

# MAXIMUM PRINCIPLE AND SYMMETRY FOR MINIMAL HYPERSURFACES IN $\mathbb{H}^n \times \mathbb{R}$

BARBARA NELLI, RICARDO SA EARP, ERIC TOUBIANA

ABSTRACT. The aim of this work is to study how the asymptotic boundary of a minimal hypersurface in  $\mathbb{H}^n \times \mathbb{R}$  determines the behavior of the hypersurface at finite points, in several geometric situations.

## 1. INTRODUCTION

In this article we discuss how, in several geometric situations, the shape at infinity of a minimal surface in  $\mathbb{H}^2 \times \mathbb{R}$  determines the shape of the surface itself.

A beautiful theorem in minimal surfaces theory is the Schoen's characterization of the catenoid [12]. It can be stated as follows. *Let  $M \subset \mathbb{R}^3$  be a complete immersed minimal surface with two annular ends. Assume that each end is a graph, then  $M$  is a catenoid.* On the other hand, there exists a complete minimal annulus immersed in a slab of  $\mathbb{R}^3$  [6].

A characterization of the catenoid in the hyperbolic space, assuming regularity at infinity, was established by G. Levitt and H. Rosenberg in [5]. In a joint work with L. Hauswirth [3], the authors of the present article proved a Schoen type theorem in  $\mathbb{H}^2 \times \mathbb{R}$ , in the class of finite total curvature surfaces.

Our first result is a new Schoen type theorem in  $\mathbb{H}^2 \times \mathbb{R}$ . Namely, we replace Schoen's assumption *each end is a graph* with the assumption *each end is a vertical graph whose asymptotic boundary is a copy of the asymptotic boundary of  $\mathbb{H}^2$*  (Theorem 2.1).

Our second result is a *maximum principle* in a vertical (closed) halfspace. Assume that  $M$  is a complete minimal surface, possibly with finite boundary, properly immersed in  $\mathbb{H}^2 \times \mathbb{R}$  and that the boundary of  $M$ , if any, is contained in the closure of a vertical halfspace  $P_+$ . Assume further that the points at finite height of the asymptotic boundary of  $M$  are contained in the asymptotic boundary of the halfspace  $P_+$ . Then  $M$  is entirely contained in the halfspace  $P_+$ , unless  $M$  is equal to the vertical halfplane  $\partial P_+$  (Theorem 3.1).

Then we generalize our results to higher dimensions.

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Theorem 2.1 and Theorem 3.1 in higher dimension are analogous to the 2-dimensional case. In order to generalize Theorem 2.1, we first need to give a characterization of the  $n$ -catenoid analogous to that of the 2-dimensional case (Theorem 4.2, see also [2]). Moreover in the higher dimensional case, it is worthwhile to state some interesting consequences of our results.

Let  $S_\infty$  be a closed set contained in an open slab of  $\partial_\infty \mathbb{H}^n \times \mathbb{R}$  with height equal to  $\pi/(n-1)$  such that the projection of  $S_\infty$  on  $\partial_\infty \mathbb{H}^n \times \{0\}$  omits an open subset.

We prove that there is no complete properly immersed minimal hypersurface  $M$  whose asymptotic boundary is  $S_\infty$  (Theorem 4.5-(2)).

Finally we prove an Asymptotic Theorem (Theorem 4.6), that implies the following non-existence result. There is no horizontal minimal graph over a bounded strictly convex domain, see [9, Equation (3)], given by a positive function  $g$  continuous up to the boundary, taking zero boundary value data (Remark 4.1).

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## 2. A CHARACTERIZATION OF THE CATENOID IN $\mathbb{H}^2 \times \mathbb{R}$

We are going to prove the characterization of the catenoid presented in the Introduction. Any surface in  $\mathbb{H}^2 \times \mathbb{R}$  with constant third coordinate is a complete totally geodesic surface called a *slice*. For any  $s \in \mathbb{R}$ , we denote by  $\Pi_s$  the slice  $\mathbb{H}^2 \times \{s\}$  and we set  $\Pi_s^+ = \{(p, t) \mid p \in \mathbb{H}^2, t > s\}$  and  $\Pi_s^- = \{(p, t) \mid p \in \mathbb{H}^2, t < s\}$ . For simplicity  $\Pi$  stands for  $\Pi_0$ .

**Lemma 2.1.** *Let  $\Gamma^+$  and  $\Gamma^-$  be two Jordan curves in  $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$  which are vertical graphs over  $\partial_\infty \mathbb{H}^2 \times \{0\}$  and such that  $\Gamma^+ \subset \partial_\infty \Pi^+$  and  $\Gamma^- \subset \partial_\infty \Pi^-$ . Assume that  $\Gamma^-$  is the symmetry of  $\Gamma^+$  with respect to  $\Pi$ .*

*Let  $M \subset \mathbb{H}^2 \times \mathbb{R}$  be an immersed, connected, complete minimal surface with two ends  $E^+$  and  $E^-$ . Assume that each end is a vertical graph and that  $\partial_\infty M = \Gamma^+ \cup \Gamma^-$ , that is  $\partial_\infty E^+ = \Gamma^+$  and  $\partial_\infty E^- = \Gamma^-$ .*

*Then  $M$  is symmetric with respect to  $\Pi$ . Furthermore, each part  $M \cap \Pi^\pm$  is a vertical graph and  $M$  is embedded.*

*Proof.* For any  $t > 0$  we set  $M_t^+ = M \cap \Pi_t^+$ . We denote by  $M_t^{+*}$  the symmetry of  $M_t^+$  with respect to the slice  $\Pi_t$ . Furthermore, we denote by  $t^+$  the highest  $t$ -coordinate of  $\Gamma^+$ . Since  $\partial_\infty M = \Gamma^+ \cup \Gamma^-$ , then  $M \cap \Pi_{t^+} = \emptyset$ , by the maximum principle.

We denote by  $E^+$  the end of  $M$  whose asymptotic boundary is  $\Gamma^+$ . As  $E^+$  is a vertical graph, there exists  $\varepsilon > 0$  such that  $M_{t^+-\varepsilon}^+$  is a vertical graph, then we can start Alexandrov reflection [1].

We keep doing the Alexandrov reflection with  $\Pi_t$ , doing  $t \searrow 0$ . By applying the interior or boundary maximum principle, we get that, for  $t > 0$ , the surface  $M_t^{+*}$  stays above  $M_t^-$ . Therefore we get that  $M_0^+$  is a vertical graph and that  $M_0^{+*}$  stays above  $M_0^-$ . Doing Alexandrov reflection with slices coming from below, one has that  $M_0^-$  is a vertical graph and that  $M_0^{-*}$  stays below  $M_0^+$ , henceforth we get  $M_0^{+*} = M_0^-$ . Thus  $M$  is symmetric with respect to  $\Pi$  and each component of  $M \setminus \Pi$  is a graph. Therefore we can show, as in the proof of [12, Theorem 2], that the whole surface  $M$  is embedded. This completes the proof.  $\square$

**Definition 2.1.** A *vertical plane* is a complete totally geodesic surface  $\gamma \times \mathbb{R}$  where  $\gamma$  is any complete geodesic of  $\mathbb{H}^2$ .

