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Vehicle Sharing System Pricing Regulation: Transit Optimization of Intractable Queuing Network

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Abstract

This paper presents polynomially solvable stochastic models to tackle the Vehicle Sharing System (VSS) pricing problem. We focus on optimizing the average number of trips sold with static demands, infinite station capacities and null transportation times. The stochastic VSS pricing problem is modeled as a closed queuing network with infinite buffers and continuous controls on transition rates for the pricing. This model is intractable for an explicit dynamic pricing optimization through the Markovian decision process framework. We study how to optimize on sub-classes of static policies, named symmetric and conservative, that have a special property: the uniform stationary distribution. For the symmetric policy optimization, we give a linear programming formulation with a size polynomial in the number of stations. We extend this formulation to provide a relaxation to the conservative policies optimization giving extremely often the optimum solution. Finally, an approximation as a simple MAXIMUM CIRCULATION problem is given and shown asymptotically optimal when the number of vehicles available tends to infinity.

1 Introduction

1.1 Context

[Shoup \(2005\)](#) reports that, based on a sample of 22 US studies, car drivers looking for a parking spot contribute to 30% of the city traffic. Moreover cars are used less than 2 hours per day on average but still occupy a parking spot the rest of the time! Could we have less vehicles and satisfy the same demand level?

Recently, the interest in Vehicle Sharing Systems (VSS) in cities has increased significantly. Indeed, urban policies intend to discourage citizens to use their personal car downtown by reducing the number of parking spots, street width, etc. VSS seem to be a promising solution to reduce jointly traffic and parking congestion, noise, and air pollution (proposing bikes or electric

cars). They offer personal mobility allowing users to pay only for the usage (sharing the cost of ownership).

We are interested in short-term one-way VSS where vehicles can be taken and returned at different places (paying by the minute). Associated with classic public transportation systems, short-term one-way VSS help to solve one of the most difficult public transit network design problem: the last kilometer issue (DeMaio, 2009). Round-trip VSS, where vehicles have to be returned at the station where they were taken, cannot address this important issue.

The first large-scale short-term one-way VSS was the bicycle VSS *Vélib'* (2007). It was implemented in Paris in 2007 and now has more than 1200 stations and 20 000 bikes selling around 110 000 trips per day. It has inspired several other cities all around the world; Now more than 300 cities have such a system, including Montréal, Beijing, Barcelona, Mexico City, Tel Aviv (DeMaio, 2009).

1.2 One-way Vehicle Sharing Systems: a management issue

One way systems increase the user freedom at the expense of a higher management complexity. In round trip rental systems, while managing the yield, the only stock that is relevant is the number of available vehicles. In one-way systems, vehicles are not the only key resource anymore: parking stations may have limited number of spots and the available parking spots become an important control leverage.

Since first bicycle VSS, problems of bikes and parking spots availability have appeared very often. Reasons are various but we can highlight two important phenomenons: the gravitational effect which indicates that a station is constantly empty or full (as Montmartre hill in *Vélib'* (2007)), and the tide phenomenon representing the oscillation of demand intensity along the day (as morning and evening flows between working and residential areas).

To improve the efficiency of the system, in the literature, different perspectives are studied. At a strategic level, some authors consider the optimal capacity and locations of stations. Shu et al. (2010) propose a stochastic network flow model to support these decisions. They use their model to design a bicycle VSS in Singapore based on demand forecast derived from current usage of the mass transit system. Lin and Ta-Hui (2011) consider a similar problem but formulate it as a deterministic mathematical model.

At a tactical level, other authors investigate the optimal number of vehicles given a set of stations. George and Xia (2011) study the fleet sizing problem with constant demand and no parking capacity. Fricker and Gast (2012), Fricker et al. (2012) consider the optimal sizing of a fleet in “toy” cities, where demand is constant over time and identical for every possible trip, and all stations have the same capacity \mathcal{K} . They show that even with an optimal fleet sizing in the most “perfect” city, if there is no operational system management, there is at least a probability of $\frac{2}{\mathcal{K}+1}$ that any given station is empty or full.

At an operational level, in order to be able to meet the demand with a reasonable standard of quality, in most bicycle VSS, trucks are used to balance the bikes among the stations. The

objective is to minimize the number of users who cannot be served, *i.e.*, the number of users who try to take a bike from an empty station or to return it to a full station. The problem is to schedule truck routes to visit stations performing pickup and delivery. In the literature many papers deal already with this problem. A static version of the bicycle VSS balancing problem is treated in [Chemla et al. \(2012\)](#) and a dynamic one in [Contardo et al. \(2012\)](#).

1.3 A study on leverage for self regulated VSS

A new type of VSS has appeared lately: one-way Car VSS with [Autolib'](#) (2011) in Paris and [Car2go](#) (2008) in more than 10 cities (Vancouver, San Diego, Lyon, Ulm...). Due to the size of cars, operational balancing optimization through relocation with trucks seems inappropriate. Another way for optimizing the system has to be found.

This study is part of a work investigating different optimization leverage for self regulation in VSS. Using operation research we want to estimate the potential impact of:

- Optimizing the system design (station capacity, fleet size);
- Using pricing techniques to influence user choices in order to drive the system towards its most efficient dynamic;
- Establishing new protocols, for instance with parking spot reservations and/or users spatial and temporal flexibility.

1.4 Assumption and goals of this study

A VSS pricing model is proposed in [Waserhole and Jost \(2012\)](#) using Markov Decision Processes (MDP). The problem with this model is that the number of states of the MDP grows exponentially with the size of the instance: number of vehicles and stations. They give a fluid approximation that computes static policies. This fluid model is polynomially solvable and gives descent results in a reasonable time. However, it is based on a deterministic approximation and the experimental results are pretty far from the upper bound. That's why in this paper, we want to consider a stochastic model but still polynomial in the size of the instance. To be able to obtain results we restrain this study to a simple model, as in [George and Xia \(2011\)](#), with a constant demand and infinite station capacities, but also with null transportation times. We optimize the average number of trips sold by the system (average throughput). Prices are the leverage, with an elastic surjective demand on price. In fact, we assume a maximum possible demand Λ and a price to obtain any demand $\lambda \in [0, \Lambda]$. Finally, to build a stochastic model polynomially solvable we only tackle two policy subclasses, the symmetric and conservative policies defined in the following. We also study an asymptotic approximation based on the MAXIMUM CIRCULATION.

