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Pricing in Vehicle Sharing Systems: Optimization in queuing networks with product forms

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Abstract

One-way Vehicle Sharing Systems (VSS) such as Vélib' Paris are flourishing. The usefulness of VSS for users is highly impacted by the availability of vehicles and parking spots. Most existing systems are ruled by the requests of users. We study the potential interest of influencing the users in order to improve the performance of the system. We focus on optimizing the number of trips taken in the system. We assume that each user is associated with a pair (O-D) of stations, and only interacts with the system if his O-D trip is available. We consider leverages that can influence the rate of user requests for each pair O-D. In order to provide exact formulas and analytical insights, VSS are modeled as closed queuing networks with infinite buffer capacity and Markovian demands. Transportation times are assumed to be null and the demand to be stationary over time. We propose a heuristic based on computing a MAXIMUM CIRCULATION together with a convex integer program solved optimally by a greedy algorithm. For M stations and N vehicles, the performance ratio of this heuristic is conjectured to be exactly $N/(N + M - 1)$ and proved to be at least $(N - M)/(N + M)$. The complexity of computing optimum policies remains open. Insights on this issue are provided in the appendix. The appendix also contains an example showing that VSS can have poor performances without regulation.

Keywords: Vehicle Sharing Systems; Closed Queuing Networks; Pricing; Product forms; Continuous-time Markov decision process; Stochastic optimization; Approximation algorithms.

1 Introduction

1.1 Context

Based on a sample of 22 US studies, [Shoup \(2005\)](#) reports that car drivers looking for a parking spot contribute to 30% of the city traffic. Moreover cars are used less than 2 hours per day on

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average but still occupy a parking spot the rest of the time! Could we have fewer vehicles and satisfy the same demand level?

Recently, the interest in Vehicle Sharing Systems (VSS) in cities has increased significantly. Indeed, urban policies intend to discourage citizens to use their personal car downtown by reducing the number of parking spots, street width, etc. VSS seem to be a promising solution to reduce jointly traffic and parking congestion, noise, and air pollution (proposing bikes or electric cars). They offer personal mobility allowing users to pay only for the usage (sharing the cost of ownership).

We are interested in short-term one-way VSS in which vehicles can be taken and returned at different places. Associated with classic public transportation systems, short-term one-way VSS help to solve one of the most difficult public transit network design problems: the last kilometer issue (DeMaio, 2009). Round-trip VSS, where vehicles have to be returned at the station where they were taken, cannot address this issue.

The first large-scale short-term one-way VSS was the Bicycle Sharing System (BSS) *Vélib'*. It was implemented in Paris in 2007. Today, it has more than 1200 stations and 20 000 bikes selling around 110 000 trips per day. It has inspired several other cities all around the world: Now more than 300 cities have such a system, including Montréal, Beijing, Barcelona, Mexico City, Tel Aviv (DeMaio, 2009).

1.2 One-way Vehicle Sharing Systems: a management issue

One-way systems increase the user freedom at the expense of a higher management complexity. In round trip rental systems, while managing the yield, the only stock that is relevant is the number of available vehicles. In one-way systems, vehicles are not the only key resource anymore: parking stations may have limited number of spots and the available parking spots become an important control leverage.

Since first BSS, problems of bikes and parking spots availability have appeared recursively. Côme (2012), among others, applies data mining to operational BSS data. He offers insights on typical usage patterns to understand causes of imbalances in the distribution of bikes. Reasons are various but we can highlight two important phenomena: the gravitational effect which indicates that a station is constantly empty or full (as Montmartre hill in *Vélib'*), and the tide phenomenon representing the oscillation of demand intensity during the day (as morning and evening flows between working and residential areas).

To improve the efficiency of the system, different perspectives are studied in the literature. At a strategic level, some authors consider the optimal capacity and locations of stations. Shu et al. (2010) propose a stochastic network flow model to support these decisions. Their model is used to design a BSS in Singapore based on demand forecast derived from current usage of the mass transit system. Lin and Yang (2011) consider a similar problem but formulate it as a deterministic mathematical model.

At a tactical level, other authors investigate the optimal number of vehicles given a set of stations. George and Xia (2011) study the fleet sizing problem with constant demand and infinite parking capacities. Fricker and Gast (2012), Fricker et al. (2012) consider the optimal sizing of a fleet in “toy” cities, where demand is constant over time and identical for every possible trip, and all stations have the same capacity \mathcal{K} . They show that even with an optimal fleet sizing in the

most “perfect” city, if there is no operational system management, there is at least a probability of $\frac{2}{\kappa+1}$ that any given station is empty or full.

At an operational level, in order to be able to meet the demand with a reasonable standard of quality, in most BSS, trucks are used to balance the bikes among the stations. The balancing problem amounts to scheduling truck routes to visit stations performing pickup and delivery. In the literature many papers deal already with this problem. A static version of the BSS balancing problem is analyzed in [Raviv et al. \(2013\)](#) and a dynamic one in [Contardo et al. \(2012\)](#).

Pricing heuristics are experimentally studied in ([Chemla et al., 2013](#); [Pfrommer et al., 2013](#)). The “power of two choices” incentive, that can be seen as a dynamic pricing is analysed by [Fricker and Gast \(2012\)](#): When a user arrives at a station to take a vehicle, he gives randomly two possible destination stations and the system is directing him toward the least loaded one. For homogeneous cities, they show that this policy allows to drastically reduce the probability to be empty or full for each station to $2^{-\frac{\kappa}{2}}$.

1.3 Towards VSS regulated with incentives

A new type of VSS has appeared recently: one-way car VSS with [Autolib’](#) in Paris and [Car2go](#) in more than 15 cities (Vancouver, San Diego, Amsterdam, Ulm...). Due to the size of cars, operational balancing optimization through relocation with trucks seems inappropriate. Another way for optimizing the system has to be found.

Pricing heuristics are experimentally studied by [Chemla et al. \(2013\)](#) and [Pfrommer et al. \(2013\)](#). The “power of two choices” incentive, that can be seen as a dynamic pricing is analysed by [Fricker and Gast \(2012\)](#): When a user arrives at a station to take a vehicle, he gives randomly two possible destination stations and the system is directing him toward the least loaded one. For homogeneous cities, they show that this policy allows to drastically reduce the probability to be empty or full for each station from $\frac{2}{\kappa+1}$ to $2^{-\frac{\kappa}{2}}$.

A VSS stochastic pricing model is proposed in [Waserhole and Jost \(2013\)](#) considering time-dependent demand and station capacities. They study a fluid approximation that provides a static heuristic policy and an upper bound. The fluid approximation is deterministic; one can wonder if a stochastic model, even considering less constraints, can have a better performance.

1.4 Contributions and structure of this paper

We investigate stochastic models allowing an analytic formula for the performance evaluation of the system.

Section 2, we consider VSS in which each user is interested with a specific O-D pair of stations, but is sensitive to the price of this trip. We discuss how prices can be made implicit when considering objectives such as the maximization of the expected number of trips sold by the system.

In Section 3, we consider VSS with stationary O-D demands and infinite station capacities, as in [George and Xia \(2011\)](#), but we also assume null transportation times. Under these assumptions, the VSS can be modeled as a closed queuing network of BCMP type. Its performance can therefore be computed analytically. We define static and dynamic stochastic pricing problems on such queuing networks.

In Section 4 we study a static heuristic policy provided by the MAXIMUM CIRCULATION on the demand graph. When the MAXIMUM CIRCULATION disconnects the city, vehicles have

to be spread among the connected components. The vehicle distribution problem amounts to maximizing a separable concave function under linear and integrality constraints. It can be solved optimally by a greedy algorithm. The guaranty of performance of our heuristic is conjectured to be $\frac{N}{N+M-1}$. We prove that this ratio of performance is at least $\frac{N-M}{N+M}$.

In Appendix A, we discuss the properties of optimal dynamic and static policies. An optimal dynamic policy can be computed with an action decomposable Markov decision process. An example shows that VSS can have poor performances without regulation.

2 Protocol, incentives and implicit pricing

A simple protocol We consider a *real-time station-to-station protocol* as defined in Figure 1. A user asks for a vehicle at station a (here and now), with destination b . The system offers a price (or rejects the user = infinite price). The user either pays the price and the vehicle is transferred from a to b , or leaves the system.