**Theorem 2.1.** *Let  $M \subset \mathbb{H}^2 \times \mathbb{R}$  be an immersed, connected, complete minimal surface with two ends. Assume that each end is a vertical graph whose asymptotic boundary is a copy of  $\partial_\infty \mathbb{H}^2$ . Then  $M$  is rotational, hence  $M$  is a catenoid.*

*Proof.* Up to a vertical translation, we can assume that the asymptotic boundary is symmetric with respect to the slice  $\Pi$ . We use the same notations as in the proof of Lemma 2.1. We know from Lemma 2.1 that  $M$  is symmetric with respect to  $\Pi$  and that  $M_0^+$  and  $M_0^-$  are vertical graphs. Therefore, at any point of  $M \cap \Pi$  the tangent plane of  $M$  is orthogonal to  $\Pi$ .

We have  $\partial_\infty M = \partial_\infty \mathbb{H}^2 \times \{t_0, -t_0\}$  for some  $t_0 > 0$ . Since  $M$  is embedded,  $M$  separates  $\mathbb{H}^2 \times [-t_0, t_0]$  into two connected components. We denote by  $U_1$  the component whose asymptotic boundary is  $\partial_\infty \mathbb{H}^2 \times [-t_0, t_0]$  and by  $U_2$  the component such that  $\partial_\infty U_2 = \partial_\infty \mathbb{H}^2 \times \{t_0, -t_0\}$ .

Let  $q_\infty \in \partial_\infty \mathbb{H}^2$  and let  $\gamma \subset \mathbb{H}^2$  be an oriented geodesic issuing from  $q_\infty$ , that is  $q_\infty \in \partial_\infty \gamma$ . Let  $q_0 \in \gamma$  be any fixed point.

For any  $s \in \mathbb{R}$ , we denote by  $P_s$  the vertical plane orthogonal to  $\gamma$  passing through the point of  $\gamma$  whose oriented distance from  $q_0$  is  $s$ . We suppose that  $s < 0$  for any point in the geodesic segment  $(q_0, q_\infty)$ .

For any  $s \in \mathbb{R}$ , we call  $M_s(l)$  the part of  $M \setminus P_s$  such that  $(q_\infty, t_0), (q_\infty, -t_0) \in \partial_\infty M_s(l)$  and let  $M_s^*(l)$  be the reflection of  $M_s(l)$  about  $P_s$ . We denote by  $M_s(r)$  the other part of  $M \setminus P_s$  and by  $M_s^*(r)$  its reflection about  $P_s$ .

It will be clear from the following two Claims, why we can start the Alexandrov reflection principle with respect to the vertical planes  $P_s$  and obtain the result.

By assumptions there exists  $s_1 < 0$  such that for any  $s < s_1$  the part  $M_s(l)$  has two connected components and both of them are vertical graphs. We deduce that  $\partial M_s(l)$  has two (symmetric) connected components, each one being a vertical graph.

We recall that  $\Pi^+ := \{t > 0\}$  and  $\Pi^- := \{t < 0\}$ .

*Claim 1.* *For any  $s < s_1$ , we have that  $M_s^*(l) \cap \Pi^+$  stays above  $M_s(r)$  and  $M_s^*(l) \cap \Pi^-$  stays below  $M_s(r)$ . Consequently  $M_s^*(l) \subset U_2$  for any  $s < s_1$ .*

Observe that  $M_s^*(l) \cap \Pi^+$  and  $M_s(r) \cap \Pi^+$  have same asymptotic boundary and that  $\partial(M_s^*(l) \cap \Pi^+) = \partial M_s(r) \cap \Pi^+$ . Therefore the asymptotic and finite boundaries of any

lifting up of  $M_s^*(l)$  is above the asymptotic and finite boundaries of  $M_s(r)$ . Hence any lifting up of  $M_s^*(l)$  is above  $M_s(r)$  by the maximum principle, which ensures that the whole  $M_s^*(l) \cap \Pi^+$  stays above  $M_s(r)$  for any  $s < s_1$ , as desired. The proof of the other assertion is analogous. Then, Claim 1 is proved.

We set

$$\sigma = \sup \{s \in \mathbb{R} \mid M_t^*(l) \cap \Pi^+ \text{ stays above } M_t(r) \cap \Pi^+ \text{ for any } t \in (-\infty, s)\}.$$

*Claim 2.* We have  $M_\sigma^*(l) = M_\sigma(r)$ . Thus, given a geodesic  $\gamma \subset \mathbb{H}^2$ , there exists a vertical plane  $P_\sigma$  orthogonal to  $\gamma$  such that  $M$  is symmetric with respect to  $P_\sigma$ .

Note that we also have

$$\sigma = \sup \{s \in \mathbb{R} \mid M_t^*(l) \subset U_2 \text{ for any } t \in (-\infty, s)\}.$$

In order to prove Claim 2, we first establish the following fact.

*Assertion.* For any  $s$  such that  $M_s^*(l) \cap \Pi \subset U_2$  then  $M_s^*(l) \subset U_2$ .

As  $M$  is symmetric with respect to  $\Pi$  the intersection  $M \cap \Pi$  is constituted of a finite number of pairwise disjoint Jordan curves  $C_1, \dots, C_k$ . Since  $M \cap \Pi^+$  is a vertical graph we deduce

$$(C_j \times \mathbb{R}) \cap M = C_j \quad \text{for any } j = 1, \dots, k.$$

Moreover, since  $M$  is connected and is symmetric about  $\Pi$ , we get that  $M \cap \Pi^+$  is connected.

Let  $D_j \subset \Pi$  be the Jordan domain bounded by  $C_j$ ,  $j = 1, \dots, k$ . Noticing that:

- $(M \cap \Pi^+) \setminus (\overline{D_j} \times \mathbb{R}) \neq \emptyset$ ,
- $M \cap \Pi^+$  is connected,
- $M \cap (C_j \times \mathbb{R}) = C_j$ ,
- $\partial_\infty M \cap \Pi^+ = \partial_\infty \mathbb{H}^2 \times \{t_0\}$ ,

we get that  $(M \cap \Pi^+) \cap (D_j \times \mathbb{R}) = \emptyset$ ,  $j = 1, \dots, k$ . Hence,  $D_i \cap D_j = \emptyset$  for any  $i \neq j$ . Therefore,  $M \cap \Pi^+$  is a vertical graph over  $\Pi \setminus \cup D_i$ .