2 Framework

In this section, we define formally the framework we use to model a Stochastic VSS.

2.1 Restriction to a simple protocol

In a real context, a user wants to use a vehicle to take a trip between an original (GPS) location a , and a final one b , during a specified time frame. On a station based VSS, he tries to find the closest station to location a with a vehicle to take and the closest station to location b with a parking spot to retrieve it. All along this process user's decision relies on several correlated inputs such as: trip total price, walking distance, public transportation competition, time frame...

A time elastic GPS to GPS demand forecast, correlated to a user's decision protocol ruling his behaviour to take a trip between two specific stations at a specific time, seems closer to reality but introduces of course a big complexity (use of utility function for instance). This is why in this study, to stay simple and be able to develop a compact model we are going to consider a station to station demand forecast, with moreover instantaneous transportation time and infinite station capacity. Finally, it amounts to consider a demand for the following simplified protocol:

1. A user asks for a vehicle at station a (here and now), with destination b ;
2. The system offers a price (or rejects the user = infinite price);
3. The user accepts the price, pay and the vehicle is transferred, or leaves the system.

2.2 A Vehicle Sharing System Stochastic Model

2.2.1 Markovian framework

In a city, there is a fleet of N vehicles along a set \mathcal{M} ($|\mathcal{M}| = M$) of stations with infinite capacity. There is an elastic demand between each station $\mathcal{D} = \mathcal{M} \times \mathcal{M}$. This demand is constant and follows a Poisson distribution of parameter $\lambda_{a,b}(p_{a,b})$ to go from station $a \in \mathcal{M}$ to station $b \in \mathcal{M}$ and is function of the proposed price $p_{a,b}$.

2.2.2 Closed queuing network model

For a given demand $\lambda_{a,b}$ for every trip $(a, b) \in \mathcal{D}$ (*i.e.* for a fixed price $p_{a,b}$), we model this stochastic VSS by a closed queuing Network see figure 1.

Each demand $(a, b) \in \mathcal{D}$ is represented by a server $(a - b)$ which has a time dependent service rate equals to the average number of clients willing to take a trip from station a to station b : $\lambda_{a,b}$. Demands with same station of origin a ($(a, b) \in \mathcal{D}, \forall b \in \mathcal{M}$) are sharing the same infinite

buffer a . When a vehicle (a job) is picked up to take the trip (a, b) (is processed by server (a, b)) it is transferred directly in station (buffer) b .

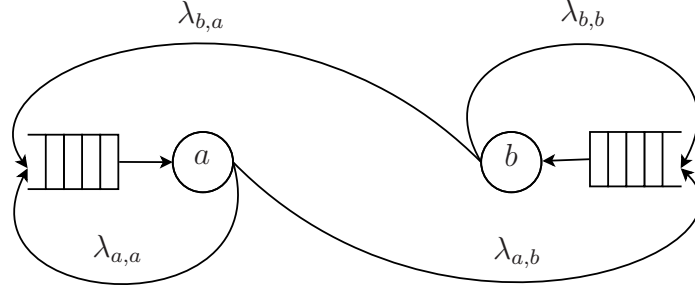


Figure 1: A closed queuing network model with servers for demands and transportation times.

2.2.3 Continuous-time Markov chain formulation

If a price $p_{a,b}$ is set for all trips $(a, b) \in \mathcal{D}$ we can model the closed queuing network by a continuous-time Markov chain on set of states \mathcal{S} :

$$\mathcal{S} = \left\{ \left(n_a : a \in \mathcal{M} \right) / \sum_{a \in \mathcal{M}} n_a = N \right\}.$$

A state $s = (n_a : a \in \mathcal{M})$ represents the vehicles distribution in the city space, n_a is the number of vehicles in station $a \in \mathcal{M}$. There is a transition rate $\lambda_{a,b}(p_{a,b})$ between state $(\dots, n_a, \dots, n_b, \dots)$ and state $(\dots, n_a - 1, \dots, n_b + 1, \dots)$.

We can note that there is an exponential number of states. For instance for a small system with $N = 150$ vehicles and $M = 50$ stations there are $\binom{N+M-1}{N} \simeq 10^{47}$ states!

2.3 Model optimization

2.3.1 Stochastic VSS pricing problem

In the previous section we model a VSS system by a closed queuing network that can be described explicitly through a continuous-time Markov chain. Using this model we want to maximize the VSS average revenue. We use for leverage the possibility to change the price to take a trip which will, assuming an elastic demand, influence the demand for such trip. We call this problem the *stochastic VSS pricing* problem. It amounts to setting a price to take every trip in order to maximize the gain of the VSS Markovian model.

Prices can be *discrete*, *i.e.* selected in a set of possibilities, or *continuous* *i.e.* chosen in a range. Pricing policies can be *dynamic*, *i.e.* dependent on system's state (vehicles distribution), or *static* *i.e.* independent on system's state, set in advance and only function of the trip.

2.3.2 Markov Decision Process curse of dimensionality

From the continuous-time Markov chain given in the previous section, we can use the well known Markov Decision Process (MDP) framework to formulate a VSS MDP model. There is a set \mathcal{Q} of possible discrete prices for each trip at each time step. A trip $(a, b) \in \mathcal{D}$ at price $p_{a,b}^q$, $q \in \mathcal{Q}$ has a demand (transition rate) $\lambda_{a,b}(p_{a,b}^q) = \lambda_{a,b}^q$.

Solving this MDP computes the best dynamic system state dependent discrete pricing policy, *i.e.* the price for a trip depends of the current state of the system (vehicles distribution). MDPs are known to be polynomially solvable in the number of states $|\mathcal{S}|$ and the number of actions $|\mathcal{A}|$ available in each state. There exists efficient solutions methods such as value iteration, policy iteration algorithm or linear programming, we refer to [Puterman \(1994\)](#) textbook.

