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# Vehicle Sharing System Pricing Regulation: Transit Optimization of Intractable Queuing Network

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## Abstract

This paper gives polynomially solvable stochastic models for a Vehicle Sharing System Pricing problem optimizing the average number of trips sold with static demands, infinite station capacities and no transportation times.

1. The Vehicle Sharing System Pricing problem is model as a Closed Queuing Network with infinite buffers and continuous controls on transition rates for the pricing;
2. This model is shown intractable for an explicit pricing optimization though Markovian Decision Process. Subclasses of static policies called symmetric and conservatives are given for which there exists a polynomially solvable formulation in the number of stations and vehicles.
3. An approximation as a simple Maximum Circulation problem is given and shown asymptotic optimal when the number of vehicles available tends to infinity.

## 1 Introduction

### 1.1 Context

Shoup (2005) reports that, based on a sample of 22 US studies, cars looking for a parking spot contribute to 30% of the city traffic. Moreover cars are used less than 2 hours per day on average but still occupy a parking spot the rest of the time! Could we have less vehicles and satisfy the same demand level?

Recently, the interest in Vehicle Sharing Systems (VSS) in cities has increased significantly. Indeed, urban policies intend to discourage citizens to use their personal car downtown by reducing the number of parking spots, street width, etc. VSS seem to be a promising solution to

reduce jointly traffic and parking congestion, noise, and air pollution (proposing bikes or electric cars). They offer personal mobility allowing users to pay only for the usage.

We are interested in short-term one-way VSS where vehicles can be taken and returned at different places (paying by the minute). Associated with classical public transportation systems, short-term one-way VSS help to solve one of the most difficult public transportation problem: the last kilometer issue (DeMaio, 2009). This is not the case for round-trip VSS where vehicles have to be returned at the station where they were taken.

The first large-scale short-term one-way VSS was the bicycle VSS Vélib'. It was implemented in Paris in 2007 and now has more than 1200 stations and 20 000 bikes selling around 110 000 trips per day. It has inspired several other cities all around the world; Now more than 300 cities have such a system, including Montréal, Beijing, Barcelona, Mexico City, Tel Aviv (DeMaio, 2009).

## 1.2 One-way Vehicle Sharing Systems: a management issue

However if freedom increases for the user in the one way model, it implies a higher complexity in its management. In round trip type rental systems, the only stock that is relevant when managing yield and reservations is the number of available vehicles. In one-way systems, a new problem occurs since vehicles aren't the only key resource any more. In practice, parking stations have a maximum number of spots, and when the total number of vehicle is comparable to the total number of parking spots, available parking spots become a new key resource.

Since first bicycle VSS, problems of bikes and parking spots availability have appeared very often. Reasons are various but we can highlight two important phenomenon: the gravitational effect which indicates that a station is constantly unbalanced (as Montmarte hill in Vélib'), and the tide phenomenon representing the oscillation of demand intensity along the day (as morning and evening flows between working and residential areas).

To improve the efficiency of the system, in the literature, different perspectives are studied. At a strategic level, some authors consider the optimal capacity and locations of bike rental stations. Shu et al. (2010) propose a stochastic network flow model to support these decisions. They use their model to design a bicycle VSS in Singapore based on demand forecast derived from current usage of the mass transit system. Lin and Ta-Hui (2011) consider a similar problem but formulate it as a deterministic mathematical model. Their model is aware of the bike path network and mode sharing with other means of public transportation.

At a tactical level, other authors investigate the optimal number of vehicles given a set of stations. George and Xia (2011) study the fleet sizing problem with constant demand and no parking capacity. Fricker and Gast (2012); Fricker et al. (2012) look into the optimal sizing of a fleet in "toy" cities, where demand is constant over time and identical for every possible trip, and all stations have the same capacity  $\mathcal{K}$ . They show that even with an optimal fleet sizing in the most "perfect" city, if there is no operational system management, there is at least a probability of  $\frac{2}{\mathcal{K}+1}$  that any given station is empty or full.

At an operational level, in order to be able to meet the demand with a reasonable standard of quality, in most bicycle VSS trucks are used to balance the bikes among the stations. The problem is to schedule vehicle routes to visit some of the stations to perform pickup and delivery so as to minimize the number of users who cannot be served, *i.e.*, the number of users who try to take a bike from an empty station or to return it to a full station. In the literature many papers deal already with this problem. A static version of the bicycle VSS balancing problem is treated in Chemla et al. (2011) and a dynamic one in Contardo et al. (2012).

A new type of VSS has appeared lately: one-way Car VSS with Autolib' in Paris and Car2go in more than 10 cities (Vancouver, San Diego, Lyon, Ulm...). With cars, operational balancing optimization through relocation seems inappropriate due to their size. We have to find another way to optimize the system.

### 1.3 Regulation through pricing

The origin of Revenue Management (RM) lies in airline industry. It started in the 1970s and 1980s with the deregulation of the market in the United States. In the early 1990s RM techniques were then applied to improve the efficiency of round trip Vehicle Rental Systems (VRS), see Carroll and Grimes (1995) and Geraghty and Johnson (1997). One way rental is now offered in many VRS, however as one can see in practice for car VRS that it is always much more expensive than round trip rental. We haven't found in the literature authors tackling the one way VRS RM problem. We can only cite Haensela et al. (2011) that model a network of only round trip car VRS but with the possibility of transferring cars between rental sites for a fixed cost. For trucks rental on the contrary, companies such as Rentn'Drop in France or Budget Truck Rental in the United States are specialized in the one way rental offering dynamic pricing. This problem is tackled by Guerriero et al. (2012) that consider the optimal managing of a fleet of trucks rented by a logistic operator, to serve customers. The logistic operator has to decide whether to accept or reject a booking request and which type of truck should be used to address it.

Anyway results for one way VRS are not directly applicable to VSS, because they differ on several points: 1) Renting are by the day in VRS and by the minute in VSS with a possible high intensity; 2) One way rental is the core in VSS, for instance only 5% of round trip rental in Bixi (Morency et al., 2011), and it is classically the opposite in car VRS. 3) There is usually no booking in advance in VSS, it is a first come first serve rule, whereas usually trips are planned several days in advance in VRS.

In this paper we are looking at VSS and optimization through pricing. Assuming that demand is elastic, we want to use prices to influence user choices in order to drive the system towards its most efficient dynamic.

This work is part of a preliminary study using operation research to 1) Establish the interest of VSS pricing regulation system 2) Give good and possibly simple pricing policies for the operational management.

