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APPROXIMATION OF A REIFENBERG-FLAT SET  
BY A SMOOTH SURFACE

Guy DAVID

**Résumé.** On montre que si  $E \subset \mathbb{R}^n$  est ensemble Reifenberg-plat de dimension  $d$  à l'échelle  $r_0$ , il existe une surface lisse  $\Sigma_0$  de dimension  $d$ , qui est proche de  $E$  à l'échelle  $r_0$ . Ceci permet d'appliquer un résultat de G. David et T. Toro [Memoirs of the AMS 215 (2012), 1012], et de montrer que  $E$  est l'image de  $\Sigma_0$  par un homéomorphisme bi-Höldérien de  $\mathbb{R}^n$ . Quand  $d = n - 1$  et  $E$  est compact et connexe,  $\Sigma_0$  est automatiquement orientable, donc  $\mathbb{R}^n \setminus E$  a exactement deux composantes connexes.

**Abstract.** We show that if  $E \subset \mathbb{R}^n$  is a Reifenberg flat set  $E$  of dimension  $d$  at scale  $r_0$ , we can find a smooth surface  $\Sigma_0$  of dimension  $d$  which is close to  $E$  at the scale  $r_0$ . Then we can apply a result of G. David and T. Toro [Memoirs of the AMS 215 (2012), 1012], and get a bi-Hölder homeomorphism of  $\mathbb{R}^n$  that sends  $\Sigma_0$  to  $E$ . When  $d = n - 1$  and  $E$  is compact and connected,  $\Sigma_0$  is automatically orientable, and  $\mathbb{R}^n \setminus E$  has exactly two connected components.

**AMS classification.** 28A75, 49Q20.

**Key words.** Reifenberg flat sets, Reifenberg topological disk theorem, parameterizations.

## 1. Introduction.

The purpose of this paper is to prove that if  $E \subset \mathbb{R}^n$  is a Reifenberg flat set of dimension  $d$  at scale  $r_0$ , we can find a smooth surface  $\Sigma_0$  of the same dimension, which is close to  $E$  at the scale  $r_0$ . This then allows us to apply one of the main results of [DT], and get a bi-Hölder homeomorphism of the ambient space  $\mathbb{R}^n$  that sends  $\Sigma_0$  to  $E$ .

Thus our result here can be seen a small preparation lemma that slightly weakens the assumptions in [DT].

In the special case of a compact connected Reifenberg-flat set  $E$  of codimension 1, we will get (or re-prove) as a corollary that  $\mathbb{R}^n \setminus E$  has exactly two connected components, and one does not need to mention this separation property as an additional assumption. This will follow easily because  $\Sigma_0$  is automatically orientable, but surely there are more direct arguments that do not use the full Reifenberg parameterization.

Fix integers  $0 < d < n$ , let  $E$  be a (nonempty) closed set in  $\mathbb{R}^n$ , and define the (bilateral P. Jones) numbers  $\gamma(x, r)$ ,  $x \in E$  and  $0 < r < +\infty$ , by

$$(1.1) \quad \gamma(x, r) = \inf \{ d_{x,r}(E, P) ; P \in \mathcal{P}(x) \},$$

where  $\mathcal{P}(x)$  denotes the set of  $d$ -dimensional affine planes that contain  $x$ , and

$$(1.2) \quad d_{x,r}(E, P) = \frac{1}{r} \sup \{ \text{dist}(y, P) ; y \in E \cap B(x, r) \} \\ + \frac{1}{r} \sup \{ \text{dist}(y, E) ; y \in P \cap B(x, r) \}$$

is a normalized local Hausdorff distance from  $E$  to  $P$ .

We shall assume that there is a radius  $r_0 > 0$  such that

$$(1.3) \quad \gamma(x, r_0) \leq \varepsilon \quad \text{for } x \in E,$$

and prove that if  $\varepsilon$  is small enough, depending only on  $n$  and  $d$ , there is a smooth  $d$ -dimensional surface  $\Sigma_0 \subset \mathbb{R}^n$  with no boundary, such that

$$(1.4) \quad \text{dist}(x, \Sigma_0) \leq C_0 \varepsilon r_0 \quad \text{for } x \in E$$

and

$$(1.5) \quad \text{dist}(x, E) \leq C_0 \varepsilon r_0 \quad \text{for } x \in \Sigma_0,$$

where  $C_0$  depends only on  $n$  and  $d$ . More precisely, we will show that there is a  $\Lambda > 0$ , that depends only on  $n$  and  $d$ , such that for each  $y \in \Sigma_0$ ,

$$(1.6) \quad \Sigma_0 \text{ coincides with a smooth } \Lambda\varepsilon\text{-Lipschitz graph in } B(y, 10^{-2}r_0).$$