The VSS MDP model is a pricing problem where the action space $\mathcal{A}(s)$ in each state $s \in \mathcal{S}$ is the Cartesian product of the available prices for each trip, *i.e.* $\mathcal{A}(s) = \mathcal{Q}^M$. However, to avoid suffering from this exponential explosion, we can model this problem as an Action Decomposable Markov Decision Process ([Waserhole et al., 2012b](#)). It is a general method based on the event-based dynamic programming ([Koole, 1998](#)) to reduce the complexity of the action space to $\mathcal{A}(s) = \mathcal{Q} \times M$.

However, there is another problem with the VSS MDP model, the explosion of the state space with the number of vehicles and stations. This phenomenon is known as the *curse of dimensionality*. We have then to look at approximations or simplifications to produce solutions in a reasonable time.

2.3.3 State of the art on this model

In the VSS literature, only simple forms of this closed queuing network model with the relationship to the underlying continuous-time Markov chain have been studied. [George and Xia \(2011\)](#) consider a VSS with only one time step, one price and infinite station capacities. Under these assumptions, they establish a compact form to compute the system performance using the BCMP network theory ([Baskett et al., 1975](#)). They solve an optimal fleet sizing problem considering a cost to maintain a vehicle and a gain to rent it.

[Fricker and Gast \(2012\)](#) consider simple cities that they call homogeneous. These cities have a unique fixed station capacity ($\mathcal{K}_a = \mathcal{K}$), a constant (one time step) arrival rate and uniform routing matrix ($\lambda_{a,b}^t = \frac{\lambda}{M}$) and a unique travel time ($\mu_{a,b}^t = \mu^{-1}$). With a mean field approximation, they obtain some asymptotic results when the number of stations tends to infinity ($M \rightarrow \infty$): if there is no operational regulation system, the optimal sizing is to have a fleet of $\frac{\mathcal{K}}{2} + \frac{\lambda}{\mu}$ vehicles per station which corresponds in filling half of the stations plus the average number of vehicles in transit ($\frac{\lambda}{\mu}$). Moreover, they show that even with an optimal fleet sizing, each station has still a probability $\frac{1}{\mathcal{K}+1}$ to be empty or full which is pretty bad since this cities are perfectly balanced. In another paper, [Fricker et al. \(2012\)](#) extend to inhomogeneous cities modeled by clusters some analytical results and verify experimentally some others.

For homogeneous cities, [Fricker and Gast \(2012\)](#) also study a heuristic using incentives called “the power of two choices” that can be seen as a dynamic pricing. When a user arrives at a station to take a vehicle, he gives randomly two possible destination stations and the system is directing him to the least loaded one. They show that this policy allows to drastically reduce the probability to be empty or full for each station to $2^{-\frac{\kappa}{2}}$.

[Waserhole and Jost \(2012\)](#) propose a stochastic VSS pricing model considering station capacity and time dependent demand. They give a fluid approximation, a deterministic solution technique, that has for best advantage to be able to deal with the tide phenomenon and to provide an upper bound on optimization gap. Through some experimentation, they show that the fluid solution gives interesting results but nevertheless has a simulation value pretty far from an upper bound. They raise the question if weather or not this was due do the deterministic relaxation or to the poor upper bound their algorithm is giving. This is where this work stands, it intends to give a tractable stochastic solution techniques for the VSS pricing problem.

3 VSS stochastic optimization

We develop now tractable stochastic models to optimize a VSS through pricing. We want to find compact forms to evaluate the system performance. In [George and Xia \(2011\)](#) they derive analytic results to compute the average throughput of the system for fixed prices in order to compute the best fleet sizing. However, to consider prices as controls the same approach doesn't seem to be working.

For general classes of pricing policies optimizing the average throughput seems rather hard, we can note that the deterministic version of the pricing problem is even shown NP-hard in [Waserhole et al. \(2012a\)](#). Nevertheless, if we restrain our study to special types of policy (symmetric and conservative) we can use a special property of the induced Markov chain (the uniform distribution over all states) to obtain a compact formulation for the average throughput of the system usable for a polynomial price optimization.

3.1 On optimizing the system

We want to compute a policy maximizing the average throughput of a system for large scale (real) systems. Thus, we look at polynomial optimization and hence policies allowing compact formulations. *Dynamic policies* can have prices to take a trip that depend of the state of the system, the vehicles distribution. However, in our model the number of states is exponential, therefore to be able to formulate dynamic policies in a tractable way we would need to have a compact formulation using threshold for instance. This problem is not present for *static policies* where only one price per trip need to be set independently of the system's state. With our assumption it amounts in setting customer arrival rates λ with $\lambda_{a,b} \leq \Lambda_{a,b}, \forall (a,b) \in \mathcal{D}$.

To simplify even more, we investigate also *pure policies* that only set for each trip $(a,b) \in \mathcal{D}$ if it is either open ($\lambda_{a,b} = \Lambda_{a,b}$), or close ($\lambda_{a,b} = 0$).

One can wonder if pure policies are not dominant when maximizing the average throughput. What is shown in the following is that it is true for dynamic policies (see lemma 1) but false for dynamic policies even (which is more surprising) when the number of vehicles tends to infinity (see lemma 2)

Lemma 1 *Pure policies are dominant among dynamic policies maximizing the average flow (number of trips sold).*

Proof: On a state graph, the following LP gives the best static policies with continuous actions, *i.e.* the best price to take each trip in each state of the system.