## 1.4 Assumption and Goals of this study

A VSS pricing model is proposed in Waserhole et al. (2012a) using Markov Decision Processes (MDP). The problem with this model is that the number of states of the MDP grows exponentially with the size of the instance: number of vehicles and stations. Therefore they give a fluid approximation that computes static policies. This fluid model is polynomially solvable and therefore gives results in a reasonable time. However it is based a deterministic approximation and the expected results are pretty far from the experiments. That's why in this paper, we want to consider a stochastic model but still polynomial in the size of the instance. To obtain some results we consider a simple model, as in George and Xia (2011), with a constant demand and infinite station capacities, moreover transportation times are considered null. We optimize the average number of trips sold by the system (average throughput). Prices are the leverage, with an elastic surjective demand on price. In fact we just assume a maximum possible demand  $\Lambda$  and then that there exists a price to obtain any demand  $\lambda \in [0, \Lambda]$ . Finally to build a stochastic model polynomially solvable we only tackle two policy subclasses, the symmetric and conservative policies.

## 2 Framework

We define formally in this section the framework we use to model the Stochastic Vehicle Sharing System.

### 2.1 Restriction to a simple protocol

In a real context, a user wants to use a vehicle to take a trip between an original (GPS) location  $a$ , and a final one  $b$ , during a specified time frame. On a station based VSS, he tries to find the closest station to location  $a$  with a vehicle to take and the closest station to location  $b$  with a parking spot to retrieve it. All along this process user's decision relies on several correlated inputs such as: trip total price, walking distance, public transportation competition, time frame...

A time elastic GPS to GPS demand forecast, correlated to a user's decision protocol ruling his behaviour to take a trip between two specific stations at a specific time, seems closer to reality but introduces of course a big complexity (use of utility function for instance). This is why in this study, to stay simple and be able to develop a compact model we are going to consider a station to station demand forecast, with moreover instantaneous transportation time and infinite station capacity. Finally it amounts to consider a demand for the following simplified protocol:

1. A user asks for a vehicle at station  $a$  (here and now), with destination  $b$ ;
2. The system offers a price (or rejects the user = infinite price);
3. The user accepts the price, pay and the vehicle is transferred, or leaves the system.

## 2.2 A Vehicle Sharing System Stochastic Model

### 2.2.1 Markovian framework

We define in this section a framework to model a stochastic Vehicle Sharing Systems with the protocol defined previous section.

In a city there is a fleet of  $N$  vehicles along a set  $\mathcal{M}$  ( $|\mathcal{M}| = M$ ) of stations with infinite capacity. There is an elastic demand between each station  $\mathcal{D} = \mathcal{M} \times \mathcal{M}$ . This demand is constant and follows a Poisson distribution of parameter  $\lambda_{a,b}(p_{a,b})$  to go from station  $a \in \mathcal{M}$  to station  $b \in \mathcal{M}$ , function of the proposed price  $p_{a,b}$ .

### 2.2.2 Closed queuing network model

For a given demand  $\lambda_{a,b}$  for every trips  $(a, b) \in \mathcal{D}$  (*i.e.* for a fixed price  $p_{a,b}$ ), we can model this stochastic Vehicle Sharing System by a Closed Queuing Network see figure 1.

Each demand  $(a, b) \in \mathcal{D}$  is represented by a server  $(a - b)$  which has a time dependent service rate equal to the average number of clients willing to take a trip from station  $a$  to station  $b$ :  $\lambda_{a,b}$ . Demands with same station of origin  $a$  ( $(a, b) \in \mathcal{D}, \forall b \in \mathcal{M}$ ) are sharing the same infinite buffer  $a$ . When a vehicle (a job) is picked up to take the trip  $(a, b)$  (is processed by server  $(a, b)$ ) it is transferred directly in station (buffer)  $b$ .

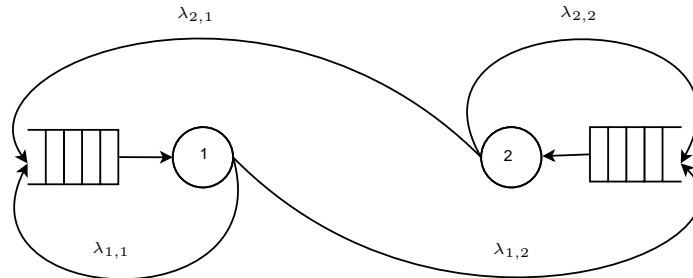


Figure 1: A closed queuing network model with servers for demands and transportation times.

### 2.2.3 Continuous-Time Markov Chain (CT-MC) formulation

If a price  $p_{a,b}$  is set for all trips  $(a, b) \in \mathcal{D}$  we can model the closed queuing network by a Continuous Time-Markov Chain on set of states  $\mathcal{S}$ :

$$\mathcal{S} = \left\{ \left( n_a : a \in \mathcal{M} \right) / \sum_{a \in \mathcal{M}} n_a = N \right\}$$

A state  $s = (n_a : a \in \mathcal{M})$  represents the repartition of the vehicles in the city space,  $n_a$  is the number of vehicles in station  $a \in \mathcal{M}$ . There is a transition rate  $\lambda_{a,b}(p_{a,b})$  between state  $(\dots, n_a, \dots, n_b, \dots)$  and state  $(\dots, n_a - 1, \dots, n_b + 1, \dots)$ .

We can note that there is an exponential number of states. For instance for a small system with  $N = 150$  vehicles and  $M = 50$  stations we have  $\binom{N+M-1}{N} \simeq 10^{47}$  states!

## 2.3 Model optimization

### 2.3.1 On optimizing the system

Previous section we modelled the system by a closed queuing network that can be describe explicitly through a Continuous-Time Markov Chain. Through this model we want now to optimize the Vehicle Sharing System in order to maximize the average revenue. To do so we have as leverage the possibility to change the price to take a trip which will, assuming an elastic demand, influence the demand for such trip. We call this problem the Stochastic Vehicle Sharing System Pricing Problem.

**Definition 1 (VSS Pricing Problem)** *The Stochastic Vehicle Sharing System Pricing Problem amounts in setting a price to take every trip in order to maximize the gain of the Vehicle Sharing System Markovian Model.*

*Prices can be Discrete, i.e. selected in a set of possibilities, or Continuous i.e. chosen in a range. Pricing policies can be Dynamic, i.e. dependent on system's state (vehicle repartition), or Static i.e. independent on system's state, set in advance and function of the trip.*

### 2.3.2 An intractable Markov Decision Process (MDP) Resolution

We can solve this queuing network model through the well know Markov Decision Process framework based on the Continuous-Time Markov Chain given previous section. To do so we define a set  $\mathcal{Q}$  of possible discrete prices for each trip. A trip  $(a, b) \in \mathcal{D}$  at price  $p_{a,b}^q$ ,  $q \in \mathcal{Q}$  have a demand (transition rate)  $\lambda_{a,b}(p_{a,b}^q) = \lambda_{a,b}^q$ .