This implies that, for any  $\varepsilon > 0$ , the vertical translation  $(M_s^*(l) \cap \Pi^+) + (0, 0, \varepsilon)$  stays above  $M$ . This proves the Assertion.

Let us continue the proof of Claim 2. The definition of  $\sigma$  implies that  $M_{\sigma+\varepsilon}^*(l) \cap U_1 \neq \emptyset$ , for  $\varepsilon$  small enough.

We deduce from the Assertion that  $M_{\sigma+\varepsilon}^*(l) \cap \Pi$  is not contained in  $U_2$  for any small enough  $\varepsilon > 0$ . Hence we infer that  $M_\sigma^*(l) \cap \Pi$  and  $M_\sigma(r) \cap \Pi$  are tangent at an interior or boundary point lying in some Jordan curve  $C_j$  contained in  $M \cap \Pi$ . Since  $M_\sigma^*(l) \subset \overline{U_2}$ ,  $M_\sigma(r) \subset \partial U_2$  and the tangent plane of  $M$  is vertical along  $M \cap \Pi$ , we are able to apply the maximum principle (possibly with boundary) to conclude that  $M_\sigma^*(l) = M_\sigma(r)$ , that is  $P_\sigma$  is a plane of symmetry of  $M$ . This proves Claim 2.

For any  $\alpha \in (0, \pi/2]$  consider a family of vertical planes making an angle  $\alpha$  with  $P_\sigma$ , generated by hyperbolic translations along the horizontal geodesic  $P_\sigma \cap \Pi$ . Now, doing the Alexandrov reflection principle with this family of planes, we find a vertical plane of symmetry of  $M$ , say  $P^\alpha$ . Hence  $M$  is invariant by the rotation of angle  $2\alpha$  around

the vertical geodesic  $P^\alpha \cap P_\sigma$ . Choosing an angle  $\alpha$  such that  $\pi/\alpha$  is not rational, we find that  $M$  is invariant by rotation around the axis  $P^\alpha \cap P_\sigma$ . This concludes the proof of Theorem 2.1, as desired.  $\square$

**Remark 2.1.** For any integer  $n$ , there exists a minimal surface in  $\mathbb{H}^2 \times \mathbb{R}$  which is a vertical graph, whose asymptotic boundary is a copy of  $\partial_\infty \mathbb{H}^2$  and whose finite boundary is constituted of  $n$  smooth Jordan curves in the slice  $\Pi$ , see [10, Theorem 5.1]. In the same article the second and third author asked about the existence of such graphs with two boundary curves in  $\Pi$  cutting orthogonally the slice  $\Pi$ . Theorem 2.1 implies that the answer to this question is negative.

### 3. MAXIMUM PRINCIPLE IN A VERTICAL HALFSPACE OF $\mathbb{H}^2 \times \mathbb{R}$ .

In this section we prove some maximum principle in a vertical halfspace. More precisely, we prove that, under some geometric assumptions, the behavior of the asymptotic boundary of  $M$  at finite height, determines the behaviour of  $M$ .

**Definition 3.1.** We call a *vertical halfspace* any of the two components of  $(\mathbb{H}^2 \times \mathbb{R}) \setminus P$ , where  $P$  is a vertical plane.

**Theorem 3.1.** *Let  $M$  be a complete minimal surface, possibly with finite boundary, properly immersed in  $\mathbb{H}^2 \times \mathbb{R}$ . Let  $P$  be a vertical plane and let  $P_+$  be one of the two halfspaces determined by  $P$ . If  $\partial M \subset \overline{P_+}$  and  $\partial_\infty M \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) \subset \partial_\infty P_+$ , then  $M \setminus \partial M \subset P_+$ , unless  $M \subset P$ .*

For the proof of Theorem 3.1 we need to consider the one parameter family of surfaces  $M_d, d > 0$ , that have origin in [7, Section 4] and whose geometry is described in [10, Proposition 2.1]. This family of surfaces was already used, for example, in [8, Example 2.1].

First we describe the asymptotic boundary of  $M_d$ , for  $d > 1$ .

Consider a horizontal geodesic  $\gamma$  in  $\mathbb{H}^2$ , with asymptotic boundary  $\{p, q\}$  and let  $\alpha$  be the closure of a connected component of  $(\partial_\infty \mathbb{H}^2 \times \{0\}) \setminus (\{p, q\} \times \{0\})$ . Let

$$H(d) = \int_{\cosh^{-1}(d)}^{+\infty} \frac{d}{\sqrt{\cosh^2 u - d^2}} du, \quad d > 1$$

be the positive number defined in (1) of [10]. Notice that  $\lim_{d \rightarrow 1} H(d) = +\infty$  and

$$\lim_{d \rightarrow +\infty} H(d) = \pi/2.$$

Let  $\alpha_d$  in  $\partial_\infty \mathbb{H}^2 \times \{H(d)\}$  and  $\alpha_{-d}$  in  $\partial_\infty \mathbb{H}^2 \times \{-H(d)\}$  be the two curves that project vertically onto  $\alpha$ . Let  $L_d, R_d$  be two vertical segments in  $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$  of height  $2H(d)$  such that the curve  $L_d \cup \alpha_d \cup R_d \cup \alpha_{-d}$  is a closed simple curve. Then  $\partial_\infty M_d = L_d \cup \alpha_d \cup R_d \cup \alpha_{-d}$ .

Now we describe the position of  $M_d$  in the ambient space, for  $d > 1$ .

Denote by  $Q_\gamma$  the halfspace determined by  $\gamma \times \mathbb{R}$ , whose asymptotic boundary contains the curve  $\alpha$ . Let  $\gamma_d$  be the curve in  $Q_\gamma \cap (\mathbb{H}^2 \times \{0\})$  at constant distance  $\cosh^{-1}(d)$  from

$\gamma$ .  $M_d$  contains the curve  $\gamma_d$ . Denote by  $Z_d$  the closure of the non mean convex side of the cylinder over the curve  $\gamma_d$ . Then,  $M_d$  is contained in  $Z_d$  which is contained in  $Q_\gamma$ . Notice that any vertical translation of the surface  $M_d$  is contained in  $Z_d$ . Moreover, any vertical translation of  $M_d$  is arbitrarily close to  $Q_\gamma$  if  $d$  is sufficiently close to 1. We observe that in the description above,  $\gamma$  can be any geodesic of  $\mathbb{H}^2$ .

*Proof of Theorem 3.1.* The proof is an application of the maximum principle between the surface  $M$  and the one parameter family of surfaces  $M_d$ .

We choose the geodesic  $\gamma$ , in order to construct the  $M_d$ 's, as follows. Let  $\gamma \subset \mathbb{H}^2$  be any geodesic such that

- P1: The halfspace  $Q_\gamma$  is strictly contained in  $(\mathbb{H}^2 \times \mathbb{R}) \setminus P_+$ .
- P2:  $\partial_\infty \gamma \cap \partial_\infty P = \emptyset$ .