Let \mathcal{D}_s be the set of trips available in state $s \in \mathcal{S}$, the LP computing the dynamic policy maximizing the throughput can be written as:

$$\begin{aligned}
\max \quad & \sum_{s \in \mathcal{S}} \sum_{(a,b) \in \mathcal{D}_s} \pi_{a,b}^s \Lambda_{a,b} \\
\text{s.t.} \quad & \sum_{(a,b) \in \mathcal{D}_s} \pi_{a,b}^s \Lambda_{a,b} = \sum_{t \in \mathcal{S}, (b,a) \in \mathcal{D}_t : t+(b,a)=s} \pi_{b,a}^t \Lambda_{b,a}, & \forall s \in \mathcal{S}, \\
& \pi_{a,b}^s \leq \pi_s, & \forall s \in \mathcal{S}, \forall (a,b) \in \mathcal{D}_s, \\
& \sum_{s \in \mathcal{S}} \pi_s = 1, \\
& \pi_{a,b}^s \geq 0, & \forall s \in \mathcal{S}, \forall (a,b) \in \mathcal{D}_s, \\
& \pi_s \geq 0, & \forall s \in \mathcal{S}.
\end{aligned}$$

We can prove that this LP has for vertexes pure policies by simply counting the number of variables and independent constraints. Indeed, the transition rate in state $s \in \mathcal{S}$ for a demand $(a,b) \in \mathcal{D}_s$ equal to $\lambda_{a,b}^s = \frac{\pi_{a,b}^s}{\pi_s} \Lambda_{a,b}$ and optimal solutions have either $\pi_{a,b}^s = 0$ or $\pi_{a,b}^s = \pi_s$. \square

Lemma 2 *Pure policies are sub optimal among static policies even when the number of vehicles tends to infinity.*

Proof: Let consider 4 stations (a,b,c,d) with maximum transition rates $\Lambda_{a,b} = \Lambda_{b,c} = 3$, $\Lambda_{c,d} = \Lambda_{d,a} = \Lambda_{c,a} = 2$ and all others equals to 0. Figure 2 represents such demands. There are only 2 possible pure static policies λ defining a connexe demand graph: $\lambda_{i,j} = \Lambda_{i,j}$, $(i,j) \neq (c,a)$ and $\lambda_{c,a} = 0$ or 2.

Look now at the availability of a trip when the number of vehicles tends to infinity. With the formula given by [George and Xia \(2011\)](#) the availability of a vehicle a station a equals $\frac{\pi_a}{\max_{b \in \mathcal{M}} \pi_b}$, where π is the stationary distribution for one vehicle.

For a $\lambda_{c,a} = 0$ policy, we have $\pi_a = \pi_b = \frac{2}{10}$ and $\pi_c = \pi_d = \frac{3}{10} = \pi_{\max}$, so the average throughput when $N \rightarrow \infty$ equals $\frac{\pi_a}{\pi_{\max}}(3+3) + \frac{\pi_c}{\pi_{\max}}(2+2) = 8$. And for $\lambda_{c,a} = 2$ policy we have

$\pi_a = \pi_b = \frac{4}{14}$ and $\pi_c = \pi_d = \frac{3}{14}$, so the average throughput when $N \rightarrow \infty$ equals 10.5 which is thus the best pure static policy.

Yet, for a mixed policy (not pure) with $\lambda_{c,a} = 1$ and $\lambda_{i,j} = \Lambda_{i,j}$, $(i,j) \neq (c,a)$ we have $\pi_a = \pi_b = \pi_c = \pi_d = \frac{1}{4}$, so the average throughput when $N \rightarrow \infty$ equals 11 > 10.5. This shows that pure policies are suboptimal even when the number of vehicles tends to infinity. \square

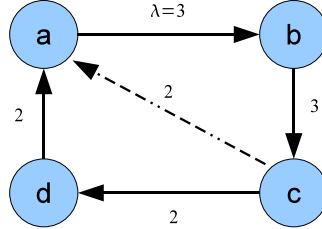


Figure 2: JW paradox: Pure policies are sub optimal even when the number of vehicles tends to infinity.

Lemma 3 *Pure policies are sub optimal among static policies when considering systems with more than 2 vehicles.*

3.2 Static symmetric policies optimization

We are now going to consider a simple and intuitive class of policy that we call *symmetric* and that consists in setting the prices in order to have as much demands to take a trip from station a to station b as in the opposite direction.

Definition 1 (Symmetric policies) *A policy is said symmetric if it sets a same demand to take a trip and its opposite, i.e. $\forall(a,b) \in \mathcal{D}$, $\lambda_{a,b} = \lambda_{b,a}$.*

In this section, we assume symmetry in the data $\Lambda_{a,b} = \Lambda_{b,a}$, $\forall(a,b) \in \mathcal{D}$, possibly using truncation as a preprocessing.

Definition 2 (USD policies) *A policy is said USD if it has a Uniform Stationary Distribution (USD) over the (reachable) states of the induced Markov chain.*

In the following we investigate symmetric policies because they are USD (see Lemma 4). Using this property we can compute the vehicle availability at a station and hence the average throughput by simply counting the number of reachable states of the system. For infinite station capacities the number of states can be computed analytically and leads a compact and simple formulation for the vehicle availability (see Lemma 5).

Lemma 4 *Symmetric policies are USD.*

Proof: Consider the continuous-time Markov chain formulation for N vehicles and M stations, with system states \mathcal{S} and transition rates A . Without loss of generality we are not considering unreachable states. The stationary distribution π over the state \mathcal{S} of the system is solution of the following equations:

$$\begin{aligned} \sum_{i \in \mathcal{S}} \pi_i &= 1, \\ \sum_{(i,j) \in A} \pi_i \lambda_{i,j} &= \sum_{(j,i) \in A} \pi_j \lambda_{j,i}, \quad \forall i \in \mathcal{S}. \end{aligned}$$

Since for symmetric policies we have $\forall (a, b) \in D$, $\lambda_{a,b} = \lambda_{b,a}$, then $\sum_{(i,j) \in A} \lambda_{i,j} = \sum_{(j,i) \in A} \lambda_{j,i}$, $\forall i \in \mathcal{S}$ and the uniform stationary distribution $\pi_i = \frac{1}{|\mathcal{S}|}$, $\forall i \in \mathcal{S}$ is trivially solution of this system. Moreover, it is the only one since the rank of the linear system is $|\mathcal{S}|$ which is the number of variables. \square

Lemma 5 *For M stations with infinite station capacity, N vehicles and an USD policy, the availability A of a vehicle at a station equals:*

$$A = \frac{N}{N - 1 + M}.$$

Proof: For M stations with infinite station capacity and N vehicles there is a exactly $\mathcal{N}_N^M = \binom{N+M-1}{N}$ system states. Therefore, for an USD policy, the availability A of a vehicle at a station equals:

$$A = \frac{\mathcal{N}_{N-1}^M}{\mathcal{N}_N^M} = 1 - \frac{\mathcal{N}_N^{M-1}}{\mathcal{N}_N^M} = \frac{N}{N - 1 + M}.$$

\square

We are now interested in optimizing symmetric static policies. We can see that pure policies are dominant over static policies (see lemma 6), which is giving a compact combinatorial formulation for the symmetric static policies optimization. However, even if this formulation is compact, it is not straightforwardly giving a polynomial algorithm to solve it. This is one of the contributions of this paper to prove that this problem is polynomial by giving a LP formulation (see Theorem 1).