Solving this MDP computes the best Dynamic System State Dependent Trip Discrete Pricing policy, *i.e.* the price for a trip depends of the current state of the system (vehicle repartition). MDP are known to be polynomially solvable on the number of states  $|\mathcal{S}|$  and the number of actions  $|\mathcal{A}|$  available in each state. For example through Value Iteration, Policy Iteration algorithm or Linear Programming techniques, *c.f.* book of Puterman (1994).

In this particular case, we are dealing with a pricing problem and the action space  $\mathcal{A}(s)$  in each state  $s \in \mathcal{S}$  is the Cartesian product of the available prices for each trip *i.e.*  $\mathcal{A}(s) = \mathcal{Q}^M$ . However to not suffer from this exponential explosion we can model this problem as a Decomposable Markov Decision Process, *c.f.* Waserhole et al. (2012b). It is a general method based on the Event Based Dynamic Programming, *c.f.* Koole (1998), to reduce the complexity of the action space to  $\mathcal{A}(s) = \mathcal{Q} \times M$  and allows to use Value Iteration, Policy Iteration algorithm or Linear Programming solution techniques.

**Lemma 1 (Dynamic Pricing MDP)** *The previous MDP gives the best Dynamic System State Dependent Discrete Pricing Policy for the stochastic VSS Pricing problem.*

Anyway the state space is still exponential so it does not give a polynomial algorithm. We have therefore to look into approximations or simplifications to tackle the problem.

### 2.3.3 State of the art on this model

In the literature only simple forms of this closed queuing network model with the relationship to the underlying CT-MC have already been studied for VSS. George and Xia (2011) consider a VSS with only one time step, one price and infinite station capacities. Under these assumptions they establish a compact form to compute the system performance using the BCMP network theory (Baskett et al., 1975). They solve an optimal fleet sizing problem considering a cost to maintain a vehicle and a gain to rent it.

Fricker and Gast (2012) consider simple cities that they call homogeneous. These cities have a unique fixed station capacity  $\mathcal{K}_a = \mathcal{K}$ , a constant (one time step) arrival rate and uniform routing matrix:  $\lambda_{a,b}^t = \frac{\lambda}{M}$ ; they also have a unique travel time following an exponential distribution of mean  $\mu_{a,b}^{t-1} = \mu^{-1}$ . With a Mean Field Approximation, they obtain some asymptotic results when the number of stations tends to infinity ( $M \rightarrow \infty$ ): If there is no operational regulation system, the optimal sizing is to have  $\frac{\mathcal{K}}{2} + \frac{\lambda}{\mu}$  vehicles per station which corresponds in filling half of the station plus the average number of vehicles in transit ( $\frac{\lambda}{\mu}$ ). Moreover they show that even with an optimal sizing each station has still a probability  $\frac{1}{\mathcal{K}+1}$  to be empty or full which is considered pretty bad. In another paper Fricker et al. (2012) extend to inhomogeneous cities modelled by clusters some analytical results and verified experimentally some others.

Fricker and Gast (2012) also study in homogeneous cities a heuristic called “The power of two choices” using incentives that can be seen as a Dynamic Station State Pricing. When a user is showing at a station and is taking a vehicle, he gives randomly two possible destination stations and the system is directing him to the least loaded one. They show that this policy allows to drastically reduce the probability to be empty or full for each station to  $2^{-\frac{\mathcal{K}}{2}}$ .

Waserhole et al. (2012a) propose a model for the Stochastic Vehicle Sharing System Pricing problem considering station capacity and time dependent demand. They give a fluid approximation, a deterministic solution technique, that have for best advantage to be able to deal with the tide phenomenon and to give an Upper Bound. Through some experimentation they show that the fluid solution gives interesting results but nevertheless has a simulation value pretty far from an upper bound. They raise the question on whether or not this was due to the deterministic relaxation or to the poor upper bound their algorithm is giving. This is where this work stands, it intends to give a tractable stochastic solution technique for the Vehicle Sharing Pricing problem



### 3 VSS stochastic optimization

We develop now tractable models to optimize a Vehicle Sharing System through pricing with stochastic consideration. We want to find compact forms to evaluate the system performance. In George and Xia (2011) they derive analytic results to compute the average throughput of the system for fixed prices in order to compute the best fleet sizing. However when we want to take the prices as controls the same approach doesn't seem to work.

For general classes of pricing policies optimizing the average throughput seems rather hard, we can note that the deterministic version of the pricing problem is even shown *NP*-hard in Wasserhole and Jost (2012). Nevertheless if we restrain our study to special types of policy (symmetric and conservative) we can use a special property of the Induced Markov Chain (the uniform distribution over all states) to obtain a compact formulation for the average throughput of the system usable for a polynomial price optimization.

#### 3.1 On optimizing the system

We want to compute a policy maximizing the average throughput of a system and to be able to compute this policy for large (reel) systems. Therefore we look at polynomial optimization and hence policies allowing compact formulation. *Dynamic* policies can have prices to take a trip depending of the state of the system, the vehicle repartition. However in our model the number of states is exponential (vehicle repartition), therefore to have tractable dynamic policy we need a compact formulation, maybe threshold policies. We don't have this problem with *Static* policies where we only need to set the price (hence the transition rate  $\lambda$ ) to take a trip between any stations independently of the system's state. *i.e.*  $\lambda_{a,b} \leq \Lambda_{a,b}, \forall (a,b) \in \mathcal{D}$ .

To simplify even more the policies we can be interested at the *Pure* policy that only set for each trip  $(a,b) \in \mathcal{D}$  if it is either open (*i.e.*  $\lambda_{a,b} = \Lambda_{a,b}$ ), or close (*i.e.*  $\lambda_{a,b} = 0$ ).

One can wonder if to maximize the average throughput pure policies are not dominant. What is shown in the following is that it is true for dynamic policies (see lemma 2) but false for dynamic policies even (which more surprising) when the number of vehicles tends to infinity (see lemma 3)

**Lemma 2 (Pure policies domination over dynamic policies)** *Pure policies are dominant over dynamic policies maximizing the average flow (number of trip sold).*

**Proof :** On a state graph, the following linear program gives the best static policies with continuous actions, *i.e.* the best price for each trip in each state of the system.