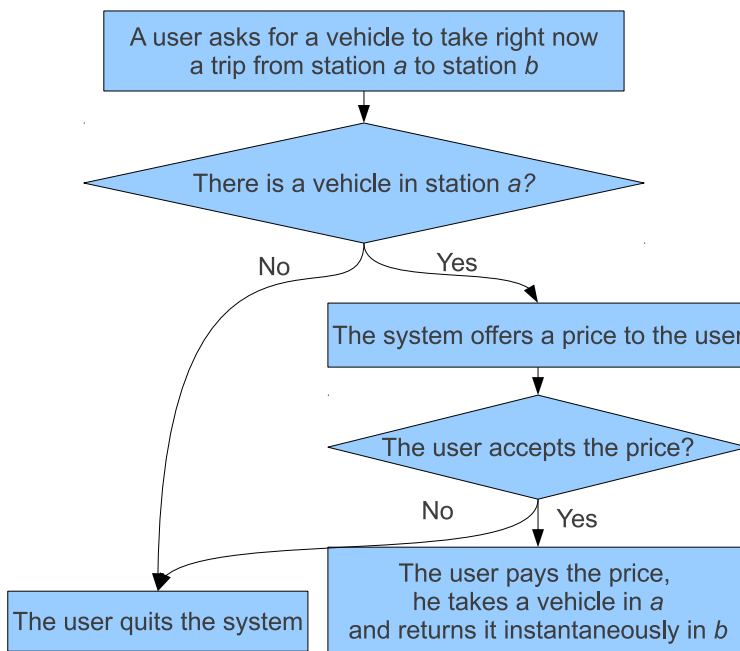


Figure 1: The real-time station-to-station protocol.

Concept of maximum potential demand We assume that for each trip (a, b) and independently of the other trips, there is a pool of potential users that may try to take trip (a, b) in the time horizon of the model. We denote this pool $\Lambda_{a,b}$, which, in this paper is interpreted as a Poisson arrival of users with intensity $\Lambda_{a,b}$ per time unit (but other deterministic interpretations of $\Lambda_{a,b}$ are discussed in (Waserhole et al., 2013b; Waserhole and Jost, 2013)).

Pricing policies and incentives We assume that there exist leverages (incentives) able to decrease the maximum demand (separately for each trip). A classic incentive is the price to take

a trip; the demand is then a function of the price: basically, the higher the price, the lower the demand. A pricing/incentive policy is *static* if the price to take each trip is independent of the state of the system. A policy is *dynamic* otherwise.

Continuous elastic demand In this study we focus on continuous pricing optimization with the following hypothesis: Let $\Lambda_{a,b}$ be the maximum demand of users who want to take a trip between stations a and b . There exists a price $p(\lambda_{a,b})$ to obtain any demand $\lambda_{a,b} \in [0, \Lambda_{a,b}]$. A price function is schemed Figure 2. Notice that, in this example, the maximum demand Λ is obtained with a minimum price $p(\Lambda)$ that is negative. Indeed it is conceivable that the system chooses to pay users to take certain trips (instead of paying trucks).

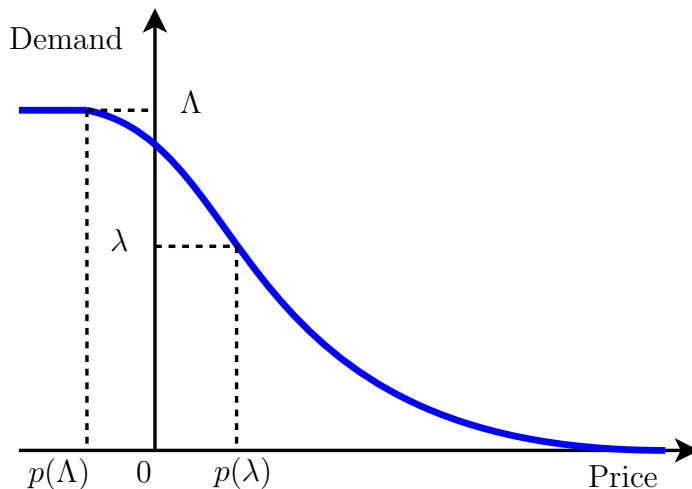


Figure 2: Continuous elastic demand $\lambda \in [0, \Lambda]$.

Implicit pricing The problem with pricing incentives is that it is hard to know the link between prices and demand. It can be a complex function, not continuous, with thresholds... Moreover, setting the proper prices to attain a fixed (optimized) demand requires the skill of an economist and experimental studies. On the other hand, there exist some objectives such as maximizing the number of trips sold (transit) or the total travel time that do not need an explicit price function. The only data necessary for such optimization is the space of the possible demand, for instance $\lambda \in [0, \Lambda]$ for continuous elastic demand.

Therefore in this study we focus on the transit optimization and do not consider prices explicitly. We talk about pricing policies but they can be seen as incentive policies or simply policies regulating demand.

3 Stochastic framework

3.1 The VSS stochastic evaluation model

Continuous-time Markov chain evaluation framework We model the VSS dynamic by a stochastic process: the *VSS stochastic evaluation model*. It measures VSS performances for a given policy (demand vector). We use this evaluation model to compare the performance of the proposed pricing policies in term of number of trips sold. We now define formally the VSS stochastic evaluation model under the real-time station-to-station protocol (defined in Figure 1).

VSS STOCHASTIC EVALUATION MODEL

- **INPUT:**

- A number N of vehicles and a set \mathcal{M} of stations:
 - A set \mathcal{S} of states: $\mathcal{S} = \left\{ \left(n_a : a \in \mathcal{M} \right) / \sum_{a \in \mathcal{M}} n_a = N \right\}$;
 - State $s = (n_a : a \in \mathcal{M})$ represents the vehicle distribution in the city space: n_a is the number of vehicles in station $a \in \mathcal{M}$.
- A policy λ :
 - $\lambda_{a,b}^s$ is the arrival rate of users to take the trip $(a, b) \in \mathcal{D} = \mathcal{M} \times \mathcal{M}$, between state $s = (\dots, n_a \geq 1, \dots, n_b, \dots) \in \mathcal{S}$ and state $(\dots, n_a - 1, \dots, n_b + 1, \dots) \in \mathcal{S}$;
 - The graph spanned by $\{(a, b \in \mathcal{M}, s \in \mathcal{S}, \lambda_{a,b}^s > 0)\}$ is supposed to be strongly connected.

- **OUTPUT:** The expected number of trips sold in the steady state behavior of the continuous-time Markov chain defined by states \mathcal{S} and transition rates λ .

The number of states is exponential in the number of vehicles and stations. For instance, for a system with $N = 150$ vehicles and $M = 50$ stations there are already $\simeq 10^{47}$ states:

Proposition 1. *The number of state of the Markov chain for N vehicles and M stations with infinite station capacities and null transportation time is equal to $\binom{N+M-1}{N}$.*

Proof. The states of the Markov chain for N vehicles and M stations are in one to one mapping with non decreasing functions from $\{1, \dots, N\}$ to $\{1, \dots, M\}$ which are in one to one mapping with strictly increasing functions from $\{1, \dots, N\}$ to $\{1, \dots, M + N - 1\}$. \square

Steady-state distribution of the continuous-time Markov chain For any strongly connected dynamic policy, the unique stationary distribution π over the state space \mathcal{S} of the

continuous-time Markov chain with transition rate λ satisfies Equations (1) (Puterman, 1994).

$$\sum_{s \in \mathcal{S}} \pi_s = 1 \quad (1a)$$

$$\sum_{\substack{(a,b) \in \mathcal{D} \\ s - e_a + e_b \in \mathcal{S}}} \pi_s \lambda_{a,b}^s = \sum_{\substack{(b,a) \in \mathcal{D}, s' \in \mathcal{S} \\ s' - e_b + e_a = s}} \pi_{s'} \lambda_{b,a}^s, \quad \forall s \in \mathcal{S}, \quad (1b)$$

$$\pi_s \geq 0, \quad \forall s \in \mathcal{S}. \quad (1c)$$

Closed queuing network model for static policies The VSS stochastic evaluation model can be represented as a closed queueing network for static policies. An example with 2 stations is schemed in Figure 3. This closed queueing network is built as follows.

Since there is a fixed number of vehicles circulating in the network, it is natural to see the system from a vehicle's perspective. Each station $a \in \mathcal{M}$ is represented by a server a with infinite capacity queue. The N vehicles are N jobs waiting in these queues for users to take them. The service rate λ_a of server a is equal to the average number of users willing to take a vehicle at station a : $\lambda_a = \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b}$. A vehicle taken by a user for a trip $(a,b) \in \mathcal{D}$ is represented by a job processed by server a with routing probability $\frac{\lambda_{a,b}}{\lambda_a}$. When a vehicle (a job) is taken for the trip (a,b) it is transferred instantaneously in station (buffer) b .

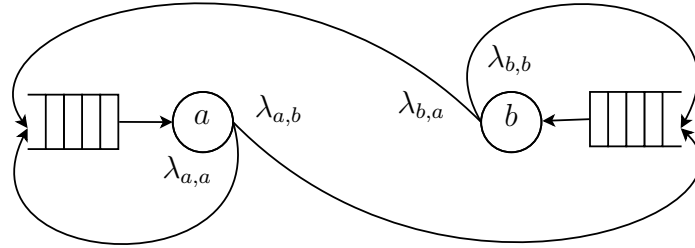


Figure 3: A closed queuing network model: servers represent users demands.

Analytic evaluation for static policies The stochastic evaluation model for static policies is the same as the one considered by George and Xia (2011) but with null transportation times. They provide a compact form to compute the system performance using the BCMP network theory (Baskett et al., 1975). In Section 4.2, we consider static policies providing demands for which the performance evaluation is slightly simpler than the formula of George and Xia (2011), see Lemma 1.

