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# Gradient schemes: a generic framework for the discretisation of linear, nonlinear and nonlocal elliptic and parabolic equations

Jérôme Droniou<sup>1</sup>, Robert Eymard<sup>2</sup>, Thierry Gallouët<sup>3</sup>, Raphaèle Herbin<sup>4</sup>.  
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## Abstract

Gradient schemes are nonconforming methods written in discrete variational formulation and based on independent approximations of functions and gradients, using the same degrees of freedom. Previous works showed that several well-known methods fall in the framework of gradient schemes. Four properties, namely coercivity, consistency, limit-conformity and compactness, are shown in this paper to be sufficient to prove the convergence of gradient schemes for linear and nonlinear elliptic and parabolic problems, including the case of nonlocal operators arising for example in image processing. We also show that the Hybrid Mixed Mimetic family, which includes in particular the Mimetic Finite Difference schemes, may be seen as gradient schemes meeting these four properties, and therefore converges for the class of above mentioned problems.

## 1 Introduction

We consider general elliptic equations of the form

$$\begin{aligned} -\operatorname{div} \mathbf{a}(\mathbf{x}, \bar{u}, \nabla \bar{u}) &= f \text{ in } \Omega, \\ \bar{u} &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.1}$$

where  $\Omega$  is an open bounded connected subset of  $\mathbb{R}^d$ , for  $d \in \mathbb{N}^*$  with a boundary denoted by  $\partial\Omega = \bar{\Omega} \setminus \Omega$ . The solution  $\bar{u}$  is sought in the space  $W_0^{1,p}(\Omega)$  for some  $p \in (1, +\infty)$ . Particular choices of  $\mathbf{a}$  include general anisotropic heterogeneous linear operators  $\mathbf{a}(\mathbf{x}, u, \boldsymbol{\xi}) = \Lambda(\mathbf{x})\boldsymbol{\xi}$ , Leray-Lions operators such as the  $p$ -Laplacian  $\mathbf{a}(\mathbf{x}, u, \boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\boldsymbol{\xi}$ , and some nonlinear and nonlocal diffusion operators  $\mathbf{a}(\mathbf{x}, u, \boldsymbol{\xi}) = \Lambda(u, \mathbf{x})\boldsymbol{\xi}$  for  $u$  in a given functional space.

We shall also consider the evolution problem associated to problem (1.1), which is the following nonlinear parabolic problem (where  $T \in (0, +\infty)$ ):

$$\begin{aligned} \partial_t \bar{u} - \operatorname{div} \mathbf{a}(\mathbf{x}, \bar{u}, \nabla \bar{u}) &= f \text{ in } \Omega \times (0, T), \\ \bar{u}(\mathbf{x}, 0) &= u_{\text{ini}}(\mathbf{x}) \text{ in } \Omega, \\ \bar{u} &= 0 \text{ on } \partial\Omega \times (0, T). \end{aligned} \tag{1.2}$$

Such evolution equations, involving non local operators, arise in particular in image processing, in the spirit of [10, 14, 29] and references therein. The linear anisotropic heterogeneous case is involved in most models used in underground engineering (oil recovery, nuclear waste disposals, etc.). In these models, computations have to be performed on meshes adapted to the geological layers, and including complex geometrical features such as faults, vanishing layers, inclined wells, highly heterogeneous permeability fields, local nonconforming refinement. Since standard finite element methods are not well adapted under such constraints, a large number of schemes have been developed for the numerical approximation of (1.1) and (1.2) in this case. Although we cannot give here an exhaustive list, let us mention only a few of them:

- the Multi-Point Flux Approximation (MPFA) schemes [1],

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- the Hybrid Mixed Mimetic family which includes the Mimetic Finite Difference schemes, the SUSHI scheme and the Mixed Finite Volume scheme, see [18] and references therein,
- the Discrete Duality Finite Volume (DDFV) schemes [25, 13, 5].

A construction and proof of convergence in the case of nonlinear Leray-Lions operators is already known for some of these methods, namely the Mixed Finite Volume method [15], the DDFV scheme [2], the SUSHI scheme in its cell centred version [19] (see also [6] for the DG scheme for the  $p$ -Laplacian).

Although the analytical tools used to study these methods are often similar, they are usually considered as different schemes whose study requires new work each time. However, as noticed in [21, 23, 22], many of these methods can be included in the unified theoretical framework of (possibly) nonconforming gradient schemes. In particular, the following methods are gradient schemes:

- some MPFA and DDFV schemes in 2D or 3D,
- the Galerkin methods, including the conforming finite element methods,
- the nonconforming P1 finite element discretisation,
- the Mixed Finite Element discretisations.

The aim of this paper is to show that gradient schemes, which can be characterised by only a small number of discrete elements, have the two following interesting properties:

1. They provide a generic framework in which, using only a small number of discrete notions, error estimate for the linear stationary case and convergence proofs for both the nonlinear stationary and transient cases can be obtained.
2. They encompass the entire Hybrid Mixed Mimetic family, and thus in particular the Mimetic Finite Difference methods. Given the success of these methods for linear problems, see e.g. [8, 9, 3, 4], we find quite exciting and remarkable to extend them to fully nonlinear problems and to prove their convergence in this setting. **Note that the implementation of the HMM methods, seen as gradient schemes, to the linear problems studied here, is identical to the standard one, using the “pressure” unknowns at the faces of the mesh (and not the flux unknowns); when applied to the nonlinear problems, it leads to a system of nonlinear algebraic equations with the same stencil as that of the linear problem.**

This paper is organised as follows. In Section 2, we present the small number of discrete elements which are needed to define a gradient scheme. In Section 3, we consider the stationary cases. We provide an error estimate in the linear case, and a convergence proof for Leray-Lions problems including a non-local dependency of the operator. In Section 4, we give a convergence proof for the time-dependent Leray-Lions problem, using a generic discrete Aubin-Lions theorem. A particularly remarkable fact is that these proofs are made under very few and generic discretisation assumptions. Finally, in Section 5, we show that all the schemes derived from the entire Hybrid Mixed Mimetic family are gradient schemes **such that the properties needed for the convergence analysis of Sections 3–4 hold**, and thus that these methods are suitable not only for linear problems but also for nonlinear equations.

## 2 Gradient discretisations and gradient schemes

### 2.1 Definitions

We present here properties which are shown in the next sections to be sufficient for the convergence of gradient schemes, considering homogeneous Dirichlet boundary conditions.

A gradient scheme can be viewed as a general formulation of several discretisations of (3.3) which are based on a nonconforming approximation of the weak formulation of the problem. The approximation

of the weak formulation of (3.3) is based on some discrete spaces and mappings, the set of which we call a gradient discretisation. Throughout this paper,  $\Omega$  is an open bounded subset of  $\mathbb{R}^d$ ,  $d \in \mathbb{N}^*$ , and  $p \in (1, +\infty)$ .

**Definition 2.1 (Gradient discretisation)** *A gradient discretisation  $\mathcal{D}$  of Problem (1.1) is defined by  $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ , where:*

1. *the set of discrete unknowns  $X_{\mathcal{D},0}$  is a finite dimensional vector space on  $\mathbb{R}$ ,*
2. *the linear mapping  $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \rightarrow L^p(\Omega)$  is the reconstruction of the approximate function,*
3. *the linear mapping  $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \rightarrow L^p(\Omega)^d$  is the discrete gradient operator. It must be chosen such that  $\|\cdot\|_{\mathcal{D}} := \|\nabla_{\mathcal{D}} \cdot\|_{L^p(\Omega)^d}$  is a norm on  $X_{\mathcal{D},0}$ .*

**Remark 2.2 (Boundary conditions.)** *The definition of  $\|\cdot\|_{\mathcal{D}}$  depends on the considered boundary conditions. Here for simplicity we only consider homogeneous Dirichlet boundary conditions, but other conditions can easily be addressed. For example, in the case of homogeneous Neumann boundary conditions, we will use the notation  $X_{\mathcal{D}}$  instead of  $X_{\mathcal{D},0}$  for the discrete space, and define  $\|\cdot\|_{\mathcal{D}} := (\|\Pi_{\mathcal{D}} \cdot\|_{L^p(\Omega)}^p + \|\nabla_{\mathcal{D}} \cdot\|_{L^p(\Omega)^d}^p)^{1/p}$ .*

The related gradient scheme is merely the discretisation of the weak formulation of (1.1) obtained by using the discrete space and mappings of the gradient discretisation.

**Definition 2.3 (Gradient scheme)** *If  $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$  is a gradient discretisation, then we define the related gradient scheme for (1.1) by*

$$\text{Find } u \in X_{\mathcal{D},0} \text{ such that, } \forall v \in X_{\mathcal{D},0}, \quad \int_{\Omega} \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}}u, \nabla_{\mathcal{D}}u(\mathbf{x})) \cdot \nabla_{\mathcal{D}}v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) \Pi_{\mathcal{D}}v(\mathbf{x}) d\mathbf{x}. \quad (2.1)$$

Since  $X_{\mathcal{D},0}$  is a finite dimensional space, there exists at least one solution to (2.1) provided that  $\mathbf{a}$  and  $f$  satisfy the usual assumptions that ensure the existence of a weak solution to (1.1) (see Section 3). For the solution of this finite dimensional problem to converge to a weak solution of (1.1), we need of course some consistency and stability properties to be satisfied. As in the framework of the finite element methods, the stability is obtained thanks to some uniform coercivity of the discrete operator which relies on a discrete Poincaré inequality.

**Definition 2.4 (Coercivity)** *Let  $\mathcal{D}$  be a gradient discretisation for Problem (1.1) in the sense of Definition 2.1, and let  $C_{\mathcal{D}}$  be the norm of the linear mapping  $\Pi_{\mathcal{D}}$ , defined by*

$$C_{\mathcal{D}} = \max_{v \in X_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}}v\|_{L^p(\Omega)}}{\|v\|_{\mathcal{D}}}. \quad (2.2)$$

*A sequence  $(\mathcal{D}_m)_{m \in \mathbb{N}}$  of gradient discretisations is said to be **coercive** if there exists  $C_P \in \mathbb{R}_+$  such that  $C_{\mathcal{D}_m} \leq C_P$  for all  $m \in \mathbb{N}$ .*

**Remark 2.5 (Discrete Poincaré inequality.)** *Equation (2.2) yields  $\|\Pi_{\mathcal{D}}v\|_{L^p(\Omega)} \leq C_{\mathcal{D}} \|\nabla_{\mathcal{D}}v\|_{L^p(\Omega)^d}$ .*

The consistency is ensured by a proper choice of the interpolation operator and discrete gradient.

**Definition 2.6 (Consistency)** *Let  $\mathcal{D}$  be a gradient discretisation for Problem (1.1) in the sense of Definition 2.1, and let  $S_{\mathcal{D}} : W_0^{1,p}(\Omega) \rightarrow [0, +\infty)$  be defined by*

$$\forall \varphi \in W_0^{1,p}(\Omega), \quad S_{\mathcal{D}}(\varphi) = \min_{v \in X_{\mathcal{D},0}} (\|\Pi_{\mathcal{D}}v - \varphi\|_{L^p(\Omega)} + \|\nabla_{\mathcal{D}}v - \nabla\varphi\|_{L^p(\Omega)^d}). \quad (2.3)$$

*A sequence  $(\mathcal{D}_m)_{m \in \mathbb{N}}$  of gradient discretisations is said to be **consistent** if, for all  $\varphi \in W_0^{1,p}(\Omega)$ ,  $S_{\mathcal{D}_m}(\varphi)$  tends to 0 as  $m \rightarrow \infty$ .*

Since we are dealing with nonconforming methods, we need that the dual of the discrete gradient be “close to” a discrete divergence.

**Definition 2.7 (Limit-conformity)** Let  $\mathcal{D}$  be a gradient discretisation for Problem (1.1) in the sense of Definition 2.1. We let  $p' = \frac{p}{p-1}$ ,  $W^{\text{div},p'}(\Omega) = \{\boldsymbol{\varphi} \in L^{p'}(\Omega)^d, \text{div}\boldsymbol{\varphi} \in L^{p'}(\Omega)\}$  and  $W_{\mathcal{D}}: W^{\text{div},p'}(\Omega) \rightarrow [0, +\infty)$  be defined by

$$\forall \boldsymbol{\varphi} \in W^{\text{div},p'}(\Omega), \quad W_{\mathcal{D}}(\boldsymbol{\varphi}) = \max_{u \in X_{\mathcal{D},0} \setminus \{0\}} \frac{1}{\|u\|_{\mathcal{D}}} \left| \int_{\Omega} (\nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x}) + \Pi_{\mathcal{D}} u(\mathbf{x}) \text{div}\boldsymbol{\varphi}(\mathbf{x})) \, d\mathbf{x} \right|. \quad (2.4)$$

A sequence  $(\mathcal{D}_m)_{m \in \mathbb{N}}$  of gradient discretisations is said to be **limit-conforming** if, for all  $\boldsymbol{\varphi} \in W^{\text{div},p'}(\Omega)$ ,  $W_{\mathcal{D}_m}(\boldsymbol{\varphi})$  tends to 0 as  $m \rightarrow \infty$ .

Dealing with generic non-linearity often requires compactness properties on the scheme.

**Definition 2.8 (Compactness)** Let  $\mathcal{D}$  be a gradient discretisation for Problem (1.1) in the sense of Definition 2.1, and let  $T_{\mathcal{D}}: \mathbb{R}^d \rightarrow \mathbb{R}^+$  be defined by

$$\forall \boldsymbol{\xi} \in \mathbb{R}^d, \quad T_{\mathcal{D}}(\boldsymbol{\xi}) = \max_{v \in X_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}} v(\cdot + \boldsymbol{\xi}) - \Pi_{\mathcal{D}} v\|_{L^p(\mathbb{R}^d)}}{\|v\|_{\mathcal{D}}} \quad (2.5)$$

where  $\Pi_{\mathcal{D}} v$  has been extended by 0 outside  $\Omega$ .

A sequence  $(\mathcal{D}_m)_{m \in \mathbb{N}}$  of gradient discretisations is said to be **compact** if the following uniform limit holds:

$$\lim_{|\boldsymbol{\xi}| \rightarrow 0} \sup_{m \in \mathbb{N}} T_{\mathcal{D}_m}(\boldsymbol{\xi}) = 0.$$

Thanks to [21, Lemma 2.4], we may check the consistency and limit-conformity properties of given gradient schemes, only using dense subsets of the test functions spaces. The following lemma, useful in Section 5, is an immediate consequence of [21, Lemma 2.4].

**Lemma 2.9 (Sufficient conditions)** Let  $\mathcal{F}$  be a family of gradient discretisations for Problem (1.1) in the sense of Definition 2.1. Assume that there exist  $C, \nu \in (0, \infty)$  and, for all  $\mathcal{D} \in \mathcal{F}$ , a real value  $h_{\mathcal{D}} \in (0, +\infty)$  such that:

$$C_{\mathcal{D}} \leq C, \quad (2.6a)$$

$$S_{\mathcal{D}}(\boldsymbol{\varphi}) \leq Ch_{\mathcal{D}} \|\boldsymbol{\varphi}\|_{W^{2,\infty}(\Omega)}, \text{ for all } \boldsymbol{\varphi} \in C_c^{\infty}(\Omega), \quad (2.6b)$$

$$W_{\mathcal{D}}(\boldsymbol{\varphi}) \leq Ch_{\mathcal{D}} \|\boldsymbol{\varphi}\|_{(W^{1,\infty}(\mathbb{R}^d))^d}, \text{ for all } \boldsymbol{\varphi} \in C_c^{\infty}(\mathbb{R}^d)^d, \quad (2.6c)$$

$$T_{\mathcal{D}}(\boldsymbol{\xi}) \leq C|\boldsymbol{\xi}|^{\nu}, \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d, \quad (2.6d)$$

where  $C_{\mathcal{D}}, S_{\mathcal{D}}, W_{\mathcal{D}}$  and  $T_{\mathcal{D}}$  are defined by (2.2)-(2.5).

Then, any sequence  $(\mathcal{D}_m)_{m \in \mathbb{N}} \subset \mathcal{F}$  such that  $h_{\mathcal{D}_m} \rightarrow 0$  as  $m \rightarrow \infty$  is coercive, consistent, limit-conforming and compact.

**Remark 2.10** In several cases,  $h_{\mathcal{D}}$  stands for the mesh size: this is for instance the case for the analysis of the Hybrid Mimetic Mixed schemes in Section 5.

## 3 Elliptic problems

### 3.1 Error estimate in the linear case

We recall an error estimate which was obtained in [21] in the linear case. We consider the following problem, corresponding to (1.1) with  $p = 2$  and  $a(\mathbf{x}, s, \boldsymbol{\xi}) = \Lambda(\mathbf{x})\boldsymbol{\xi}$ :

$$\begin{aligned} -\text{div}(\Lambda(\mathbf{x})\nabla \bar{u}) &= f \text{ in } \Omega, \\ \bar{u} &= 0 \text{ on } \partial\Omega, \end{aligned} \quad (3.1)$$

with

$$\begin{aligned} \Lambda : \Omega \rightarrow \mathcal{S}_d(\mathbb{R}) \text{ measurable s.t. } \Lambda(\mathbf{x}) \text{ has eigenvalues in } (\underline{\lambda}, \bar{\lambda}) \subset (0, +\infty) \text{ for a.e. } \mathbf{x} \in \Omega, \\ f \in L^2(\Omega), \end{aligned} \quad (3.2)$$

( $\mathcal{S}_d(\mathbb{R})$  is the set of  $d \times d$  symmetric matrices). Under these hypotheses, the weak solution of (1.1) is the unique function  $\bar{u}$  satisfying:

$$\begin{cases} \bar{u} \in H_0^1(\Omega), \\ \int_{\Omega} \Lambda(\mathbf{x}) \nabla \bar{u}(\mathbf{x}) \cdot \nabla \bar{v}(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) \bar{v}(\mathbf{x}) d\mathbf{x} \quad \forall \bar{v} \in H_0^1(\Omega). \end{cases} \quad (3.3)$$

Problem (3.3) is approximated by Scheme (2.1) with  $a(\mathbf{x}, s, \boldsymbol{\xi}) = \Lambda(\mathbf{x})\boldsymbol{\xi}$ . The following lemma, proved in [21], is in the spirit of the results given in [28].