Linear Program for dynamic policies:

$$\begin{aligned}
\max \quad & \sum_{s \in \mathcal{S}} \sum_{(a,b) \in \mathcal{D}_s} \pi_{a,b}^s \times \Lambda_{a,b} \\
\text{s.t.} \quad & \sum_{(a,b) \in \mathcal{D}_s} \pi_{a,b}^s \times \Lambda_{a,b} = \sum_{t \in \mathcal{S}, (b,a) \in \mathcal{D}_t : t+(b,a)=s} \pi_{b,a}^t \times \Lambda_{b,a} && \forall s \in \mathcal{S} \\
& \pi_{a,b}^s \leq \pi_s && \forall s \in \mathcal{S}, \forall (a,b) \in \mathcal{D}_s \\
& \sum_{s \in \mathcal{S}} \pi_s = 1 \\
& \pi_{a,b}^s \geq 0 && \forall s \in \mathcal{S}, \forall (a,b) \in \mathcal{D}_s \\
& \pi_s \geq 0 && \forall s \in \mathcal{S}
\end{aligned}$$

We can prove that this LP has for vertexes pure policies by a simple counting of the number of variables and independent constraints, same proof used in Waserhole et al. (2012b). Indeed here the transition rate in state  $s \in \mathcal{S}$  for a demand  $(a,b) \in \mathcal{D}_s$  equals  $\lambda_{a,b}^s = \frac{\pi_{a,b}^s}{\pi_s} \times \Lambda_{a,b}$  and optimal solutions have either  $\pi_{a,b}^s = 0$  or  $\pi_{a,b}^s = \pi_s$ .  $\square$

**Lemma 3 (Pure policies sub optimality over static policies)** *Pure policies are sub optimal over static policies even when the number of vehicles tends to infinity.*

**Proof :** Let consider 4 stations  $(a,b,c,d)$  with maximum transition rates  $\Lambda_{a,b} = \Lambda_{b,c} = 3$ ,  $\Lambda_{c,d} = \Lambda_{d,a} = \Lambda_{c,a} = 2$  and all others equals to 0. Figure 2 represent such demands. There is only 2 possible pure static policies  $\lambda$  defining a connexe demand graph:  $\lambda_{i,j} = \Lambda_{i,j}$ ,  $(i,j) \neq (c,a)$  and  $\lambda_{c,a} = 0$  or 2.

Let's now look at the availability of a trip when the number of vehicles tends to infinity, with the formula given by George and Xia (2011) the availability of a vehicle a station  $a$  equals  $\frac{\pi_a}{\max_{b \in \mathcal{M}} \pi_b}$ , where  $\pi$  is the stationary distribution for one vehicle.

For  $\lambda_{c,a} = 0$  policy we have  $\pi_a = \pi_b = \frac{2}{10}$  and  $\pi_c = \pi_d = \frac{3}{10} = \pi_{\max}$ , so the average throughput when  $N \rightarrow \infty$  equals  $\frac{\pi_a}{\pi_{\max}}(3+3) + \frac{\pi_c}{\pi_{\max}}(2+2) = 8$ . And for  $\lambda_{c,a} = 2$  policy we have  $\pi_a = \pi_b = \frac{4}{14}$  and  $\pi_c = \pi_d = \frac{3}{14}$ , so the average throughput when  $N \rightarrow \infty$  equals 10.5 which is thus the best pure static policy.

Yet for a mixed policy (not pure) with  $\lambda_{c,a} = 1$  and  $\lambda_{i,j} = \Lambda_{i,j}$ ,  $(i,j) \neq (c,a)$  we have  $\pi_a = \pi_b = \pi_c = \pi_d = \frac{1}{4}$ , so the average throughput when  $N \rightarrow \infty$  equals 11  $>$  10.5. This shows us that pure policies are suboptimal even when the number of vehicles tends to infinity.  $\square$

## 3.2 Symmetric policies optimization

We are now going to consider a simple and intuitive class of policy that we call symmetric which consists in setting the prices in order to have as many people wanting to take a trip in each direction.

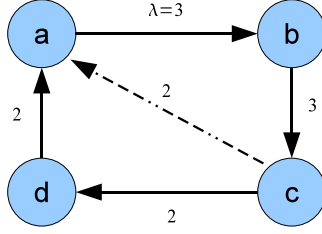


Figure 2: JW Parafuxe: Pure policies sub optimal even when the number of vehicles tends to infinity.

**Definition 2 (Symmetric policies)** A policy is said symmetric if it sets a same demand to take a trip and its opposite, i.e.  $\forall(a, b) \in \mathcal{D}, \lambda_{a,b} = \lambda_{b,a}$ .

In this section for simplification reasons we assume symmetry in the data  $\Lambda_{a,b} = \Lambda_{b,a}, \forall(a, b) \in \mathcal{D}$ , possibly using truncation as a preprocessing.

We want to study symmetric policies because they got a special feature: the Uniform Stationary Distribution (USD) over the states of the system (see lemma 4). The trick we use is that assuming a USD, computing the availability of vehicle at a station and hence the average throughput can be done by only counting the number of states of the system. What is nice is that with infinite station capacity the number of states can be computed analytically and that we can finally have a compact and simple form for the vehicle availability (see lemma 5).

**Lemma 4 (Uniform Stationary Distribution of Symmetric policies)** Markov Chains induced by symmetric policies have a uniform stationary distribution over the states.