An important concept that we use for a static policy (with demand λ) is the *availability* A_a of (a vehicle at) station $a \in \mathcal{M}$ which is the probability that station a contains at least one vehicle. Availabilities satisfy steady-state equations:

$$\sum_{b \in \mathcal{M}} A_a \lambda_{a,b} = \sum_{b \in \mathcal{M}} A_b \lambda_{b,a} \quad \forall a \in \mathcal{M}. \quad (2)$$

Notice that availabilities are not totally determined by (2) because they also depend on the number of vehicles.

3.2 The VSS stochastic pricing problem

We want to maximize the VSS performance using pricing as leverage. The efficiency of a pricing policy is measured by the VSS stochastic evaluation model. We call this problem the *VSS stochastic pricing* problem.

VSS STOCHASTIC CONTINUOUS PRICING TRANSIT MAXIMIZATION

- **INSTANCE:**

- A number N of vehicles available;
- A set \mathcal{M} of stations with infinite capacities;
- The maximum demand per time unit $\Lambda_{a,b}$ to take every trip $(a, b) \in \mathcal{D}$.

- **SOLUTION:**

[**Dynamic Policy**] A demand $\lambda_{a,b}(s) \in [0, \Lambda_{a,b}]$, to take each trip $(a, b) \in \mathcal{D}$ function of the system's state $s \in \mathcal{S}$.

[**Static Policy**] A tuple $(\lambda, k, \vec{\mathcal{M}}, \vec{N})$, where:

- $\lambda_{a,b} \in [0, \Lambda_{a,b}]$ is the demand to take each trip $(a, b) \in \mathcal{D}$,
- λ defines a set of k strongly connected components $\vec{\mathcal{M}} = (\mathcal{M}_1, \dots, \mathcal{M}_k)$,
- $\vec{N} = (N_1, \dots, N_k)$ is the vehicle distribution over $\vec{\mathcal{M}}$, ($\sum_{i=1}^k N_i = N$).

- **MEASURE:** The expected number of trips sold of the pricing policy measured by the stochastic evaluation model.

We restrict the study of dynamic policies to the (dominant) class for which the graph spanned by $\{(a, b \in \mathcal{M}, s \in \mathcal{S}, \lambda_{a,b}^s > 0)\}$ has only one strongly connected component. Otherwise, the stationary distribution on the state graph is not unique: it depends on the initial state of the system.

Sometimes optimal static policies need more than one strongly connected components on the station graph. An example is given in Proposition 9 Section A.3. The k strongly connected components of the static policy graph $G(\mathcal{M}, \lambda)$ divides the city into k independent VSS, sharing a number N of vehicles. The vehicle distribution has then to be explicitly specified since it impacts the policy performance. For dynamic policies, the vehicle distribution is explicit (defined by the system states for single component policies). That is why for ease of notations the stochastic evaluation model is defined for dynamic policies (any static policy can be represented as a dynamic one).

Complexity in a stochastic framework The previous formal problem definition enables to define *tractability*, *polynomiality* or simply *efficiency* for VSS stochastic pricing optimization. To tackle large scale (real-world) systems, we need solution methods that have computational time polynomial in N and M . The solutions (pricing policies) produced (output) need also to be of moderate size. Notice that the state graph (of exponential size) representing all possible

vehicle distributions (system’s states) is not part of the input. The explicit representation of dynamic policies is hence not tractable.

For static policies, measuring exactly the stochastic evaluation model is polynomial in M and N : [George and Xia \(2011\)](#) provide a product form formula and algorithms to compute the stochastic evaluation model for a static pricing policy. However, we are able to prove that the decision version of the above static pricing problem is in NP only under further assumptions ([Waserhole, 2013](#)).

We discuss in [Appendix A](#) the problem of characterizing dynamic and static optimal policies. However, the complexity is unknown for both classes of policies. The deterministic version of the stochastic pricing problem is shown NP-hard in [Waserhole et al. \(2013b\)](#). Nevertheless there is no obvious reduction between these problems.

4 MAXIMUM CIRCULATION approximation

In this section we study an approximation algorithm based on the MAXIMUM CIRCULATION problem ([Edmonds and Karp, 1972](#)): a network flow problem with flow conservation at all nodes (no source no sink).

4.1 MAXIMUM CIRCULATION Upper Bound

A vector λ is called a *circulation* if it is solution of the following LP.

MAXIMUM CIRCULATION LP

$$\begin{aligned} \max \quad & \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} \\ \text{s.t.} \quad & \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda_{b,a}, & \forall a \in \mathcal{M}, \\ & 0 \leq \lambda_{a,b} \leq \Lambda_{a,b}, & \forall (a,b) \in \mathcal{D}. \end{aligned}$$

Theorem 1. *The objective value of MAXIMUM CIRCULATION on the demand graph is an upper bound on any dynamic policy for any number of vehicles.*

Proof. From any dynamic policy, with transition rate $\lambda_{a,b}^s \leq \Lambda_{a,b}$ in state $s \in \mathcal{S}$ for trip $(a,b) \in \mathcal{D}$, we construct a circulation on the demand graph with same value. Under this policy, the stationary distribution π over the state space \mathcal{S} of the continuous-time Markov chain defined by λ satisfies [Equations \(1\)](#). Let $\lambda'_{a,b}$ be the expected transit for any trip $(a,b) \in \mathcal{D}$: $\lambda'_{a,b} = \sum_{s \in \mathcal{S}} \pi_s \lambda_{a,b}^s$. We show that λ' is a circulation. The capacity constraints are satisfied since $\sum_{s \in \mathcal{S}} \pi_s = 1$ and hence:

$$\lambda'_{a,b} = \sum_{s \in \mathcal{S}} \pi_s \lambda_{a,b}^s \leq \sum_{s \in \mathcal{S}} \pi_s \Lambda_{a,b} = \Lambda_{a,b}, \quad \forall (a,b) \in \mathcal{D}.$$

Flow conservation constraints are satisfied because in the steady state of a dynamic policy, a station receives as many vehicles as it is sending. Finally, the expected transit of the system is equal to $\sum_{(a,b) \in \mathcal{D}} \lambda'_{a,b}$ which is the value of circulation λ' . \square

4.2 MAXIMUM CIRCULATION static policy

The MAXIMUM CIRCULATION outputs a demand vector $\lambda \leq \Lambda$. It is natural to try to use this demand vector as a static policy. However, whenever the MAXIMUM CIRCULATION is not strongly connected, one has to specify a vehicle distribution \vec{N} over the k strongly connected component $\vec{M} = \{\mathcal{M}_1, \dots, \mathcal{M}_k\}$. This issue may indeed occur (Proposition 2). We call a static policy $\phi = (\lambda, k, \vec{\mathcal{M}}, \vec{N})$ a *circulation policy* if λ is a circulation.

Proposition 2. *The optimal solution(s) of MAXIMUM CIRCULATION might consist of more than one strongly connected component.*

Proof. Consider the demand graph in Figure 4 consisting of $\Lambda = 1$ for all drawn arcs (both dotted and straight). The unique MAXIMUM CIRCULATION sets $\lambda = 1$ for straight arcs and 0 elsewhere. Its policy demand graph is not strongly connected. \square

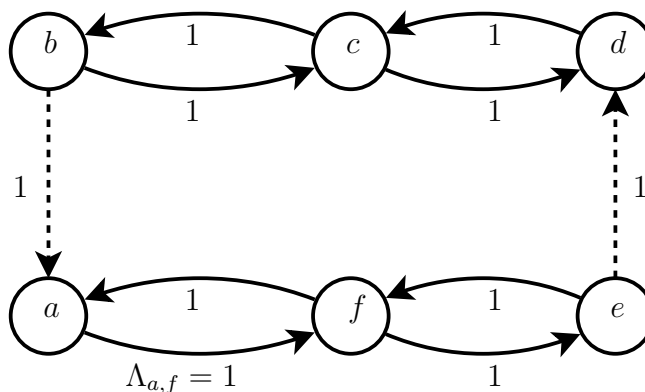


Figure 4: MAXIMUM CIRCULATION can consist of several strongly connected components.

4.2.1 Evaluation for a given distribution

Recall that for a static policy ϕ , the availability $A_a(\phi)$ of (a vehicle at) station $a \in \mathcal{M}$ is the probability that station a contains at least one vehicle. Lemma 1 essentially says that the availability of a station is $\frac{N}{N+M-1}$ for a circulation spanning only one strongly connected component with M stations.

Lemma 1. *For any circulation λ and any vehicle distribution \vec{N} , the expected transit $T(\phi)$ of the circulation policy $\phi = (\lambda, k, \vec{\mathcal{M}}, \vec{N})$ is equal to:*

$$T(\phi) = \sum_{i=1}^k \left(\frac{N_i}{N_i + |\mathcal{M}_i| - 1} \sum_{a,b \in \mathcal{M}_i} \lambda_{a,b} \right).$$

The remaining of Section 4.2.1 is devoted to a proof of Lemma 1, which is done by expressing relations between transit, availability and the continuous-time Markov chain formulation.