That is, we can find a  $d$ -plane  $P_y$  and a  $\Lambda\varepsilon$ -Lipschitz mapping  $F_y : P_y \rightarrow P_y^\perp$  such that, if

$$(1.7) \quad \mathcal{G}(F_y) = \{w + F_y(w); w \in P_y\}$$

denotes the graph of  $F_y$ , then

$$(1.8) \quad \Sigma_0 \cap B(y, 10^{-2}r_0) = \mathcal{G}(F_y) \cap B(y, 10^{-2}r_0).$$

In addition, the  $F_y$  are smooth, and there exist constants  $\Lambda_k, k \geq 1$ , such that

$$(1.9) \quad \|D^k F_y\|_\infty \leq \Lambda_k \varepsilon r_0^{1-k}.$$

(For  $k = 1$ , we already knew this with  $\Lambda_1 = \Lambda$ .) Let us summarize all this officially.

**Theorem 1.10.** *There exist constants  $\varepsilon_0 > 0$  and  $\Lambda_k, k \geq 1$ , that depend only on  $n$  and  $d$ , such that if  $E \subset \mathbb{R}^n$  is a nonempty closed set such that (1.3) holds for some  $r_0 > 0$  and some  $\varepsilon \in (0, \varepsilon_0)$ , then we can find a smooth  $d$ -dimensional surface  $\Sigma_0$  with the properties (1.4)-(1.9).*

Notice that we do not require  $E$  to be flat at all scales smaller than  $r_0$ , as in the next result.

Theorem 1.10 is designed so that we can apply Theorem 12.1 in [DT], and get the following statement, where we decided to work with  $r_0 = 1$  for convenience.

**Corollary 1.11.** *There exist constants  $\varepsilon_1 \leq \varepsilon_0$  and  $C_2 > 1$ , that depend only on  $n$  and  $d$ , such that if  $E \subset \mathbb{R}^n$  is a nonempty closed set such that*

$$(1.12) \quad \gamma(x, r) \leq \varepsilon \quad \text{for } x \in E \text{ and } 0 < r \leq 1,$$

and if  $\Sigma_0$  is the smooth surface provided by Theorem 1.10, then there is a bijective mapping  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$(1.13) \quad g(x) = x \quad \text{when } \text{dist}(x, E) \geq 10^{-1},$$

$$(1.14) \quad |g(x) - x| \leq C_2 \varepsilon \quad \text{for } x \in \mathbb{R}^n,$$

$$(1.15) \quad C_2^{-1} |x' - x|^{1+C_2 \varepsilon} \leq |g(x) - g(x')| \leq C_2 |x' - x|^{1-C_2 \varepsilon}$$

for  $x, x' \in \mathbb{R}^n$  such that  $|x' - x| \leq 1$ , and

$$(1.16) \quad g(\Sigma_0) = E.$$

Let us check this. We want to apply Theorem 12.1 in [DT] to the set  $E' = AE$ , where we choose  $A = 2 \cdot 10^4$ , and with the open set

$$(1.17) \quad U = \{x \in \mathbb{R}^n ; \text{dist}(x, E') > 3\}.$$

The assumption (12.1) of [DT], relative to the smooth set  $\Sigma'_0 = A\Sigma_0$ , is satisfied because of (1.6)-(1.9) (although only with the constant  $C\varepsilon$ , which does not matter). The proximity condition (12.3) follows from (1.4) and (1.5), and so Theorem 12.1 in [DT] gives a mapping  $g'$  that satisfies (12.4)-(12.8) in [DT].

Here we take  $g(x) = A^{-1}g'(Ax)$ , (1.13) holds because (12.4) in [DT] says that  $g'(x) = x$  when  $\text{dist}(x, U) \geq 13$ , (1.14) and (1.15) easily follow from (12.6) and (12.7) in [DT]. Finally, (12.8) in [DT] says that  $E' \cap U = g'(\Sigma'_0 \cap U)$ . But  $E' = E' \cap U$  by (1.17), and  $\Sigma'_0 \subset U$  by (1.5), so in fact  $E' = g'(\Sigma'_0)$  and  $E = g(\Sigma_0)$ , as needed. So Corollary 1.11 follows from Theorem 1.10 and Theorem 12.1 in [DT].  $\square$

**Corollary 1.18.** *Suppose  $d = n - 1$ , and let  $E$  be a compact connected set that satisfies the assumptions of Corollary 1.11. Then  $\mathbb{R}^n \setminus E$  has exactly two connected components.*

Probably there are more direct proofs of this, where one would count the number of apparent components of  $\mathbb{R}^n \setminus E$  at the scale  $r > 0$ , and control it for smaller and smaller values of  $r$ . There is an argument in [DS], Chapter II.4, where something like this is done. Here we can give a shorter proof, because Corollary 1.11 uses the full force of the Reifenberg construction. The main point is only that the result is easier to prove for smooth sets  $E$ , and we reduce to that case.