Lemma 6 *Pure policies are dominant among static symmetric policies maximizing the throughput.*

Proof: If we had an optimal static symmetry with two open stations a and b , and a demand rate $\lambda_{b,a} = \lambda_{a,b} < \Lambda_{a,b}$. Adjusting the demand rate to its maximum value $\lambda_{b,a} = \lambda_{a,b} = \Lambda_{a,b}$ preserves the symmetric property, so the stationary distribution that is still uniform (see lemma 4). This modification strictly increases the overall throughput and it finally would means that this solution is not optimal. \square

Corollary 1 *Station opening policies are dominant over pure static symmetric policies. Hence, looking for the best static symmetric policy amounts to looking for a sub-graph induced by a set of open stations.*

STATIC SYMMETRIC POLICIES OPTIMIZATION

- **INPUT:** For N vehicles, a set \mathcal{M} of M stations and a maximum demand $\Lambda_{a,b}$ for every pair of stations $(a,b) \in \mathcal{D}$;
- **OUTPUT:** A demand $\lambda_{a,b}$ for every pair of stations $(a,b) \in \mathcal{D}$, such that $\lambda_{a,b} = \lambda_{b,a}$ and $0 \leq \lambda_{a,b} \leq \Lambda_{a,b}$;
- **OBJECTIVE:** Maximize the average number of trips sold;
- **COMBINATORIAL FORMULATION:** From Corollary 1, Lemma 4 and 5 we have:

$$\max_{T \subseteq \mathcal{M}} \sum_{a,b \in T} \Lambda_{a,b} \frac{N}{N + |T| - 1}.$$

Theorem 1 (LP for symmetric policies optimization) *The following LP gives the optimal static symmetric policy for a system with a set \mathcal{M} of stations, N vehicles and a demand \mathcal{D}_u of undirected trips.*

$$\begin{aligned} \max \quad & 2 \sum_{(a,b) \in \mathcal{D}_u} A_{a,b} \Lambda_{a,b} \\ \text{s.t.} \quad & A_{a,b} \leq A_a, & \forall (a,b) \in \mathcal{D}_u, \\ & A_{a,b} \leq A_b, & \forall (a,b) \in \mathcal{D}_u, \\ & (N-1)A^{\max} + \sum_{a \in \mathcal{S}} A_a = N, \\ & A_a \leq A^{\max}, & \forall a \in \mathcal{M}, \\ & A_{a,b} \geq 0, & \forall (a,b) \in \mathcal{D}_u, \\ & A_a \geq 0, & \forall a \in \mathcal{M}, \\ & A^{\max} \geq 0. \end{aligned}$$

Its variables can be interpreted as follows: A_a is the availability of a vehicle at station a . A^{\max} is the maximum availability of all station. $A_{a,b}$ is the availability of trip (a,b) that set its demand rate $\lambda_{a,b} = \frac{A_{a,b}}{A_a} \Lambda_{a,b}$.

Proof: Let U^k be the USD for k open stations: $U^k = \frac{N}{N-1+k}$. We prove that the vertices of this LP are USD policies or USD solutions: $\exists T \subseteq \mathcal{M}$ such that $\forall a \in T, \mathcal{A}_a = U^{|T|}$ and $\forall a \in \mathcal{M} \setminus T, \mathcal{A}_a = 0$. We show that a non USD optimal solution worth the same value as a linear combination of USD solutions.

Consider a non USD optimal solution S^* with p different values for variables $\mathcal{A}_a, \forall a \in \mathcal{M}$. Without loss of generality, in the following, we only consider the set $T \subseteq \mathcal{M}$ of reachable stations,

i.e. that have an associated variable $\mathcal{A}_a > 0$. Let $\{\psi_1, \dots, \psi_p\}$ be the ordered set of the A_a values, *i.e.* $\psi_i < \psi_j$ if $i < j$. Let \mathcal{M}_i be the set of stations a with an associated availability \mathcal{A}_a equals to ψ_i and let \mathcal{M}_i^+ be the set of stations with an associated availability greater or equals to ψ_i , *i.e.* $\mathcal{M}_i^+ = \bigcup_{j=i}^p \mathcal{M}_j$.

For any given objective, if a solution is optimal, its variables A_a totally determine its the cost because there is only one possible value for variable A^{\max} and one unique optimal value for variables $A_{a,b}$ present in the objective. Therefore, to ensure that two optimal solutions worth the same value, we only need to check that their variables A_a are equal.

We now construct a linear combination of p USD solutions, $\{S^1, \dots, S^p\}$ with associated vector variables A^i and coefficient α_i , $i \in \{1, \dots, p\}$, such that it builds a solution equal to the variables A^* of the non USD optimal solution S^* , *i.e.* $\sum_{i=1}^p \alpha_i A^i = A^*$. USD solution S^i is composed with stations set \mathcal{M}_i^+ and has a uniform distribution among this stations equal to $U^{|\mathcal{M}_i^+|}$, *i.e.* $A_a^i = U^{|\mathcal{M}_i^+|}$ if $a \in \mathcal{M}_i^+$ and $A_a^i = 0$ if $a \in \mathcal{T} \setminus \mathcal{M}_i^+$. Its linear combination coefficient is $\alpha_i = \frac{\psi_i - \psi_{i-1}}{U^{|\mathcal{M}_i^+|}}$ with $\psi_0 = 0$. We can verify that

$$\begin{aligned} \sum_{i=1}^p \alpha_i &= \frac{1}{N} \left(\sum_{i=1}^{p-1} \psi_i (|\mathcal{M}_i^+| - |\mathcal{M}_{i+1}^+|) + (N - 1 + |\mathcal{M}_p^+|) \psi_p \right) \\ &= \frac{1}{N} \left(\sum_{i=1}^p \psi_i |\mathcal{M}_i^+| + (N - 1) \psi_p \right) \\ &= \frac{1}{N} \left(\sum_{a \in \mathcal{M}} A_a^* + (N - 1) A^{\max*} \right) = 1. \end{aligned}$$