A static policy $\phi = (\lambda, k, \vec{\mathcal{M}}, \vec{N})$ can be evaluated by a Continuous-Time Markov Chain: CTMC(ϕ). The availability $A_a(\phi)$ of station $a \in \mathcal{M}$ is equal to the sum of the stationary distributions π_s of the states $s \in \mathcal{S}$ of CTMC(ϕ) where there is at least one vehicle in station a :

$$A_a(\phi) = \sum_{s=(\dots, n_a \geq 1, \dots) \in \mathcal{S}} \pi_s.$$

The expected transit $T(\phi)$ of static policy ϕ is then:

$$T(\phi) = \sum_{a \in \mathcal{M}} \left(A_a(\phi) \sum_{b \in \mathcal{M}} \lambda_{a,b} \right).$$

We now state two lemmas, Lemma 2 and 3, that imply Lemma 1.

Lemma 2. *For a static policy ϕ , CTMC(ϕ) is the product of k independent CTMC(ϕ^i), where $\phi^i = (\lambda_{(a,b) \in \mathcal{M}_i^2}, 1, \{\mathcal{M}_i\}, \{N_i\})$ is a static policy with one strongly connected component. The expected transit $T(\phi)$ is then decomposed as follows:*

$$T(\phi) = \sum_{a \in \mathcal{M}} \left(A_a(\phi) \sum_{b \in \mathcal{M}} \lambda_{a,b} \right) = \sum_{i=1}^k \sum_{a \in \mathcal{M}_i} \left(A_a(\phi^i) \sum_{b \in \mathcal{M}_i} \lambda_{a,b} \right).$$

□

Lemma 3. *For a circulation policy ϕ and for any strongly connected component \mathcal{M}_i , the availability $A(\phi^i)$ of a vehicle at any station is equal to:*

$$A(\phi^i) = \frac{N_i}{N_i + |\mathcal{M}_i| - 1}.$$

Proof. Recall availabilities steady-state Equations (2):

$$A_a \sum_{b \in \mathcal{M}} \lambda_{a,b} = \sum_{b \in \mathcal{M}} A_b \lambda_{b,a}, \quad \forall a \in \mathcal{M}.$$

A circulation satisfies $\sum_{b \in \mathcal{M}} \lambda_{a,b} = \sum_{b \in \mathcal{M}} \lambda_{b,a}$, $\forall a \in \mathcal{M}$, therefore $A_a = u$, $\forall a \in \mathcal{M}$, is solution of Equations (2) for any $u \in \mathbb{R}$. Hence for a circulation policy ϕ with one single strongly connected component, the availability is the same for all stations:

$$A_a = A(\phi), \quad \forall a \in \mathcal{M}.$$

For a static policy ϕ with one single strongly connected component $\phi = (\lambda, 1, \{\mathcal{M}\}, \{N\})$, CMTC(ϕ) has exactly $\mathcal{N}_N^M = \binom{N+M-1}{N}$ states (Proposition 1). For any station $a \in \mathcal{M}$, there are \mathcal{N}_{N-1}^M states with at least one vehicle available in station a . Finally, since each state of CMTC(ϕ) has the same stationary distribution, the availability $A(\phi)$ of a vehicle at any station is equal to:

$$A(\phi) = \frac{\mathcal{N}_{N-1}^M}{\mathcal{N}_N^M} = \frac{\binom{N+M-2}{N-1}}{\binom{N+M-1}{N}} = \frac{\frac{(N+M-2)!}{(N-1)!(M-1)!}}{\frac{(N+M-1)!}{(N)!(M-1)!}} = \frac{N}{N+M-1}. \quad \square$$

4.2.2 Optimality of the greedy distribution of vehicles

Let $\{\mathcal{M}_1, \dots, \mathcal{M}_k\}$ be the set of the k strongly connected components of a circulation λ . If we allocate N_i vehicles to component i , the expected transit of the policy $\phi^i = (\lambda_{(a,b) \in \mathcal{M}_i^2}, 1, \{\mathcal{M}_i\}, \{N_i\})$ is:

$$T(\phi^i) = f_i(N_i) = \frac{N_i}{N_i + M_i - 1} \sum_{a,b \in \mathcal{M}_i} \lambda_{a,b}. \quad (3)$$

For a distribution $\vec{N} = \{N_1, \dots, N_k\}$ of the N vehicles, the expected transit of policy $\phi = (\lambda, k, \vec{\mathcal{M}}, \vec{N})$ is hence:

$$T(\phi) = f(\vec{N}) = \sum_{i=1}^k f_i(N_i). \quad (4)$$

The optimal distribution \vec{N}^* of the N vehicles among the k strongly connected components is then solution of the following problem:

$$\begin{aligned} \vec{N}^* &= \max f(\vec{N}) \\ \text{s.t. } &\sum_{i=1}^k N_i = N, \\ &\vec{N} \in \mathbb{Z}_+^k. \end{aligned}$$

Consider the following algorithm for finding a feasible solution to the previous problem:

Algorithm 1 Greedy algorithm for load distribution

- 1: $\vec{N} := (0, \dots, 0)$
 - 2: **for** $n = 1$ to N **do**
 - 3: Choose $j \in \arg \max_{i \in \{1, \dots, k\}} f(\vec{N} + e_i)$;
 - 4: $\vec{N} := \vec{N} + e_j$;
 - 5: **end for**
 - 6: **return** \vec{N} .
-

In general Algorithm 1 may not provide an optimal solution. A function $f(\vec{N})$ for which there exist functions f_i such that $\forall \vec{N}, f(\vec{N}) = \sum_{i=1}^k f_i(N_i)$, is called separable. Moreover if each f_i is concave, f is called concave separable.

Separable concave functions are of interest in mathematical economics, an example is the gain function (4). It turns out that separable concavity is enough for the greedy algorithm to find an optimal solution under the constraint $\sum_{i=1}^k N_i = N$ (see Theorem 2). Maximizing separable concave functions can also be done over more complex feasible spaces, such as polymatroids (Glebov, 1973; Shenmaier, 2003).

Theorem 2. *Let k be a positive integer, $\{f_i\}_{i \in \{1, \dots, k\}}$ be concave functions and $N \in \mathbb{Z}_+$. Also denote $f(\vec{N}) := \sum_i f_i(N_i)$. Then the solution of the following integer program is attained by*

greedy Algorithm 1.

$$\begin{aligned} & \max \sum_{i=1}^k f_i(N_i) \\ & \text{s.t. } \sum_{i=1}^k N_i = N, \\ & \quad \vec{N} \in \mathbb{Z}_+^k. \end{aligned}$$

Proof. We give a proof by induction on N . The case $N = 0$ is trivial since $\vec{N} = (0, \dots, 0)$ is the only feasible solution. Assume case N is correct: the greedy algorithm provides an optimal solution, say \vec{N}^* for N . Now, let \vec{N}' be an optimal solution for $N + 1$. Choose $j \in \{1, \dots, k\}$ such that $N'_j > N_j^*$. By induction hypothesis, $f(\vec{N}^*) \geq f(\vec{N}' - e_j)$. Also, by concavity of f_j and because $N'_j - 1 \geq N_j^*$, one has:

$$\begin{aligned} f(\vec{N}^* + e_j) &= f(\vec{N}^*) + f_j(N_j^* + 1) - f_j(N_j^*) \\ &\geq f(\vec{N}^*) + f_j(N'_j) - f_j(N'_j - 1) \\ &\geq f(\vec{N}' - e_j) + f_j(N'_j) - f_j(N'_j - 1) = f(\vec{N}'). \end{aligned}$$

A solution found by the greedy algorithm is hence at least as good as $f(\vec{N}^* + e_j)$ which is at least as good as $f(\vec{N}')$. \square

Corollary 1. *For any fixed λ and any $N \in \mathbb{Z}_+$, a vehicle distribution $\vec{N} \in \mathbb{Z}_+^{k(\lambda)}$ maximizing the expected transit under the constraint $\sum_{i=1}^k N_i = N$ can be computed with greedy Algorithm 1.*

Proof. Let $\{\mathcal{M}_1, \dots, \mathcal{M}_k\}$ be the set of the strongly connected components of the static policy graph $G(\mathcal{M}, \lambda)$. For any static policy, the expected transit of the system is the sum of the expected transit of each component, hence the gain function is separable. The concavity of the gain function in each component can be deduced from (3) for circulation policies, and is proved in (George and Xia, 2011, Theorem 2) for general policies. \square

4.3 Performance evaluation

We study the performance of the MAXIMUM CIRCULATION static policy together with its optimal vehicle distribution.

4.3.1 An upper bound on the approximation ratio

The expected transit of the MAXIMUM CIRCULATION static policy together with its optimal vehicle distribution can be arbitrarily close to $\frac{N}{N+M-1}$ times the value of a static policy:

Proposition 3. *For any number $M \geq 2$ of stations and any number N of vehicles, the ratio between the value of MAXIMUM CIRCULATION policy and a static policy can be arbitrary close to $\frac{N}{N+M-1}$.*

Proof. We consider instances with N vehicles, $M \geq 2$ stations $\mathcal{M} = \{1, \dots, M\}$ and demand graph consisting of a circuit $\{1, \dots, M, 1\}$ with maximum demand $\Lambda_{i,i+1} = k$, $i \in \{1, \dots, M-1\}$ and $\Lambda_{M,1} = 1$ (all other demands are equal to 0).

