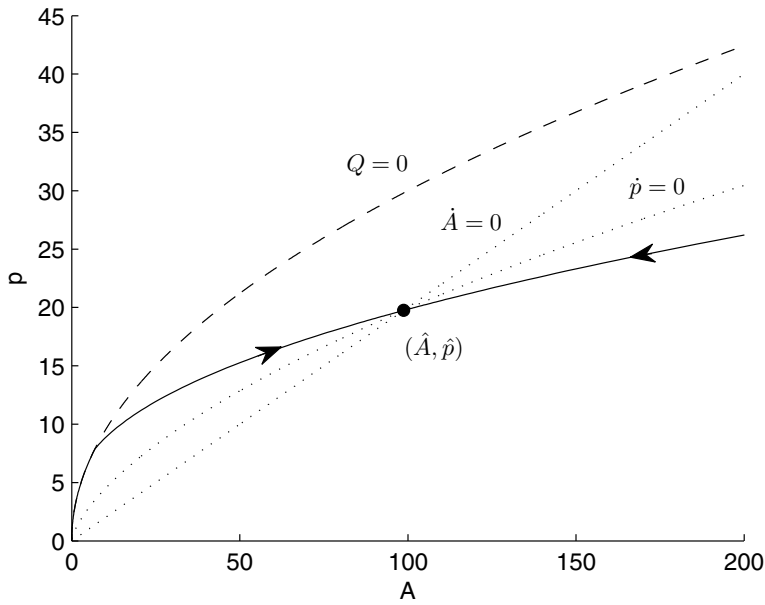


Figure 1: Optimal trajectories in Stage 2.

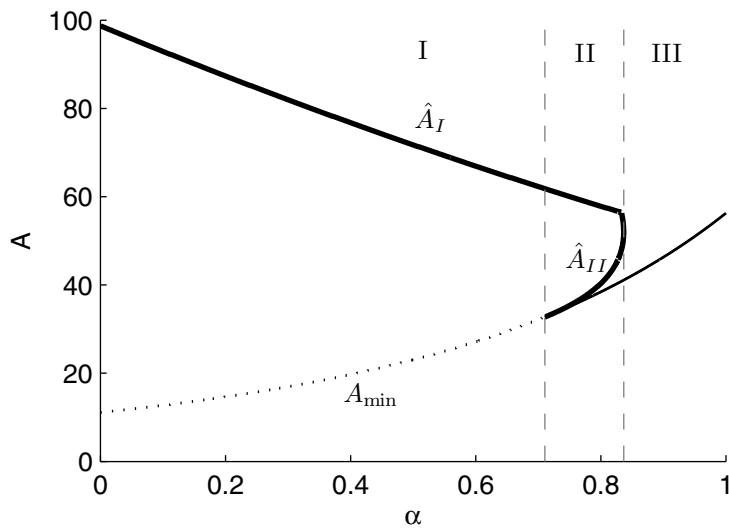


Figure 2: Bifurcation diagram for parameter α showing the values of the brand image in the steady state and the minimum possible reputation. The dotted line means that A_{\min} cannot be an end point of an optimal solution with $T > 0$.

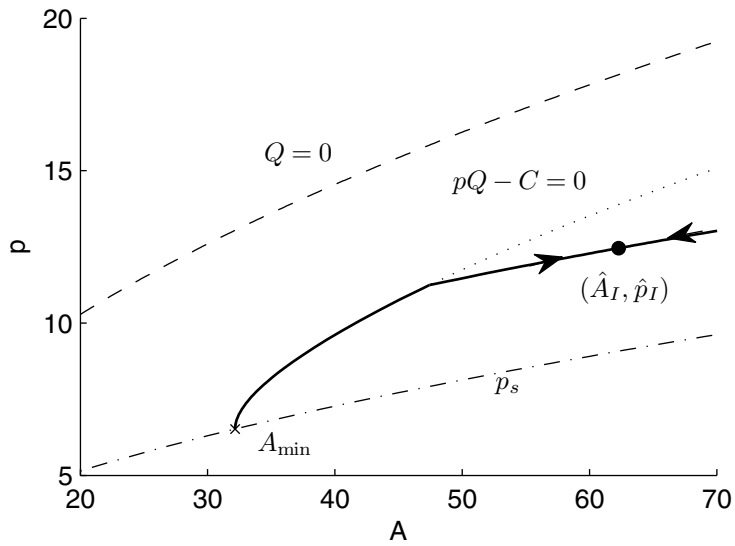


Figure 3: The firm's optimal trajectories in Stage 1 for the parameter values of (5) with $\alpha = 0.7$.