**Lemma 3.1 (Control of the approximation error)** *Under Hypotheses (3.2), let  $\bar{u} \in H_0^1(\Omega)$  be the solution of (3.3) (remark that since  $f \in L^2(\Omega)$ , one has  $\Lambda \nabla \bar{u} \in W^{\text{div},2}(\Omega)$ ).*

*Let  $\mathcal{D}$  be a gradient discretisation in the sense of Definition 2.1 with  $p = 2$ . Then there exists one and only one  $u_{\mathcal{D}} \in X_{\mathcal{D},0}$  solution to the gradient scheme (2.1). This solution moreover satisfies the following inequalities:*

$$\begin{aligned} \|\nabla \bar{u} - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} &\leq \frac{1}{\underline{\lambda}} (W_{\mathcal{D}}(\Lambda \nabla \bar{u}) + (\bar{\lambda} + \underline{\lambda}) S_{\mathcal{D}}(\bar{u})), \\ \|\bar{u} - \Pi_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)} &\leq \frac{1}{\underline{\lambda}} (C_{\mathcal{D}} W_{\mathcal{D}}(\Lambda \nabla \bar{u}) + (C_{\mathcal{D}} \bar{\lambda} + \underline{\lambda}) S_{\mathcal{D}}(\bar{u})), \end{aligned}$$

where  $C_{\mathcal{D}}$ ,  $S_{\mathcal{D}}$  and  $W_{\mathcal{D}}$  are defined by (2.2)-(2.4).

**Remark 3.2** *In this linear case, compactness of the gradient discretisation is not necessary to obtain the convergence of the scheme (see also Remark 3.6).*

## 3.2 Convergence in the nonlinear case

We now study the convergence for the more general nonlinear framework of Problem (1.1) under the following assumptions:

$$\mathbf{a} : \Omega \times L^p(\Omega) \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \text{ with } p \in (1, +\infty), \text{ is a Caratheodory function,} \quad (3.4a)$$

(i.e. a function such that for a.e.  $\mathbf{x} \in \Omega$ ,  $(u, \boldsymbol{\xi}) \mapsto \mathbf{a}(\mathbf{x}, u, \boldsymbol{\xi})$  is continuous, and for any  $(u, \boldsymbol{\xi}) \in L^p(\Omega) \times \mathbb{R}^d$  the function  $\mathbf{x} \mapsto \mathbf{a}(\mathbf{x}, u, \boldsymbol{\xi})$  is measurable)

$$\exists \underline{a} \in (0, +\infty) : \mathbf{a}(\mathbf{x}, u, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq \underline{a} |\boldsymbol{\xi}|^p, \text{ for a.e. } \mathbf{x} \in \Omega, \forall u \in L^p(\Omega), \forall \boldsymbol{\xi} \in \mathbb{R}^d, \quad (3.4b)$$

$$(\mathbf{a}(\mathbf{x}, u, \boldsymbol{\xi}) - \mathbf{a}(\mathbf{x}, u, \boldsymbol{\chi})) \cdot (\boldsymbol{\xi} - \boldsymbol{\chi}) \geq 0, \text{ for a.e. } \mathbf{x} \in \Omega, \forall u \in L^p(\Omega), \forall \boldsymbol{\xi}, \boldsymbol{\chi} \in \mathbb{R}^d, \quad (3.4c)$$

$$\exists \bar{a} \in L^{p'}(\Omega), \exists \mu \in (0, +\infty) :$$

$$|\mathbf{a}(\mathbf{x}, u, \boldsymbol{\xi})| \leq \bar{a}(\mathbf{x}) + \mu |\boldsymbol{\xi}|^{p-1}, \text{ for a.e. } \mathbf{x} \in \Omega, \forall u \in L^p(\Omega), \forall \boldsymbol{\xi} \in \mathbb{R}^d, \quad (3.4d)$$

and

$$f \in L^{p'}(\Omega) \text{ where } p' = \frac{p}{p-1}. \quad (3.5)$$

**Remark 3.3** *Note that the dependence of  $\mathbf{a}$  on  $u$  is assumed to be non-local:  $\mathbf{a}(\mathbf{x}, u, \cdot)$  depends on all the values of  $u \in L^p(\Omega)$ , not only on  $u(\mathbf{x})$ . These assumptions cover for example the case where  $\mathbf{a}(\mathbf{x}, u, \nabla u(\mathbf{x})) = \Lambda[u](\mathbf{x}) \nabla u(\mathbf{x})$  with  $\Lambda : L^p(\Omega) \rightarrow L^\infty(\Omega; \mathcal{S}_d(\mathbb{R}))$  as in [10, 14, 29].*

*These assumptions (in particular (3.4a)) do not allow to cover usual local dependencies  $\mathbf{a}(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x}))$  as in the non-monotone operators studied in [26]. However, the adaptation of the following results to this case is quite easy and more classical. See e.g. [15] for an adaptation of the original Leray-Lions method to a numerical scheme (based on the Mixed Finite Volume method) for local non-monotone operators.*

If a function  $\mathbf{a}$  satisfies (3.4), then the mapping  $u \mapsto -\operatorname{div}\mathbf{a}(\cdot, u, \nabla u)$  is called a generalised Leray-Lions operator. A classical example is the  $p$ -Laplacian operator, obtained by setting  $\mathbf{a}(\mathbf{x}, u, \boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\boldsymbol{\xi}$ . Note that the existence of at least one solution to (1.1) is shown in [26] under hypotheses (3.4) in the case where  $\mathbf{a}$  does not depend on  $u$ . In our framework, we say that a function  $\bar{u}$  is a weak solution to (1.1) if:

$$\begin{cases} \bar{u} \in W_0^{1,p}(\Omega), \\ \int_{\Omega} \mathbf{a}(\mathbf{x}, \bar{u}, \nabla \bar{u}(\mathbf{x})) \cdot \nabla \bar{v}(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) \bar{v}(\mathbf{x}) d\mathbf{x}, \quad \forall \bar{v} \in W_0^{1,p}(\Omega). \end{cases} \quad (3.6)$$

**Remark 3.4** *Note that, even if  $\mathbf{a}$  does not depend on  $u \in L^p(\Omega)$ , the solution to (3.6) is not necessarily unique. Consider the case where  $p = 2$ ,  $d = 1$ ,  $\Omega = ]-1, 2[$ ,  $f(x) = 0$  for  $x \in (-1, 0) \cup (1, 2)$ ,  $f(x) = 2$  for  $x \in (0, 1)$ , and*

$$\mathbf{a}(\mathbf{x}, u, \boldsymbol{\xi}) = (\min(|\boldsymbol{\xi}|, 1) + \max(|\boldsymbol{\xi}| - 2, 0)) \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}, \quad \forall \boldsymbol{\xi} \in \mathbb{R}, \quad \forall u \in L^2(\Omega).$$

*Then (3.4b) is satisfied with  $\underline{a} = \frac{1}{2}$ , (3.4c) is satisfied since  $\mathbf{a}$  is non decreasing and (3.4d) is satisfied with  $\bar{a}(\mathbf{x}) = 0$  and  $\mu = 1$ . Then the function  $u(x) = \alpha(x + 1)$  for  $x \in (-1, 0)$ ,  $\alpha + x(1 - x)$  for  $x \in (0, 1)$ ,  $\alpha(2 - x)$  for  $x \in (1, 2)$  is solution to (3.6) for any value  $\alpha \in [1, 2]$ .*

The hypothesis that  $\mathbf{a}$  is strictly monotone, which may be expressed by

$$(\mathbf{a}(\mathbf{x}, u, \boldsymbol{\xi}) - \mathbf{a}(\mathbf{x}, u, \boldsymbol{\chi})) \cdot (\boldsymbol{\xi} - \boldsymbol{\chi}) > 0, \quad \text{for a.e. } \mathbf{x} \in \Omega, \quad \forall u \in L^p(\Omega), \quad \forall \boldsymbol{\xi}, \boldsymbol{\chi} \in \mathbb{R}^d \text{ with } \boldsymbol{\xi} \neq \boldsymbol{\chi}, \quad (3.7)$$

is only used to prove the strong convergence of the approximate gradient (see theorem below).

**Theorem 3.5 (Convergence)** *Under assumptions (3.4)-(3.5), let  $(\mathcal{D}_m)_{m \in \mathbb{N}}$  be a sequence of gradient discretisations in the sense of Definition 2.1, which is coercive, consistent, limit-conforming and compact in the sense of Definitions 2.4, 2.6, 2.7 and 2.8.*

*Then, for any  $m \in \mathbb{N}$ , there exists at least one  $u_{\mathcal{D}_m} \in X_{\mathcal{D}_m, 0}$  solution to the gradient scheme (2.1) and, up to a subsequence,  $\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}$  converges strongly in  $L^p(\Omega)$  to a solution  $\bar{u}$  of (3.6) and  $\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}$  converges weakly in  $L^p(\Omega)^d$  to  $\nabla \bar{u}$  as  $m \rightarrow \infty$ . Moreover, if we assume that the Leray-Lions operator  $\mathbf{a}$  is strictly monotone in the sense of (3.7), then  $\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}$  converges strongly in  $L^p(\Omega)^d$  to  $\nabla \bar{u}$  as  $m \rightarrow \infty$ .*

*In the case where the solution  $\bar{u}$  of (3.6) is unique, then the whole sequence converges to  $\bar{u}$  as  $m \rightarrow \infty$  in the above senses.*

**Remark 3.6** *In the case where  $\mathbf{a}$  does not depend on  $u \in L^p(\Omega)$ , the proof of the weak convergence of  $\Pi_{\mathcal{D}_m} u$  to a solution of (3.6) does not require the compactness of the sequence of gradient discretisations. In this case the strong convergence results from (3.7) (which gives the strong convergence of the approximate gradient) and from the coercivity of the sequence  $(\mathcal{D}_m)_{m \in \mathbb{N}}$ . **In the particular case where the  $p$ -Laplacian case  $\mathbf{a}(\mathbf{x}, u, \boldsymbol{\chi}) = |\boldsymbol{\chi}|^{p-2}\boldsymbol{\chi}$  is considered, an error estimate, following the spirit of Lemma 3.1, can be the object of further works.***

### Proof

This proof follows the same ideas as that of [15, 19].

#### Step 1: existence of a solution to the scheme

Let  $\mathcal{D}$  be a gradient discretisation in the sense of Definition 2.1. We endow the finite dimensional space  $X_{\mathcal{D}, 0}$  with an inner product  $\langle \cdot, \cdot \rangle$  and we denote by  $|\cdot|$  the norm coming from this inner product. We define  $F : X_{\mathcal{D}, 0} \rightarrow X_{\mathcal{D}, 0}$  as the function such that, if  $u \in X_{\mathcal{D}, 0}$ ,  $F(u)$  is the unique element in  $X_{\mathcal{D}, 0}$  which satisfies

$$\forall v \in X_{\mathcal{D}, 0}, \quad \langle F(u), v \rangle = \int_{\Omega} \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}} u, \nabla_{\mathcal{D}} u(\mathbf{x})) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x}.$$

Likewise, we denote by  $w \in X_{\mathcal{D},0}$  the unique element such that

$$\forall v \in X_{\mathcal{D},0}, \quad \langle w, v \rangle = \int_{\Omega} f(\mathbf{x}) \Pi_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x}.$$

The properties of  $\mathbf{a}$  show that  $F$  is continuous and that, for all  $u \in X_{\mathcal{D},0}$ ,  $\langle F(u), u \rangle \geq \underline{a} \|\nabla_{\mathcal{D}} u\|_{L^p(\Omega)^d}^p$ . By equivalence of the norms  $|\cdot|$  and  $\|\nabla_{\mathcal{D}} \cdot\|_{L^p(\Omega)^d}$  on  $X_{\mathcal{D},0}$ , we deduce that  $\langle F(u), u \rangle \geq C_1 |u|^p$  with  $C_1$  not depending on  $u$ . This shows that  $\lim_{|u| \rightarrow \infty} \frac{\langle F(u), u \rangle}{|u|} = +\infty$  and thus that  $F$  is surjective (see [26] or [12, Theorem 3.3, page 19]). There exists therefore  $u_{\mathcal{D}} \in X_{\mathcal{D},0}$  such that  $F(u_{\mathcal{D}}) = w$ , and this  $u_{\mathcal{D}}$  is a solution to (2.1).

### Step 2: convergence to a solution of the continuous problem

Letting  $v = u_{\mathcal{D}_m}$  in (2.1) with  $\mathcal{D} = \mathcal{D}_m$  and using (2.2) and Hypothesis (3.4b), we get

$$\underline{a} \|\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}\|_{L^p(\Omega)^d}^{p-1} \leq C_{\mathcal{D}_m} \|f\|_{L^{p'}(\Omega)}.$$

Thanks to the coercivity of the sequence of gradient discretisations, this provides an estimate on  $\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}$  in  $L^p(\Omega)^d$  and on  $\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}$  in  $L^p(\Omega)$ . By Hypothesis (3.4d), the sequence of functions  $\mathcal{A}_{\mathcal{D}_m}(\mathbf{x}) = \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}(\mathbf{x}))$  remains bounded in  $L^{p'}(\Omega)^d$ . Extending  $\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}$  and  $\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}$  by 0 outside  $\Omega$ , we infer the existence of  $\bar{u} \in L^p(\mathbb{R}^d)$ ,  $\bar{\mathbf{G}} \in L^p(\mathbb{R}^d)^d$  and  $\mathbf{A} \in L^{p'}(\Omega)^d$  such that, up to a subsequence again denoted by  $(\mathcal{D}_m)_{m \in \mathbb{N}}$ ,  $\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}$  weakly converges to  $\bar{\mathbf{G}}$  in  $L^p(\mathbb{R}^d)^d$ ,  $\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}$  weakly converges to  $\bar{u}$  in  $L^p(\mathbb{R}^d)$  and  $\mathcal{A}_{\mathcal{D}_m}$  weakly converges to  $\mathbf{A}$  in  $L^{p'}(\Omega)^d$ , as  $m \rightarrow \infty$ .

Thanks to the limit-conformity of the sequence of discretisations, passing to the limit in (2.4) we get that

$$\int_{\Omega} (\bar{\mathbf{G}}(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x}) + \bar{u}(\mathbf{x}) \operatorname{div} \boldsymbol{\varphi}(\mathbf{x})) d\mathbf{x} = 0, \quad \forall \boldsymbol{\varphi} \in W^{\operatorname{div}, p'}(\Omega).$$

Since  $\bar{\mathbf{G}} = 0$  and  $\bar{u} = 0$  outside of  $\Omega$ , the above relation may be written for all  $\boldsymbol{\varphi} \in W^{\operatorname{div}, p'}(\mathbb{R}^d)$  with integration on  $\mathbb{R}^d$ , which proves that  $\bar{\mathbf{G}} = \nabla \bar{u}$  and that the restriction of  $\bar{u}$  to  $\Omega$ , again denoted by  $\bar{u}$ , belongs to  $W_0^{1,p}(\Omega)$ . Finally, the compactness of the sequence of gradient discretisations and Kolmogorov's theorem give, up to the extraction of a subsequence again denoted by  $(\mathcal{D}_m)_{m \in \mathbb{N}}$ , the strong convergence of  $\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}$  to  $\bar{u}$  in  $L^p(\mathbb{R}^d)$  (this strong convergence property is only necessary for coping with the dependence of  $\mathbf{a}$  with respect to  $u$ ).