Now, notice that

- (1) The intersection of  $\partial_\infty M$  with  $\partial_\infty(\mathbb{H}^2 \times \mathbb{R}) \setminus \partial_\infty P_+$  contains no points at finite height.
- (2) The asymptotic boundary of any vertical translation  $M_d$  is contained in the asymptotic boundary of  $Q_\gamma \subset \mathbb{H}^2 \times \mathbb{R} \setminus P_+$ .

We claim that  $M_d$  and  $M$  are disjoint for any  $d > 1$ . Indeed, letting  $p \rightarrow q$  (recall that  $p, q$  are the endpoints of the geodesic  $\gamma$ ), one has that  $M_d$  collapses to a vertical segment in  $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ . Suppose that, when  $p \rightarrow q$ , the surfaces  $M_d$  always have a nonempty intersection with  $M$ . Then, there would exist a point of the asymptotic boundary of  $M$  at finite height in  $\partial_\infty(\mathbb{H}^2 \times \mathbb{R}) \setminus \partial_\infty P_+$ , giving a contradiction with (1). Then, if  $M \cap M_d \neq \emptyset$ , we would obtain a last intersection point between  $M$  and some modified  $M_d$  letting  $p \rightarrow q$ , contradicting the maximum principle.

Therefore, by the maximum principle, any vertical translation of  $M_d$  and  $M$  are disjoint.

Let  $d \rightarrow 1$ . By the maximum principle, there is no first point of contact between  $M_d$  and  $M$ . As we can apply the maximum principle between any vertical translation of  $M_d$  and  $M$ , one has that  $M$  is contained in the closed halfspace  $\mathbb{H}^2 \times \mathbb{R} \setminus Q_\gamma$  for any geodesic  $\gamma$  satisfying the properties P1 and P2. Therefore,  $M$  is included in the closure of  $P_+$ .

Now we have one of the following possibilities:

- Some points of the interior of  $M$  touches  $\partial P_+ = P$ , then, by the maximum principle,  $M \subset P$ .
- $M \setminus \partial M$  is contained in the halfspace  $P_+$ .

The result is thus proved. □

Let us give a definition, before stating some consequences of Theorem 3.1.

**Definition 3.2.** We say that  $L \subset \partial_\infty(\mathbb{H}^2 \times \mathbb{R})$  is a *line* if  $L = \{p\} \times \mathbb{R}$  for some  $p \in \partial_\infty \mathbb{H}^2$ .

Given vertical lines  $L_1, \dots, L_k$  in  $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ , we define the set  $P(L_1, \dots, L_k)$  as follows. Let  $P_i$  the vertical plane such that  $\partial_\infty P_i \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) = L_i \cup L_{i+1}$  (with the convention



that  $L_{k+1} = L_1$ ). Denote by  $\tilde{P}_i$  the halfspace determined by the vertical plane  $P_i$  such that  $\bigcup_j L_j \subset \partial_\infty \tilde{P}_i$ . Then, we set  $P(L_1, \dots, L_k) := \bigcap_i \tilde{P}_i$ .

**Corollary 3.1.** *Let  $M$  be a complete minimal surface, possibly with finite boundary, properly immersed in  $\mathbb{H}^2 \times \mathbb{R}$  and let  $\Gamma = \partial_\infty M \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R})$ . Let  $L_1, \dots, L_k$  be vertical lines in  $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ . If  $\Gamma \subset L_1 \cup \dots \cup L_k$  and  $\partial M \subset \overline{P(L_1, \dots, L_k)}$ , then  $M \setminus \partial M$  is contained in  $P(L_1, \dots, L_k)$ , unless  $M$  is contained in one of the  $P_i$ .*

*Proof.* By Theorem 3.1,  $M$  is contained in every halfspace  $\tilde{P}_i$  determined by the vertical plane  $P_i$  such that  $\bigcup_j L_j \subset \partial_\infty \tilde{P}_i$ , unless it is contained in one of the  $P_i$ . Hence it is contained in  $P(L_1, \dots, L_k)$ , by definition, unless it is contained in one of the  $P_i$ .  $\square$

**Corollary 3.2.** *Let  $M$  be a complete minimal surface properly immersed in  $\mathbb{H}^2 \times \mathbb{R}$ . Let  $P$  be a vertical plane. If  $\partial_\infty M \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) \subset \partial_\infty P$ , then  $M = P$ .*

*Proof.* By Theorem 3.1,  $M$  is contained in the closure of both halfspaces determined by  $P$ , hence it is contained in  $P$ . Then  $M = P$  because it is complete.  $\square$

**Corollary 3.3.** *Let  $M$  be a complete minimal surface properly immersed in  $\mathbb{H}^2 \times \mathbb{R}$ . Suppose that the asymptotic boundary of  $M$  is contained in the asymptotic boundary of a totally geodesic plane  $S$  of  $\mathbb{H}^2 \times \mathbb{R}$ . Then  $M = S$ .*

*Proof.* The proof is a simple consequence of the maximum principle and of the previous results. We do it for completeness. First assume that the asymptotic boundary of  $M$  is contained in the asymptotic boundary of a slice, say  $\{t = 0\}$ . Then, for  $n$  sufficiently large, the slice  $\{t = n\}$  is disjoint from  $M$ . Now, we translate the slice  $\{t = n\}$  down. The first contact point, cannot be interior because of the maximum principle, hence  $M$  must stay below the slice  $\{t = 0\}$ . One can do the same reasoning with slices coming from the bottom, and  $M$  must stay above the slice  $\{t = 0\}$ . Hence  $M$  coincides with the slice  $\{t = 0\}$ .

If the the asymptotic boundary of  $M$  is contained is the asymptotic boundary of a vertical plane, the result follows by Corollary 3.2.  $\square$

**Corollary 3.4.** *Let  $M$  be a complete minimal surface properly immersed in  $\mathbb{H}^2 \times \mathbb{R}$ . Assume that the projection of the asymptotic boundary of  $M$  into  $\partial_\infty \mathbb{H}^2$  omits a closed interval  $\alpha$  joining two points  $p$  and  $q$ . Let  $\gamma$  be the horizontal geodesic in  $\mathbb{H}^2$  whose the asymptotic boundary is  $\{p, q\}$  and let  $Q_\gamma$  be the halfspace determined by  $\gamma \times \mathbb{R}$  whose asymptotic boundary contains  $\alpha$ . Then  $M$  is contained in  $\mathbb{H}^2 \times \mathbb{R} \setminus \overline{Q_\gamma}$ .*

*Proof.* By hypothesis  $\partial_\infty M \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R})$  is contained in the asymptotic boundary of  $(\mathbb{H}^2 \times \mathbb{R}) \setminus Q_\gamma$ . The result follows by Theorem 3.1 with  $P_+ = (\mathbb{H}^2 \times \mathbb{R}) \setminus \overline{Q_\gamma}$ .  $\square$

**Remark 3.1.** There exist examples of minimal surfaces with asymptotic boundary equal to two vertical halflines, lines and a curve at finite height, see [7, Equation (32)] and [10, Proposition 2.1 (2)].