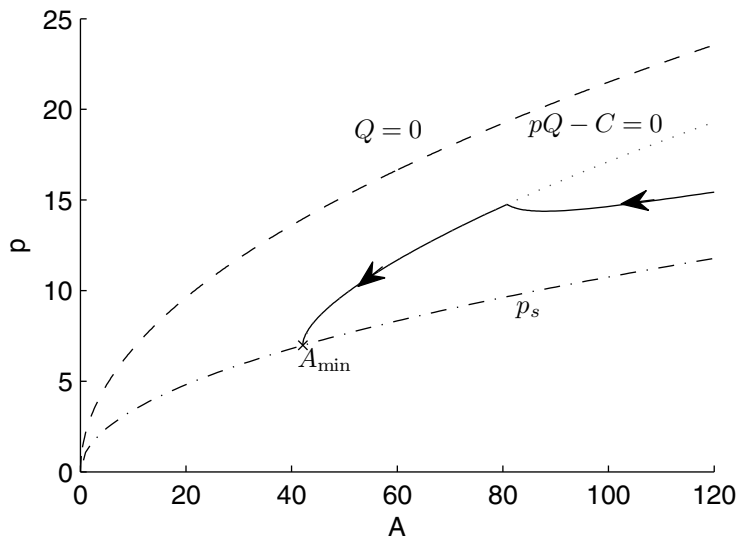


Figure 4: The firm's optimal trajectories in Stage 1 for the parameter values of (5) with $\alpha = 0.85$.

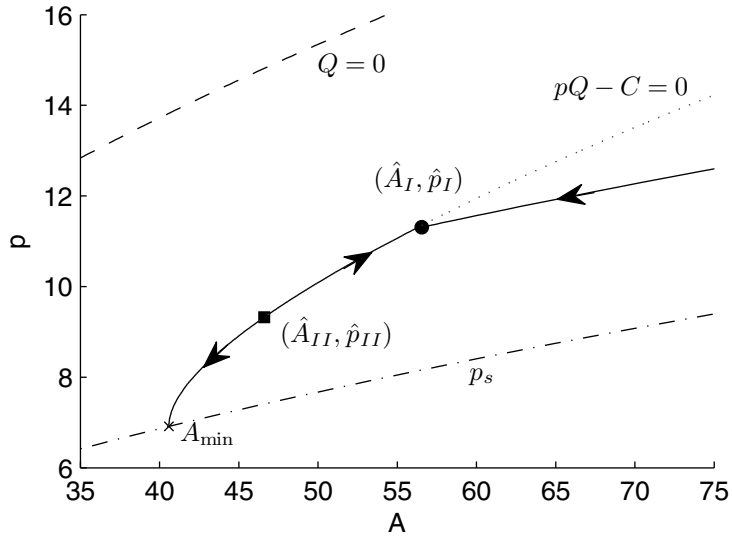


Figure 5: The firm's optimal trajectories in Stage 1 for the parameter values of (5) with $\alpha = 0.83$.

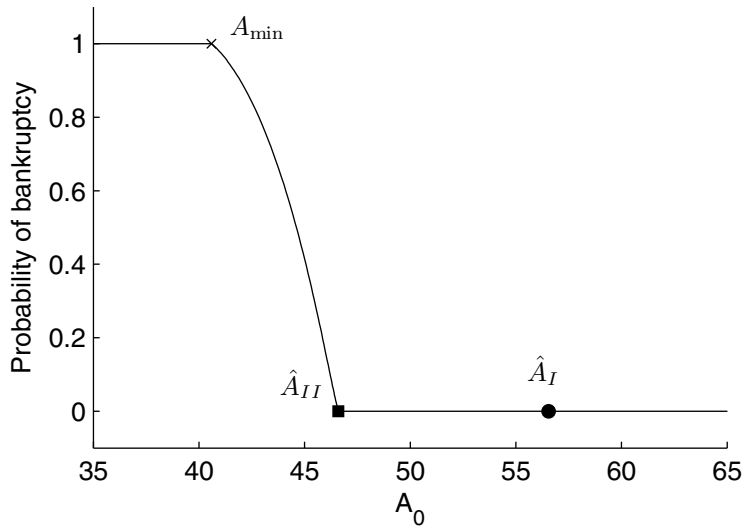


Figure 6: Bankruptcy probability for the parameter values of (5) with $\alpha = 0.83$, indicating a recession of intermediate severity.