Let  $\Sigma_0$  and  $g$  be as in Theorem 1.10 and then Corollary 1.11. We know that  $\Sigma_0$  is a smooth hypersurface without boundary, and it is compact by (1.5). We shall soon check that it is connected; let us conclude from here. By [AH], page 440 (for nonsmooth manifolds), or rather [S] (a shorter argument for the smooth case, using transversality),  $\Sigma_0$  is orientable and  $\mathbb{R}^n \setminus \Sigma_0$  has exactly two connected components, which we denote by  $\Omega_+$  and  $\Omega_-$ . Recall that  $g$  is smooth, bijective, and equal to the identity near infinity; then we get that  $\mathbb{R}^n \setminus E = g(\mathbb{R}^n \setminus \Sigma_0)$  has two components, namely the  $g(\Omega_{\pm})$ .

We still need to check that  $\Sigma_0$  is connected. Let  $a_1, a_2 \in \Sigma_0$  be given. By (1.5), we can find  $b_1, b_2 \in E$  such that  $|b_i - a_i| \leq C\varepsilon$ . Because  $E$  is connectedness, we can find a chain of points  $w_i \in E$ ,  $0 \leq i \leq m$ , with  $w_0 = b_1$ ,  $w_m = b_2$ , and  $|w_i - w_{i-1}| \leq C\varepsilon$  for  $1 \leq i \leq m$ . (Otherwise, the set of points of  $E$  that can be connected to  $b_1$  by such a chain, which is open and closed in  $E$ , would contain  $b_1$  but not  $b_2$ .) By (1.4), we can find  $y_i \in \Sigma_0$  such that  $|y_i - w_i| \leq C\varepsilon$ . By definition of  $b_1$  and  $b_2$ , we can take  $y_0 = a_1$  and  $y_m = a_2$ . Finally, by (1.7)-(1.8), applied to the points  $y_i$ , we get that for  $1 \leq i \leq m$ , there is a curve in  $\Sigma_0$  that connects  $y_{i-1}$  to  $y_i$ . Thus  $\Sigma_0$  is connected, and Corollary 1.18 follows.  $\square$

**Remark 1.19.** For the proof we only need to assume that  $E$  is  $\frac{1}{20}$ -connected. That is, that any two points  $b_1, b_2$  in  $E$  can be connected by a chain  $\{w_i\}$  in  $E$  as above, with  $|w_i - w_{i-1}| \leq \frac{1}{20}$  for  $1 \leq i \leq m$ .

If we do not suppose that, we only get that  $\mathbb{R}^n \setminus E$  has at least two component (pick any point  $x_0 \in E$ , and apply the result to the set of points  $x \in E$  that can be  $\frac{1}{20}$ -connected to  $x_0$ . This set alone separates  $\mathbb{R}^n$  into two components.

We do not really need to assume that  $E$  is bounded either. Then we do not get that  $\Sigma_0$  is compact, but its image under an inversion of  $\mathbb{R}^n$  (centered away from  $E$  and  $\Sigma$  is compact, and smooth except at one point (the image of  $\infty$ ). Even then, it separates  $\mathbb{R}^n$  into two components (the argument of [S] goes through, just manage to avoid the bad point), and we can conclude as before.

We mentioned Corollary 1.18 here because in some papers, one feels tempted to add the fact that  $E$  separates  $\mathbb{R}^n$  into two connected components to the assumptions that  $E$  is connected and Reifenberg-flat. This extra assumption is thus unneeded.

**Remark 1.20.** In [DT] the authors also consider situations where instead of requiring the bilateral control (1.12), one merely assumes that the points of  $E$  lie close to  $d$ -planes  $P(x, r)$ , with some control on how fast they depend on  $(x, r)$ . Here we can try to construct  $\Sigma_0$  with similar data; our construction will only give a smooth surface  $\Sigma_0$ , but with a boundary, and we shall not try to see whether it is contained in a smooth surface without boundary. See Remark 2.31.

The rest of this paper will be devoted to a proof of Theorem 1.10, which will be obtained by a simple construction where we start from a net of points and use a covering to fill the holes in a finite number of steps.

The author wishes to thank T. D. Luu for discussions about Theorem 1.10 (we thought at some time that a result like this would be needed for his PhD thesis, but it turned out not to be the case), and A. Lemenant and A. Chambolle, in particular for asking about Corollary 1.18 and discussions about orientability.