The MAXIMUM CIRCULATION policy opens all trips of the circuit to 1. Its value P_{Circ^*} is equal to: $P_{Circ^*} = \frac{NM}{N+M-1}$.

Consider the generous static policy opening all trips to their maximum value: $\lambda = \Lambda$. The generous static policy demand graph is a circuit, hence the expected transit ($A_a \times \Lambda_{a,b}$) is the same for all trips (a, b) of the circuit. Availabilities A satisfy Equations (2) hence:

$$A_M \times 1 = A_i \times k, \quad \forall i \in \{1, \dots, M-1\}, \text{ so:}$$

$$\sum_{a \in \mathcal{M}} A_a = A_M \left(1 + \frac{M-1}{k}\right).$$

Since $\sum_{a \in \mathcal{M}} A_a = 1$ for one vehicle, and $\forall a \in \mathcal{M}$, A_a is a non decreasing function of the number of vehicles (George and Xia, 2011), we have that $\sum_{a \in \mathcal{M}} A_a \geq 1$. Hence, $\lim_{k \rightarrow \infty} A_M(k) = 1$ and $\lim_{k \rightarrow \infty} A_i(k) = 0$, $\forall i \in \{1, \dots, M-1\}$. When $k \rightarrow \infty$, the value of the generous static policy is then $\lim_{k \rightarrow \infty} P_{Gen}(k) = M$.

The ratio between the static generous policy and the MAXIMUM CIRCULATION static policy can then be arbitrary close to:

$$\frac{N}{N+M-1} = \lim_{k \rightarrow \infty} \frac{P_{Gen}(k)}{P_{Circ^*}(k)}. \quad \square$$

4.3.2 The conjectured tight guaranty of performance

We think that the $\frac{N}{N+M-1}$ upper bound of Proposition 3 is actually the tight ratio of performance of MAXIMUM CIRCULATION static policy together with its optimal vehicle distribution. However we have not been able to prove it:

Conjecture 1. MAXIMUM CIRCULATION static policy together with its optimal vehicle distribution is a tight $\frac{N}{N+M-1}$ -approximation on both static and dynamic optimal policies.

As it's not easy to perform extensive numerical validations of Conjecture 1, we studied simpler and deeper versions. Conjecture 2 settles a refinement providing a probabilistic proof of Conjecture 1 (see Proposition 4 below). It has been verified experimentally for all (M, N) such that $M \leq 1000$ and $N \leq 3500$.

Conjecture 2. For any static circulation policy spanning k strongly connected components, let \vec{N}^U be the random Uniform vehicle distribution defined by assigning each of the N vehicles independently to a strongly connected component, with probability $\frac{M_i}{M}$ for component $i \in \{1, \dots, k\}$. The expected availability $\mathbb{E}[A(\vec{N}^U)]$ of a vehicle at any station satisfies:

$$\mathbb{E}[A(\vec{N}^U)] \geq \frac{N}{N+M-1}.$$

Remark 1. To verify experimentally Conjecture 2 we have used the following property. Let $\{\mathcal{M}_1, \dots, \mathcal{M}_k\}$ be the set of the strongly connected components of a circulation policy graph $G(\mathcal{M}, \lambda)$. Let \vec{N}^U be the random uniform vehicle distribution. The expected availability of a vehicle at any station in component i with M_i stations is:

$$\mathbb{E}\left[A(\vec{N}^U, M_i)\right] = \sum_{n=1}^N \left(\frac{M_i}{M}\right)^n \left(\frac{M-M_i}{M}\right)^{N-n} \binom{N}{n} \frac{n}{n+M_i-1}$$

which only depends on M_i , M and N . Therefore Conjecture 2 amounts to:

$$\mathbb{E}\left[A(\vec{N}^U, M_1)\right] \geq \frac{N}{N+M-1}, \quad \forall M_1 \in \{1, \dots, M\}, \quad \forall N, M \geq 0.$$

Moreover, to speed-up the tests, we used the properties:

$$\left(\frac{M_i}{M}\right)^{(n+1)} \left(\frac{M-M_i}{M}\right)^{N-(n+1)} = \left(\frac{M_i}{M}\right)^{(n)} \left(\frac{M-M_i}{M}\right)^{N-(n)} \times \frac{M_i}{M-M_i},$$

$$\text{as well as } \binom{N}{n+1} = \binom{N}{n} \times \frac{N-n}{n+1}.$$

Proposition 4. If Conjecture 2 is true, then Conjecture 1 is true.

Proof. Let $Circ^*$ be the optimal value of MAXIMUM CIRCULATION with k strongly connected components $\{\mathcal{M}_1, \dots, \mathcal{M}_k\}$. Component \mathcal{M}_i is composed with M_i stations and contributes to a value C_i^* in the optimal MAXIMUM CIRCULATION: $\sum_{i=1}^k C_i^* = Circ^*$.

Let $P_{Circ}^{\vec{N}}$ be the value of the circulation policy with vehicle distribution \vec{N} . Let \vec{N}^* be the optimal vehicle distribution for the MAXIMUM CIRCULATION static policy. Let \vec{N}^U be the random Uniform vehicle distribution, defined by assigning each of the N vehicles independently to a strongly connected component, with probability $\frac{M_i}{M}$ for component $i \in \{1, \dots, k\}$.

Assume that Conjecture 2 is true. For random uniform vehicle distribution \vec{N}^U , the expected uniform stationary distribution $\mathbb{E}[A(\vec{N}^U, M_i)]$ of a vehicle at any station belonging to component i satisfies: $\mathbb{E}[A(\vec{N}^U, M_i)] \geq \frac{N}{N+M-1}$. Therefore:

$$P_{Circ}^{\vec{N}^*} \geq \mathbb{E}\left[P_{Circ}^{\vec{N}^U}\right] = \mathbb{E}\left[\sum_{i=1}^k A(\vec{N}^U, M_i)C_i^*\right] = \sum_{i=1}^k \mathbb{E}\left[A(\vec{N}^U, M_i)\right]C_i^* \geq \frac{N}{N+M-1}Circ^*.$$

Let P_{dyn}^* be the value of an optimum dynamic policy. We have finally:

$$\frac{N}{N+M-1}P_{Dyn}^* \leq \frac{N}{N+M-1}Circ^* \leq P_{Circ}^{\vec{N}^*}. \quad \square$$

4.3.3 Weak guaranties of performance

The complete demand graph case We consider the case of complete demand graph. With this property MAXIMUM CIRCULATION contains only one single strongly connected component. Therefore no vehicle distribution has to be specified.

Lemma 4. *For a complete demand graph, MAXIMUM CIRCULATION opens all stations and all trips. There is hence only one strongly connected component: $k = 1$ and $\mathcal{M}_1 = \mathcal{M}$.*

Proof. Assume there exists a MAXIMUM CIRCULATION λ with value $Circ^*$ with at least two strongly connected components \mathcal{M}_1 and \mathcal{M}_2 . Since the demand graph is complete, choose any $a \in \mathcal{M}_1$, $b \in \mathcal{M}_2$ we have $\Lambda_{a,b} > \lambda_{a,b} = 0$ and $\Lambda_{b,a} > \lambda_{b,a} = 0$. The vector

$$\lambda' = \left((\lambda_{c,d}, \forall (c,d) \in \mathcal{D} \setminus \{(a,b), (b,a)\}), \lambda'_{a,b} = \lambda'_{b,a} = \min \{ \Lambda_{a,b}, \Lambda_{b,a} \} \right)$$

is a circulation with a value $Circ'$ strictly better than the value of the supposed MAXIMUM CIRCULATION λ : $Circ' = Circ^* + 2 \min \{ \Lambda_{a,b}, \Lambda_{b,a} \}$. \square

Proposition 5. *For complete demand graphs, MAXIMUM CIRCULATION static policy is a tight $\frac{N}{N+M-1}$ -approximation on both static and dynamic optimal policies.*

Proof. Let $Circ^*$ be the optimal value of MAXIMUM CIRCULATION and P_{Circ^*} be the value of the static policy provided by MAXIMUM CIRCULATION. For a complete demand graph, MAXIMUM CIRCULATION policy has a single strongly connected component containing the M stations (Lemma 4). From Lemma 1 we have $P_{Circ^*} = \frac{N}{N+M-1} Circ^*$. As shown in Theorem 1, $Circ^*$ is an upper bound on the optimal dynamic policies of value $P_{D_{dyn}^*}$. Therefore, $P_{D_{dyn}^*} \leq Circ^*$ and

$$\frac{N}{N+M-1} P_{D_{dyn}^*} \leq \frac{N}{N+M-1} Circ^* = P_{Circ^*}.$$

The example of Proposition 3 can be extended to the complete demand graph: Consider a circuit with M stations and maximum demand $\{k, \dots, k, 1\}$. Complete the maximum demand vector replacing null demands by $\frac{1}{k}$. For any number $M \geq 2$ of stations and any number N of vehicles, the limit (as $k \rightarrow \infty$) of the ratio between the value of MAXIMUM CIRCULATION and the generous static policy is $\frac{N}{N+M-1}$. \square

A weaker guaranty of performance for general demand graphs

Theorem 3. *MAXIMUM CIRCULATION static policy together with its optimal vehicle distribution is a $\frac{N-M}{N+M}$ -approximation on optimal dynamic policies.*

Proof. Let P_{dyn^*} be the value of a optimal dynamic policy. Let $Circ^*$ be the optimal value of MAXIMUM CIRCULATION with k strongly connected components $\{\mathcal{M}_1, \dots, \mathcal{M}_k\}$. Component \mathcal{M}_i is composed with M_i stations and contributes to a value C_i^* in the MAXIMUM CIRCULATION: $\sum_{i=1}^k C_i^* = Circ^*$. If we allocate N_i vehicles to component \mathcal{M}_i , the expected transit is $\frac{N_i}{N_i+M_i-1} C_i^*$ (Lemma 3).