Moreover, for each station $a \in T$ there exists j such that $A_a^* = \psi_j$ and $\sum_{i=1}^p \alpha_i A_a^i = \sum_{i=1}^j \alpha_i A_a^i = \sum_{i=1}^j \frac{\psi_i - \psi_{i-1}}{U^{|\mathcal{M}_i^+|}} U^{|\mathcal{M}_i^+|} = \psi_j$. We have then $\sum_{i=1}^p \alpha_i A^i = A^*$ and since variables A_a determine the cost of a solution we have a linear combination of USD solutions forming a solution with same value as the non USD optimal solution S^* : $val(\sum_{i=1}^p \alpha_i S^i) = val(S^*)$. Non USD solutions are hence not vertices of the polygon. \square

Remark 1 Notice that for $N = 1$ the symmetric policies optimization amounts to solving the densest weighted subgraph problem. Our LP generalized the one given by [Charikar \(2000\)](#) that he proves optimal through another type of proof, by the absurd.

3.3 Static conservative policies optimization

In the previous section, we introduce the symmetric policies that have the useful advantage to be USD. However, this class of policies seems very restrictive and maybe the USD property can be found in a wider class. This is the case of the conservative policies that are also USD (Lemma 7) and in fact containing the symmetric policies.

Definition 3 (Conservative policies) A policy is said conservative if it sets in each station a same amount of demand to take a vehicle as to drop one, i.e. $\forall a \in \mathcal{M}, \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda_{b,a}$.

Lemma 7 A static policy is conservative if and only if it is USD.

Proof: Consider the Continuous-Time Markov Chain formulation for N vehicle and M stations, with system states \mathcal{S} and transition rates A . Without loss of generality we are not considering unreachable states. The USD distribution π over the state \mathcal{S} of the system is solution of the following equations:

$$\begin{aligned} \sum_{i \in \mathcal{S}} \pi_i &= 1, \\ \sum_{(i,j) \in A} \pi_i \lambda_{i,j} &= \sum_{(j,i) \in A} \pi_j \lambda_{j,i}, \quad \forall i \in \mathcal{S}. \end{aligned}$$

On the one hand, conservative policies have $\sum_{b \in \mathcal{M}} \lambda_{a,b} = \sum_{b \in \mathcal{M}} \lambda_{b,a}, \forall a \in \mathcal{M}$, therefore $\sum_{(i,j) \in A} \lambda_{i,j} = \sum_{(j,i) \in A} \lambda_{j,i}$ and $\pi_i = \frac{1}{|\mathcal{S}|}, \forall i \in \mathcal{S}$ is trivially solution of this system. It is the only one possible since the rank of the linear system is $|\mathcal{S}|$ which is the number of variables.

On the other hand, a USD policy have $\pi_i = \frac{1}{|\mathcal{S}|}, \forall i \in \mathcal{S}$ and static USD policies should then satisfy:

$$\begin{aligned} \sum_{(i,j) \in A} \lambda_{i,j} &= \sum_{(j,i) \in A} \lambda_{j,i}, \quad \forall i \in \mathcal{S}, \\ \lambda_{i,j} &= \lambda_{a,b}, \quad \forall (a,b) \in \mathcal{D} \rightarrow (i,j) \in A. \end{aligned}$$

$\sum_{b \in \mathcal{M}} \lambda_{a,b} = \sum_{b \in \mathcal{M}} \lambda_{b,a}, \forall a \in \mathcal{M}$ is trivially the only solution of this system. \square

Corollary 2 A dynamic policy is conservative if and only if it is USD.

Lemma 5 is also applicable for conservative policy optimization since they are USD. We have hence a compact combinatorial formulation for the the conservative policies optimization. The complexity of this new problem remains unsolved. However, the approach we used for symmetric policies can be extended to give a LP relaxation for conservative policy optimization (Theorem 2) that seems to provide solutions often optimal (Remark 2).

STATIC CONSERVATIVE POLICIES OPTIMIZATION

- **INPUT:** For N vehicle, a set \mathcal{M} of M stations and a maximum demand $\Lambda_{a,b}$ for every pair of stations $(a,b) \in \mathcal{D}$;
- **OUTPUT:** A demand $\lambda_{a,b}$ for every pair of stations $(a,b) \in \mathcal{D}$, such that $\forall a \in \mathcal{M}, \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda_{b,a}$ with $0 \leq \lambda_{a,b} \leq \Lambda_{a,b}$;
- **OBJECTIVE:** Maximize the average number of trips sold;

- **COMBINATORIAL FORMULATION:** From Lemma 7 and 5 we have:

$$\begin{aligned}
& \max_{F \subseteq \mathcal{D}} \sum_{(a,b) \in F} \lambda_{a,b} \frac{N}{N-1 + |\mathcal{M}(F)|} \\
& \text{s.t.} \quad \sum_{(a,b) \in F} \lambda_{a,b} = \sum_{(b,a) \in F} \lambda_{b,a}, & \forall a \in \mathcal{M}(F), \\
& \quad \quad 0 \leq \lambda_{a,b} \leq \Lambda_{a,b}, & \forall (a,b) \in F,
\end{aligned}$$

With $M(F) = \{a \in \mathcal{M} : (a,b) \in F \parallel (b,a) \in F\}$.