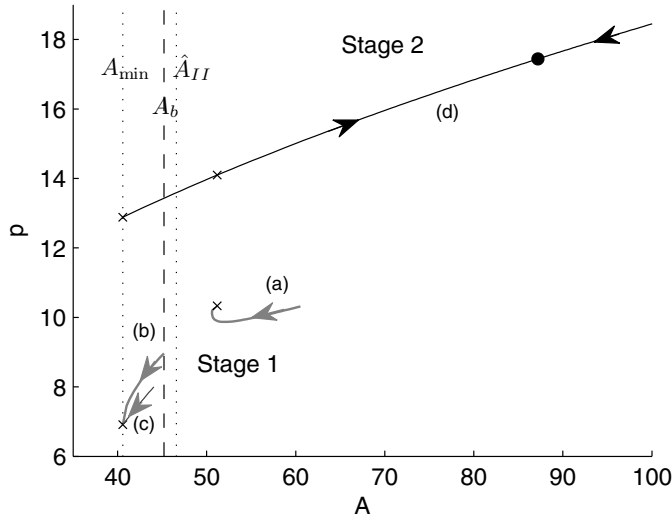


Figure 7: Optimal trajectories in Stage 1 and Stage 2 for the parameter values of (5) but with a fixed recession duration T , and with $\alpha = 0.83$. (a) and (b) depict optimal solution paths of the first stage of the two-stage problem for two different initial values, where (d) shows the corresponding Stage 2 solution paths. \times denotes the switching points between the two stages. A_b is the smallest initial reputation where a firm is able to survive; (c) depicts the case where $A_0 < A_b$ and is thus found by solving only the Stage 1 problem with free end time. \hat{A}_{II} corresponds to the steady state in the original model.

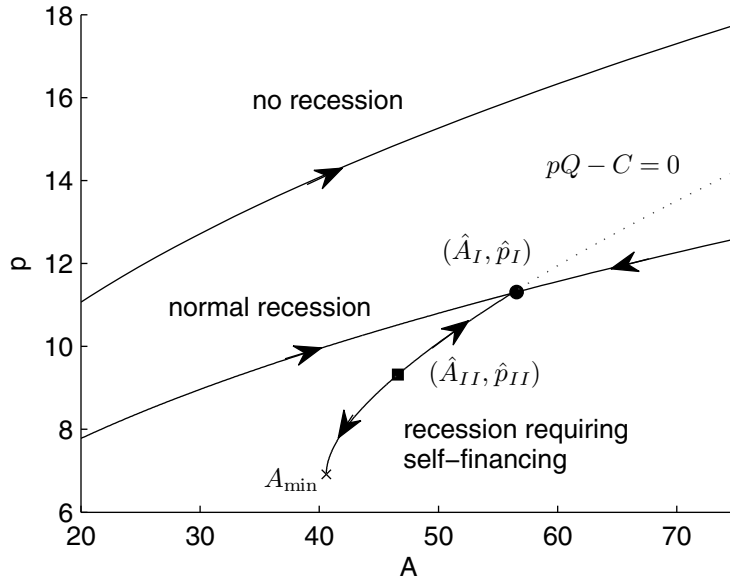


Figure 8: Phase portrait for $\alpha = 0.83$ in a recession with and without self-financing constraint.

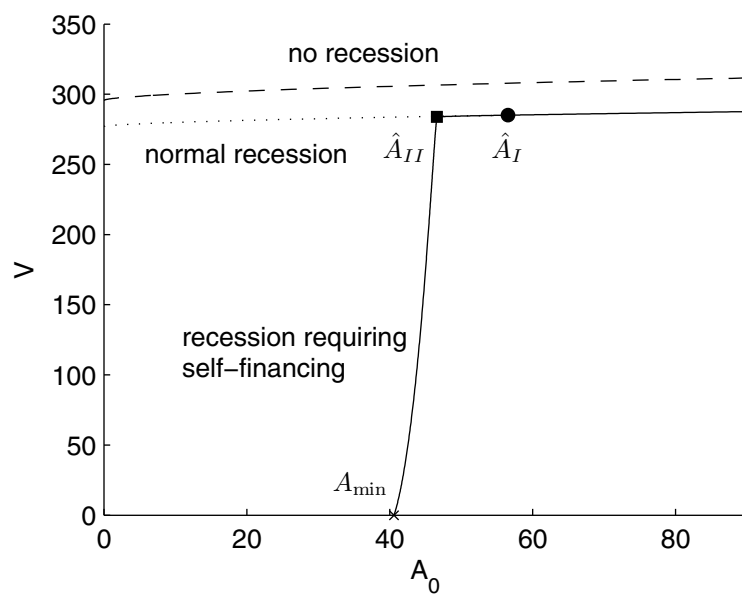


Figure 9: Value of the objective function for $\alpha = 0.83$ in a recession with and without self-financing constraint.

Recession	Region	Parameter α
Mild	I	$0, m - 2 \left(\gamma^\beta C^{\beta-1} \right)^{\frac{1}{\beta-2}}$
Intermediate	II	$\left(m - 2 \left(\gamma^\beta C^{\beta-1} \right)^{\frac{1}{\beta-2}}, m - 3 \left(\frac{C}{4\gamma} \right)^{\frac{1}{3}} \right)$
Severe	III	$\left(m - 3 \left(\frac{C}{4\gamma} \right)^{\frac{1}{3}}, m \right)$

Table 1: Characterization of the severity of the recession

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