Let $P_{Circ}^{\vec{N}}$ be the value of the MAXIMUM CIRCULATION policy with a vehicle distribution \vec{N} among the k components. Let \vec{N}^* be the optimal vehicle distribution and let $\overrightarrow{N/M}$ be the vehicle distribution setting $\lfloor \frac{N}{M} M_i \rfloor$ vehicles in each component \mathcal{M}_i . We have:

$$P_{Circ}^{\vec{N}^*} \geq P_{Circ}^{\overrightarrow{N/M}} = \sum_{i=1}^k \frac{\lfloor \frac{N}{M} M_i \rfloor}{\lfloor \frac{N}{M} M_i \rfloor + M_i - 1} C_i^* \geq \sum_{i=1}^k \frac{\frac{N}{M} M_i - 1}{\frac{N}{M} M_i + M_i} C_i^* = \sum_{i=1}^k \frac{\frac{N}{M} - \frac{1}{M_i}}{\frac{N}{M} + 1} C_i^*.$$

$\forall 1 \leq i \leq k, M_i \geq 1$, hence:

$$P_{Circ}^{\vec{N}^*} \geq \frac{N - M}{N + M} \sum_{i=1}^k C_i^* = \frac{N - M}{N + M} C_{Circ}^*.$$

Since C_{Circ}^* is an upper bound on the optimal dynamic policy with value P_{dyn}^* (Theorem 1), we have:

$$P_{Circ}^{\vec{N}^*} \geq \frac{N - M}{N + M} P_{dyn}^*. \quad \square$$

5 Conclusion

We investigated an optimization/control problem of queuing networks, and used it to model regulation through pricing of vehicle sharing systems. Micro-economical and non-linearity issues related to the elasticity of demand can be avoided for some objectives, including the maximization of the number of trips sold. We proposed a heuristic combining MAXIMUM CIRCULATION and a greedy algorithm and studied its performance ratio for the transit maximization. Evidences for the validity of the conjectured tight bound $\frac{N}{N+M-1}$ of this ratio are provided.

Several extensions are natural for this work. We believe that adding transportation times has a minor impact on our results. Moreover, since circulation policies spread vehicles very well among the stations, adding capacities to the stations may still allow these policies to be efficient.

On the other hand, demands that are not stationary over time (such as house-work commute) usually do not benefit from naive steady-state goals: stations in residential areas are better off being full in mornings and empty after work. However, MAXIMUM CIRCULATION heuristics can be generalized to optimize over non-stationary demands, as discussed in [Waserhole and Jost \(2013\)](#), although no guaranty of performance is provided.

Nevertheless, in dense networks of stations such as Vélib's Paris, some users have flexibilities in their origin and destination stations. The classical (BCMP) queuing network results fall apart under such generalization. Different theoretical tools might be required. Numerical analysis through simulations requires data on the demand. However, the demand is hard to estimate since available data only relate the trips sold and not unsatisfied users.

A Toward computing optimal policies

In this appendix, we discuss structures of optimal policies in order to develop tractable stochastic models to optimize a VSS through pricing. We discuss in Section [A.2](#) the problem of characterizing dynamic optimal policies and in Section [A.3](#) the problem of characterizing static ones. Simple classes of policies, easier to optimize, are shown suboptimal.

A.1 Markov Decision Process – The curse of dimensionality

Computing optimal dynamic policies The continuous-time Markov chain formulation of the VSS stochastic evaluation model leads directly to a Markov Decision Process (MDP), named the *VSS MDP model*. This model considers, in each state $s \in \mathcal{S}$, a set \mathcal{Q} of discrete prices for each possible trip. Solving the VSS MDP model computes the optimal dynamic discrete pricing policy.

MDPs are known to be polynomially solvable in the number of states $|\mathcal{S}|$ and actions $|\mathcal{A}|$ available in each state. To solve an MDP, efficient solution methods exist such as value iteration, policy iteration algorithm or linear programming; see [Puterman \(1994\)](#) textbook. In each state $s \in \mathcal{S}$, the VSS MDP model’s action space $\mathcal{A}(s)$ is the Cartesian product of the available prices for each trip, *i.e.* $\mathcal{A}(s) = \mathcal{Q}^{|\mathcal{M}|^2}$. The action space size is then exponential in the number of stations. However, to avoid suffering from this explosion, we can model this problem as an action decomposable Markov decision process; see [\(Waserhole et al., 2013a\)](#). Thanks to this general framework, based on the event-based dynamic programming ([Koole, 1998](#)), the complexity of solving the VSS MDP model becomes polynomial in $|\mathcal{S}|$ and $|\mathcal{Q}||\mathcal{M}|^2$ (that is far less than $|\mathcal{Q}|^{|\mathcal{M}|^2}$). Nevertheless, the VSS MDP model has another problem: the explosion of its state space \mathcal{S} with the number of vehicles and stations. This phenomenon is known as the *curse of dimensionality* ([Bellman, 1953](#)).

A.2 Structures of optimal dynamic policies

Recall that *Dynamic policies* have prices to take a trip that depend on the state of the system, *i.e.* the vehicle distribution. Unfortunately, even with homogeneous demand ($\Lambda_{a,b} = \Lambda$) optimal dynamic policies seem hard to describe.

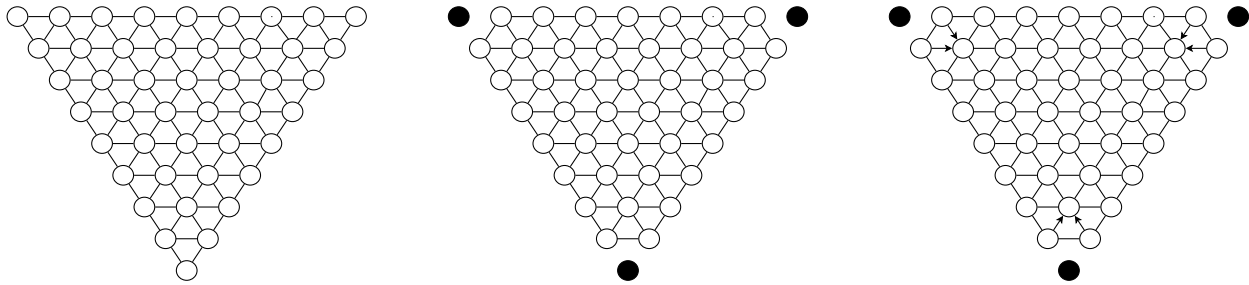
Since the number of states is exponential, we would like to restrict to dynamic policies allowing a compact description. *Capacity policies* amount to specifying a virtual station capacity \mathcal{K} , and to accept a trip from station a to station b if only if the number of vehicles in b is not exceeding \mathcal{K}_b .

We show in the next proposition that capacity policies are suboptimal among dynamic policies for the VSS stochastic pricing optimization problem.

Proposition 6. *Capacities policies are suboptimal among dynamic policies, even in homogeneous cities.*

Proof. Figure 5 compares the induced Markov chain (state graph) of three policies in an homogeneous city ($\Lambda = 1$) with 3 stations and 8 vehicles. An edge represents that the trip is open to its maximum in both directions, an arc indicates that it is open only in one way. Figure 5a represents the generous policy opening all trips and expects to sell 4.8 trips per time unit. Figure 5b represents the optimal dynamic capacity policy and increases the gain to ≈ 4.857 . Finally, the optimal dynamic policy is represented in Figure 5c, and increases the number of trips sold to ≈ 4.865 . \square

Figure 5 shows that using dynamic pricing policies can increase the number of trips sold by the system even in homogeneous cities (perfectly balanced). Figure 6 represents the optimal dynamic policies in an homogeneous cities with 3 stations when the number of vehicles increases:



(a) Policy opening all trips, value 4.8. (b) Optimal dynamic capacity policy, value ≈ 4.857 . (c) Optimal dynamic policy, value ≈ 4.865 .