**Proof :** Let consider the Continuous Time Markov Chain formulation for  $N$  vehicle and  $M$  stations, with system states  $\mathcal{S}$  and transition rate  $A$ . Without loss of generality we don't consider unreachable states. The stationary distribution  $\pi$  over the state  $\mathcal{S}$  of the system is solution of the following equations:

$$\begin{aligned} \sum_{i \in \mathcal{S}} \pi_i &= 1 \\ \sum_{(i,j) \in A} \pi_i \times \lambda_{i,j} &= \sum_{(j,i) \in A} \pi_j \times \lambda_{j,i} \quad \forall i \in \mathcal{S} \end{aligned}$$

Or if  $\forall(a, b) \in \mathcal{D}, \lambda_{a,b} = \lambda_{b,a}$ , we have  $\sum_{(i,j) \in A} \lambda_{i,j} = \sum_{(j,i) \in A} \lambda_{j,i}, \forall i \in \mathcal{S}$  and the uniform stationary distribution  $\pi_i = \frac{1}{|\mathcal{S}|}, \forall i \in \mathcal{S}$  is trivially solution of this system. Moreover it is the only one since the rank of the linear system is  $|\mathcal{S}|$  which is the number of variables.  $\square$

**Lemma 5 (Availability for USD with infinite station capacity)** For  $M$  stations with infinite station capacity,  $N$  vehicles and an Uniform Stationary Distribution over the system's states, the availability  $A$  of a vehicle at a station equals:

$$A = \frac{N}{N - 1 + M}$$

**Proof :** For  $M$  stations with infinite station capacity and  $N$  vehicles there is exactly  $\mathcal{N}_N^M = \binom{N+M-1}{N}$  states for the system. Therefore if there is an UDS over these states, the availability  $A$  of a vehicle at a station equals:

$$A = \frac{\mathcal{N}_{N-1}^M}{\mathcal{N}_N^M} = 1 - \frac{\mathcal{N}_N^{M-1}}{\mathcal{N}_N^M} = \frac{N}{N-1+M}$$

□

We are now interested in optimizing symmetric solution with static policies. We can see that pure policies are dominant over static policies (see lemma 6), which is giving us a compact combinatorial formulation for our optimization problem (see definition 3). However even if this formulation is compact it is not straightforwardly giving a polynomial algorithm to solve it and the question of its complexity can be therefore be raise. This is one of the contributions of this paper to prove that this problem is polynomial by giving a Linear Programming formulation (see lemma 7).

**Lemma 6 (Pure policies domination over static symmetric policies)** *Pure policies are dominant over all static symmetric policies maximizing the flow.*

**Proof :** By reduction to the absurd: If we have an optimal static symmetry with two open stations  $a$  and  $b$ , with a demand rate set at  $\lambda_{a,b} < \Lambda_{a,b}$ . Adjusting the demand rate to the maximum possible  $\lambda_{a,b} = \Lambda_{a,b}$  is not changing the stationary distribution and which is still USD over the states (see lemma 4), but is strictly increasing the overall flow. It means in the end mean that this solution is not optimal. □

**Corollary 1** *Station opening policies are dominant over all pure static symmetric policies. Hence looking for the best static symmetric policy amount in looking for sub-graph induced by a set of opened stations.*

**Definition 3 (Symmetric static policies optimization)**

- **INPUT:** For  $N$  vehicle, a set  $\mathcal{M}$  of  $M$  stations and a maximum demand  $\Lambda_{a,b}$  for every pair of stations  $(a,b) \in \mathcal{D}$ ;
- **OUTPUT:** A demand  $\lambda(a,b)$  for every pair of stations  $(a,b) \in \mathcal{D}$ , such that  $\lambda_{a,b} = \lambda_{b,a}$  and  $0 \leq \lambda_{a,b} \leq \Lambda_{a,b}$ ;
- **OBJECTIVE:** Maximize the average number of trip sold.

From corollary 1, lemma 4 and 5 we can state the following combinatorial formulation:

$$\max_{T \subseteq \mathcal{M}} \sum_{a,b \in T} \Lambda_{a,b} \times \frac{N}{N + |T| - 1}$$

**Lemma 7 (LP for Symmetric policies optimization)** *The following Linear Program gives the optimal symmetric static policy for a system with  $N$  vehicles and a set  $\mathcal{M}$  of stations with a set  $\mathcal{D}_u$  of undirected trips.*

$$\max 2 \times \sum_{(a,b) \in \mathcal{D}_u} \pi_{a,b} \times \Lambda_{a,b} \quad (1a)$$

$$s.t. \pi_{a,b} \leq \pi_a \quad \forall (a,b) \in \mathcal{D}_u \quad (1b)$$

$$\pi_{a,b} \leq \pi_b \quad \forall (a,b) \in \mathcal{D}_u \quad (1c)$$

$$(N-1) \times z + \sum_{a \in \mathcal{S}} \pi_a = N \quad (1d)$$

$$\pi_a \leq z \quad \forall a \in \mathcal{M} \quad (1e)$$

$$\pi_{a,b} \geq 0 \quad \forall (a,b) \in \mathcal{D}_u \quad (1f)$$

$$\pi_a \geq 0 \quad \forall a \in \mathcal{M} \quad (1g)$$

$$z \geq 0 \quad (1h)$$

**Proof :** Let's prove that the vertices of the linear program are an USD on a subset of opened stations. *i.e.*  $\exists T \subseteq \mathcal{M}$  such that  $\forall a \in T, \pi_a = \pi$  and  $\forall a \in \mathcal{M} \setminus T, \pi_a = 0$ , with  $\pi = \frac{N}{N-1+|T|}$ .

To do so we are going to prove that a solution with more than one value not equal to 0 for variables  $\pi_a, a \in \mathcal{M}$  is a linear combination of USD solutions.

Let's consider a non uniform optimal solution  $S^*$  with  $p$  different values for the variables  $\pi_a, a \in \mathcal{M}$ . Without loss of generality we are going to consider only stations with associated variable ( $\pi_a, a \in \mathcal{M}$ ) with values greater than 0. Let  $\Psi = [\psi_1 \dots \psi_p]$  be the ordered set of these values, with  $\forall 1 \leq i < j \leq p, \psi_i < \psi_j$ . Value  $\psi_i$  appears  $x_i$  times in the solution and we hence have a vector  $\chi = [x_1 \dots x_p]$  with  $\sum_{i=1}^p x_i = M$ : the number of variables  $\pi_a$  (number of opened stations).

If  $S^*$  is optimal, we have  $z = \psi_p$  and  $\pi_{a,b} = \min\{\pi_a^*, \pi_b^*\}, \forall (a,b \in \mathcal{D}_u)$ , otherwise we could strictly increase the solution. Therefore to ensure two optimal solutions are equal we just need to check variables  $\pi_a, a \in \mathcal{M}$ .

Let  $x_i^{\geq}$  be the number of variables  $\pi_a, a \in \mathcal{M}$  with value greater or equal than  $\psi_i$ :  $x_i^{\geq} = \sum_{j \geq i} x_j$  and  $\pi(k)$  be the USD for  $k$  stations open (associated variable  $\pi_a > 0$ ):  $\pi(k) = \frac{N}{N-1+k}$ .