4. SOME GENERALIZATIONS TO  $\mathbb{H}^n \times \mathbb{R}$ .

Let us recall the construction and the properties of the  $n$ -catenoids in  $\mathbb{H}^n \times \mathbb{R}$ ,  $n \geq 3$ , established, by P. Bérard and the second author in [2, Proposition 3.2]. Given any  $a > 0$  we denote by  $(I_a, f(a, \cdot))$ , where  $I_a \subset \mathbb{R}$  is an interval, the maximal solution of the following Cauchy problem:

$$\begin{cases} f_{tt} = (n-1)(1+f_t^2) \coth(f), \\ f(0) = a > 0, \\ f_t(0) = 0. \end{cases}$$

**Theorem 4.1** ([2]). *For  $a > 0$ , the maximal solution  $(I_a, f(a, \cdot))$  gives rise to the generating curve  $C_a$ , parametrized by  $t \mapsto (\tanh(f(a, t)), t)$ , of a complete minimal rotational hypersurface  $\mathcal{C}_a$  ( $n$ -catenoid) in  $\mathbb{H}^n \times \mathbb{R}$ , with the following properties.*

- (1) *The interval  $I_a$  is of the form  $I_a = ]-T(a), T(a)[$  where*

$$T(a) = \sinh^{n-1}(a) \int_a^\infty (\sinh^{2n-2}(u) - \sinh^{2n-2}(a))^{-1/2} du.$$

- (2)  *$f(a, \cdot)$  is an even function of the second variable.*  
(3) *For all  $t \in I_a$ ,  $f(a, t) \geq a$ .*  
(4) *The derivative  $f_t(a, \cdot)$  is positive on  $]0, T(a)[$ , negative on  $] -T(a), 0[$ .*  
(5) *The function  $f(a, \cdot)$  is a bijection from  $]0, T(a)[$  onto  $[a, \infty[$ , with inverse function  $\lambda(a, \cdot)$  given by*

$$\lambda(a, \rho) = \sinh^{n-1}(a) \int_a^\rho (\sinh^{2n-2}(u) - \sinh^{2n-2}(a))^{-1/2} du.$$

- (6) *The catenoid  $\mathcal{C}_a$  has finite vertical height  $h_R(a) := 2T(a)$ ,*  
(7) *The function  $a \mapsto h_R(a)$  increases from 0 to  $\frac{\pi}{(n-1)}$  when  $a$  increases from 0 to infinity. Furthermore, given  $a \neq b$ , the generating catenaries  $C_a$  and  $C_b$  intersect at exactly two symmetric points.*

For later use, we need the following result. Although we believe that the result is classical, we give a proof for the sake of completeness. The reader is referred to [4, chapter VII] or [13, chapter 9, addendum 3] for the proof of the analogous statement in Euclidean space.

**Proposition 4.1.** *Let  $S \subset \mathbb{H}^n$  be a finite union of connected, closed and embedded  $(n-1)$ -submanifolds  $C_j$ ,  $j = 1, \dots, k$ , such that the bounded domains whose boundary are the  $C_j$  are pairwise disjoint. Assume that for any geodesic  $\gamma \subset \mathbb{H}^n$ , there exists a  $(n-1)$ -geodesic plane  $\pi_\gamma \subset \mathbb{H}^n$  of symmetry of  $S$  which is orthogonal to  $\gamma$ . Then  $S$  is a  $(n-1)$ -geodesic sphere of  $\mathbb{H}^n$ .*

*Proof.* We will proceed the proof by induction on  $n \geq 2$ .

First assume that  $n = 2$ . By hypothesis, there exist two geodesics  $c_1, c_2 \subset \mathbb{H}^2$  of symmetry of the closed curve  $S$  intersecting at some point  $p \in \mathbb{H}^2$  and making an angle



$\alpha \neq 0$  such that  $\pi/\alpha$  is not rational. For any  $q \in S$ , denote by  $C_q$  the circle centered at  $p$  passing through  $q$ . Then  $C_q$  is contained in  $S$ . Let  $\tilde{q} \neq q$  be points of  $S$ . If  $C_q \neq C_{\tilde{q}}$  then the geodesic disks bounded by  $C_q$  and  $C_{\tilde{q}}$  are not disjoint, since they have the same center, which contradicts the hypothesis. Consequently, we get  $C_q = C_{\tilde{q}}$  and we conclude that  $S$  is a circle.

Let  $n \in \mathbb{N}$ ,  $n \geq 3$ . Assume that the statement holds for  $k = 2, \dots, n-1$ .

Let  $\pi_0 \subset \mathbb{H}^n$  be a  $(n-1)$ -geodesic plane of symmetry of  $S$ .

*Claim 1.*  $S \cap \pi_0$  is a  $(n-2)$ -geodesic sphere of  $\pi_0$ .

Indeed, let  $\gamma \subset \pi_0$  be a geodesic. By hypothesis there exists a  $(n-1)$ -geodesic plane  $\pi_\gamma \subset \mathbb{H}^n$  orthogonal to  $\gamma$  which is a plane of symmetry of  $S$ . Since  $\pi_\gamma$  is orthogonal to  $\pi_0$ , then  $S \cap \pi_0$  is symmetric about  $\pi_\gamma \cap \pi_0$  (which is a  $(n-2)$ -geodesic plane of  $\pi_0$ ), see [11, Lemme 3.3.15]. As  $\pi_0$  is a  $(n-1)$  hyperbolic space,  $S \cap \pi_0$  satisfies the assumptions of the statement in  $\mathbb{H}^{n-1}$ .

By the induction hypothesis we deduce that  $S \cap \pi_0$  is a  $(n-2)$ -geodesic sphere of  $\pi_0$ . This proves Claim 1.

Let  $p_0 \in \pi_0$  and  $\rho_0 > 0$  be respectively the center and the radius of the  $(n-2)$ -geodesic sphere  $S \cap \pi_0$ .

*Claim 2.* Let  $\pi_1 \subset \mathbb{H}^n$  be a  $(n-1)$ -geodesic plane of symmetry of  $S$  orthogonal to  $\pi_0$ . Then  $S \cap \pi_1$  is a  $(n-2)$ -geodesic sphere of  $\pi_1$  with center  $p_0$  and radius  $\rho_0$ .