Theorem 2 (LP for a conservative policies optimization relaxation) *The following LP (2) is a relaxation of the static conservative policies optimization for a system with a set \mathcal{M} of stations N vehicles and a demand \mathcal{D} of directed trips.*

$$\begin{aligned}
& \max \sum_{a,b \in \mathcal{M}} A_{a,b} \Lambda_{a,b} & (2a) \\
& \text{s.t.} \quad A_{a,b} \leq A_a \Lambda_{a,b}, & \forall (a,b) \in \mathcal{D}, & (2b) \\
& \quad \quad A_{a,b} \leq A_b \Lambda_{a,b}, & \forall (a,b) \in \mathcal{D}, & (2c) \\
& \quad \quad \sum_{(a,b) \in \mathcal{D}} A_{a,b} \Lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} A_{a,b} \Lambda_{b,a}, & \forall a \in \mathcal{M}, & (2d) \\
& \quad \quad A_a \leq A^{\max}, & \forall a \in \mathcal{M}, & (2e) \\
& \quad \quad (N-1)A^{\max} + \sum_{a \in \mathcal{M}} A_a = N, & & (2f) \\
& \quad \quad A_{a,b} \geq 0, & \forall (a,b) \in \mathcal{D}, & (2g) \\
& \quad \quad A_a \geq 0, & \forall a \in \mathcal{M}, & (2h) \\
& \quad \quad A^{\max} \geq 0. & & (2i)
\end{aligned}$$

Its variables can be interpreted as follows: A_a is the availability of a vehicle at station a . A^{\max} is the maximum availability of all station. $A_{a,b}$ is the availability of trip (a,b) that set its demand rate $\lambda_{a,b} = \frac{A_{a,b}}{A_a} \Lambda_{a,b}$.

Proof: Conservative policies are trivially solution of LP (2). □

Corollary 3 1) LP (2) optimal solution value is an upper bound on conservative policies optimization.

2) If LP (2) gives a conservative policy, it is the optimal conservative policy.

Remark 2 *Although LP (2) does not solve the conservative policies optimization, in practice, it is a good approximation that provides extremely often the best conservative policies as the system grows.*

The smallest example we found where LP (2) does not produce the best conservative policies has 3 stations, 2 vehicles and a demand graph defining a cycle $1 - 2 - 2$, i.e. with for non-null demand rates $\Lambda_{a,b} = 1$, $\Lambda_{b,c} = \Lambda_{c,a} = 2$.

On 10 000 random instances with integer demand rates λ drawn uniformly in $[0, 10]$ for each trip, Table 1 gives the percentage of time LP (2) did not provide an (optimal) USD policy.

(m,n)	(3,2)	(3,6)	(3,10)	(4,10)	(5,20)	(20,100)
LP (2) not optimal USD	3.40%	1.51%	1.01%	0.43%	0.06%	0,00%

Table 1: LP (2) relaxation provides extremely often the best conservative policies as the system grows.

4 Maximum circulation approximation

In the previous section we figured out new LP formulations, computing explicitly the USD over the system's states in function of the number of vehicles and the number of open stations. However, looking at the conservative policies optimization, we see that if we want to open all stations it amounts to solving a MAXIMUM CIRCULATION problem (Edmonds and Karp, 1972): generalization of network flow problems with flow conservation at all nodes (no source no sink).

MAXIMUM CIRCULATION LP

$$\begin{aligned}
 \max \quad & \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} \\
 \text{s.t.} \quad & \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda_{b,a}, & \forall a \in \mathcal{M}, \\
 & 0 \leq \lambda_{a,b} \leq \Lambda_{a,b}, & \forall (a,b) \in \mathcal{D}.
 \end{aligned}$$

4.1 An asymptotic approximation for infinite number of vehicles

In the following, we show that MAXIMUM CIRCULATION gives the best dynamic policy when the number of vehicles tends to infinity. In Lemma 8 we first prove that MAXIMUM CIRCULATION gives the best static policies when $N \rightarrow \infty$. Then, in Theorem 3, we prove that static policies are themselves dominant over dynamic policies.

Lemma 8 *The MAXIMUM CIRCULATION policy is asymptotically optimal among static policies when the number of vehicles tends to infinity.*

Proof: Let π be the stationary distribution vector for $N = 1$ vehicle and M stations without transportation time. Let $\pi_{\max} = \max_{a \in \mathcal{M}} \pi_a$, from Theorem 1 in [George and Xia \(2011\)](#), we have that the availability A_a of a vehicle at a station $a \in \mathcal{M}$ when the number N of vehicles tends to infinity equals:

$$\lim_{N \rightarrow \infty} (A(N)) = \frac{\pi_a}{\pi_{\max}}.$$

Note that [George and Xia \(2011\)](#) states this theorem with π being the stationary distribution for $N = 1$ vehicle and M stations with transportation time. However, for one vehicle the ratio $\frac{\pi_a}{\pi_b}$, $\forall a, b \in \mathcal{M}$ is independent from the transportation times. Therefore we can restrain to the computation of π without transportation times. We have hence that the best static policy is solution of the following program:

$$\begin{aligned} \max \quad & \sum_{(a,b) \in \mathcal{D}} \frac{\pi_a}{\pi_{\max}} \lambda_{a,b} \\ \text{s.t.} \quad & \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} \pi_a = \sum_{(b,a) \in \mathcal{D}} \lambda_{b,a} \pi_b, & \forall a \in \mathcal{M}, \\ & 0 \leq \lambda_{a,b} \leq \Lambda_{a,b}, & \forall (a,b) \in \mathcal{D}, \\ & \sum_{a \in \mathcal{M}} \pi_a = 1, \\ & \pi_{\max} = \max_{a \in \mathcal{M}} \pi_a. \end{aligned}$$

With a variable substitution $\lambda_{a,b} = \frac{\pi_{\max}}{\pi_a} \lambda'_{a,b}$, we obtain:

$$\max \quad \sum_{(a,b) \in \mathcal{D}} \lambda'_{a,b} \tag{3a}$$

$$\text{s.t.} \quad \sum_{(a,b) \in \mathcal{D}} \lambda'_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda'_{b,a}, \quad \forall a \in \mathcal{M}, \tag{3b}$$

$$0 \leq \lambda'_{a,b} \leq \Lambda_{a,b} \frac{\pi_a}{\pi_{\max}}, \quad \forall (a,b) \in \mathcal{D}, \tag{3c}$$

$$\sum_{a \in \mathcal{M}} \pi_a = 1, \tag{3d}$$

$$\pi_{\max} = \max_{a \in \mathcal{M}} \pi_a. \tag{3e}$$

Note that $\frac{\pi_a}{\pi_{\max}} \leq 1$, $\forall a \in \mathcal{M}$. So taking $\pi_a = \frac{1}{|\mathcal{M}|}$, $\forall a \in \mathcal{M}$ implies $\frac{\pi_a}{\pi_{\max}} = 1$ which does not reduce the solution space of constraints (3c) and satisfies moreover constraint (3d). Finally, it means that the previous program is equivalent to solve MAXIMUM CIRCULATION with $\lambda' = \lambda$. \square