Let us now show that  $\bar{u}$  is solution to (3.6), using the well-known Minty trick [27]. For a given  $\varphi \in W_0^{1,p}(\Omega)$  and for any gradient discretisation  $\mathcal{D}$  belonging to the sequence  $(\mathcal{D}_m)_{m \in \mathbb{N}}$ , we introduce

$$P_{\mathcal{D}} \varphi = \operatorname{argmin}_{v \in X_{\mathcal{D},0}} (\|\Pi_{\mathcal{D}} v - \varphi\|_{L^p(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla \varphi\|_{L^p(\Omega)^d})$$

as a test function in (2.1). By the consistency of  $(\mathcal{D}_m)_{m \in \mathbb{N}}$ , letting  $m \rightarrow \infty$  we get

$$\int_{\Omega} \mathbf{A}(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) \varphi(\mathbf{x}) d\mathbf{x}, \quad \forall \varphi \in W_0^{1,p}(\Omega). \quad (3.8)$$

On the other hand, we may let  $m \rightarrow \infty$  in (2.1) with  $u_{\mathcal{D}_m}$  as a test function. Using (3.8) with  $\varphi = \bar{u}$ , this leads to

$$\lim_{m \rightarrow \infty} \int_{\Omega} \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}(\mathbf{x})) \cdot \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) \bar{u}(\mathbf{x}) d\mathbf{x} = \int_{\Omega} \mathbf{A}(\mathbf{x}) \cdot \nabla \bar{u}(\mathbf{x}) d\mathbf{x}. \quad (3.9)$$

For any  $\mathbf{G} \in L^p(\Omega)^d$ , we write, thanks to Hypothesis (3.4c),

$$\int_{\Omega} (\mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}(\mathbf{x})) - \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \mathbf{G}(\mathbf{x}))) \cdot (\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}(\mathbf{x}) - \mathbf{G}(\mathbf{x})) d\mathbf{x} \geq 0.$$



Developing the preceding inequality and using (3.9) for the only “weak-weak” term, we may let  $m \rightarrow \infty$ . We then get

$$\int_{\Omega} (\mathbf{A}(\mathbf{x}) - \mathbf{a}(\mathbf{x}, \bar{u}, \mathbf{G}(\mathbf{x}))) \cdot (\nabla \bar{u}(\mathbf{x}) - \mathbf{G}(\mathbf{x})) d\mathbf{x} \geq 0, \quad \forall \mathbf{G} \in L^p(\Omega)^d.$$

We then set  $\mathbf{G} = \nabla \bar{u} + \alpha \boldsymbol{\varphi}$  in the preceding inequality, where  $\boldsymbol{\varphi} \in C_c^\infty(\Omega)^d$  and  $\alpha > 0$ . Dividing by  $\alpha$ , we get

$$- \int_{\Omega} (\mathbf{A}(\mathbf{x}) - \mathbf{a}(\mathbf{x}, \bar{u}, \nabla \bar{u}(\mathbf{x}) + \alpha \boldsymbol{\varphi}(\mathbf{x}))) \cdot \boldsymbol{\varphi}(\mathbf{x}) d\mathbf{x} \geq 0, \quad \forall \boldsymbol{\varphi} \in C_c^\infty(\Omega)^d, \quad \forall \alpha > 0.$$

We then let  $\alpha \rightarrow 0$  and use the dominated convergence theorem, which leads to

$$- \int_{\Omega} (\mathbf{A}(\mathbf{x}) - \mathbf{a}(\mathbf{x}, \bar{u}, \nabla \bar{u}(\mathbf{x}))) \cdot \boldsymbol{\varphi}(\mathbf{x}) d\mathbf{x} \geq 0, \quad \forall \boldsymbol{\varphi} \in C_c^\infty(\Omega)^d.$$

Changing  $\boldsymbol{\varphi}$  into  $-\boldsymbol{\varphi}$ , we deduce that

$$\int_{\Omega} (\mathbf{A}(\mathbf{x}) - \mathbf{a}(\mathbf{x}, \bar{u}, \nabla \bar{u}(\mathbf{x}))) \cdot \boldsymbol{\varphi}(\mathbf{x}) d\mathbf{x} = 0, \quad \forall \boldsymbol{\varphi} \in C_c^\infty(\Omega)^d,$$

and therefore that

$$\mathbf{A}(\mathbf{x}) = \mathbf{a}(\mathbf{x}, \bar{u}, \nabla \bar{u}(\mathbf{x})), \quad \text{for a.e. } \mathbf{x} \in \Omega. \quad (3.10)$$

In addition to (3.8), this shows that  $\bar{u}$  is a solution to (3.6). This concludes the proof of the convergence of  $\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}$  to  $\bar{u}$  in  $L^p(\Omega)$  and of  $\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}$  to  $\nabla \bar{u}$  weakly in  $L^p(\Omega)^d$  as  $m \rightarrow \infty$ .

**Step 3: Assuming now hypothesis (3.7), strong convergence of the approximate gradient**

We follow here the ideas of [26]. Thanks to (3.9) and (3.10), we get

$$\lim_{m \rightarrow \infty} \int_{\Omega} (\mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}(\mathbf{x})) - \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla \bar{u}(\mathbf{x}))) \cdot (\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}(\mathbf{x}) - \nabla \bar{u}(\mathbf{x})) d\mathbf{x} = 0.$$

Since  $(\mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}(\mathbf{x})) - \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla \bar{u}(\mathbf{x}))) \cdot (\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}(\mathbf{x}) - \nabla \bar{u}(\mathbf{x})) \geq 0$  for a.e.  $\mathbf{x} \in \Omega$ , we then have

$$(\mathbf{a}(\cdot, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}) - \mathbf{a}(\cdot, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla \bar{u})) \cdot (\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} - \nabla \bar{u}) \rightarrow 0 \text{ in } L^1(\Omega), \quad (3.11)$$

and therefore a.e. for a sub-sequence. Then, thanks to the strict monotony assumption (3.7), we may use Lemma 3.7 given below to show that  $\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} \rightarrow \nabla \bar{u}$  a.e. as  $m \rightarrow \infty$ , at least for the same sub-sequence. This shows the a.e. convergence of  $\mathbf{a}(\cdot, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}) \cdot \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}$  to  $\mathbf{a}(\cdot, \bar{u}, \nabla \bar{u}) \cdot \nabla \bar{u}$ . We next recall that, by (3.9) and (3.10),

$$\lim_{m \rightarrow \infty} \int_{\Omega} \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}(\mathbf{x})) \cdot \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}(\mathbf{x}) d\mathbf{x} = \int_{\Omega} \mathbf{a}(\mathbf{x}, \bar{u}, \nabla \bar{u}(\mathbf{x})) \cdot \nabla \bar{u}(\mathbf{x}) d\mathbf{x}. \quad (3.12)$$

Since  $\mathbf{a}(\cdot, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}) \cdot \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} \geq 0$ , we can apply Lemma 3.8 to get  $\mathbf{a}(\cdot, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}) \cdot \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} \rightarrow \mathbf{a}(\cdot, \bar{u}, \nabla \bar{u}) \cdot \nabla \bar{u}$  in  $L^1(\Omega)$  as  $m \rightarrow \infty$ . This  $L^1$ -convergence gives the equi-integrability of the sequence of functions  $\mathbf{a}(\cdot, \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}) \cdot \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}$ , which gives in turn, thanks to (3.4b), the equi-integrability of  $(|\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}|^p)_{m \in \mathbb{N}}$ . The strong convergence of  $\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}$  to  $\nabla \bar{u}$  in  $L^p(\Omega)^d$  is then a consequence of Vitali's theorem.  $\square$

**Lemma 3.7** *Let  $B$  be a metric space, let  $\mathbf{b}$  be a continuous function from  $B \times \mathbb{R}^d$  to  $\mathbb{R}^d$  such that*

$$(\mathbf{b}(u, \delta) - \mathbf{b}(u, \gamma)) \cdot (\delta - \gamma) > 0, \quad \forall \delta \neq \gamma \in \mathbb{R}^d, \quad \forall u \in B.$$

*Let  $(u_m, \beta_m)_{m \in \mathbb{N}}$  be a sequence in  $B \times \mathbb{R}^d$  and  $(u, \beta) \in B \times \mathbb{R}^d$  such that  $(\mathbf{b}(u_m, \beta_m) - \mathbf{b}(u_m, \beta)) \cdot (\beta_m - \beta) \rightarrow 0$  and  $u_m \rightarrow u$  as  $m \rightarrow \infty$ . Then,  $\beta_m \rightarrow \beta$  as  $m \rightarrow \infty$ .*

**Proof** We begin the proof with a preliminary remark. Let  $\delta \in \mathbb{R}^d \setminus \{0\}$ . We define, for all  $m \in \mathbb{N}$ , the function  $h_{\delta,m}$  from  $\mathbb{R}$  to  $\mathbb{R}$  by  $h_{\delta,m}(s) = (\mathbf{b}(u_m, \beta + s\delta) - \mathbf{b}(u_m, \beta)) \cdot \delta$ . The hypothesis on  $\mathbf{b}$  gives that  $h_{\delta,m}$  is an increasing function since, for  $s > s'$ , one has :

$$h_{\delta,m}(s) - h_{\delta,m}(s') = (\mathbf{b}(u_m, \beta + s\delta) - \mathbf{b}(u_m, \beta + s'\delta)) \cdot \delta > 0.$$

We prove now, by contradiction, that  $\lim_{m \rightarrow \infty} \beta_m = \beta$ . If the sequence  $(\beta_m)_{m \in \mathbb{N}}$  does not converge to  $\beta$ , there exists  $\varepsilon > 0$  and a subsequence, still denoted by  $(\beta_m)_{m \in \mathbb{N}}$ , such that  $s_m := |\beta_m - \beta| \geq \varepsilon$ , for all  $m \in \mathbb{N}$ . Then, we set  $\delta_m = \frac{\beta_m - \beta}{|\beta_m - \beta|}$  and we can assume, up to a subsequence, that  $\delta_m \rightarrow \delta$  as  $m \rightarrow \infty$ , for some  $\delta \in \mathbb{R}^d$  with  $|\delta| = 1$ . We then have, since  $s_m \geq \varepsilon$  :

$$(\mathbf{b}(u_m, \beta_m) - \mathbf{b}(u_m, \beta)) \cdot \frac{\beta_m - \beta}{s_m} = h_{\delta_m,m}(s_m) \geq h_{\delta_m,m}(\varepsilon) = (\mathbf{b}(u_m, \beta + \varepsilon\delta_m) - \mathbf{b}(u_m, \beta)) \cdot \delta_m.$$

Then, passing to the limit as  $m \rightarrow \infty$ ,

$$0 = \lim_{m \rightarrow \infty} \frac{1}{s_m} (\mathbf{b}(u_m, \beta_m) - \mathbf{b}(u_m, \beta)) \cdot (\beta_m - \beta) \geq (\mathbf{b}(u, \beta + \varepsilon\delta) - \mathbf{b}(u, \beta)) \cdot \delta > 0,$$

which is impossible.  $\square$

The following result is classical (see [26]). Its proof is given for the sake of completeness.

**Lemma 3.8** *Let  $(F_m)_{m \in \mathbb{N}}$  be a sequence non-negative functions in  $L^1(\Omega)$ . Let  $F \in L^1(\Omega)$  be such that  $F_m \rightarrow F$  a.e. in  $\Omega$  and  $\int_{\Omega} F_m(\mathbf{x}) d\mathbf{x} \rightarrow \int_{\Omega} F(\mathbf{x}) d\mathbf{x}$ , as  $m \rightarrow \infty$ . Then,  $F_m \rightarrow F$  in  $L^1(\Omega)$  as  $m \rightarrow \infty$ .*

**Proof** The proof of this lemma is very classical. Applying the Dominated Convergence Theorem to the sequence  $(F - F_m)^+$  leads to  $\int_{\Omega} (F(\mathbf{x}) - F_m(\mathbf{x}))^+ d\mathbf{x} \rightarrow 0$  as  $m \rightarrow \infty$ . Then, since  $|F - F_m| = 2(F - F_m)^+ - (F - F_m)$ , we conclude that  $F_m \rightarrow F$  in  $L^1(\Omega)$  as  $m \rightarrow \infty$ .  $\square$

## 4 Evolution problems

In this section, we consider the evolution problem (1.2) under Hypotheses (3.4) and

$$\begin{aligned} T &\in (0, +\infty), \\ u_{\text{ini}} &\in L^2(\Omega), \\ f &\in L^{p'}(\Omega \times (0, T)) \text{ where } p' = \frac{p}{p-1}. \end{aligned} \quad (4.1)$$

The precise mathematical notion of solution to (1.2) that we consider is the following:

$$\left\{ \begin{aligned} &\bar{u} \in L^p(0, T; W_0^{1,p}(\Omega)) \cap C^0([0, T]; L^2(\Omega)), \partial_t \bar{u} \in L^{p'}(0, T; W^{-1,p'}(\Omega)), \\ &\bar{u}(\cdot, 0) = u_{\text{ini}}, \\ &\int_0^T \langle \partial_t \bar{u}(\cdot, t), \bar{v}(\cdot, t) \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} dt + \int_0^T \int_{\Omega} \mathbf{a}(\mathbf{x}, \bar{u}(\cdot, t), \nabla \bar{u}(\mathbf{x}, t)) \cdot \nabla \bar{v}(\mathbf{x}, t) d\mathbf{x} dt \\ &= \int_0^T \int_{\Omega} f(\mathbf{x}, t) \bar{v}(\mathbf{x}, t) d\mathbf{x} dt, \quad \forall \bar{v} \in L^p(0, T; W_0^{1,p}(\Omega)). \end{aligned} \right. \quad (4.2)$$

**Remark 4.1** *The derivative  $\partial_t \bar{u}$  is to be understood in the usual sense of distributions on  $\Omega \times (0, T)$ . Since the set  $\mathcal{T} = \{\sum_{i=1}^q \varphi_i(t) \gamma_i(\mathbf{x}) : q \in \mathbb{N}, \varphi_i \in C_c^\infty(0, T), \gamma_i \in C_c^\infty(\Omega)\}$  of tensorial functions in  $C^\infty(\Omega \times (0, T))$  is dense in  $L^p(0, T; W_0^{1,p}(\Omega))$ , one can ensure that this distribution derivative  $\partial_t \bar{u}$  belongs to  $L^{p'}(0, T; W^{-1,p'}(\Omega)) = (L^p(0, T; W_0^{1,p}(\Omega)))'$  by checking that the linear form*

$$\varphi \in \mathcal{T} \mapsto \langle \partial_t \bar{u}, \varphi \rangle_{\mathcal{D}', \mathcal{D}} = - \int_0^T \int_{\Omega} \bar{u}(\mathbf{x}, t) \partial_t \varphi(\mathbf{x}, t) d\mathbf{x} dt$$

*is continuous for the norm of  $L^p(0, T; W_0^{1,p}(\Omega))$ .*

**Definition 4.2 (Space-time gradient discretisation)** Let  $p \in (1, +\infty)$  let  $\Omega$  be an open subset of  $\mathbb{R}^d$ , with  $d \in \mathbb{N}^*$  and let  $T > 0$  be given. We say that  $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, (t^{(n)})_{n=0,\dots,N})$  is a space-time gradient discretisation if

- $(X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$  is a gradient discretisation of  $\Omega$ , in the sense of Definition 2.1, which satisfies  $\Pi_{\mathcal{D}}(X_{\mathcal{D},0}) \subset L^{\max(p,2)}(\Omega)$ ,
- $t^{(0)} = 0 < t^{(1)} \dots < t^{(N)} = T$ .

We then set  $\delta t^{(n+\frac{1}{2})} = t^{(n+1)} - t^{(n)}$ , for  $n = 0, \dots, N-1$ , and  $\delta t_{\mathcal{D}} = \max_{n=0,\dots,N-1} \delta t^{(n+\frac{1}{2})}$ .

Let  $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, (t^{(n)})_{n=0,\dots,N})$  be a space-time gradient discretisation in the sense of Definition 4.2. We define, for a given  $\alpha \in [\frac{1}{2}, 1]$ , the following scheme for the discretisation of Problem (1.2): we take  $u^{(0)} \in X_{\mathcal{D},0}$  and consider a sequence  $(u^{(n)})_{n=0,\dots,N} \subset X_{\mathcal{D},0}$  such that, for all  $n = 0, \dots, N-1$ ,

$$\left\{ \begin{array}{l} \text{Setting } u^{(n+\alpha)} = \alpha u^{(n+1)} + (1-\alpha)u^{(n)} \text{ and } \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u = \frac{u^{(n+1)} - u^{(n)}}{\delta t^{(n+\frac{1}{2})}}, \text{ we have:} \\ \int_{\Omega} \left[ \Pi_{\mathcal{D}} \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\mathbf{x}) \Pi_{\mathcal{D}} v(\mathbf{x}) + \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}} u^{(n+\alpha)}, \nabla_{\mathcal{D}} u^{(n+\alpha)}(\mathbf{x})) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) \right] d\mathbf{x} \\ = \frac{1}{\delta t^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_{\Omega} f(\mathbf{x}, t) \Pi_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} dt, \quad \forall v \in X_{\mathcal{D},0}. \end{array} \right. \quad (4.3)$$

Note that the choice  $\alpha \geq \frac{1}{2}$  is required for stability reasons and that the choice  $\alpha = 1$  leads to the implicit scheme. We again use the notations  $\Pi_{\mathcal{D}}$  and  $\nabla_{\mathcal{D}}$  for the definition of space-time dependent functions, defining

$$\begin{aligned} \Pi_{\mathcal{D}} u(\mathbf{x}, t) &= \Pi_{\mathcal{D}} u^{(n+\alpha)}(\mathbf{x}) \text{ and } \nabla_{\mathcal{D}} u(\mathbf{x}, t) = \nabla_{\mathcal{D}} u^{(n+\alpha)}(\mathbf{x}), \\ &\text{for a.e. } (\mathbf{x}, t) \in \Omega \times (t^{(n)}, t^{(n+1)}), \quad \forall n = 0, \dots, N-1. \end{aligned} \quad (4.4)$$

We also let

$$\tilde{\Pi}_{\mathcal{D}} u(\mathbf{x}, t) = \Pi_{\mathcal{D}} u^{(n+1)}(\mathbf{x}), \text{ for a.e. } (\mathbf{x}, t) \in \Omega \times (t^{(n)}, t^{(n+1)}), \quad \forall n = 0, \dots, N-1 \quad (4.5)$$

and

$$\delta_{\mathcal{D}} u(\mathbf{x}, t) = \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\mathbf{x}), \text{ for a.e. } (\mathbf{x}, t) \in \Omega \times (t^{(n)}, t^{(n+1)}), \quad \forall n = 0, \dots, N-1. \quad (4.6)$$

**Lemma 4.3** ( $L^\infty(0, T; L^2(\Omega))$  estimate, discrete  $L^p(0, T; W_0^{1,p}(\Omega))$  estimate and existence of a discrete solution)