Claim 1 yields that  $S \cap \pi_1$  is a  $(n-2)$ -geodesic sphere of  $\pi_1$ . Since  $\pi_0$  and  $\pi_1$  are orthogonal, then the geodesic sphere  $S \cap \pi_0$  is symmetric about  $\pi_1$ . Therefore  $p_0 \in \pi_1$ . If  $n > 3$ , then  $(S \cap \pi_0) \cap \pi_1$  is  $(n-3)$ -geodesic sphere with center  $p_0$  and radius  $\rho_0$  of  $\pi_0 \cap \pi_1$  (which is a  $(n-2)$  hyperbolic space). If  $n = 3$ , then  $(S \cap \pi_0) \cap \pi_1$  is constituted of two points whose the distance is  $2\rho_0$ . In both cases we infer that  $\text{diam}_{\mathbb{H}^n}(S \cap \pi_1) \geq 2\rho_0$  and then the radius of the geodesic sphere  $S \cap \pi_1$  is  $\rho_1 \geq \rho_0$ . Analogously we can show that  $\rho_0 \geq \rho_1$ . We deduce that  $\rho_1 = \rho_0$ , that is  $S \cap \pi_0$  and  $S \cap \pi_1$  have both center at  $p_0$  and radius  $\rho_0$ . This proves Claim 2.

*Claim 3.* Let  $\pi_2 \subset \mathbb{H}^n$  be any  $(n-1)$ -geodesic plane of symmetry of  $S$ . Then  $S \cap \pi_2$  is a  $(n-2)$ -geodesic sphere of  $\pi_2$  with center  $p_0$  and radius  $\rho_0$ .

Since  $S$  is symmetric with respect to  $\pi_0$  and  $\pi_2$ ,  $\pi_0$  and  $\pi_2$  are distinct and  $S$  is compact, then the  $(n-1)$ -geodesic planes  $\pi_0$  and  $\pi_2$  cannot be disjoint.

Then, we find a third  $(n-1)$ -geodesic plane  $\pi_3$  of symmetry of  $S$ , orthogonal to both  $\pi_0$  and  $\pi_2$ . Claim 2 implies that  $S \cap \pi_2$  is a  $(n-2)$ -geodesic sphere of  $\pi_2$  with center  $p_0$  and radius  $\rho_0$ . This proves Claim 3.

Now we finish the proof of the Proposition as follows. Let  $p \in S$  and let  $\pi \subset \mathbb{H}^n$  be any  $(n-1)$ -geodesic plane passing through  $p$  and  $p_0$ . Let  $\gamma \subset \mathbb{H}^n$  be the geodesic through  $p_0$  orthogonal to  $\pi$ . By Claim 2, there exists a  $(n-1)$ -geodesic plane  $\pi_\gamma$  of symmetry of  $S$  and orthogonal to  $\gamma$ . Claim 3 ensures that  $p_0 \in \pi_\gamma$ , then  $\pi_\gamma = \pi$ . Claim 3 yields also that  $S \cap \pi$  is  $(n-2)$ -geodesic sphere of  $\pi$  with center  $p_0$  and radius  $\rho_0$ ,

thus  $d_{\mathbb{H}^n}(p, p_0) = \rho_0$ . This shows that  $S$  is the  $(n - 1)$ -geodesic sphere of  $\mathbb{H}^n$  of radius  $\rho_0$  and center  $p_0$ .  $\square$

Now we establish a characterization of the  $n$ -catenoid, that is a generalization to higher dimension of Theorem 2.1.

**Theorem 4.2.** *Let  $M \subset \mathbb{H}^n \times \mathbb{R}$  be an immersed, connected, complete minimal hypersurface with two ends. Assume that each end is a vertical graph whose asymptotic boundary is a copy of  $\partial_\infty \mathbb{H}^n$ . Then  $M$  is a  $n$ -catenoid.*

*Proof.* Up to a vertical translation, we can assume that the asymptotic boundary of  $M$  is symmetric with respect to  $\Pi := \mathbb{H}^n \times \{0\}$ . We set  $\Gamma^+ := \partial_\infty M \cap \{t > 0\}$  and recall that  $\Gamma^+$  is a copy of  $\partial_\infty \mathbb{H}^n$ . As usual we set  $M^+ := M \cap \{t > 0\}$ .

Next Claim can be shown in the same fashion as in  $\mathbb{H}^2 \times \mathbb{R}$  (see Lemma 2.1 and the proof of Claim 2 of Theorem 2.1). For this reason we just state it.

*Claim.*  $M$  is symmetric about  $\Pi$ , and each connected component of  $M \setminus \Pi$  is a vertical graph. Moreover, for any geodesic  $\gamma \subset \Pi$  there exists a vertical hyperplane  $P_\gamma \subset \mathbb{H}^n \times \mathbb{R}$  orthogonal to  $\gamma$  which is a  $n$ -plane of symmetry of  $M$ . Therefore,  $\pi_\gamma := P_\gamma \cap \Pi$  is a  $(n - 1)$ -plane of symmetry of  $\Sigma := M \cap \Pi$ .

Using the result of the Claim we get that  $\Sigma$  satisfies the assumptions of Proposition 4.1. Then  $\Sigma$  is a  $(n - 1)$ -geodesic sphere of  $\Pi$ , since  $\Pi = \mathbb{H}^n \times \{0\}$ .

Let  $\mathcal{C} \subset \mathbb{H}^n \times \mathbb{R}$  be the catenoid through  $\Sigma$  and orthogonal to  $\Pi$ . We set  $\mathcal{C}^+ := \mathcal{C} \cap \{t > 0\}$ .

Both  $\mathcal{C}^+$  and  $M^+$  are vertical along their common finite boundary  $\Sigma$ , hence they are tangent along  $\Sigma$ .

Let  $t_{\mathcal{C}}$  (resp.  $t_M$ ) the height of the asymptotic boundary of  $\mathcal{C}^+$  (resp.  $M^+$ ).

Suppose for example that  $t_{\mathcal{C}} \leq t_M$ . Then, lifting upward and downward  $M^+$ , we obtain that  $M^+$  is above  $\mathcal{C}^+$ . Therefore we deduce that  $M^+ = \mathcal{C}^+$  by applying the boundary maximum principle. The case  $t_M \leq t_{\mathcal{C}}$  is analogous.

We conclude that  $M = \mathcal{C}$  and the proof is completed.  $\square$

In order to establish the generalization in higher dimension of Theorem 3.1, we need to state some existence results, established for  $n \geq 3$ , in [2, Theorem 3.8], inspired by [10, Proposition 2.1]. In fact, we only need the  $d > 1$  case, but we state the whole result for the sake of completeness.