Figure 5: Induced Markov chain of 3 policies evaluated in an homogeneous city with 8 vehicles and 3 stations. Legend: (\circ) reachable state; (\bullet) unreachable state; ($-$) trip between two states open in both directions; (\rightarrow) trip open in only one direction.

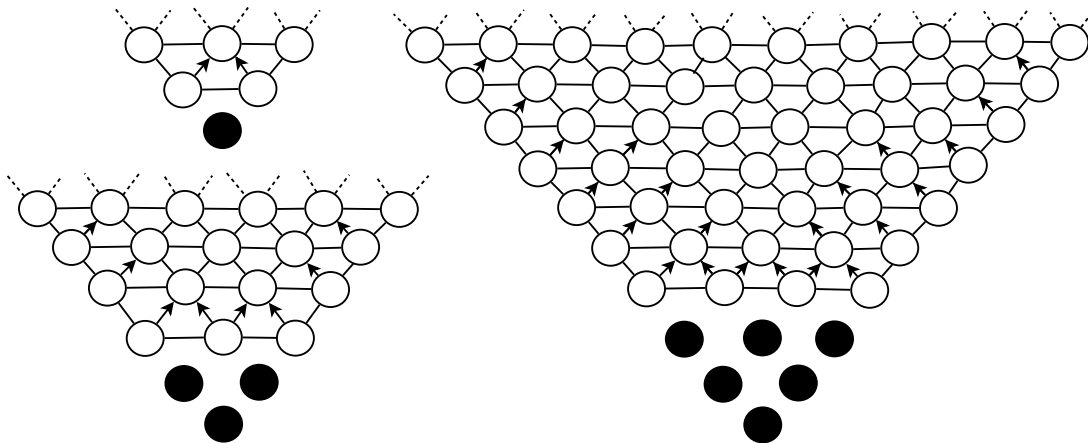


Figure 6: “Spikes” of optimal dynamic policies’ state graph for an homogeneous city with 3 stations and $N=8, 14$ or 30 vehicles.

from 8 vehicles (as in Figure 5b), to 14 and 30 vehicles. Only the “spikes” of the dynamic policies’ induced Markov chain are represented since, the solution is invariant under the group S_3 of permutation of the stations. These solutions are the unique optimum. The optimal dynamic policy is solved with the VSS (decomposed) MDP model. This model is of exponential size in N and $|\mathcal{M}|$ but still solvable for the size of these 3 instances. The solution uniqueness has been checked greedily solving several decomposed MDPs. It seems hard to find a compact description of optimal solutions in general.

A.3 Suboptimal classes of static policies

A.3.1 Generous policies / No regulation

When investigating (pricing) policies, the most important practical issue is the trade-off between the simplicity (and in particular, the readability for users) and the performance. The first practical question might always be whether “unoptimized” policies perform well.

The (static) *generous* policy sets all demands to their maximum value ($\lambda = \Lambda$). To the best of our understanding, the generous policy is the most natural and relevant to compare with in theoretical studies, as long as the objective function is in terms of service quality and not in terms of monetary gain.

In Proposition 7, provides an example in which the number of trips sold by the generous policy can be arbitrarily far from an optimal static policy. It contains a “gravitational” phenomenon, which occurs in particular for bike sharing systems in non-flat cities.

Proposition 7. *The ratio between the number of trips sold by the (static) generous policy ($\lambda = \Lambda$) and the static optimal policy is unbounded.*

Proof. Consider a complete demand graph where all trip maximum demands are equal to 1 except the trips from a special station $z \in \mathcal{M}$ to any other station that are worth L^{-1} : $\Lambda_{a,b} = 1$, $\Lambda_{z,a} = 1$, $\forall a \in \mathcal{M}$, $\forall b \in \mathcal{M} \setminus \{z\}$.

For any number of vehicle, when $L \rightarrow \infty$ the expected number of trips sold $T(G)$ for the generous policy G tends to 0: The stationary distribution for one vehicle is $\pi_a = \frac{1}{L+M-1}$, $\forall a \in \mathcal{M} \setminus \{z\}$ and $\pi_z = \frac{L}{L+M-1}$, hence $\lim_{L \rightarrow \infty} \pi_a = 0$, $\forall a \in \mathcal{M} \setminus \{z\}$ and $\pi_z = 1$. Since for all N , the availability vector A satisfies $A = \alpha_N \pi$ for some scalar α_N , we have:

$$\forall N \geq 1, \quad \lim_{L \rightarrow \infty} A_a = 0, \quad \forall a \in \mathcal{M} \setminus \{z\} \quad \text{and} \quad \lim_{L \rightarrow \infty} A_z = 1,$$

hence

$$\forall N \geq 1, \quad T(G) = \sum_{a \in \mathcal{M}} A_a (M-1) + A_z L^{-1} (M-1) \quad \Rightarrow \quad \lim_{L \rightarrow \infty} T(G) = 0.$$

On the other hand, the static circulation policy C closing only trips to and from station a has a expected number of trips sold $T(C) > 1$ that is independent of L :

$$\forall L > 0, \quad \forall N \geq 1, \quad A_b = \frac{N}{N+M-2}, \quad \forall b \in \mathcal{M} \setminus \{a\} \quad \text{and} \quad A_a = 0,$$

hence independently of L , and for all $N \geq 1$ and $M \geq 3$

$$T(C) = \sum_{a \in \mathcal{M} \setminus \{z\}} A_a (M-2) = \frac{N(M-1)(M-2)}{N+M-2} \geq 1. \quad \square$$

A.3.2 Bang bang policies

Static policies directly have a compact representation: only one price per trip needs to be set, independently of the system’s state.

However, a compact formulation does not directly lead to a polynomial optimization. When considering only two possible prices per trip, a brute force solution method still needs $2^{|\mathcal{M}|^2}$ calls to the stochastic evaluation model. We need to exhibit structures to design efficient algorithms.

With the continuous demand assumption, static policies optimization amounts to setting the user arrival rates λ with $0 \leq \lambda_{a,b} \leq \Lambda_{a,b}$, $\forall (a,b) \in \mathcal{D}$. We investigate *bang-bang policies* (all or nothing) that set each trip $(a,b) \in \mathcal{D}$ to be either open ($\lambda_{a,b} = \Lambda_{a,b}$), or closed ($\lambda_{a,b} = 0$).

One can wonder if bang-bang policies are dominant for the transit maximization. It is true for dynamic policies: bang-bang dynamic policies optimization can be reduced to a discrete price dynamic policies optimization in which deterministic policies are dominant (classic MDP results (Puterman, 1994)). Nevertheless, we show that bang-bang policies are not dominant among static policies even (which is more surprising) when the number of vehicles tends to infinity.

Proposition 8. *Bang-bang policies are suboptimal among static policies even when the number of vehicles tends to infinity.*

Proof. Figure 7 exhibits a counter example with 4 stations (a, b, c, d) and maximum trip demands $\Lambda_{a,b} = \Lambda_{b,c} = 3$, $\Lambda_{c,d} = \Lambda_{d,a} = \Lambda_{c,a} = 2$, all others are equal to 0. There are only 2 bang-bang static policies λ defining a strongly connected demand graph: $\lambda_{i,j} = \Lambda_{i,j}$, $(i, j) \neq (c, a)$ and either $\lambda_{c,a} = 0$ or $\lambda_{c,a} = 2$. When the number of vehicles tends to infinity, the availability of a vehicle at station a equals $\frac{\pi_a}{\max_{b \in \mathcal{M}} \pi_b}$, where π is the stationary distribution for one vehicle (George and Xia, 2011). For the $\lambda_{c,a} = 0$ policy, we have $\pi_a = \pi_b = \frac{2}{10}$ and $\pi_c = \pi_d = \frac{3}{10} = \pi_{\max}$, so the expected transit when $N \rightarrow \infty$ is worth $\frac{\pi_a}{\pi_{\max}}(3 + 3) + \frac{\pi_c}{\pi_{\max}}(2 + 2) = 8$. For the $\lambda_{c,a} = 2$, policy we have $\pi_a = \pi_b = \frac{4}{14}$ and $\pi_c = \pi_d = \frac{3}{14}$, so the expected transit when $N \rightarrow \infty$ is worth 10.5 which is thus the optimal bang-bang static policy. Yet, for the non bang-bang policy with $\lambda_{c,a} = 1$ and still $\lambda_{i,j} = \Lambda_{i,j}$, $(i, j) \neq (c, a)$, we have $\pi_a = \pi_b = \pi_c = \pi_d = \frac{1}{4}$, so the expected transit when $N \rightarrow \infty$ is worth $11 > 10.5$. Hence, bang-bang policies are suboptimal even when the number of vehicles tends to infinity. \square

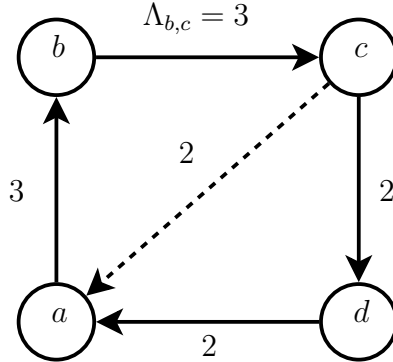


Figure 7: Bang-bang policies are suboptimal even when the number of vehicles tends to infinity.