Under Hypotheses (3.4) and (4.1), let  $\mathcal{D}$  be a space-time gradient discretisation in the sense of Definition 4.2. Then there exists at least one solution to Scheme (4.3), and there exists  $C_2 > 0$ , only depending on  $p$ ,  $C_P \geq C_{\mathcal{D}}$ ,  $C_{\text{ini}} \geq \|u_{\text{ini}} - \Pi_{\mathcal{D}} u^{(0)}\|_{L^2(\Omega)}$ ,  $u_{\text{ini}}$ ,  $f$ ,  $\underline{a}$  such that, for any solution  $u$  to this scheme,

$$\|\tilde{\Pi}_{\mathcal{D}} u\|_{L^\infty(0, T; L^2(\Omega))} \leq C_2, \quad \|\Pi_{\mathcal{D}} u\|_{L^\infty(0, T; L^2(\Omega))} \leq C_2 \text{ and } \|\nabla_{\mathcal{D}} u\|_{L^p(\Omega \times (0, T))^d} \leq C_2. \quad (4.7)$$

**Proof** Let us first prove the estimates. We let  $v = \delta t^{(n+\frac{1}{2})} u^{(n+\alpha)}$  in (4.3). Since

$$\delta t^{(n+\frac{1}{2})} \Pi_{\mathcal{D}} \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u \Pi_{\mathcal{D}} u^{(n+\alpha)} = \frac{1}{2} ((\Pi_{\mathcal{D}} u^{(n+1)})^2 - (\Pi_{\mathcal{D}} u^{(n)})^2) + \left( \alpha - \frac{1}{2} \right) (\Pi_{\mathcal{D}} u^{(n+1)} - \Pi_{\mathcal{D}} u^{(n)})^2,$$

we get, by summing on  $n = 0, \dots, m-1$  for a given  $m = 1, \dots, N$ ,

$$\begin{aligned} \frac{1}{2} \|\Pi_{\mathcal{D}} u^{(m)}\|_{L^2(\Omega)}^2 + \underline{a} \int_0^{t^{(m)}} \|\nabla_{\mathcal{D}} u(\cdot, t)\|_{L^p(\Omega)^d}^p dt \\ \leq \|f\|_{L^{p'}(\Omega \times (0, t^{(m)}))} \|\Pi_{\mathcal{D}} u\|_{L^p(\Omega \times (0, t^{(m)}))} + \frac{1}{2} \|\Pi_{\mathcal{D}} u^{(0)}\|_{L^2(\Omega)}^2, \end{aligned}$$

which provides, thanks to the Young inequality,

$$\begin{aligned} & \frac{1}{2} \|\Pi_{\mathcal{D}} u^{(m)}\|_{L^2(\Omega)}^2 + \underline{a} \int_0^{t^{(m)}} \|\nabla_{\mathcal{D}} u(\cdot, t)\|_{L^p(\Omega)^d}^p dt \\ & \leq \frac{2^{1/(p-1)} C_{\mathcal{D}}^{p'}}{(\underline{p}\underline{a})^{1/(p-1)} p'} \|f\|_{L^{p'}(\Omega \times (0, t^{(m)}))}^{p'} + \frac{\underline{a}}{2C_{\mathcal{D}}^p} \|\Pi_{\mathcal{D}} u\|_{L^p(\Omega \times (0, t^{(m)}))}^p + \frac{1}{2} \|\Pi_{\mathcal{D}} u^{(0)}\|_{L^2(\Omega)}^2. \end{aligned}$$

Applying (2.2) proves the estimates on  $\tilde{\Pi}_{\mathcal{D}} u$  and  $\nabla_{\mathcal{D}} u$ . The estimate on  $\Pi_{\mathcal{D}} u$  follows from the inequality  $\|\Pi_{\mathcal{D}} u^{(n+\alpha)}\|_{L^2(\Omega)} \leq \alpha \|\Pi_{\mathcal{D}} u^{(n+1)}\|_{L^2(\Omega)} + (1-\alpha) \|\Pi_{\mathcal{D}} u^{(n)}\|_{L^2(\Omega)}$ .

The existence for each  $n = 0, \dots, N-1$  of at least one solution to (4.3) follows the same proof as that of Theorem 3.5, reasoning on  $u^{(n+\alpha)}$  rather than  $u^{(n+1)}$  and using the above estimates.  $\square$

The following semi-norm on  $X_{\mathcal{D},0}$  will be useful to apply Theorem 5.13 in the appendix.

**Definition 4.4 (Dual semi-norm)** *Under Hypotheses (3.4), let  $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$  be a gradient discretisation of  $\Omega$  in the sense of Definition 2.1. We define the following dual semi-norm on  $X_{\mathcal{D},0}$ :*

$$\forall w \in X_{\mathcal{D},0}, |w|_{\star, \mathcal{D}} = \sup \left\{ \int_{\Omega} \Pi_{\mathcal{D}} w(\mathbf{x}) \Pi_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} : v \in X_{\mathcal{D},0}, \|v\|_{\mathcal{D}} = 1 \right\}. \quad (4.8)$$

**Lemma 4.5 (Estimate on the dual semi-norm of the discrete time derivative)**

*Under Hypotheses (3.4) and (4.1), let  $\mathcal{D}$  be a space-time gradient discretisation in the sense of Definition 4.2. Let  $u$  be a solution to Scheme (4.3). Then there exists  $C_3$ , only depending on  $p, \mu, \bar{a}, \underline{a}, C_{\text{ini}} \geq \|u_{\text{ini}} - \Pi_{\mathcal{D}} u^{(0)}\|_{L^2(\Omega)}, u_{\text{ini}}, f, T$  and  $C_P \geq C_{\mathcal{D}}$ , such that*

$$\int_0^T |\delta_{\mathcal{D}} u(\cdot, t)|_{\star, \mathcal{D}}^{p'} dt \leq C_3. \quad (4.9)$$

**Proof** Let us take  $v \in X_{\mathcal{D},0}$  as test function in Scheme (4.3). We have, thanks to Assumption (3.4d) on  $\mathbf{a}$ ,

$$\begin{aligned} \int_{\Omega} \Pi_{\mathcal{D}} \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\mathbf{x}) \Pi_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} & \leq \int_{\Omega} (\bar{a}(\mathbf{x}) + \mu |\nabla_{\mathcal{D}} u^{(n+\alpha)}(\mathbf{x})|^{p-1}) |\nabla_{\mathcal{D}} v(\mathbf{x})| d\mathbf{x} \\ & \quad + \frac{1}{\delta_{\mathcal{D}}^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_{\Omega} f(\mathbf{x}, t) \Pi_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} dt, \end{aligned}$$

which leads, thanks to (2.2), to the existence of  $C_4 > 0$  only depending on  $p, \mu$  such that

$$\begin{aligned} & \int_{\Omega} \Pi_{\mathcal{D}} \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\mathbf{x}) \Pi_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} \\ & \leq C_4 \left( \|\bar{a}\|_{L^{p'}(\Omega)}^{p'} + \|\nabla_{\mathcal{D}} u^{(n+\alpha)}\|_{L^p(\Omega)}^p + \frac{C_{\mathcal{D}}}{\delta_{\mathcal{D}}^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \|f(\cdot, t)\|_{L^{p'}(\Omega)}^{p'} dt \right)^{(p-1)/p} \|\nabla_{\mathcal{D}} v\|_{L^p(\Omega)}. \end{aligned}$$

Taking the supremum on  $v \in X_{\mathcal{D},0}$  such that  $\|\nabla_{\mathcal{D}} v\|_{L^p(\Omega)^d} = 1$  gives an estimate on  $|\delta_{\mathcal{D}}^{(n+\frac{1}{2})} u|_{\star, \mathcal{D}}$ . The proof is concluded by raising this estimate to the power  $p'$ , multiplying by  $\delta_{\mathcal{D}}^{(n+\frac{1}{2})}$ , summing on  $n$  and estimating  $\|\nabla_{\mathcal{D}} u\|_{L^p(\Omega \times (0, T))}^p$  thanks to Lemma 4.3.  $\square$

In order to prove the convergence, we shall use the assumptions of coercivity, limit-conformity and compactness that were already used for the steady state problems. However, in order to pass safely to the limit on the time term, we need a modified consistency property for the sequence of gradient discretisations.

**Definition 4.6 (Space-time consistency)** Let  $\mathcal{D}$  be a space-time gradient discretisation for Problem (1.2) in the sense of Definition 4.2, and let  $\widehat{S}_{\mathcal{D}} : W_0^{1,p}(\Omega) \cap L^2(\Omega) \rightarrow [0, +\infty)$  be defined by

$$\forall \varphi \in W_0^{1,p}(\Omega), \quad \widehat{S}_{\mathcal{D}}(\varphi) = \min_{v \in X_{\mathcal{D},0}} (\|\Pi_{\mathcal{D}}v - \varphi\|_{L^{\max(p,2)}(\Omega)} + \|\nabla_{\mathcal{D}}v - \nabla\varphi\|_{L^p(\Omega)^d}). \quad (4.10)$$

A sequence  $(\mathcal{D}_m)_{m \in \mathbb{N}}$  of space-time gradient discretisations is said to be **consistent** if:

- for all  $\varphi \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ ,  $\widehat{S}_{\mathcal{D}_m}(\varphi)$  tends to 0 as  $m \rightarrow \infty$ ,
- $\delta_{\mathcal{D}_m}$  tends to 0 as  $m \rightarrow \infty$ .

We cannot provide here, as in the elliptic case, an error estimate similar to Lemma 3.1, which could be expressed in terms of a  $L^\infty(0, T; L^2(\Omega))$  norm. This is due to both the minimal regularity assumptions on the continuous problem and the general gradient scheme framework, which does not allow to estimate for example time-space derivatives error on the continuous solution. But we show the following convergence result.

**Theorem 4.7 (Convergence of the scheme)** Under assumptions (3.4) and (4.1), let  $(\mathcal{D}_m)_{m \in \mathbb{N}}$  be a consistent sequence of space-time gradient discretisations in the sense of Definition 4.6, such that the associated sequence of gradient discretisations is coercive (Definition 2.4), limit-conforming (Definition 2.7) and compact (Definition 2.8). Let  $\alpha \in [\frac{1}{2}, 1]$  be given. For any  $m \in \mathbb{N}$ , let  $u_{\mathcal{D}_m}$  be a solution to Scheme (4.3), such that  $\|u_{\text{ini}} - \Pi_{\mathcal{D}_m}u^{(0)}\|_{L^2(\Omega)} \rightarrow 0$  as  $m \rightarrow \infty$ .

Then, up to a subsequence,  $\Pi_{\mathcal{D}_m}u_{\mathcal{D}_m}$  converges strongly in  $L^1(0, T; L^p(\Omega))$  and in  $L^2(\Omega \times (0, T))$  to a solution  $\bar{u}$  of (4.2),  $\tilde{\Pi}_{\mathcal{D}_m}u_{\mathcal{D}_m}$  converges strongly in  $L^2(\Omega \times (0, T))$  to  $\bar{u}$  and  $\nabla_{\mathcal{D}_m}u_{\mathcal{D}_m}$  converges weakly in  $L^p(\Omega \times (0, T))^d$  to  $\nabla\bar{u}$  as  $m \rightarrow \infty$ .

Moreover, if we assume that the Leray-Lions operator  $\mathbf{a}$  is strictly monotone in the sense of (3.7), then  $\nabla_{\mathcal{D}}u_{\mathcal{D}}$  converges strongly in  $L^p(\Omega \times (0, T))^d$  to  $\nabla\bar{u}$  and  $\Pi_{\mathcal{D}_m}u_{\mathcal{D}_m}$  converges strongly in  $L^p(\Omega \times (0, T))$  to  $\bar{u}$  as  $m \rightarrow \infty$ .

In the case where the solution  $\bar{u}$  of (3.6) is unique, then the whole sequence converges to  $\bar{u}$  as  $m \rightarrow \infty$  in the above senses.

**Proof** We shall simply denote by  $u_m$  instead of  $u_{\mathcal{D}_m}$  a solution to Scheme (4.3) using the space-time gradient discretisation  $\mathcal{D}_m$ . In this proof, some indices  $m$  are omitted in the expressions which are developed.

**Step 1** Proof that hypotheses (h1)-(h2)-(h3)-(h4) of Theorem 5.13 hold, and consequences.

In our setting, the space  $B$  of Theorem 5.13 is  $L^p(\Omega)$ . We take  $B_m = \Pi_{\mathcal{D}_m}(X_{\mathcal{D}_m,0})$ . We define the norm  $\|\cdot\|_{X_m}$  by

$$\|v\|_{X_m} = \inf\{\|w\|_{\mathcal{D}_m}, w \in X_{\mathcal{D}_m,0} \text{ such that } \Pi_{\mathcal{D}_m}w = v\},$$

(note that, for all  $v \in B_m$ , there exists one and only one  $w \in X_{\mathcal{D}_m,0}$  such that  $\Pi_{\mathcal{D}_m}w = v$  and  $\|w\|_{\mathcal{D}_m} = \|v\|_{X_m}$ ) and the norm  $\|\cdot\|_{Y_m}$  is defined from Definition 4.4 by

$$\begin{aligned} \|v\|_{Y_m} &= |w|_{\star, \mathcal{D}_m} \text{ for any } w \in X_{\mathcal{D}_m,0} \text{ such that } \Pi_{\mathcal{D}_m}w = v \\ &= \sup \left\{ \int_{\Omega} v(\mathbf{x}) \Pi_{\mathcal{D}_m}z(\mathbf{x}) d\mathbf{x}, z \in X_{\mathcal{D}_m,0}, \|z\|_{\mathcal{D}_m} = 1 \right\}. \end{aligned}$$

We remark that  $\|\cdot\|_{Y_m}$  is indeed a norm (if  $v \neq 0$ , then  $v = \Pi_{\mathcal{D}_m}w$  with  $w \neq 0$ , and taking  $z = w/\|w\|_{\mathcal{D}_m}$  shows that  $\|v\|_{Y_m} > 0$ ).

Let  $(v_m)_{m \in \mathbb{N}}$  be a sequence of functions of  $B_m$  such that  $\|v_m\|_{X_m} \leq C$  for some  $C \in \mathbb{R}_+$ . Then, taking  $w_m \in X_{\mathcal{D}_m,0}$  such that  $v_m = \Pi_{\mathcal{D}_m}(w_m)$  and  $\|v_m\|_{X_m} = \|w_m\|_{\mathcal{D}_m}$ , we get that the norm  $\|w_m\|_{\mathcal{D}_m}$  remains bounded. Thanks to the coercivity and the compactness of the sequence of discretisations, the sequence  $(\Pi_{\mathcal{D}_m}w_m)_{\mathcal{D}_m}$  converges in  $L^p(\Omega)$  to some  $v \in L^p(\Omega)$ . Thus, assumption (h1) of Theorem 5.13 is satisfied.

Let us then show that assumption (h2) is also satisfied. Let  $(v_m)_{m \in \mathbb{N}}$  be a sequence of functions of  $L^p(\Omega)$  such that  $v_m \in B_m$ ,  $\|v_m\|_{X_m} \leq C$  for some  $C \in \mathbb{R}_+$ , and such that there exists  $v \in B$  with  $v_m \rightarrow v$  in  $B$  and  $\|v_m\|_{Y_m} \rightarrow 0$  as  $m \rightarrow \infty$ . Taking  $w_m \in X_{\mathcal{D}_m,0}$  such that  $v_m = \Pi_{\mathcal{D}_m}(w_m)$  and  $\|v_m\|_{X_m} = \|w_m\|_{\mathcal{D}_m}$ ,

$$\int_{\Omega} (v(\mathbf{x}))^2 d\mathbf{x} = \int_{\Omega} (\Pi_{\mathcal{D}_m} w_m(\mathbf{x}))^2 d\mathbf{x} \leq |w_m|_{\star, \mathcal{D}_m} \|w_m\|_{\mathcal{D}_m} \leq C \|v_m\|_{Y_m} \rightarrow 0 \text{ as } m \rightarrow \infty,$$

which shows that  $v = 0$ .

Estimates (4.7) and (4.9) show that hypotheses (h3) and (h4) of Theorem 5.13 are satisfied as well. Therefore, we deduce that there exists  $\bar{u} \in L^1(0, T; L^p(\Omega))$  and a subsequence of  $(\mathcal{D}_m)_{m \in \mathbb{N}}$ , again denoted by  $(\mathcal{D}_m)_{m \in \mathbb{N}}$ , such that  $\Pi_{\mathcal{D}_m} u_m$  converges in  $L^1(0, T; L^p(\Omega))$  to  $\bar{u}$  as  $m \rightarrow \infty$ .