**Theorem 4.3** ([2]). *There exists a one parameter family  $\{\mathcal{M}_d, d > 0\}$  of complete embedded minimal hypersurfaces in  $\mathbb{H}^n \times \mathbb{R}$  invariant under hyperbolic translations.*

- (1) *If  $d > 1$ , then  $\mathcal{M}_d$  consists of the union of two symmetric vertical graphs over the exterior of an equidistant hypersurface in the slice  $\mathbb{H}^n \times \{0\}$ .*

*The asymptotic boundary of  $\mathcal{M}_d$  is topologically an  $(n - 1)$ -sphere which is homologically trivial in  $\partial_\infty \mathbb{H}^n \times \mathbb{R}$ . More precisely, we set for  $d > 1$ :*

$$S(d) = \cosh(a) \int_1^\infty (t^{2n-2} - 1)^{-1/2} (\cosh^2(a)t^2 - 1)^{-1/2} dt, \quad \text{where } d =: \cosh^{n-1}(a).$$

Then, the asymptotic boundary of  $\mathcal{M}_d$  consists of the union of two copies of an hemisphere  $S_+^{n-1} \times \{0\}$  of  $\partial_\infty \mathbb{H}^n \times \{0\}$  in parallel slices  $t = \pm S(d)$ , glued with the finite cylinder  $\partial S_+^{n-1} \times [-S(d), S(d)]$

The vertical height of  $\mathcal{M}_d$  is  $2S(d)$ . The height of the family  $\mathcal{M}_d$  is a decreasing function of  $d$  and varies from infinity (when  $d \rightarrow 1$ ) to  $\pi/(n-1)$  (when  $d \rightarrow \infty$ ).

- (2) If  $d = 1$ , then  $\mathcal{M}_1$  consists of a complete (non-entire) vertical graph over a halfspace in  $\mathbb{H}^n \times \{0\}$ , bounded by a totally geodesic hyperplane  $P$ . It takes infinite boundary value data on  $P$  and constant asymptotic boundary value data. The asymptotic boundary of  $\mathcal{M}_1$  is the union of a spherical cap  $S$  of  $\partial_\infty \mathbb{H}^n \times \{c\}$  with a half vertical cylinder over  $\partial S$ .
- (3) If  $d < 1$ , then  $\mathcal{M}_d$  is an entire vertical graph with finite vertical height. Its asymptotic boundary consists of a homologically non-trivial  $(n-1)$ -sphere in  $\partial_\infty \mathbb{H}^n \times \mathbb{R}$ .

The hypersurfaces  $\mathcal{M}_d$  are the analogous in higher dimension of the surfaces  $M_d$  in  $\mathbb{H}^2 \times \mathbb{R}$ . Also, as in  $\mathbb{H}^2 \times \mathbb{R}$ , by (vertical) *hyperplane* we mean a complete totally geodesic hypersurface  $\Pi \times \mathbb{R}$ , where  $\Pi$  is any totally geodesic hyperplane of  $\mathbb{H}^n \times \{0\}$ . Moreover, we call a *vertical halfspace* any component of  $(\mathbb{H}^n \times \mathbb{R}) \setminus P$  where  $P$  is a vertical hyperplane. Thus, working with the hypersurfaces  $\mathcal{M}_d$  exactly in the same way as in Theorem 3.1, we obtain the following result.

**Theorem 4.4.** *Let  $M$  be a complete minimal hypersurface properly immersed in  $\mathbb{H}^n \times \mathbb{R}$ , possibly with finite boundary. Let  $P$  be a vertical geodesic hyperplane and  $P_+$  one of the two halfspaces determined by  $P$ . If  $\partial M \subset \overline{P_+}$  and  $\partial_\infty M \cap (\partial_\infty \mathbb{H}^n \times \mathbb{R}) \subset \partial_\infty P_+$ , then  $M \setminus \partial M \subset P_+$ , unless  $M \subset P$ .*

Obviously, the analogous in higher dimension of Corollaries 3.1, 3.2, 3.3 hold as well. Part (1) of next Theorem is a generalization in higher dimension of Corollary 3.4, while part (2) was proved, for  $n = 2$  by the second and the third authors [10, Corollary 2.2]

**Theorem 4.5.** *Let  $S_\infty \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$  be a closed set whose the vertical projection on  $\partial_\infty \mathbb{H}^n \times \{0\}$  omits an open subset  $U$ .*

- (1) *Let  $M$  be a complete minimal hypersurface properly immersed in  $\mathbb{H}^n \times \mathbb{R}$  such that  $\partial_\infty M = S_\infty$ . Let  $Q \subset \mathbb{H}^n \times \mathbb{R}$  be a vertical halfspace whose asymptotic boundary is contained in  $U \times \mathbb{R}$ . Then  $M$  is contained in  $\mathbb{H}^n \times \mathbb{R} \setminus \overline{Q}$ .*
- (2) *Assume that  $S_\infty$  is contained in an open slab whose height is equal to  $\frac{\pi}{n-1}$ . Then, there is no complete connected properly immersed minimal hypersurface  $M$  in  $\mathbb{H}^n \times \mathbb{R}$  with asymptotic boundary  $S_\infty$ .*

*Proof.* The first statement is a consequence of Theorem 4.4 and the proof is analogous to that of Corollary 3.4.

Let us prove the second statement. Assume, by contradiction, that there is such a minimal hypersurface  $M$  with asymptotic boundary  $S_\infty$ . Then, up to a vertical

translation, we can assume that  $M$  is contained in the slab  $\mathcal{S} := \{\varepsilon < t < \frac{\pi}{n-1} - \varepsilon\}$  for some  $\varepsilon > 0$ , and thus  $S_\infty \subset \partial_\infty \mathcal{S}$ . By assumption, there exists a  $(n-1)$ -geodesic plane  $\pi \subset \mathbb{H}^n \times \{0\}$  such that a component  $\pi^+$  of  $\mathbb{H}^n \times \{0\} \setminus \pi$  satisfies:

- (1)  $\partial_\infty \pi^+ \subset U$ .
- (2)  $M \cap (\pi^+ \times \mathbb{R}) = \emptyset$ .

Let  $C \subset \mathbb{H}^n \times (0, \frac{\pi}{n-1})$  be any  $n$ -catenoid such that a component of its asymptotic boundary stays strictly above  $\partial_\infty \mathcal{S}$  and the other component stays strictly below  $\partial_\infty \mathcal{S}$ . We take a connected and compact piece  $K$  of  $C$  such that its boundary lies in the boundary of the slab  $\mathcal{S}$ .

Let  $q \in M$  be a point and let  $q_0 \in \mathbb{H}^n \times \{0\}$  be the vertical projection of  $q$ . Let  $p_\infty \in \partial_\infty \pi^+$  be an asymptotic point. Denote by  $\tilde{\gamma} \subset \partial_\infty \mathbb{H}^n \times \{0\}$  the complete geodesic passing through  $q_0$  such that  $p_\infty \in \partial_\infty \tilde{\gamma}$ . We can translate  $K$  along  $\tilde{\gamma}$  such that the translated  $K$  is contained in the halfspace  $\pi^+ \times \mathbb{R}$ .