Theorem 3 MAXIMUM CIRCULATION *gives the best dynamic policy when the number of vehicles tends to infinity.*

Proof: We first prove that MAXIMUM CIRCULATION optimal solution gives an upper bound on dynamic policies for any number of vehicles. To do so, we show that we can construct from any dynamic policies a MAXIMUM CIRCULATION solution with same value.

Consider a dynamic pricing policies with transition rate $\lambda_{a,b}^s \leq \Lambda_{a,b}$ in state $s \in \mathcal{S}$ (state dependent) for trip $(a,b) \in \mathcal{D}$. Under this policy, the system is stationary and ergodic under very general conditions. Therefore, we can look at its stationary distribution π on its state space \mathcal{S} that satisfies the following equations:

$$\begin{aligned} \sum_{(a,b) \in \mathcal{D}} \pi_s \lambda_{a,b}^s &= \sum_{(b,a) \in \mathcal{D}} \pi_{s-\{(b,a)\}} \lambda_{b,a}^s, & \forall s \in \mathcal{S}, \\ \sum_{s \in \mathcal{S}} \pi_s &= 1. \end{aligned}$$

Let $\lambda'_{a,b}$ be the average throughput for a trip $(a,b) \in \mathcal{D}$:

$$\lambda'_{a,b} = \sum_{s \in \mathcal{S}} \pi_s \lambda_{a,b}^s \leq \Lambda_{a,b}, \quad \forall (a,b) \in \mathcal{D}.$$

The average throughput of the system equal to $\sum_{(a,b) \in \mathcal{D}} \lambda'_{a,b}$. λ' satisfies the flow conservation constraints and the capacity constraints of MAXIMUM CIRCULATION:

$$\begin{aligned} \sum_{(a,b) \in \mathcal{D}} \lambda'_{a,b} &= \sum_{(b,a) \in \mathcal{D}} \lambda'_{b,a}, & \forall a \in \mathcal{M}, \\ 0 \leq \lambda'_{a,b} &\leq \Lambda_{a,b}, & \forall (a,b) \in \mathcal{D}. \end{aligned}$$

Flow conservation constraints are respected because otherwise it would mean that in the dynamic policy's steady state, a station would receive more vehicles that it is sending which is absurd. The capacity constraints are also respected since $\sum_{s \in \mathcal{S}} \pi_s = 1$. Therefore, $\lambda'_{a,b}$ is solution of MAXIMUM CIRCULATION with the same objective value. It proves that MAXIMUM CIRCULATION optimal solution is an upper bound on any dynamic policies.

From Lemma 8, we know that MAXIMUM CIRCULATION is asymptotically giving the best static policy when the number of vehicles tends to infinity. Therefore, when $N \rightarrow \infty$, static policies are asymptotically dominant over dynamic policies and MAXIMUM CIRCULATION solves the optimal dynamic policy. \square

Remark 3 MAXIMUM CIRCULATION approximation works also when considering transportation times. Indeed, when the number of vehicle tends to infinity, transportation times have no impact on optimal solution (proof given in Lemma 8).

4.2 An approximation algorithm for finite number of vehicles

Theorem 4 *The static conservative policy provided by MAXIMUM CIRCULATION is a $\frac{N+M-1}{N}$ -approximation on static policies.*

Proof: Let $Circ^*$ be the value of MAXIMUM CIRCULATION and $Cons_{Circ}$ be the value of the static conservative policy provided by MAXIMUM CIRCULATION. Simply applying the USD property we have that $Cons_{Circ} = \frac{N}{N+M-1}Circ^*$. As shown in Theorem 3, $Circ^*$ is an upper bound on the optimal static policies of gain $Stat^*$. Therefore, $Stat^* \leq Circ^*$ and

$$\frac{N}{N+M-1}Stat^* \leq \frac{N}{N+M-1}Circ^* = Cons_{Circ}.$$

□

Corollary 4 *For M stations and N vehicles, the optimal conservative static policy is a tight $\frac{N+M-1}{N}$ -approximation on the optimal static policies.*

Proof: From Theorem 4, MAXIMUM CIRCULATION static policy (of value $Cons_{Circ}$) is a $\frac{N+M-1}{N}$ -approximation on static policies. The best conservative policy of value $Cons^* \geq Cons_{Circ}$ is therefore also a $\frac{N+M-1}{N}$ -approximation on static policies.

To prove that this approximation is tight, we consider an instance with N vehicles, M stations and a demand graph induced by a circuit with M vertices and maximum demand $\Lambda = (1, k, \dots, k)$. Let a be the station with the 1 outgoing maximum demand with rate 1. The best conservative policy opens all possible trips with $\lambda = 1$. Because it is USD we can easily compute its value: $Cons^* = \frac{NM}{N+M-1}$. The best static policy opens all trips to their maximum: $\lambda = \Lambda$. The availability A_a of station a is $A_a = O(\frac{k}{M+k-1})$ and $\lim_{k \rightarrow \infty} A_a = 1$. The availability of station $b \neq a$ is $A_b = O(\frac{1}{M+k-1})$ and $\lim_{k \rightarrow \infty} A_b = 0$. When $k \rightarrow \infty$, the value of the best static policy is hence $Stat^* = M$, and the ratio between the optimal conservative static policy and the optimal static policy is:

$$\frac{Cons^*}{Stat^*} = \frac{N}{N+M-1}.$$

□

Note that Corollary 4 implies that MAXIMUM CIRCULATION policy is a tight $\frac{N+M-1}{N}$ -approximation on static policies.

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