**Step 2** Proof that  $\bar{u}$  is solution to (4.2).

Thanks to Lemma 4.3, the convergence of  $\Pi_{\mathcal{D}_m} u_m$  to  $\bar{u}$  also holds in  $L^\infty(0, T; L^2(\Omega))$  weak-\*. The same lemma allows us to assume that  $\tilde{\Pi}_{\mathcal{D}_m} u_m$  converges for the weak-\* topology of  $L^\infty(0, T; L^2(\Omega))$ . Let us take  $\varphi \in C_c^\infty(\Omega \times (0, T))$ , define

$$P_{\mathcal{D}} v = \operatorname{argmin}_{w \in X_{\mathcal{D},0}} (\|\Pi_{\mathcal{D}} w - v\|_{L^{\max(p,2)}(\Omega)} + \|\nabla_{\mathcal{D}} w - \nabla v\|_{L^p(\Omega)^d}), \quad \forall v \in W_0^{1,p}(\Omega) \cap L^2(\Omega) \quad (4.11)$$

and let  $P_{\mathcal{D}} \varphi(t) = P_{\mathcal{D}}(\varphi(\cdot, t))$ . Using the fact that  $0 \in X_{\mathcal{D}_m,0}$ , we get that  $\widehat{S}_{\mathcal{D}}(\varphi(t)) \leq \|\varphi(t)\|_{L^{\max(p,2)}(\Omega)} + \|\nabla \varphi(t)\|_{L^p(\Omega)^d}$ , which leads, thanks to the triangular inequality, to

$$\|\Pi_{\mathcal{D}_m} P_{\mathcal{D}_m} \varphi(t)\|_{L^{\max(p,2)}(\Omega)} + \|P_{\mathcal{D}_m} \varphi(t)\|_{\mathcal{D}_m} \leq 2(\|\varphi(t)\|_{L^{\max(p,2)}(\Omega)} + \|\nabla \varphi(t)\|_{L^p(\Omega)^d}). \quad (4.12)$$

We then write

$$\begin{aligned} & \left| \int_0^T \int_{\Omega} (\tilde{\Pi}_{\mathcal{D}_m} u_m(\mathbf{x}, t) - \Pi_{\mathcal{D}_m} u_m(\mathbf{x}, t)) \Pi_{\mathcal{D}_m} P_{\mathcal{D}_m} \varphi(\mathbf{x}, t) d\mathbf{x} dt \right| \\ & \leq (1 - \alpha) \delta_{\mathcal{D}_m} \int_0^T |\delta_{\mathcal{D}_m} u(\cdot, t)|_{\star, \mathcal{D}_m} \|P_{\mathcal{D}_m} \varphi(t)\|_{\mathcal{D}_m} dt \end{aligned}$$

and use Lemma 4.5 and Estimate (4.12) to see that the right-hand side of this inequality tends to 0 as  $m \rightarrow \infty$ . Thanks to the consistency of the sequence of approximations, for all  $t \in \mathbb{R}$  we have  $\Pi_{\mathcal{D}_m} P_{\mathcal{D}_m} \varphi(t) \rightarrow \varphi(t)$  in  $L^2(\Omega)$  as  $m \rightarrow \infty$ . Hence, the dominated convergence theorem and (4.12) show that  $\Pi_{\mathcal{D}_m} P_{\mathcal{D}_m} \varphi \rightarrow \varphi$  in  $L^2(0, T; L^2(\Omega))$ . Since  $\Pi_{\mathcal{D}_m} u_m$  and  $\tilde{\Pi}_{\mathcal{D}_m} u_m$  are bounded in  $L^2(0, T; L^2(\Omega))$ , we deduce

$$\begin{aligned} 0 &= \lim_{m \rightarrow \infty} \int_0^T \int_{\Omega} (\tilde{\Pi}_{\mathcal{D}_m} u_m(\mathbf{x}, t) - \Pi_{\mathcal{D}_m} u_m(\mathbf{x}, t)) \Pi_{\mathcal{D}_m} P_{\mathcal{D}_m} \varphi(\mathbf{x}, t) d\mathbf{x} dt \\ &= \lim_{m \rightarrow \infty} \int_0^T \int_{\Omega} (\tilde{\Pi}_{\mathcal{D}_m} u_m(\mathbf{x}, t) - \Pi_{\mathcal{D}_m} u_m(\mathbf{x}, t)) \varphi(\mathbf{x}, t) d\mathbf{x} dt. \end{aligned} \quad (4.13)$$

This proves that the weak-\* limits of  $\tilde{\Pi}_{\mathcal{D}_m} u_m$  and  $\Pi_{\mathcal{D}_m} u_m$  in  $L^\infty(0, T; L^2(\Omega))$  are identical (equal to  $\bar{u}$ ). By (4.7) and the coercivity of the sequence of gradient discretisations, we also have  $\Pi_{\mathcal{D}_m} u_m \rightarrow \bar{u}$  weakly in  $L^p(0, T; L^p(\Omega))$  and we can assume that  $\nabla_{\mathcal{D}_m} u_m$  weakly converges to some  $\bar{\mathbf{G}}$  in  $L^p(0, T; L^p(\Omega)^d)$  as  $m \rightarrow \infty$ . Extending  $\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}$ ,  $\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}$ ,  $\bar{u}$  and  $\bar{\mathbf{G}}$  by 0 outside  $\Omega$ , the consistency of the underlying gradient discretisation gives, as in the proof of Theorem 3.5,  $\bar{\mathbf{G}} = \nabla \bar{u}$  and  $\bar{u} \in L^p(0, T; W_0^{1,p}(\Omega))$ . Finally, we notice that, by Assumption (3.4d), the functions  $\mathcal{A}_{\mathcal{D}_m}(\mathbf{x}, t) = \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}_m} u_m(\cdot, t), \nabla_{\mathcal{D}_m} u_m(\mathbf{x}, t))$  remain bounded in  $L^{p'}(\Omega \times (0, T))^d$  and converges therefore, up to a subsequence, to some  $\mathbf{A}$  weakly in  $L^{p'}(\Omega \times (0, T))^d$  as  $m \rightarrow \infty$ .

Let  $\varphi \in C_c^1(-\infty, T)$  and  $v \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ . We introduce  $\delta^{(n+\frac{1}{2})} \varphi(t^{(n)}) P_{\mathcal{D}} v$  as test function in (4.3). We get  $T_1^{(m)} + T_2^{(m)} = T_3^{(m)}$ , with

$$T_1^{(m)} = \sum_{n=0}^{N-1} \varphi(t^{(n)}) \delta^{(n+\frac{1}{2})} \int_{\Omega} \Pi_{\mathcal{D}} \delta_{\mathcal{D}}^{(n+\frac{1}{2})} u(\mathbf{x}) \Pi_{\mathcal{D}} P_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x},$$

$$T_2^{(m)} = \sum_{n=0}^{N-1} \varphi(t^{(n)}) \delta t^{(n+\frac{1}{2})} \int_{\Omega} \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}} u^{(n+\alpha)}, \nabla_{\mathcal{D}} u^{(n+\alpha)}(\mathbf{x})) \cdot \nabla_{\mathcal{D}} P_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x},$$

$$T_3^{(m)} = \sum_{n=0}^{N-1} \varphi(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} \int_{\Omega} f(\mathbf{x}, t) \Pi_{\mathcal{D}} P_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} dt.$$

Using discrete integrate-by-parts formula to transform the terms  $\varphi(t^{(n)})(\Pi_{\mathcal{D}} u^{(n+1)} - \Pi_{\mathcal{D}} u^{(n)})$  appearing in  $T_1^{(m)}$  into  $(\varphi(t^{(n)}) - \varphi(t^{(n+1)}))\Pi_{\mathcal{D}} u^{(n+1)}$ , we have

$$T_1^{(m)} = - \int_0^T \varphi'(t) \int_{\Omega} \tilde{\Pi}_{\mathcal{D}} u(\mathbf{x}) \Pi_{\mathcal{D}} P_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} dt - \varphi(0) \int_{\Omega} \Pi_{\mathcal{D}} u^{(0)}(\mathbf{x}) \Pi_{\mathcal{D}} P_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x}.$$

Letting  $\varphi_{\mathcal{D}}(t) = \varphi(t^{(n)})$  for  $t \in (t^{(n)}, t^{(n+1)})$ , we have

$$T_2^{(m)} = \int_0^T \varphi_{\mathcal{D}}(t) \int_{\Omega} \mathbf{A}_{\mathcal{D}}(\mathbf{x}, t) \cdot \nabla_{\mathcal{D}} P_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} dt, \quad T_3^{(m)} = \int_0^T \varphi_{\mathcal{D}}(t) \int_{\Omega} f(\mathbf{x}, t) \Pi_{\mathcal{D}} P_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} dt.$$

We may then let  $m \rightarrow \infty$  and see that  $\bar{u}$  satisfies

$$\begin{cases} \bar{u} \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \\ - \int_0^T \varphi'(t) \int_{\Omega} \bar{u}(\mathbf{x}, t) v(\mathbf{x}) d\mathbf{x} dt - \varphi(0) \int_{\Omega} u_{\text{ini}}(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \\ + \int_0^T \varphi(t) \int_{\Omega} \mathbf{A}(\mathbf{x}, t) \cdot \nabla v(\mathbf{x}) d\mathbf{x} dt = \int_0^T \varphi(t) \int_{\Omega} f(\mathbf{x}, t) v(\mathbf{x}) d\mathbf{x} dt, \\ \forall v \in W_0^{1,p}(\Omega) \cap L^2(\Omega), \forall \varphi \in C_c^\infty(-\infty, T). \end{cases} \quad (4.14)$$

Linear combinations of this relation show that it also holds with  $\varphi(t)v(\mathbf{x})$  replaced by a tensorial functions in  $C_c^\infty(\Omega \times (0, T))$  and, by Remark 4.1, allows to prove that  $\partial_t \bar{u} \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ . Standard arguments then show that  $\bar{u}$  can be identified with an element of  $C^0([0, T]; L^2(\Omega))$ , with the property  $\bar{u}(\cdot, 0) = u_{\text{ini}}$ . Using the density of tensorial functions in  $L^p(0, T; W_0^{1,p}(\Omega))$ , we then see that  $\bar{u}$  satisfies

$$\begin{cases} \int_0^T \langle \partial_t \bar{u}(\cdot, t), \bar{v}(\cdot, t) \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} dt \\ + \int_0^T \int_{\Omega} \mathbf{A}(\mathbf{x}, t) \cdot \nabla \bar{v}(\mathbf{x}, t) d\mathbf{x} dt = \int_0^T \int_{\Omega} f(\mathbf{x}, t) \bar{v}(\mathbf{x}, t) d\mathbf{x} dt, \quad \forall \bar{v} \in L^p(0, T; W_0^{1,p}(\Omega)). \end{cases} \quad (4.15)$$

It remains to prove that

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{a}(\mathbf{x}, \bar{u}(\cdot, t), \nabla \bar{u}(\mathbf{x}, t)), \text{ for a.e. } (\mathbf{x}, t) \in \Omega \times (0, T). \quad (4.16)$$

We start by writing (4.15) with  $\bar{v} = \varphi(t)\bar{u}$ , for a given  $\varphi \in C_c^\infty(-\infty, T)$  such that  $\varphi'(t) \leq 0$  for all  $t \geq 0$  (which implies  $\varphi(t) \geq 0$  for all  $t \geq 0$ ). Using

$$\int_0^T \langle \partial_t \bar{u}(\cdot, t), \varphi(t)\bar{u}(\cdot, t) \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} dt = -\frac{1}{2} \int_0^T \varphi'(t) \int_{\Omega} \bar{u}(\mathbf{x}, t)^2 d\mathbf{x} dt - \frac{1}{2} \varphi(0) \int_{\Omega} \bar{u}(\mathbf{x}, 0)^2 d\mathbf{x},$$

we get

$$\begin{cases} -\frac{1}{2} \int_0^T \varphi'(t) \int_{\Omega} \bar{u}(\mathbf{x}, t)^2 d\mathbf{x} dt - \frac{1}{2} \varphi(0) \int_{\Omega} u_{\text{ini}}(\mathbf{x})^2 d\mathbf{x} \\ + \int_0^T \varphi(t) \int_{\Omega} \mathbf{A}(\mathbf{x}, t) \cdot \nabla \bar{u}(\mathbf{x}, t) d\mathbf{x} dt = \int_0^T \varphi(t) \int_{\Omega} f(\mathbf{x}, t) \bar{u}(\mathbf{x}, t) d\mathbf{x} dt. \end{cases} \quad (4.17)$$

We then introduce the test function  $v = \varphi(t^{(n)})u^{(n+\alpha)}$  in (4.3), and we sum on  $n = 0, \dots, N-1$ . Since we have

$$\begin{aligned} \sum_{n=0}^{N-1} \Pi_{\mathcal{D}}(u^{(n+1)} - u^{(n)})\varphi(t^{(n)})\Pi_{\mathcal{D}}(\alpha u^{(n+1)} + (1-\alpha)u^{(n)}) &= -\frac{1}{2} \sum_{n=0}^{N-1} (\Pi_{\mathcal{D}}u^{(n+1)})^2(\varphi(t^{(n+1)}) - \varphi(t^{(n)})) \\ &- \frac{1}{2}(\Pi_{\mathcal{D}}u^{(0)})^2\varphi(0) + \left(\alpha - \frac{1}{2}\right) \sum_{n=0}^{N-1} (\Pi_{\mathcal{D}}u^{(n+1)} - \Pi_{\mathcal{D}}u^{(n)})^2\varphi(t^{(n)}), \end{aligned}$$

we may write, setting  $\varphi_{\mathcal{D}}(t) = \varphi(t^{(n)})$  for  $t \in (t^{(n)}, t^{(n+1)})$ ,

$$\begin{aligned} &-\frac{1}{2} \int_0^T \varphi'(t) \int_{\Omega} (\tilde{\Pi}_{\mathcal{D}}u(\mathbf{x}, t))^2 d\mathbf{x} dt - \frac{1}{2}\varphi(0) \int_{\Omega} (\Pi_{\mathcal{D}}u^{(0)}(\mathbf{x}))^2 d\mathbf{x} \\ &+ \int_0^T \varphi_{\mathcal{D}}(t) \int_{\Omega} \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}}u(\cdot, t), \nabla_{\mathcal{D}}u(\mathbf{x}, t)) \cdot \nabla_{\mathcal{D}}u(\mathbf{x}, t) d\mathbf{x} dt \\ &\leq \int_0^T \varphi_{\mathcal{D}}(t) \int_{\Omega} f(\mathbf{x}, t)\Pi_{\mathcal{D}}u(\mathbf{x}, t) d\mathbf{x} dt. \end{aligned} \quad (4.18)$$

Since  $\tilde{\Pi}_{\mathcal{D}_m}u_m$  converges to  $\bar{u}$  weakly in  $L^2(\Omega \times (0, T))$  and recalling that  $\varphi'(t) \leq 0$ , we have

$$\liminf_{m \rightarrow \infty} -\frac{1}{2} \int_0^T \varphi'(t) \int_{\Omega} (\tilde{\Pi}_{\mathcal{D}_m}u_m(\mathbf{x}, t))^2 d\mathbf{x} dt \geq -\frac{1}{2} \int_0^T \varphi'(t) \int_{\Omega} (\bar{u}(\mathbf{x}, t))^2 d\mathbf{x} dt.$$

This shows, thanks to (4.17), that

$$\begin{aligned} \limsup_{m \rightarrow \infty} \int_0^T \varphi_{\mathcal{D}_m}(t) \int_{\Omega} \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}_m}u_m(\cdot, t), \nabla_{\mathcal{D}_m}u_m(\mathbf{x}, t)) \cdot \nabla_{\mathcal{D}_m}u_m(\mathbf{x}, t) d\mathbf{x} dt \\ \leq \int_0^T \varphi(t) \int_{\Omega} \mathbf{A}(\mathbf{x}, t) \cdot \nabla \bar{u}(\mathbf{x}, t) d\mathbf{x} dt. \end{aligned} \quad (4.19)$$

It is now possible to apply Minty's trick, considering, for  $\mathbf{G} \in L^p(0, T; L^p(\Omega)^d)$ ,

$$\int_0^T \varphi_{\mathcal{D}}(t) \int_{\Omega} (\mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}}u(\cdot, t), \nabla_{\mathcal{D}}u(\mathbf{x}, t)) - \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}}u(\cdot, t), \mathbf{G}(\mathbf{x}, t))) \cdot (\nabla_{\mathcal{D}}u(\mathbf{x}, t) - \mathbf{G}(\mathbf{x}, t)) d\mathbf{x} dt \geq 0.$$

Developing this inequality and using (4.19) for the only "weak-weak" term, we may take the lim sup as  $m \rightarrow \infty$  to find

$$\int_0^T \varphi(t) \int_{\Omega} (\mathbf{A}(\mathbf{x}, t) - \mathbf{a}(\mathbf{x}, \bar{u}(\cdot, t), \mathbf{G}(\mathbf{x}, t))) \cdot (\nabla \bar{u}(\mathbf{x}, t) - \mathbf{G}(\mathbf{x}, t)) d\mathbf{x} \geq 0, \quad \forall \mathbf{G} \in L^p(0, T; L^p(\Omega)^d).$$

By taking  $\varphi$  which approximates the characteristic function of  $(-\infty, T)$ , we can remove it from this equation. Application of Minty's method then shows that (4.16) holds and concludes the proof that  $\bar{u}$  is a weak solution to (4.2).