We are now going to construct a linear combination of  $p$  USD solutions,  $[S^1 \dots S^p]$ , to build the non USD solution  $S^*$ : USD solution  $S^i$  will be composed with stations with associated variables with values in  $[\psi_i \dots \psi_p]$ , *i.e.* with  $M_i = a \in \mathcal{M}$  such that  $\pi_a \geq \psi_i$ . The USD for this solution will be equal to  $\pi(x_i^{\geq})$ , hence we have  $\forall a, b \in \mathcal{M}_i, \pi_a^i = \pi_{a,b}^i = \pi(x_i^{\geq})$ . The coefficient in the linear combination will be  $\alpha_i = \frac{\psi_i - \psi_{i-1}}{\pi(x_i^{\geq})}$ .

We have  $\pi_a^* = \sum_{j=1}^i \alpha_j \times \pi_a^i = \sum_{j=1}^i \alpha_j \times \pi(x_j^{\geq})$  with  $\sum_i \alpha_i = 1$ . Therefore we have a linear combination of USD solutions forming the non USD solution:  $\sum_{i=1}^p \alpha_i \times S^i = S^*$ . Which means in the end that non USD solutions are not vertices of the polygon.  $\square$

**Remark 1** Notice that for  $N = 1$  we have the well known densest weighted subgraph problem that corresponds to the Linear Formulation given by Charikar (2000). We can also notice that he proved it to be optimal through another type of proof, by the absurd.

### 3.3 Conservative policies optimization

Previous section we introduce and solve in the static case the symmetric policies that have the useful advantage to give a Uniform Stationary Distribution (USD) over the state of the induced Markov Chain. However this class of policies seems really restrictive and maybe the USD property can be found in other ones. This is the case of the conservative policies that are in fact containing the symmetric policies and that still have the USD property (see lemma 8).

**Definition 4 (Conservative policies)** A policy is said conservative if it sets in each station a same amount of demand take a vehicle as to drop one, i.e.  $\forall a \in \mathcal{M}, \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda_{b,a}$ .

**Lemma 8 (USD of Conservative policies)** Markov Chains induced by conservative policies have an USD the system states  $(\pi_i = \frac{1}{|\mathcal{S}|})$ .

**Proof :** Let consider the Continuous Time Markov Chain formulation for  $N$  vehicle and  $M$  stations, with system states  $\mathcal{S}$  and stochastic transitions  $A$ . Without loss of generality we don't consider unreachable states. The USD distribution  $\pi$  over the state  $\mathcal{S}$  of the system is solution of the following equations:

$$\begin{aligned} \sum_{i \in \mathcal{S}} \pi_i &= 1 \\ \sum_{(i,j) \in A} \pi_i \times \lambda_{i,j} &= \sum_{(j,i) \in A} \pi_j \times \lambda_{j,i} \quad \forall i \in \mathcal{S} \end{aligned}$$

Yet if  $\forall a \in \mathcal{M}, \sum_{b \in \mathcal{M}} \lambda_{a,b} = \sum_{b \in \mathcal{M}} \lambda_{b,a}$ , we have  $\sum_{(i,j) \in A} \lambda_{i,j} = \sum_{(j,i) \in A} \lambda_{j,i}, \forall i \in \mathcal{S}$  and the USD  $\pi_i = \frac{1}{|\mathcal{S}|}, \forall i \in \mathcal{S}$  is trivially solution of this system. Moreover it is the only one since the rank of the linear system is  $|\mathcal{S}|$  which is the number of variables.  $\square$

Lemma 5 is also applicable for conservative policy optimization since they have an USD. We can therefore have a compact combinatorial formulation for the optimization of conservative policies (see definition 5). The complexity of this new problem is again unclear. However we can extend the approach we used for symmetric policies to give a Linear Programming formulation (see lemma 9) and show that this problem is also polynomial.

**Definition 5 (Conservative static policies optimization)**

- **INPUT:** For  $N$  vehicle, a set  $\mathcal{M}$  of  $M$  stations and a maximum demand  $\Lambda_{a,b}$  for every pair of stations  $(a,b) \in \mathcal{D}$ ;

- **OUTPUT:** A demand  $\lambda(a, b)$  for every pair of stations  $(a, b) \in \mathcal{D}$ , such that  $\forall a \in \mathcal{M}$ ,  $\sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda_{b,a}$  with  $0 \leq \lambda_{a,b} \leq \Lambda_{a,b}$ ;
- **OBJECTIVE:** Maximize the average number of trip sold.

From lemma 8 and 5 we can state the following combinatorial formulation:

$$\begin{aligned}
& \max_{F \subseteq \mathcal{D}} \sum_{(a,b) \in F} \lambda_{a,b} \times \frac{N}{N-1 + |\mathcal{M}(F)|} \\
& \text{s.t.} \quad \sum_{(a,b) \in F} \lambda_{a,b} = \sum_{(b,a) \in F} \lambda_{b,a} \quad \forall a \in \mathcal{M}(F) \\
& \quad \quad 0 \leq \lambda_{a,b} \leq \Lambda_{a,b} \quad \forall (a,b) \in F
\end{aligned}$$

With  $\mathcal{M}(F) = \{a \in \mathcal{M} : (a, b) \in F \parallel (b, a) \in F\}$ .

**Lemma 9 (LP for conservative policies optimization)** *The following Linear Program gives the optimal conservative static policy for  $N$  vehicles for system with a set  $\mathcal{M}$  of stations and  $\mathcal{D}$  directed trip.*

$$\max \sum_{a,b \in \mathcal{M}} \pi_{a,b} \times \Lambda_{a,b} \quad (2a)$$

$$\text{s.t.} \quad \sum_{(a,b) \in \mathcal{D}} \pi_{a,b} \times \Lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \pi_{b,a} \times \Lambda_{b,a} \quad \forall a \in \mathcal{M} \quad (2b)$$

$$(N-1) \times z + \sum_{a \in \mathcal{M}} \pi_a = N \quad (2c)$$

$$\pi_a \leq z \quad \forall a \in \mathcal{M} \quad (2d)$$

$$\pi_{a,b} \leq \pi_a \quad \forall (a,b) \in \mathcal{D} \quad (2e)$$

$$\pi_{a,b} \leq \pi_b \quad \forall (a,b) \in \mathcal{D} \quad (2f)$$

$$\pi_{a,b} \geq 0 \quad \forall (a,b) \in \mathcal{D} \quad (2g)$$

$$\pi_a \geq 0 \quad \forall a \in \mathcal{M} \quad (2h)$$

$$z \geq 0 \quad (2i)$$

**Proof :** We can use the same proof as for symmetric policies optimization, making a linear combination of  $p$  USD solution to form any optimal non USD solution  $S^*$ :  $S^* = \sum_{i=1}^p \alpha_i \times S^i$ , except that here we have to take a special care to variable  $\pi_{a,b}$ ,  $\forall (a,b) \in \mathcal{D}$ .