A.3.3 Single component policies

One may wonder whether it is useful to have a policy dividing the city. Notice that when considering static pricing policies with more than one strongly connected component, one should explicitly consider the vehicle distribution among these components. In fact, dividing the city sometimes lead to better performances: It is a leverage to prevent the system from being unprofitable (unbalanced) states.

Proposition 9. *Static policies with one single strongly connected component are suboptimal among static policies.*

Proof. An example is schemed Figure 8 with 4 stations and a symmetric demand matrix. For two vehicles, the optimal static policies in this case is to close the trips (b, c) and (c, b) and open all other trips to their maximum value, *i.e.* $\lambda = \Lambda$ except $\lambda_{b,c} = \lambda_{c,b} = 0$. The demand graph of this policy has two strongly connected components. The optimal vehicle distribution is to put one vehicle on each of them. With such distribution it expects to sell 200 trips per time unit. The optimal static policy with a single strongly connected component opens all trips to their maximum value, $\lambda = \Lambda$. It expects to sell 160.8 trips per time unit. \square

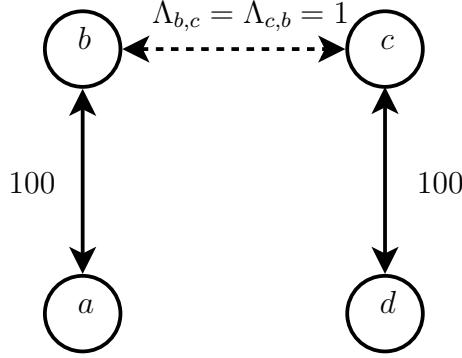


Figure 8: Static policies with a single strongly connected component are suboptimal.

A.3.4 Circulation policies

A static policy ϕ is said *USD* if $\text{CTMC}(\phi)$ has a Uniform Stationary Distribution over its states, *i.e.* $\pi_s = \frac{1}{|\mathcal{S}|}, \forall s \in \mathcal{S}$. The broader class of static policies where the USD property holds are the static circulation policies. Static circulation policies are suboptimal among static policies.

Lemma 5. *A static policy is USD if and only if it is a circulation.*

Proof. First, we prove that static circulation policies are USD. A static circulation policy λ satisfies: $\lambda_{a,b}^s = \lambda_{a,b}, \forall s - e_a + e_b \in \mathcal{S}$, and $\sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda_{b,a}, \forall a \in \mathcal{M}$. Therefore, the stationary distribution π of policy λ satisfies:

$$\sum_{\substack{(a,b) \in \mathcal{D} \\ s - e_a + e_b \in \mathcal{S}}} \pi_s \lambda_{a,b}^s = \sum_{\substack{(b,a) \in \mathcal{D}, s' \in \mathcal{S} \\ s' - e_b + e_a = s}} \pi_{s'} \lambda_{b,a}^s, \quad \forall s \in \mathcal{S},$$

and $\pi_s = \frac{1}{|\mathcal{S}|}, \forall s \in \mathcal{S}$, is the unique solution of this system together with $\sum_{s \in \mathcal{S}} \pi_s = 1$, since its rank is $|\mathcal{S}|$ which is the number of variables.

Secondly, we prove that any static USD policy λ is a circulation. The stationary distribution π of static USD policy λ satisfies $\pi_s = \frac{1}{|\mathcal{S}|}, \forall s \in \mathcal{S}$, and then also:

$$\sum_{\substack{(a,b) \in \mathcal{D} \\ s - e_a + e_b \in \mathcal{S}}} \lambda_{a,b}^s = \sum_{\substack{(b,a) \in \mathcal{D}, s' \in \mathcal{S} \\ s' - e_b + e_a = s}} \lambda_{b,a}^s, \quad \forall s \in \mathcal{S}, \quad \text{and} \quad \lambda_{a,b}^s = \lambda_{a,b}, \quad \forall s - e_a + e_b \in \mathcal{S}.$$

The unique solution of this system is $\sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda_{b,a}, \forall a \in \mathcal{M}$. \square

Proposition 10. *Circulation policies are suboptimal among static policies.*

Proof. Consider two stations a and b with maximum demand $\Lambda_{a,b} = 99$ and $\Lambda_{b,a} = 1$. For one vehicle, the optimal static circulation policy ($\lambda = 1$) expects to sell 1 trip per time unit while the optimal static policy ($\lambda = \Lambda$) expects to sell 1.98 trips per time unit. \square

References

- Autolib'. <http://www.autolib-paris.fr>, Paris, France, 2011.
- F. Baskett, K.M. Chandy, R.R. Muntz, and F. Palacios-Gomez. Open, closed, and mixed networks of queues with different classes of customers. *Journal of the Association for Computing Machinery*, 22, 1975.
- R. Bellman. Bottleneck problems and dynamic programming. *Proceedings of the National Academy of Sciences of the United States of America*, 39(9):947, 1953.
- Car2go. <http://www.car2go.com>, Ulm, Germany – Vancouver, Canada – San Diego, USA – ..., 2008.
- D. Chemla, F. Meunier, T. Pradeau, R. Wolfer Calvo, and H. Yahiaoui. Self-service bike sharing systems: Simulation, repositioning, pricing. 2013.
- E. Côme. Model-based clustering for BSS usage mining: a case study with the vélib' system of paris. In *International workshop on spatio-temporal data mining for a better understanding of people mobility: The Bicycle Sharing System (BSS) case study. Dec 2012*, 2012.
- C. Contardo, C. Morency, and L-M. Rousseau. Balancing a dynamic public bike-sharing system. Technical Report 09, CIRRELT, 2012.
- P. DeMaio. Bike-sharing: History, impacts, models of provision, and future. *Journal of Public Transportation*, 12(4):41–56, 2009.
- J. Edmonds and R.M. Karp. Theoretical improvements in algorithmic efficiency for network flow problems. *Journal of the ACM (Association for Computing Machinery)*, 19(2):248–264, 1972.
- C. Fricker and N. Gast. Incentives and regulations in bike-sharing systems with stations of finite capacity. *arXiv :1201.1178v1*, January 2012.
- C. Fricker, N. Gast, and H. Mohamed. Mean field analysis for inhomogeneous bike sharing systems. In *23rd Intern. Meeting on Probabilistic, Combinatorial, and Asymptotic Methods for the Analysis of Algorithms (AofA'12)*, 2012.
- D. K. George and C. H. Xia. Fleet-sizing and service availability for a vehicle rental system via closed queueing networks. *European Journal of Operational Research*, 211(1):198 – 207, 2011.

- N.I. Glebov. About one class of convex integer programs. *Upravlyaemye Sistemy, Vol. 11, in: Institute of Mathematics of Siberian Branch of Academy of Sciences of the USSR, Novosibirsk*, pages 38–42 (in Russian), 1973.
- G. M. Koole. Structural results for the control of queueing systems using event-based dynamic programming. *Queueing Systems*, 1998.
- J.-R. Lin and T.-H. Yang. Strategic design of public bicycle sharing systems with service level constraints. *Transportation research part E: logistics and transportation review*, 47(2):284–294, 2011.
- J. Pfrommer, J. Warrington, G. Schildbach, and M. Morari. Dynamic vehicle redistribution and online price incentives in shared mobility systems. *arXiv preprint arXiv:1304.3949*, 2013.
- M. L. Puterman. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley and Sons, New York, NY, 1994.
- T. Raviv, M. Tzur, and I. A. Forma. Static repositioning in a bike-sharing system: models and solution approaches. *EURO Journal on Transportation and Logistics*, 2(3):187–229, 2013.
- VV Shenmaier. A greedy algorithm for some classes of integer programs. *Discrete applied mathematics*, 133(1):93–101, 2003.
- D.C. Shoup. *The High Cost of Free-Parking*. Planners Press, Chicago, 2005.
- J. Shu, M. Chou, O. Liu, C.P Teo, and I-L Wang. Bicycle-sharing system: Deployment, utilization and the value of re-distribution. unpublished paper. 2010. URL <http://bschool.nus.edu/Staff/bizteocp/BS2010.pdf>.
- Vélib'. <http://www.velib.paris.fr>, Paris, France, 2007.
- A. Wasserhole. *Vehicle Sharing Systems Pricing Optimization*. PhD thesis, Université Joseph Fourier, Grenoble, France, 2013.
- A. Wasserhole and V. Jost. Vehicle sharing system pricing regulation: A fluid approximation. 2013. URL <http://hal.archives-ouvertes.fr/hal-00727041>.
- A. Wasserhole, J. P. Gayon, and V. Jost. Linear programming formulations for queueing control problems with action decomposability. 2013a. URL <http://hal.archives-ouvertes.fr/hal-00727039>.
- A. Wasserhole, V. Jost, and N. Brauner. Vehicle sharing system optimization: Scenario-based approach. 2013b. URL <http://hal.archives-ouvertes.fr/hal-00727040>.