**Step 3:** Strong  $L^2(\Omega \times (0, T))$ -convergence of the approximate solution.

In view of (3.4c), (4.19) and (4.16), we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_0^T \varphi_{\mathcal{D}}(t) \int_{\Omega} (\mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}}u(\cdot, t), \nabla_{\mathcal{D}}u_{\mathcal{D}}(\mathbf{x}, t)) - \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}}u(\cdot, t), \nabla \bar{u}(\mathbf{x}, t))) \\ \cdot (\nabla_{\mathcal{D}}u_{\mathcal{D}}(\mathbf{x}, t) - \nabla \bar{u}(\mathbf{x}, t)) d\mathbf{x} dt = 0, \end{aligned} \quad (4.20)$$

which proves that

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_0^T \varphi_{\mathcal{D}_m}(t) \int_{\Omega} \mathbf{a}(\mathbf{x}, \Pi_{\mathcal{D}_m}u(\cdot, t), \nabla_{\mathcal{D}_m}u_{\mathcal{D}_m}(\mathbf{x}, t)) \cdot \nabla_{\mathcal{D}_m}u_m(\mathbf{x}, t) d\mathbf{x} dt \\ = \int_0^T \varphi(t) \int_{\Omega} \mathbf{a}(\mathbf{x}, \bar{u}(\cdot, t), \nabla \bar{u}(\mathbf{x}, t)) \cdot \nabla \bar{u}(\mathbf{x}, t) d\mathbf{x} dt. \end{aligned} \quad (4.21)$$



As a result, from (4.17), (4.18) and (4.21) and letting  $\varphi(t) \rightarrow T - t$ , we obtain

$$\limsup_{m \rightarrow \infty} \frac{1}{2} \int_0^T \int_{\Omega} (\tilde{\Pi}_{\mathcal{D}_m} u_m(\mathbf{x}, t))^2 d\mathbf{x} dt \leq \frac{1}{2} \int_0^T \int_{\Omega} (\bar{u}(\mathbf{x}, t))^2 d\mathbf{x} dt,$$

which shows that the weak convergence of  $\tilde{\Pi}_{\mathcal{D}_m} u_m$  to  $\bar{u}$  in  $L^2(\Omega \times (0, T))$  is in fact strong. We then remark that

$$\begin{aligned} & \left| \int_0^T \int_{\Omega} (\Pi_{\mathcal{D}_m} u_m(\mathbf{x}, t))^2 d\mathbf{x} dt - \int_0^T \int_{\Omega} \tilde{\Pi}_{\mathcal{D}_m} u_m(\mathbf{x}, t) \Pi_{\mathcal{D}_m} u_m(\mathbf{x}, t) d\mathbf{x} dt \right| \\ & \leq (1 - \alpha) \delta_{\mathcal{D}_m} \int_0^T |\delta_{\mathcal{D}_m} u_m(\cdot, t)|_{\star, \mathcal{D}_m} \|u_m(\cdot, t)\|_{\mathcal{D}_m} dt \\ & \leq (1 - \alpha) \delta_{\mathcal{D}_m} |\delta_{\mathcal{D}_m} u_m|_{L^{p'}(0, T; Y_m)} \|\nabla_{\mathcal{D}_m} u_m\|_{L^p(0, T; L^p(\Omega)^d)}. \end{aligned}$$

By Lemmas 4.3 and 4.5, the right-hand side tends to 0 as  $m \rightarrow \infty$ . Moreover, by strong/weak convergence, we have

$$\lim_{m \rightarrow \infty} \frac{1}{2} \int_0^T \int_{\Omega} \tilde{\Pi}_{\mathcal{D}_m} u_m(\mathbf{x}, t) \Pi_{\mathcal{D}_m} u_m(\mathbf{x}, t) d\mathbf{x} dt = \frac{1}{2} \int_0^T \int_{\Omega} (\bar{u}(\mathbf{x}, t))^2 d\mathbf{x} dt.$$

Therefore,

$$\lim_{m \rightarrow \infty} \int_0^T \int_{\Omega} (\Pi_{\mathcal{D}_m} u_m(\mathbf{x}, t))^2 d\mathbf{x} dt = \int_0^T \int_{\Omega} (\bar{u}(\mathbf{x}, t))^2 d\mathbf{x} dt$$

and the weak convergence of  $\Pi_{\mathcal{D}_m} u_m$  to  $\bar{u}$  in  $L^2(\Omega \times (0, T))$  is in fact also strong.

**Step 4:** Assuming now hypothesis (3.7), strong convergence of the approximate gradient.

Using (4.20), the proof is similar to that of Theorem 3.5.  $\square$

## 5 The Hybrid Mixed Mimetic family of schemes

In [21], we showed that the SUSHI scheme, among others, is a gradient scheme. Moreover, it was proved in [18] that the SUSHI scheme [20], the Mimetic Finite Difference methods [7, 8] and the Mixed Finite Volume methods [16, 17] can all be gathered under a same generalising framework, the Hybrid Mixed Mimetic (HMM) methods. We show here that, in the case of the linear problem (3.1), any scheme of the HMM family can be viewed as a gradient scheme, and we then prove, applying Lemma 2.9, that the resulting family of gradient discretisations is coercive, consistent, limit-conforming and compact. Therefore, by Theorems 3.5 and 4.7, they lead to convergent approximations for the nonlinear elliptic and parabolic problems (1.1) and (1.2). Note that some schemes adapting HMM ideas and variants to some non-linear equations and systems have already been proposed and analysed in [15, 11, 19], but they are not gradient schemes and do not fully take advantage of the coercive gradient provided by HMM methods.

In this section, we assume  $\Omega$  to be an open polygonal bounded and connected subset of  $\mathbb{R}^d$  with Lipschitz-continuous boundary  $\partial\Omega$ .

### 5.1 Polygonal meshes

Let us first give the definition of the meshes which are used for the HMM family, see Figure 1 for some notations.

**Definition 5.1 (Pointed strictly star-shaped polygonal mesh)** *A pointed strictly star-shaped polygonal mesh of  $\Omega$  is given by the triplet  $(\mathcal{M}, \mathcal{E}, \mathcal{P})$ , where:*

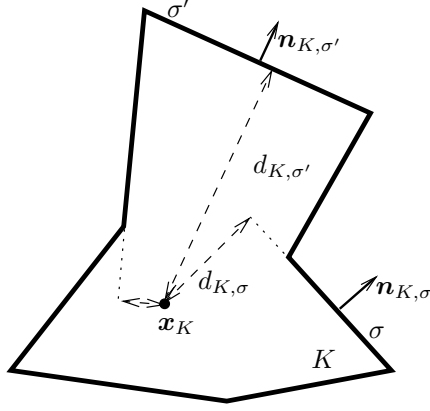


Figure 1: A control volume  $K$  of a pointed strictly star-shaped polygonal mesh

1.  $\mathcal{M}$  is a finite family of non empty connected open disjoint subsets of  $\Omega$  (the “control volumes” or “cells”) such that  $\bar{\Omega} = \cup_{K \in \mathcal{M}} \bar{K}$ . For any  $K \in \mathcal{M}$ , let  $\partial K = \bar{K} \setminus K$  be the boundary of  $K$ ,  $|K| > 0$  be the measure of  $K$  and  $h_K$  be the diameter of  $K$ .
2.  $\mathcal{E} = \mathcal{E}_{\text{int}} \cup \mathcal{E}_{\text{ext}}$  is a finite family of disjoint subsets of  $\bar{\Omega}$  (the “edges” of the mesh) such that any  $\sigma \in \mathcal{E}$  is a non-empty open subset of an hyperplane of  $\mathbb{R}^d$ . We denote by  $\mathcal{E}_{\text{int}}$  the set of edges included in  $\Omega$  and by  $\mathcal{E}_{\text{ext}}$  the set of edges included in  $\partial\Omega$ . The  $(d-1)$ -dimensional measure and the center of gravity of  $\sigma \in \mathcal{E}$  are respectively denoted by  $|\sigma|$  and  $\bar{\mathbf{x}}_\sigma$ .  
We assume that, for all  $K \in \mathcal{M}$ , there exists a subset  $\mathcal{E}_K$  of  $\mathcal{E}$  such that  $\partial K = \cup_{\sigma \in \mathcal{E}_K} \bar{\sigma}$ . We then set  $\mathcal{M}_\sigma = \{K \in \mathcal{M}, \sigma \in \mathcal{E}_K\}$  and we assume that, for all  $\sigma \in \mathcal{E}$ , either  $\mathcal{M}_\sigma$  has exactly one element and then  $\sigma \in \mathcal{E}_{\text{ext}}$  or  $\mathcal{M}_\sigma$  has exactly two elements and then  $\sigma \in \mathcal{E}_{\text{int}}$ . For all  $K \in \mathcal{M}$  and any  $\sigma \in \mathcal{E}_K$ , we denote by  $\mathbf{n}_{K,\sigma}$  the unit vector normal to  $\sigma$  outward to  $K$ .
3.  $\mathcal{P} = ((\mathbf{x}_K)_{K \in \mathcal{M}}, (\mathbf{x}_\sigma)_{\sigma \in \mathcal{E}})$  is a family of points of  $\Omega$  indexed by  $\mathcal{M}$  and  $\mathcal{E}$  such that for all  $K \in \mathcal{M}$ ,  $\mathbf{x}_K \in K$  and for all  $\sigma \in \mathcal{E}$ ,  $\mathbf{x}_\sigma \in \sigma$ . We then denote by  $d_{K,\sigma}$  the signed distance between  $\mathbf{x}_K$  and  $\sigma$  (see Figure 1), that is:

$$d_{K,\sigma} = (\mathbf{x} - \mathbf{x}_K) \cdot \mathbf{n}_{K,\sigma}, \quad \mathbf{x} \in \sigma. \quad (5.1)$$

(Note that  $(\mathbf{x} - \mathbf{x}_K) \cdot \mathbf{n}_{K,\sigma}$  is constant for  $\mathbf{x} \in \sigma$ .)

We further assume that all cells  $K \in \mathcal{M}$  are strictly  $\mathbf{x}_K$ -star-shaped, which means that, for all  $\mathbf{x} \in K$ , the line segment  $[\mathbf{x}_K, \mathbf{x}]$  is included in  $K$  or, equivalently, that  $d_{K,\sigma} > 0$  for all  $\sigma \in \mathcal{E}_K$ .

The size of the discretisation is defined by  $h_{\mathcal{D}} = \max\{h_K, K \in \mathcal{M}\}$ . For all  $K \in \mathcal{M}$  and  $\sigma \in \mathcal{E}_K$ , we denote by  $D_{K,\sigma}$  the cone with vertex  $\mathbf{x}_K$  and basis  $\sigma$ :  $D_{K,\sigma} = \{t\mathbf{x}_K + (1-t)\mathbf{y}, t \in (0,1), \mathbf{y} \in \sigma\}$ . We have

$$|D_{K,\sigma}| = \frac{1}{d} |\sigma| d_{K,\sigma} \quad \text{and} \quad \sum_{\sigma \in \mathcal{E}_K} |\sigma| d_{K,\sigma} = d|K|. \quad (5.2)$$

The following lemma, which directly results from the Stokes formula, is classically used in the construction of consistent approximate gradients.

**Lemma 5.2** *Let  $K$  be a nonempty polyhedral subset of  $\mathbb{R}^d$ . For  $\sigma \in \mathcal{E}_K$ , we denote by  $\bar{\mathbf{x}}_\sigma$  the barycentre of  $\sigma$  and by  $\mathbf{n}_{K,\sigma}$  the unit vector normal to  $\sigma$  outward to  $K$ . Let  $\mathbf{x}_K$  be any point of  $\mathbb{R}^d$ . Then:*

$$\sum_{\sigma \in \mathcal{E}_K} |\sigma| \mathbf{n}_{K,\sigma} (\bar{\mathbf{x}}_\sigma - \mathbf{x}_K)^t = |K| \text{Id}, \quad (5.3)$$

where  $(\bar{\mathbf{x}}_\sigma - \mathbf{x}_K)^t$  is the transpose of  $\bar{\mathbf{x}}_\sigma - \mathbf{x}_K \in \mathbb{R}^d$  and  $\text{Id}$  is the  $d \times d$  identity matrix.

## 5.2 Hybrid Mixed Mimetic methods

Let us recall the construction of an HMM method. Let  $(\mathcal{M}, \mathcal{E}, \mathcal{P})$  be a strictly star-shaped pointed polygonal mesh of  $\Omega$ , in the sense of Definition 5.1. Let

$$X_{\mathcal{D},0} = \{v = ((v_K)_{K \in \mathcal{M}}, (v_\sigma)_{\sigma \in \mathcal{E}}) : v_K \in \mathbb{R}, v_\sigma \in \mathbb{R}, v_\sigma = 0 \text{ for all } \sigma \in \mathcal{E}_{\text{ext}}\}, \quad (5.4)$$

and define, for  $v \in X_{\mathcal{D},0}$ ,  $K \in \mathcal{M}$  and  $\sigma \in \mathcal{E}_K$ ,

$$\nabla_K v = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (v_\sigma - v_K) \mathbf{n}_{K,\sigma} \quad \text{and} \quad R_{K,\sigma}(v) = v_\sigma - v_K - \nabla_K v \cdot (\bar{\mathbf{x}}_\sigma - \mathbf{x}_K).$$

We also define  $R_K(v) = (R_{K,\sigma}(v))_{\sigma \in \mathcal{E}_K}$  and  $\Lambda_K =$  value of  $\Lambda$  on  $K$  ( $\Lambda$  is assumed to be constant in each cell). With these notations, any Hybrid Mixed Mimetic method for the weak form (3.3) of Problem (3.1) can be written

$$\begin{aligned} &\text{Find } u \in X_{\mathcal{D},0} \text{ such that, for all } v \in X_{\mathcal{D},0}, \\ &\sum_{K \in \mathcal{M}} |K| \Lambda_K \nabla_K u \cdot \nabla_K v + \sum_{K \in \mathcal{M}} R_K(v)^T \mathbb{B}_K R_K(u) = \sum_{K \in \mathcal{M}} v_K \int_K f(\mathbf{x}) d\mathbf{x} \end{aligned} \quad (5.5)$$

with  $\mathbb{B}_K \in \mathbb{R}^{\mathcal{E}_K \times \mathcal{E}_K}$  symmetric positive definite matrices with suitable properties (see Remark 5.7).

**Remark 5.3 (A generalisation with barycentric edge unknowns)** *We could, as in [20], express some values  $v_\sigma$  as barycentric combinations of the values  $v_K$ , thus obtaining a cell-centred scheme. All the properties given here can be extended to this case.*

## 5.3 All HMM methods are gradient schemes

### 5.3.1 Definition of the gradient discretisation

Let  $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \rightarrow L^2(\Omega)$  be defined by

$$\forall v \in X_{\mathcal{D},0}, \forall K \in \mathcal{M}, \Pi_{\mathcal{D}}(v) = v_K \text{ on } K. \quad (5.6)$$

The right-hand side of (5.5) is therefore equal to  $\int_\Omega f \Pi_{\mathcal{D}} v$ . In order to prove that any HMM method is a gradient scheme we now have to find, for any choice of  $(\mathbb{B}_K)_{K \in \mathcal{M}}$ , a discrete gradient  $\nabla_{\mathcal{D}}$  such that, for all  $(u, v) \in X_{\mathcal{D},0}^2$ ,

$$\sum_{K \in \mathcal{M}} |K| \Lambda_K \nabla_K u \cdot \nabla_K v + \sum_{K \in \mathcal{M}} R_K(v)^T \mathbb{B}_K R_K(u) = \int_\Omega \Lambda(\mathbf{x}) \nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x}. \quad (5.7)$$

For any  $v \in X_{\mathcal{D},0}$ , we look for a piecewise constant gradient  $\nabla_{\mathcal{D}} v$  defined by:

$$\forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K, \forall \mathbf{x} \in D_{K,\sigma} : \nabla_{\mathcal{D}} v(\mathbf{x}) = \nabla_{K,\sigma} v := \nabla_K v + \frac{\sqrt{d}}{d_{K,\sigma}} (A_K \tilde{R}_K(V_K))_\sigma \mathbf{n}_{K,\sigma}, \quad (5.8)$$

where  $V_K = (v_\sigma - v_K)_{\sigma \in \mathcal{E}_K}$ ,  $\tilde{R}_K : \mathbb{R}^{\mathcal{E}_K} \rightarrow \mathbb{R}^{\mathcal{E}_K}$  is the linear mapping defined by  $\tilde{R}_K(\xi) = (\tilde{R}_{K,\sigma}(\xi))_{\sigma \in \mathcal{E}_K}$  with

$$\tilde{R}_{K,\sigma}(\xi) = \xi_\sigma - \left( \frac{1}{|K|} \sum_{\sigma' \in \mathcal{E}_K} |\sigma'| \xi_{\sigma'} \mathbf{n}_{K,\sigma'} \right) \cdot (\bar{\mathbf{x}}_\sigma - \mathbf{x}_K),$$

and  $A_K$  is an isomorphism, to be defined, of the vector space  $\text{Im}(\tilde{R}_K) \subset \mathbb{R}^{\mathcal{E}_K}$ . We prove below that, for any symmetric positive definite matrix  $\mathbb{B}_K \in \mathbb{R}^{\mathcal{E}_K \times \mathcal{E}_K}$ , we can find  $A_K$  such that for all  $(u, v) \in X_{\mathcal{D},0}^2$ ,

$$\sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \Lambda_K \nabla_{K,\sigma} u \cdot \nabla_{K,\sigma} v = |K| \Lambda_K \nabla_K u \cdot \nabla_K v + R_K(v)^T \mathbb{B}_K R_K(u). \quad (5.9)$$

With such  $A_K$ 's, the gradient  $\nabla_{\mathcal{D}}$  defined by (5.8) satisfies (5.7) and the HMM method (5.5) is exactly the gradient scheme given by the gradient discretisation  $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ .

To find  $A_K : \text{Im}(\tilde{R}_K) \rightarrow \text{Im}(\tilde{R}_K)$  we first notice that, by Lemma 5.2, the linear mapping

$$G_K : \eta \in \mathbb{R}^{\mathcal{E}_K} \rightarrow \sum_{\sigma \in \mathcal{E}_K} |\sigma| \eta_{\sigma} \mathbf{n}_{K,\sigma} \in \mathbb{R}^d \quad (5.10)$$

vanishes on  $\text{Im}(\tilde{R}_K)$ . Since  $A_K$  takes its values in this space, using (5.2) we deduce

$$\forall \xi \in \mathbb{R}^{\mathcal{E}_K}, \quad \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \frac{\sqrt{d}}{d_{K,\sigma}} (A_K \tilde{R}_K(\xi))_{\sigma} \mathbf{n}_{K,\sigma} = \frac{1}{\sqrt{d}} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (A_K \tilde{R}_K(\xi))_{\sigma} \mathbf{n}_{K,\sigma} = 0. \quad (5.11)$$

Hence, from the definition (5.8) and since  $\sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| = |K|$ , we infer

$$\begin{aligned} & \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \Lambda_K \nabla_{K,\sigma} u \cdot \nabla_{K,\sigma} v \\ &= |K| \Lambda_K \nabla_K u \cdot \nabla_K v + \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \frac{d}{d_{K,\sigma}^2} \Lambda_K \mathbf{n}_{K,\sigma} \cdot \mathbf{n}_{K,\sigma} (A_K \tilde{R}_K(U_K))_{\sigma} (A_K \tilde{R}_K(V_K))_{\sigma} \\ &= |K| \Lambda_K \nabla_K u \cdot \nabla_K v + (A_K \tilde{R}_K(V_K))^T \mathbb{D}_K (A_K \tilde{R}_K(U_K)) \end{aligned}$$

where  $U_K = (u_{\sigma} - u_K)_{\sigma \in \mathcal{E}_K}$  and  $\mathbb{D}_K = \text{diag}(\frac{|\sigma|}{d_{K,\sigma}} \Lambda_K \mathbf{n}_{K,\sigma} \cdot \mathbf{n}_{K,\sigma})$  is diagonal definite positive. Since  $R_K(u) = \tilde{R}_K(U_K)$  and  $R_K(v) = \tilde{R}_K(V_K)$ , (5.9) is satisfied provided that, for all  $(\xi, \eta) \in (\mathbb{R}^{\mathcal{E}_K})^2$ ,

$$(A_K \tilde{R}_K(\xi))^T \mathbb{D}_K (A_K \tilde{R}_K(\eta)) = \tilde{R}_K(\xi)^T \mathbb{B}_K \tilde{R}_K(\eta). \quad (5.12)$$

We now apply Lemma 5.4 below with  $E = \text{Im}(\tilde{R}_K) \subset \mathbb{R}^{\mathcal{E}_K}$  and the inner products  $\langle x, y \rangle_1 = y^T \mathbb{B}_K x$  and  $\langle x, y \rangle_2 = y^T \mathbb{D}_K x$ . The isomorphism  $A_K : \text{Im}(\tilde{R}_K) \rightarrow \text{Im}(\tilde{R}_K)$  given by Lemma 5.4 satisfies (5.13) which, applied with  $x = \tilde{R}_K(\eta)$  and  $y = \tilde{R}_K(\xi)$ , is precisely (5.12).

**Lemma 5.4** *Let  $E$  be a finite-dimensional vector space endowed with two inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ . Then there exists an isomorphism  $A : E \rightarrow E$  such that*

$$\text{for all } (x, y) \in E^2, \quad \langle x, y \rangle_1 = \langle Ax, Ay \rangle_2. \quad (5.13)$$

**Proof** Let  $e$  be an orthonormal basis for  $\langle \cdot, \cdot \rangle_2$  and  $M_e$  be the (symmetric definite positive) matrix of  $\langle \cdot, \cdot \rangle_1$  in this basis. If  $X_e$  and  $Y_e$  are the coordinates of  $x$  and  $y$  in  $e$  then  $\langle x, y \rangle_1 = Y_e^T M_e X_e$ . Let then  $A_e = \sqrt{M_e}$  and define  $A$  as the isomorphism whose matrix relative to the basis  $e$  is  $A_e$ . Since  $e$  is orthonormal for  $\langle \cdot, \cdot \rangle_2$ , the relation  $Y_e^T M_e X_e = (A_e Y_e)^T (A_e X_e)$  translates into  $\langle x, y \rangle_1 = \langle Ax, Ay \rangle_2$ .  $\square$

**Remark 5.5** *Since  $(\mathbf{n}_{K,\sigma})_{\sigma \in \mathcal{E}_K}$  spans  $\mathbb{R}^d$ , the mapping  $G_K$  defined by (5.10) has rank  $d$ , which implies  $\dim(\ker G_K) = \text{Card}(\mathcal{E}_K) - d$ . It is easy to see that  $\ker(\tilde{R}_K) = \{\xi \in \mathbb{R}^{\mathcal{E}_K} ; \exists Z_{\xi} \in \mathbb{R}^d \text{ such that } \xi_{\sigma} = Z_{\xi} \cdot (\bar{\mathbf{x}}_{\sigma} - \mathbf{x}_K)\}$  and thus that  $Z \in \mathbb{R}^d \rightarrow (Z \cdot (\bar{\mathbf{x}}_{\sigma} - \mathbf{x}_K))_{\sigma \in \mathcal{E}_K} \in \ker(\tilde{R}_K)$  is an isomorphism (the one to one property comes from the fact that  $(\bar{\mathbf{x}}_{\sigma} - \mathbf{x}_K)_{\sigma \in \mathcal{E}_K}$  spans  $\mathbb{R}^d$ ). Hence,  $\dim(\text{Im}(\tilde{R}_K)) = \text{Card}(\mathcal{E}_K) - d = \dim(\ker(G_K))$ . Since  $\text{Im}(\tilde{R}_K) \subset \ker(G_K)$ , we infer that  $\text{Im}(\tilde{R}_K) = \ker(G_K)$ .*

*Thus,  $A_K$  can be indifferently searched as an isomorphism of  $\text{Im}(\tilde{R}_K)$  or as an isomorphism of  $\ker(G_K)$ .*

### 5.3.2 Coercivity, consistency, limit-conformity and compactness

We prove here that a gradient discretisation corresponding to an HMM method is, under the usual assumptions on HMM methods, coercive, consistent, limit-conforming and compact. The proof is based on the characterisation of these properties given in Lemma 2.9.

**Lemma 5.6 (Coercivity)** *Let  $p \in [1, +\infty)$ ,  $(\mathcal{M}, \mathcal{E}, \mathcal{P})$  be a strictly star-shaped pointed polygonal mesh of  $\Omega$  in the sense of Definition 5.1 and let  $(X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$  be given by (5.4), (5.6) and (5.8), for some  $(A_K)_{K \in \mathcal{M}}$ . We take  $\theta > 0$  such that*

$$\max \left( \max_{\sigma \in \mathcal{E}_{\text{int}}, K, L \in \mathcal{M}_\sigma} \frac{d_{K,\sigma}}{d_{L,\sigma}}, \max_{K \in \mathcal{M}, \sigma \in \mathcal{E}_K} \frac{h_K}{d_{K,\sigma}} \right) \leq \theta \quad (5.14)$$

and

$$\forall K \in \mathcal{M}, \forall \xi \in \mathbb{R}^{\mathcal{E}_K},$$

$$\frac{1}{\theta} \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{\tilde{R}_{K,\sigma}(\xi)}{d_{K,\sigma}} \right|^p \leq \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{(A_K \tilde{R}_K(\xi))_\sigma}{d_{K,\sigma}} \right|^p \leq \theta \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{\tilde{R}_{K,\sigma}(\xi)}{d_{K,\sigma}} \right|^p. \quad (5.15)$$

Then there exists  $C_5 > 1$  only depending on  $\Omega$ ,  $p$  and  $\theta$  such that:

$$\frac{1}{C_5} \|u\|_{1,p} \leq \|\nabla_{\mathcal{D}} u\|_{L^p(\Omega)} \leq C_5 \|u\|_{1,p} \quad \forall u \in X_{\mathcal{D},0}, \quad (5.16)$$

where we define

$$\|u\|_{1,p}^p = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} |\sigma| d_{K,\sigma} \left| \frac{u_\sigma - u_K}{d_{K,\sigma}} \right|^p.$$

Consequently, there exists  $C_6 > 0$  only depending on  $\Omega$ ,  $p$  and  $\theta$  such that:

$$C_{\mathcal{D}} \leq C_6, \quad (5.17)$$

where  $C_{\mathcal{D}}$  is defined by (2.2).

**Remark 5.7** *To ensure the convergence of the HMM method, matrices  $\mathbb{B}_K$  are assumed to satisfy the following (see [18]): there exist  $s_* > 0$  and  $S_* > 0$  independent of the mesh such that, for all  $K \in \mathcal{M}$  and all  $\xi \in \mathbb{R}^{\mathcal{E}_K}$ ,*

$$s_* \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{d_{K,\sigma}} (\tilde{R}_{K,\sigma}(\xi))^2 \leq \tilde{R}_K(\xi)^T \mathbb{B}_K \tilde{R}_K(\xi) \leq S_* \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{d_{K,\sigma}} (\tilde{R}_{K,\sigma}(\xi))^2.$$

If  $A_K$  is chosen so that (5.12) holds then, since  $\mathbb{D}_K = \text{diag}(\frac{|\sigma|}{d_{K,\sigma}} \Lambda_K \mathbf{n}_{K,\sigma} \cdot \mathbf{n}_{K,\sigma})$ , there exists  $C_7$  only depending on  $d$ ,  $s_*$ ,  $S_*$  and the coercivity constant of  $\Lambda$  such that

$$\frac{1}{C_7} \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{d_{K,\sigma}} (\tilde{R}_{K,\sigma}(\xi))^2 \leq \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{d_{K,\sigma}} (A_K \tilde{R}_K(\xi))_\sigma^2 \leq C_7 \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{d_{K,\sigma}} (\tilde{R}_{K,\sigma}(\xi))^2,$$

which is precisely (5.15) for  $p = 2$ . Hence, in the linear framework (the only one in which generic HMM methods are defined up to now), the gradient scheme corresponding to an HMM method satisfies the assumptions in Lemma 5.6 with  $\theta$  not depending on the mesh.

**Remark 5.8** *An easy way to choose  $A_K$  such that (5.15) holds is to take  $A_K = \beta_K \text{Id}_{\text{Im}(\tilde{R}_K)}$  for some  $\beta_K \in [\frac{1}{\theta}, \theta]$ . The corresponding HMM method is then the SUSHI scheme of [20].*

**Proof**

In this proof, the notation  $\mathcal{A}(u) \sim \mathcal{B}(u)$  means that there exists  $C_8$  only depending on  $\Omega$ ,  $p$  and  $\theta$  such that, for all  $u \in X_{\mathcal{D},0}$ ,  $C_8^{-1} \mathcal{A}(u) \leq \mathcal{B}(u) \leq C_8 \mathcal{A}(u)$ .

The key ingredient in the proof is to notice that  $\nabla_K u$  is a convex combination of  $(\nabla_{K,\sigma} u)_{\sigma \in \mathcal{E}_K}$ . Precisely, (5.11) shows that  $\nabla_K u = \sum_{\sigma \in \mathcal{E}_K} \frac{|D_{K,\sigma}|}{|K|} \nabla_{K,\sigma} u$ . By convexity of  $|\cdot|^p$  on  $\mathbb{R}^d$ , we infer

$$|\nabla_K u|^p \leq \sum_{\sigma \in \mathcal{E}_K} \frac{|D_{K,\sigma}|}{|K|} |\nabla_{K,\sigma} u|^p. \quad (5.18)$$

From  $\sqrt{d} \frac{(A_K \tilde{R}_K(U_K))_\sigma}{d_{K,\sigma}} \mathbf{n}_{K,\sigma} = \nabla_{K,\sigma} u - \nabla_K u$ , we also have

$$d^{p/2} \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{(A_K \tilde{R}_K(U_K))_\sigma}{d_{K,\sigma}} \right|^p \leq 2^{p-1} |K| |\nabla_K u|^p + 2^{p-1} \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| |\nabla_{K,\sigma} u|^p. \quad (5.19)$$

Together with (5.15), Estimates (5.18) and (5.19) show that

$$|K| |\nabla_K u|^p + \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{\tilde{R}_{K,\sigma}(U_K)}{d_{K,\sigma}} \right|^p \sim \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| |\nabla_{K,\sigma} u|^p. \quad (5.20)$$

Since  $\frac{\tilde{R}_{K,\sigma}(U_K)}{d_{K,\sigma}} = \frac{u_\sigma - u_K}{d_{K,\sigma}} - \nabla_K u \cdot \frac{\bar{\mathbf{x}}_\sigma - \mathbf{x}_K}{d_{K,\sigma}}$  and  $\frac{|\bar{\mathbf{x}}_\sigma - \mathbf{x}_K|}{d_{K,\sigma}} \leq \theta$  thanks to (5.14), we see that

$$|K| |\nabla_K u|^p + \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{\tilde{R}_{K,\sigma}(U_K)}{d_{K,\sigma}} \right|^p \sim |K| |\nabla_K u|^p + \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{u_\sigma - u_K}{d_{K,\sigma}} \right|^p. \quad (5.21)$$

Finally, the definition of  $\nabla_K u$  and (5.2) show that  $\nabla_K u = d \sum_{\sigma \in \mathcal{E}_K} \frac{|D_{K,\sigma}|}{|K|} \frac{u_\sigma - u_K}{d_{K,\sigma}} \mathbf{n}_{K,\sigma}$  and thus that

$$|\nabla_K u|^p \leq d^p \sum_{\sigma \in \mathcal{E}_K} \frac{|D_{K,\sigma}|}{|K|} \left| \frac{u_\sigma - u_K}{d_{K,\sigma}} \right|^p. \quad (5.22)$$

Since  $\|\nabla_{\mathcal{D}} u\|_{L^p(\Omega)^d}^p = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| |\nabla_{K,\sigma} u|^p$ , Estimate (5.16) follows from (5.20), (5.21) and (5.22). To deduce (5.17) from (5.16), we notice that, whenever  $\sigma$  is a common edge between  $K$  and  $L$ ,

$$\frac{|u_L - u_K|}{d_{K,\sigma} + d_{L,\sigma}} \leq \frac{|u_L - u_\sigma|}{d_{L,\sigma}} + \frac{|u_\sigma - u_K|}{d_{K,\sigma}}$$

and we conclude by [20, Lemma 5.4].  $\square$

**Lemma 5.9 (Limit-conformity)** *Let  $p \in [1, +\infty)$ ,  $(\mathcal{M}, \mathcal{E}, \mathcal{P})$  be a strictly star-shaped pointed polygonal mesh of  $\Omega$  in the sense of Definition 5.1 and let  $(X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$  be given by (5.4), (5.6) and (5.8), for some  $(A_K)_{K \in \mathcal{M}}$ . We take  $\theta > 0$  such that (5.14) and (5.15) hold. Then there exists  $C_9$  only depending on  $\Omega$ ,  $p$  and  $\theta$  such that*

$$W_{\mathcal{D}}(\varphi) \leq C_9 h_{\mathcal{D}} \|\varphi\|_{(W^{1,\infty}(\mathbb{R}^d))^d}, \quad \forall \varphi \in (C_c^\infty(\mathbb{R}^d))^d. \quad (5.23)$$

**Proof** Let  $\varphi \in C_c^\infty(\mathbb{R}^d)^d$  and  $u \in X_{\mathcal{D},0}$ . Let us define the terms  $T_1^{\mathcal{D}}$  and  $T_2^{\mathcal{D}}$  by

$$T_1^{\mathcal{D}} = \int_{\mathbb{R}^d} \nabla_{\mathcal{D}} u_{\mathcal{D}}(\mathbf{x}) \cdot \varphi(\mathbf{x}) d\mathbf{x} \quad \text{and} \quad T_2^{\mathcal{D}} = - \int_{\mathbb{R}^d} \Pi_{\mathcal{D}} u_{\mathcal{D}}(\mathbf{x}) \operatorname{div} \varphi(\mathbf{x}) d\mathbf{x}.$$

Since  $\mathbf{n}_{K,\sigma} = -\mathbf{n}_{L,\sigma}$  whenever  $\sigma$  is an edge between  $K$  and  $L$ , and since  $u_\sigma = 0$  if  $\sigma \in \mathcal{E}_{\text{ext}}$ , letting  $\varphi_\sigma = \frac{1}{|\sigma|} \int_\sigma \varphi(\mathbf{x}) d\gamma(\mathbf{x})$  we have

$$T_2^{\mathcal{D}} = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (u_\sigma - u_K) \mathbf{n}_{K,\sigma} \cdot \varphi_\sigma.$$

By (5.8), we have  $T_1^{\mathcal{D}} = T_3^{\mathcal{D}} + T_4^{\mathcal{D}}$  with

$$T_3^{\mathcal{D}} = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (u_\sigma - u_K) \mathbf{n}_{K,\sigma} \cdot \varphi_K \quad \text{and} \quad T_4^{\mathcal{D}} = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \frac{\sqrt{d}}{d_{K,\sigma}} (A_K \tilde{R}_K(U_K))_\sigma \mathbf{n}_{K,\sigma} \cdot \int_{D_{K,\sigma}} \varphi(\mathbf{x}) d\mathbf{x},$$

where  $\varphi_K = \frac{1}{|K|} \int_K \varphi(\mathbf{x}) d\mathbf{x}$ . Since  $|\varphi_K - \varphi_\sigma| \leq h_{\mathcal{D}} \|\varphi\|_{(W^{1,\infty}(\mathbb{R}^d))^d}$  whenever  $\sigma \in \mathcal{E}_K$ , using Hölder's inequality we get  $|T_3^{\mathcal{D}} - T_2^{\mathcal{D}}| \leq h_{\mathcal{D}} \|\varphi\|_{(W^{1,\infty}(\mathbb{R}^d))^d} \|u\|_{1,p} (d|\Omega|)^{(p-1)/p}$ . By (5.16), we therefore find  $C_{10}$  only depending on  $\Omega$ ,  $p$  and  $\theta$  such that

$$|T_3^{\mathcal{D}} - T_2^{\mathcal{D}}| \leq C_{10} h_{\mathcal{D}} \|\varphi\|_{(W^{1,\infty}(\mathbb{R}^d))^d} \|\nabla_{\mathcal{D}} u\|_{L^p(\Omega)^d}. \quad (5.24)$$