We now have in USD solution  $S^i$  variables  $\pi_{a,b}^i = \alpha_i \times \pi_{a,b}$ ,  $\forall (a,b) \in \mathcal{D}$ ,  $\forall i \in [1, p]$ . Indeed it is making a feasible solution since that flow constraints are respected:

$$\begin{aligned}
& \sum_{(a,b) \in \mathcal{D}} \alpha_i \times \pi_{a,b} \times \Lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \alpha_i \times \pi_{b,a}^i \times \Lambda_{b,a} \quad \forall a \in \mathcal{M}, \forall i \in [1, p] \\
& \Leftrightarrow \sum_{(a,b) \in \mathcal{D}} \pi_{a,b} \times \Lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \pi_{b,a} \times \Lambda_{b,a} \quad \forall a \in \mathcal{M}, \forall i \in [1, p]
\end{aligned}$$

And that the capacity constraints are also respected:

$$\pi_{a,b}^i = \alpha_i \times \pi_{a,b} \leq \alpha_i \times \pi_a = \pi_a^i \quad \forall (a,b) \in \mathcal{D}, \forall i \in [1,p]$$

□

One can think that optimizing conservative static policies gives a good approximation on any static policies, however as shown in conjecture 1 in theory it is at least a  $M$ -Approximation which is pretty bad. Finally in practice we think that it can give some interesting results.

**Remark 2** *In a demand graph for  $M$  stations induced by a circuit with  $M$  vertexes and transition rates  $(1, k, \dots, k)$ . For one vehicle, the average flow equals  $\frac{M \times k}{(M-1)+k}$ . The best conservative policy open all trips with  $\lambda = 1$  has an average flow equals to 1 while the best static policy opening all trips to their maximum has a flow  $M$  when  $k \rightarrow \infty$ .*

**Conjecture 1** *Optimal conservative static policies on  $M$  stations, gives an  $M$ -Approximation on all static policies. Asymptotically tight with a circuit with transition rate  $(1, k, \dots, k)$  with  $k \rightarrow \infty$ .*

**Lemma 10** *For a given number of vehicles the optimal conservative policy gives a  $M$ -approximation. (with  $M$  the number of stations).*

**Lemma 11** *For a given demand the optimal conservative policy is asymptotically tight when the number of vehicles tends to infinity.*

## 4 Maximum circulation asymptotic approximation

Previous sections we had to figure out some complex new Linear Programming formulations, computing explicitly the USD over the system's states in function of the number of vehicles and the number of stations finally opened. However looking at the conservative policies optimization, we can see that if finally all stations are opened, the solution given is the same as a classical combinatorial problem MAXIMUM CIRCULATION, known polynomial (see Edmonds and Karp (1972)) with a straightforward Linear Programming formulation.

**Definition 6 (Maximum Circulation)** *The Circulation problem is a generalisation of network flow problems with flow conservation also being required for the source and sink. The following Linear Programming formulation solves the maximum flow for a circulation: the MAXIMUM CIRCULATION problem.*

$$\max \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} \tag{3a}$$

$$s.t. \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda_{b,a} \quad \forall a \in \mathcal{M} \tag{3b}$$

$$0 \leq \lambda_{a,b} \leq \Lambda_{a,b} \quad \forall (a,b) \in \mathcal{D} \tag{3c}$$



In this section we are going to study the approximation given by the circulation optimization. This approximation is independent from the number of vehicles but we show in the following that it is in fact giving the best policy when the number of vehicles tends to infinity.

**Remark 3 (A maximum circulation opens all stations)** *If the demand graph is complete, i.e.  $\forall a, b \in \mathcal{D}$ ,  $\Lambda_{a,b} > 0$ , then all stations are open in the optimal circulation flow solution (i.e.  $\forall a \in \mathcal{M} \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} > 0$ ). Indeed, if station  $a$  was close it would increase the overall circulation flow to open a trip to any station  $b$  with symmetric demand:  $\lambda_{a,b} = \lambda_{b,a} = \min\{\Lambda_{a,b}, \Lambda_{b,a}\}$ .*

We are now proving that when the number of vehicles tends to infinity ( $N \rightarrow \infty$ ), MAXIMUM CIRCULATION gives the best dynamic policy. To do so we are first proving that MAXIMUM CIRCULATION gives the best conservative policies when  $N \rightarrow \infty$  (see lemma 12). Then we prove that the conservative policies are dominant over static policies when  $N \rightarrow \infty$  (see lemma 13). And finally that the static policies are themselves dominant over dynamic policies (see lemma 14).

**Remark 4 (Extension to consider transportation times)** *When the number of vehicle tends to infinity transportation times have no impact on optimal solution (proof given in lemma 13), therefore we can extend the following lemma to the case with transportation times.*

**Lemma 12 (Maximum Circulation asymptotic optimal over conservative policies)** *When the number of vehicles tends to infinity,  $N \rightarrow \infty$ , the probability  $\frac{N}{N-1+|\mathcal{M}(F)|}$  to have a vehicle available in any station tends to 1 and therefore it amounts in solving MAXIMUM CIRCULATION.*

**Proof :** Recall of the static conservative pricing problem:

$$\begin{aligned} \max_{F \subseteq \mathcal{D}} \quad & \sum_{(a,b) \in F} \lambda_{a,b} \times \frac{N}{N-1+|\mathcal{M}(F)|} \\ \text{s.t.} \quad & \sum_{(a,b) \in F} \lambda_{a,b} = \sum_{(b,a) \in F} \lambda_{b,a} && \forall a \in \mathcal{M}(F) \\ & 0 \leq \lambda_{a,b} \leq \Lambda_{a,b} && \forall (a,b) \in F \end{aligned}$$

When the number of vehicles tends to infinity,  $N \rightarrow \infty$ , the probability  $\frac{N}{N-1+|\mathcal{M}(F)|}$  to have a vehicle available in any station tends to 1 and therefore it amounts in solving MAXIMUM CIRCULATION.  $\square$

**Lemma 13 (Maximum Circulation asymptotically optimal over conservative policies)** *When the number of vehicles tends to infinity,  $N \rightarrow \infty$ , MAXIMUM CIRCULATION gives the optimal static policy.*

**Proof :** Let  $\pi$  be the stationary distribution for  $N = 1$  vehicle and  $M$  stations without transportation time.