Now we come back translating  $K$  towards  $M$  along  $\tilde{\gamma}$ . Observe that the boundary of the translated copies of  $K$  does not touch  $M$ . Therefore, doing the translations of  $K$  along  $\tilde{\gamma}$  we find a first interior point of contact between  $M$  and a translated copy of  $K$ . Hence,  $M = C$  by the maximum principle, which leads to a contradiction. This completes the proof.  $\square$

Now we state a generalization of the Asymptotic Theorem proved in [10, Theorem 2.1]. Our result establishes some obstruction for the asymptotic boundary of a complete properly immersed minimal hypersurface in  $\mathbb{H}^n \times \mathbb{R}$ .

**Theorem 4.6** (Asymptotic Theorem). *Let  $\Gamma \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$  be a connected  $(n-1)$ -submanifold with boundary. Let  $\text{Pr} : \partial_\infty \mathbb{H}^n \times \mathbb{R} \rightarrow \partial_\infty \mathbb{H}^n$  be the projection on the first factor. Assume that:*

- (1) *There is some point  $q_\infty \in \partial \text{Pr}(\Gamma)$  such that  $q_\infty \notin \text{Pr}(\partial \Gamma)$ .*
- (2)  *$\Gamma \subset \partial_\infty \mathbb{H}^n \times (t_0, t_0 + \frac{\pi}{n-1})$  for some real number  $t_0$ .*

*Then, there is no properly and completely immersed minimal hypersurface (maybe with finite boundary)  $M \subset \mathbb{H}^n \times \mathbb{R}$  such that  $\partial_\infty M = \Gamma$  and  $M \cup \Gamma$  is a continuous  $n$ -manifold with boundary.*

*Proof.* Assume, by contradiction, that there is such a minimal hypersurface  $M$ . Since  $q_\infty \in \partial \text{Pr}(\Gamma)$  and  $q_\infty \notin \text{Pr}(\partial \Gamma)$ , there exists a  $(n-1)$ -geodesic plane  $\omega \subset \mathbb{H}^n \times \{0\}$  such that a component  $\omega^+$  of  $\mathbb{H}^n \times \{0\} \setminus \omega$  satisfies:

- (1)  $q_\infty \in \partial_\infty \omega^+$ ,  $q_\infty \notin \partial_\infty \omega$  and  $\partial_\infty \omega^+ \cap \text{Pr}(\partial \Gamma) = \emptyset$ .
- (2) If  $M_0$  denotes the component of  $M \cap (\omega^+ \times \mathbb{R})$  containing  $q_\infty$  in its asymptotic boundary, then
  - (a)  $M_0 \subset \mathbb{H}^n \times (t_0, t_0 + \frac{\pi}{n-1})$  for some real number  $t_0$ .
  - (b)  $\partial M_0 \subset \omega \times (t_0 + 2\varepsilon, t_0 - 2\varepsilon + \frac{\pi}{n-1})$  for some  $\varepsilon > 0$ .

Again, since  $q_\infty \in \partial \text{Pr}(\Gamma)$  and  $q_\infty \notin \text{Pr}(\partial \Gamma)$ , there exists a  $(n-1)$ -geodesic plane  $\pi \subset \mathbb{H}^n \times \{0\}$  such that a component  $\pi^+$  of  $\mathbb{H}^n \times \{0\} \setminus \pi$  satisfies:

- (1)  $\pi^+ \subset \omega^+$ .
- (2)  $\partial_\infty \pi^+ \cap \text{Pr}(\Gamma) = \emptyset$ .
- (3)  $M_0 \cap (\pi^+ \times \mathbb{R}) = \emptyset$ .

Therefore we can find a compact part  $K$  of a  $n$ -catenoid satisfying:

- (1)  $K$  is connected.
- (2)  $K \subset \pi^+ \times (t_0 + \varepsilon, t_0 - \varepsilon + \frac{\pi}{n-1})$ .
- (3)  $\partial K \subset \mathbb{H}^n \times \{t_0 + \varepsilon, t_0 - \varepsilon + \frac{\pi}{n-1}\}$ .

We deduce consequently that  $M_0 \cap K = \emptyset$ . Then, considering the horizontal translated copies of  $K$  and arguing as in the proof of Theorem 4.5, we get a contradiction with the maximum principle, which concludes the proof.  $\square$

The following result is an immediate consequence of Theorem 4.6.

**Corollary 4.1.** *Let  $S_\infty \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$  be a  $(n-1)$ -closed continuous submanifold. Considering the halfspace model for  $\mathbb{H}^n$ , we can assume that  $S_\infty \subset \mathbb{R}^{n-1} \times \mathbb{R}$ . If  $S_\infty$  is strictly convex in Euclidean sense, then there is no complete connected properly immersed minimal hypersurface  $M$  in  $\mathbb{H}^n \times \mathbb{R}$ , possibly with finite boundary, with asymptotic boundary  $S_\infty$  and such that  $M \cup S_\infty$  is a continuous  $n$ -manifold with boundary.*

**Remark 4.1.** It follows from Corollary 4.1 that there is no horizontal minimal graph in  $\mathbb{H}^n \times \mathbb{R}$ , [9, Equation (3)], given by a positive function  $g \in C^2(\Omega) \cap C^0(\overline{\Omega})$ , where  $\Omega \subset \mathbb{R}^{n-1} \times \mathbb{R} \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$  is a bounded strictly convex domain in Euclidean sense, assuming zero value on  $\partial\Omega$ .

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BARBARA NELLI  
DIPARTIMENTO INGEGNERIA E SCIENZE DELL'INFORMAZIONE E MATEMATICA  
UNIVERSITÀ DI L'AQUILA  
VIA VETOIO - LOC. COPPITO  
67010 (L'AQUILA)  
ITALY  
*E-mail address:* nelli@univaq.it

RICARDO SA EARP  
DEPARTAMENTO DE MATEMÁTICA  
PONTIFÍCIA UNIVERSIDADE CATÓLICA DO RIO DE JANEIRO  
RIO DE JANEIRO  
22453-900 RJ  
BRAZIL  
*E-mail address:* earp@mat.puc-rio.br

ERIC TOUBIANA  
UNIVERSITÉ PARIS DIDEROT - PARIS 7  
INSTITUT DE MATHÉMATIQUES DE JUSSIEU, UMR CNRS 7586  
UFR DE MATHÉMATIQUES, CASE 7012  
BÂTIMENT CHEVALERET  
75205 PARIS CEDEX 13  
FRANCE  
*E-mail address:* toubiana@math.jussieu.fr