Invoking (5.11), we see that

$$T_4^{\mathcal{D}} = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \frac{\sqrt{d}}{d_{K,\sigma}} (A_K \tilde{R}_K(U_K))_\sigma \mathbf{n}_{K,\sigma} \mathbf{n}_{K,\sigma} \cdot \int_{D_{K,\sigma}} (\varphi(\mathbf{x}) - \varphi_K) d\mathbf{x}.$$

Since  $|\int_{D_{K,\sigma}} (\varphi(\mathbf{x}) - \varphi_K) d\mathbf{x}| \leq h_{\mathcal{D}} \|\varphi\|_{(W^{1,\infty}(\mathbb{R}^d))^d} |D_{K,\sigma}|$ , this leads to

$$\begin{aligned} |T_4^{\mathcal{D}}| &\leq \sqrt{d} h_{\mathcal{D}} \|\varphi\|_{(W^{1,\infty}(\mathbb{R}^d))^d} \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{(A_K \tilde{R}_K(U_K))_\sigma}{d_{K,\sigma}} \right| \\ &\leq \sqrt{d} |\Omega|^{(p-1)/p} h_{\mathcal{D}} \|\varphi\|_{(W^{1,\infty}(\mathbb{R}^d))^d} \left( \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{(A_K \tilde{R}_K(U_K))_\sigma}{d_{K,\sigma}} \right|^p \right)^{1/p}. \end{aligned}$$

Recalling (5.15) and (5.20) in the proof of Lemma 5.6, we deduce the existence of  $C_{11}$  only depending on  $\Omega$ ,  $p$  and  $\theta$  such that

$$|T_4^{\mathcal{D}}| \leq C_{11} h_{\mathcal{D}} \|\varphi\|_{(W^{1,\infty}(\mathbb{R}^d))^d} \|\nabla_{\mathcal{D}} u\|_{L^p(\Omega)^d}. \quad (5.25)$$

Equations (5.24) and (5.25) prove that there exists  $C_{12}$  only depending on  $\Omega$ ,  $p$  and  $\theta$  such that  $|T_1^{\mathcal{D}} - T_2^{\mathcal{D}}| = |T_3^{\mathcal{D}} - T_2^{\mathcal{D}} + T_4^{\mathcal{D}}| \leq C_{12} h_{\mathcal{D}} \|\varphi\|_{(W^{1,\infty}(\mathbb{R}^d))^d} \|\nabla_{\mathcal{D}} u\|_{L^p(\Omega)^d}$  and the proof is complete.  $\square$

**Lemma 5.10 (Consistency)** *Let  $p \in [1, +\infty)$ ,  $(\mathcal{M}, \mathcal{E}, \mathcal{P})$  be a strictly star-shaped pointed polygonal mesh of  $\Omega$  in the sense of Definition 5.1 and let  $(X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$  be given by (5.4), (5.6) and (5.8), for some  $(A_K)_{K \in \mathcal{M}}$ . We take  $\theta > 0$  such that (5.14) and (5.15) hold. Then there exists  $C_{13}$  only depending on  $\Omega$ ,  $p$  and  $\theta$  such that:*

$$S_{\mathcal{D}}(\varphi) \leq C h_{\mathcal{D}} \|\varphi\|_{W^{2,\infty}(\Omega)}, \quad \forall \varphi \in C_c^\infty(\Omega). \quad (5.26)$$

**Proof** For all  $\varphi \in C_c^\infty(\Omega)$ , let  $v \in X_{\mathcal{D},0}$  such that  $v_K = \varphi(\mathbf{x}_K)$  for all  $K \in \mathcal{M}$  and  $v_\sigma = \varphi(\bar{\mathbf{x}}_\sigma)$  for all  $\sigma \in \mathcal{E}$ . We clearly have  $\|\Pi_{\mathcal{D}} v - \varphi\|_{L^\infty(\Omega)} \leq h_{\mathcal{D}} \|\varphi\|_{W^{1,\infty}(\Omega)}$ . Regarding the gradients, we first write

$$|\nabla_{K,\sigma} v - \nabla \varphi(\mathbf{x}_K)| \leq |\nabla_K v - \nabla \varphi(\mathbf{x}_K)| + \left| \frac{\sqrt{d}}{d_{K,\sigma}} (A_K \tilde{R}_K(V_K))_\sigma \right|. \quad (5.27)$$

Let  $Z_{K,\sigma} = \varphi(\bar{\mathbf{x}}_\sigma) - \varphi(\mathbf{x}_K) - \nabla \varphi(\mathbf{x}_K) \cdot (\bar{\mathbf{x}}_\sigma - \mathbf{x}_K)$ . We have, by Lemma 5.2,

$$\begin{aligned} \nabla \varphi(\mathbf{x}_K) &= \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| [\nabla \varphi(\mathbf{x}_K) \cdot (\bar{\mathbf{x}}_\sigma - \mathbf{x}_K)] \mathbf{n}_{K,\sigma} \\ &= \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\varphi(\bar{\mathbf{x}}_\sigma) - \varphi(\mathbf{x}_K)) \mathbf{n}_{K,\sigma} - \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| Z_{K,\sigma} \mathbf{n}_{K,\sigma} \\ &= \nabla_K v - \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| Z_{K,\sigma} \mathbf{n}_{K,\sigma}. \end{aligned}$$

Since  $h_K \leq \theta d_{K,\sigma}$  and  $|Z_{K,\sigma}| \leq h_K^2 \|\varphi\|_{W^{2,\infty}(\Omega)}$ , we deduce

$$|\nabla \varphi(\mathbf{x}_K) - \nabla_K v| \leq d \theta h_K \|\varphi\|_{W^{2,\infty}(\Omega)}. \quad (5.28)$$

We have  $\tilde{R}_{K,\sigma}(V_K) = v_\sigma - v_K - \nabla_K v \cdot (\bar{\mathbf{x}}_\sigma - \mathbf{x}_K) = Z_{K,\sigma} - (\nabla_K v - \nabla\varphi(\mathbf{x}_K)) \cdot (\bar{\mathbf{x}}_\sigma - \mathbf{x}_K)$  and thus  $|\frac{\tilde{R}_{K,\sigma}(V_K)}{d_{K,\sigma}}| \leq C_{14}h_K\|\varphi\|_{W^{2,\infty}(\Omega)}$  with  $C_{14}$  only depending on  $d, p$  and  $\theta$ . Using (5.15), we infer

$$\sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{(A_K \tilde{R}_K(V_K))_\sigma}{d_{K,\sigma}} \right|^p \leq \theta C_{14}^p h_K^p \|\varphi\|_{W^{2,\infty}(\Omega)}^p |K|.$$

Using this estimate and (5.28) in (5.27) we obtain  $\|\nabla_{\mathcal{D}} v - \nabla\varphi\|_{L^p(\Omega)} \leq C_{15}h_{\mathcal{D}}\|\varphi\|_{W^{2,\infty}(\Omega)}$  with  $C_{15}$  only depending on  $\Omega, p$  and  $\theta$ . The proof is complete.  $\square$

**Remark 5.11** *Since we estimated  $\|\Pi_{\mathcal{D}} v - \varphi\|_{L^\infty(\Omega)}$  (and not only the  $L^p$  norm), the same proof shows that the space-time consistency, in the sense of Definition 4.6, also holds.*

**Lemma 5.12 (Compactness)** *Let  $p \in [1, +\infty)$ ,  $(\mathcal{M}, \mathcal{E}, \mathcal{P})$  be a strictly star-shaped pointed polygonal mesh of  $\Omega$  in the sense of Definition 5.1 and let  $(X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$  be given by (5.4), (5.6) and (5.8), for some  $(A_K)_{K \in \mathcal{M}}$ . We take  $\theta > 0$  such that (5.14) and (5.15) hold.*

*Then there exists  $\nu > 0$  and  $C_{16}$  only depending on  $\Omega, p$  and  $\theta$  such that, for all  $\boldsymbol{\xi} \in \mathbb{R}^d$ ,  $T_{\mathcal{D}}(\boldsymbol{\xi}) \leq C_{16}|\boldsymbol{\xi}|^\nu$ .*

**Proof**

Let  $v \in X_{\mathcal{D},0}$ . Noticing that, if  $\sigma$  is an edge between  $K$  and  $L$ ,  $\frac{|v_K - v_L|}{d_{K,\sigma} + d_{L,\sigma}} \leq \frac{|v_K - v_\sigma|}{d_{K,\sigma}} + \frac{|v_L - v_\sigma|}{d_{L,\sigma}}$ , [20, Lemma 5.5] and (5.16) show that there exists  $C_{17}$  only depending on  $\Omega$  and  $\theta$  such that

$$\|\Pi_{\mathcal{D}} v(\cdot + \boldsymbol{\xi}) - \Pi_{\mathcal{D}} v\|_{L^1(\mathbb{R}^d)} \leq \sqrt{d}|\boldsymbol{\xi}| \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} |\sigma| d_{K,\sigma} \left| \frac{v_\sigma - v_K}{d_{K,\sigma}} \right| \leq C_{17}|\boldsymbol{\xi}| \|\nabla_{\mathcal{D}} v\|_{L^p(\Omega)^d}. \quad (5.29)$$

Moreover, by [20, Lemma 5.4] and (5.16) we can find  $r > p$  and  $C_{18}$  only depending on  $\Omega, p$  and  $\theta$  such that

$$\|\Pi_{\mathcal{D}} v\|_{L^r(\mathbb{R}^d)} \leq C_{18} \|\nabla_{\mathcal{D}} v\|_{L^p(\Omega)^d}. \quad (5.30)$$

We now write, thanks to Hölder's inequality,  $\|\cdot\|_{L^p(\mathbb{R}^d)} \leq \|\cdot\|_{L^1(\mathbb{R}^d)}^\nu \|\cdot\|_{L^r(\mathbb{R}^d)}^{1-\nu}$  where  $\nu > 0$  is such that  $\frac{\nu}{1} + \frac{1-\nu}{r} = \frac{1}{p}$ , and (5.29) and (5.30) conclude the proof.  $\square$

## Appendix

The proof of the following theorem is inspired from [24].

**Theorem 5.13 (Discrete Aubin-Simon lemma)** *Let  $T > 0$  and let  $B$  be a Banach space. Let  $(B_m)_{m \in \mathbb{N}}$  be a sequence of finite dimensional subspaces of  $B$ . For any  $m \in \mathbb{N}$ , let  $N_m \in \mathbb{N}^*$ ,  $t_m^{(0)} = 0 < t_m^{(1)} < \dots < t_m^{(N_m)} = T$  and  $\delta_m^{(n)} = t_m^{(n)} - t_m^{(n-1)}$ ,  $n = 1, \dots, N_m$ . Let  $\{u_m^{(n)}, n = 0, \dots, N_m\} \subset B_m$  and let  $u_m \in L^1(0, T; B_m)$  be defined, for a given real family  $(\alpha_m^{(n)})_{n=1, \dots, N_m}$ , by*

$$u_m(t) = \tilde{u}_m^{(n)} := (1 - \alpha_m^{(n)})u_m^{(n-1)} + \alpha_m^{(n)}u_m^{(n)} \in B_m, \quad (5.31)$$

for a.e.  $t \in (t_m^{(n-1)}, t_m^{(n)})$ , and  $n \in \{1, \dots, N_m\}$ .

Let  $\delta_m u_m$  be the “discrete time derivative”, defined by:

$$\delta_m u_m(t) = \delta_m^{(n)} u_m := \frac{1}{\delta_m^{(n)}}(u_m^{(n)} - u_m^{(n-1)}) \text{ for a.e. } t \in (t_m^{(n-1)}, t_m^{(n)}), n \in \{1, \dots, N_m\}.$$

Let  $\|\cdot\|_{X_m}$  and  $\|\cdot\|_{Y_m}$  be two norms on  $B_m$ . We denote by  $X_m$  the space  $B_m$  endowed with the norm  $\|\cdot\|_{X_m}$  and by  $Y_m$  the space  $B_m$  endowed with the norm  $\|\cdot\|_{Y_m}$ . We assume that

(h1) *For any sequence  $(w_m)_{m \in \mathbb{N}}$  such that  $w_m \in B_m$  and  $(\|w_m\|_{X_m})_{m \in \mathbb{N}}$  is bounded, then, up to a subsequence, there exists  $w \in B$  such that  $w_m \rightarrow w$  in  $B$  as  $m \rightarrow +\infty$ .*



(h2) For any sequence  $(w_m)_{m \in \mathbb{N}}$  such that  $w_m \in B_m$ ,  $(\|w_m\|_{X_m})_{m \in \mathbb{N}}$  is bounded, there exists  $w \in B$  such that  $w_m \rightarrow w$  in  $B$  and  $\|w_m\|_{Y_m} \rightarrow 0$  as  $m \rightarrow +\infty$ , then  $w = 0$ .

(h3) The family  $(\alpha_m^{(n)})_{n=1, \dots, N_m, m \in \mathbb{N}}$  and the sequence  $(\|u_m\|_{L^1(0, T; X_m)})_{m \in \mathbb{N}}$  are bounded.

(h4) The sequence  $(\|\delta_m u_m\|_{L^1(0, T; Y_m)})_{m \in \mathbb{N}}$  is bounded.

Then there exists  $u \in L^1(0, T; B)$  such that, up to a subsequence,  $u_m \rightarrow u$  in  $L^1(0, T; B)$  as  $m \rightarrow +\infty$ .

**Proof** The first step is to apply [24, Lemma 3.2], which states that, under Hypothesis (h1), there exists  $C_X > 0$  such that, for all  $m \in \mathbb{N}$ , for all  $u \in B_m$ ,  $\|u\|_B \leq C_X \|u\|_{X_m}$ . Following the proof of [24, Lemma 3.1], thanks to (h1) and (h2) we remark that the following variant of Lions' lemma holds: for all  $\varepsilon > 0$  there exists  $C(\varepsilon) > 0$  such that

$$\forall m \in \mathbb{N}, \forall u \in B_m, \|u\|_B \leq \varepsilon \|u\|_{X_m} + C(\varepsilon) \|u\|_{Y_m}, \quad (5.32)$$

Let us now notice, using (h4), that there exists  $C_t > 0$  such that,  $\forall m \in \mathbb{N}$ ,  $\sum_{n=1}^{N_m} \delta_m^{(n)} \|\delta_m^{(n)} u_m\|_{Y_m} \leq C_t$ . Therefore we get from (h3) that the sequence  $(\|u_m\|_{BV(0, T; Y_m)})_{m \in \mathbb{N}}$  is bounded since we can write

$$\begin{aligned} \sum_{n=1}^{N_m-1} \|\tilde{u}_m^{(n+1)} - \tilde{u}_m^{(n)}\|_{Y_m} &\leq \sum_{n=1}^{N_m-1} \left( |\alpha_m^{(n+1)}| \delta_m^{(n+1)} \|\delta_m^{(n+1)} u_m\|_{Y_m} + |1 - \alpha_m^{(n)}| \delta_m^{(n)} \|\delta_m^{(n)} u_m\|_{Y_m} \right) \\ &\leq (1 + 2C_\alpha) C_t, \end{aligned}$$

denoting by  $C_\alpha$  a bound of the family  $(\alpha_m^{(n)})_{n=1, \dots, N_m, m \in \mathbb{N}}$ . We then extend  $u_m$  by symmetry on  $(-T, 2T)$ , setting  $u_m(-t) = u_m(t)$  and  $u_m(T+t) = u_m(T-t)$  for a.e.  $t \in (0, T)$ . We then get that  $u_m \in BV(-T, 2T; Y_m)$  with  $\|u_m\|_{BV(-T, 2T; Y_m)} \leq 3(1 + 2C_\alpha) C_t$ . We also get, using (h3), that the sequence  $(\|u_m\|_{L^1(-T, 2T; X_m)})_{m \in \mathbb{N}}$  is bounded, say by  $C_0$ . Using (5.32), for any  $\tau \in (0, T)$  and  $t \in (-T, 2T - \tau)$  we can write

$$\|u_m(t + \tau) - u_m(t)\|_B \leq \varepsilon \|u_m(t + \tau) - u_m(t)\|_{X_m} + C(\varepsilon) \|u_m(t + \tau) - u_m(t)\|_{Y_m},$$

which provides

$$\int_{-T}^{2T-\tau} \|u_m(t + \tau) - u_m(t)\|_B dt \leq \varepsilon 2C_0 + \tau C(\varepsilon) 3(1 + 2C_\alpha) C_t.$$

This proves that  $\int_{-T}^{2T-\tau} \|u_m(t + \tau) - u_m(t)\|_B dt$  tends to 0 with  $\tau$ , uniformly with respect to  $m \in \mathbb{N}$ . Multiplying  $u_m$  by a function  $\psi \in C_c^\infty(-T, 2T)$  equal to 1 on  $(0, T)$ , we may then apply the compactness theorem [24, Theorem 2.1] to obtain the relative compactness of the family  $(u_m)_{m \in \mathbb{N}}$  in  $L^1(0, T; B)$ .  $\square$

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