Let  $\pi_{\max} = \max_{a \in \mathcal{M}} \pi_a$ , from Theorem 1 in George and Xia (2011), we have that the availability  $A_a$  of a vehicle at a station  $a \in \mathcal{M}$  when the number  $N$  of vehicles tends to infinity equals:

$$\lim_{N \rightarrow \infty} (A(N)) = \lim_{N \rightarrow \infty} \left( \pi_a \frac{G(N-1)}{G(N)} \right) = \frac{\pi_a}{\pi_{\max}}$$

Note that George and Xia (2011) stated it with  $\pi$  the stationary distribution for  $N = 1$  vehicle and  $M$  stations WITH transportation time. But simply looking at the linear equation with transportation time for one vehicle, we can see that the ratio  $\frac{\pi_a}{\pi_b}$ ,  $\forall a, b \in \mathcal{M}$  is independent from the transportation times. Therefore we can restrain to the computation of  $\pi$  without transportation time.

The static policy giving the performance of the system when  $N \rightarrow \infty$  is thus given by the following program:

$$\begin{aligned} \max \quad & \sum_{(a,b) \in \mathcal{D}} \frac{1}{\pi_{\max}} \lambda_{a,b} \\ \text{s.t.} \quad & \sum_{(a,b) \in \mathcal{D}} \lambda_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda_{b,a} && \forall a \in \mathcal{M} \\ & 0 \leq \lambda_{a,b} \leq \Lambda_{a,b} \times \pi_a && \forall (a,b) \in \mathcal{D} \\ & \sum_{a \in \mathcal{M}} \pi_a = 1 \end{aligned}$$

Making a change of variable  $\lambda'_{a,b} = \frac{\lambda_{a,b}}{\pi_{\max}}$  i.e.  $\lambda_{a,b} = \pi_{\max} \times \lambda'_{a,b}$  we have:

$$\max \quad \sum_{(a,b) \in \mathcal{D}} \lambda'_{a,b} \tag{4a}$$

$$\text{s.t.} \quad \sum_{(a,b) \in \mathcal{D}} \lambda'_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda'_{b,a} \quad \forall a \in \mathcal{M} \tag{4b}$$

$$0 \leq \lambda'_{a,b} \leq \Lambda_{a,b} \times \frac{\pi_a}{\pi_{\max}} \quad \forall (a,b) \in \mathcal{D} \tag{4c}$$

$$\sum_{a \in \mathcal{M}} \pi_a = 1 \tag{4d}$$

Yet  $\forall a \in \mathcal{M}$ ,  $\frac{\pi_a}{\pi_{\max}} \leq 1$ , so taking  $\forall a \in \mathcal{M}$ ,  $\pi_a = \frac{1}{|\mathcal{M}|} \rightarrow \frac{\pi_a}{\pi_{\max}} = 1$  does not reduce the solution space of constraints (4c) and more over satisfies constraint (4d). It finally gives us MAXIMUM CIRCULATION:

$$\begin{aligned} \max \quad & \sum_{(a,b) \in \mathcal{D}} \lambda'_{a,b} \\ \text{s.t.} \quad & \sum_{(a,b) \in \mathcal{D}} \lambda'_{a,b} = \sum_{(b,a) \in \mathcal{D}} \lambda'_{b,a} && \forall a \in \mathcal{M} \\ & 0 \leq \lambda'_{a,b} \leq \Lambda_{a,b} && \forall (a,b) \in \mathcal{D} \end{aligned}$$

□

**Lemma 14 (Maximum Circulation asymptotically optimal over dynamic policies)** *When the number of vehicles tends to infinity, MAXIMUM CIRCULATION gives the best dynamic policy.*

**Proof :** The system under such a dynamic pricing scheme can be shown to be stationary and ergodic under very general conditions. Therefore we can look at the stationary distribution.

Let's prove first that the maximum circulation is an upper bound on dynamic policies for any number of vehicles. For a stationary distribution  $\pi$  of any dynamic pricing policies giving a transition rate  $\lambda_{a,b}^s \leq \Lambda_{a,b}$  in state  $s$  (state dependent) for trip  $(a, b) \in \mathcal{D}$  we have the following equations of optimality satisfied:

$$\begin{aligned} \sum_{(a,b) \in \mathcal{D}} \pi_s \times \lambda_{a,b}^s &= \sum_{(b,a) \in \mathcal{D}} \pi_{s-\{(b,a)\}} \times \lambda_{b,a}^s & \forall s \in \mathcal{S} \\ \sum_{s \in \mathcal{S}} \pi_s &= 1 \end{aligned}$$

For an average throughput of the system equal to:

$$\sum_{s \in \mathcal{S}} \sum_{(a,b) \in \mathcal{D}} \pi_s \times \lambda_{a,b}^s = \sum_{(a,b) \in \mathcal{D}} \lambda'_{a,b}$$

With  $\lambda'_{a,b}$  the average throughput for a trip  $(a, b) \in \mathcal{D}$  we have:

$$\sum_{s \in \mathcal{S}} \pi_s \times \lambda_{a,b}^s = \lambda'_{a,b} \leq \Lambda_{a,b} \quad \forall (a, b) \in \mathcal{D}$$

Yet  $\lambda'$  satisfy also the flow conservation constraints and the capacity constraints in the circulation problem

$$\begin{aligned} \sum_{(a,b) \in \mathcal{D}} \lambda'_{a,b} &= \sum_{(b,a) \in \mathcal{D}} \lambda'_{b,a} & \forall a \in \mathcal{M} \\ 0 \leq \lambda'_{a,b} &\leq \Lambda_{a,b} & \forall (a, b) \in \mathcal{D} \end{aligned}$$

Finally the flow conservation constraints are respected because otherwise it would mean that in the stationary state of the dynamic policy, a station would receive more vehicles that it is sending which is absurd. The capacity constraints is respected since  $\sum_{s \in \mathcal{S}} \pi_s = 1$ .

In the end we have that  $\lambda'_{a,b}$  is solution of MAXIMUM CIRCULATION, moreover with the same objective value. Hence it proves that MAXIMUM CIRCULATION is an upper bound on any dynamic policies. And when the number of vehicles tends to infinity,  $N \rightarrow \infty$ , MAXIMUM CIRCULATION is asymptotically giving the best static policies from lemma 13. Therefore static policies are asymptotically dominant over dynamic policies and MAXIMUM CIRCULATION gives the optimal dynamic policy. □

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