## 2. A proof of Theorem 1.10.

In this section we prove Theorem 1.10. Since the statement is invariant under dilations, it will be enough to prove the theorem when  $r_0 = 1$ .

Let  $E$  be as in the theorem, with  $r_0 = 1$ , set  $a = \frac{1}{32}$ , and choose a maximal collection  $X$  of points of  $E$  such that  $|x - x'| \geq a$  for  $x, x' \in X$  such that  $x' \neq x$ . Thus

$$(2.1) \quad \text{dist}(x, X) \leq a \quad \text{for } x \in E,$$

because otherwise we could add  $x$  to  $X$ . Next decompose  $X$  as the disjoint union

$$(2.2) \quad X = \bigcup_{1 \leq j \leq N} X_j,$$

where for each  $j$ ,

$$(2.3) \quad |x - x'| \geq 16a = \frac{1}{2} \text{ when } x \text{ and } x' \text{ lie in } X_j \text{ and } x \neq x'.$$

Let us check that we can do this with an  $N$  that depends only on  $n$ . Define  $X_j$  by induction, to be a maximal set contained in  $X \setminus \cup_{i < j} X_i$  and with the property (2.3); it is easy to see that we can stop as soon as  $j$  is larger than the maximal number of points in a ball of radius  $16a$  that lie at mutual distances at least  $a$ . This number is in turn estimated by saying that the balls of radius  $a/2$  centered at these points are all disjoint and contained in a ball of radius  $17a$ , which yields  $N \leq 34^n$  by comparing Lebesgue measures.

For each  $x \in X$ , use (1.3) to find an affine  $d$ -plane  $P_x$  through  $x$  such that

$$(2.4) \quad d_{x,1}(E, P_x) \leq \varepsilon$$

(see the definitions (1.1) and (1.2)). Then denote by  $\pi_x : \mathbb{R}^n \rightarrow P_x$  the orthogonal projection on  $P_x$ , and by  $\pi_x^\perp$  the orthogonal projection on the vector space  $P_x^\perp$  of dimension  $n - d$  which is orthogonal to  $P_x$ . Let us check that

$$(2.5) \quad d_{x,1/4}(P_x, P_y) \leq 8\varepsilon \text{ for } x, y \in X \text{ such that } |x - y| \leq \frac{1}{2}.$$

Indeed, if  $z \in P_x \cap B(x, 1/4)$ , we can use (2.4) to find  $w \in E$  such that  $|w - z| \leq \varepsilon$ ; then  $w \in E \cap B(y, 1)$  and (2.4) for  $y$  says that we can find  $z' \in P_y$  such that  $|z' - w| \leq \varepsilon$  and hence  $|z' - z| \leq 2\varepsilon$ . By a similar argument, for each  $z' \in P_y \cap B(x, 1/4)$  we can find  $z \in P_x$  such that  $|z - z'| \leq 2\varepsilon$ , and (2.5) follows.

Thus in (2.5),  $P_x$  and  $P_y$  make a small angle; this will be useful because we want small Lipschitz graphs over  $P_x$  to be small Lipschitz graphs over  $P_y$  as well.

We shall now construct a nondecreasing sequence of sets  $S_j$ ,  $0 \leq j \leq N$ . We start with  $S_0 = X$ , and our final set  $S_N$  will be a good choice of  $\Sigma_0$ , except for the fact that we shall not immediately take care of the many derivatives in (1.9). Notice that  $X$  gives the general position of  $\Sigma_0$ , so our problem will essentially consist in completing  $S_0$  into a smooth surface that contains it; we shall only use the set  $E$  marginally, to prove that  $\Sigma_0$  has no boundary.

We shall construct the  $S_j$  by induction, with the property that for each  $x \in X$ , there is an  $A_j\varepsilon$ -Lipschitz mapping  $F_{j,x} : P_x \rightarrow P_x^\perp$  such that

$$(2.6) \quad S_j \cap B(x, 4a) \subset \mathcal{G}(F_{j,x}),$$

where  $\mathcal{G}(F_{j,x})$  denotes the graph of  $F_{j,x}$  over  $P_x$ , defined as in (1.7). The constants  $A_j$  will be chosen larger and larger, but since we can take  $\varepsilon$  as small as we want, the  $A_j\varepsilon$  will stay small.

In order to prove (2.6), we shall first check that for each  $x \in X$ ,

$$(2.7) \quad |\pi_x^\perp(y) - \pi_x^\perp(z)| \leq A'_j \varepsilon |\pi_x(y) - \pi_x(z)| \quad \text{for } y, z \in S_j \cap B(x, 4a)$$

for some constant  $A'_j$  (that will depend on  $A_{j-1}$  if  $j \geq 1$ ). As soon as we have (2.7), we observe that  $\pi_x$  is injective on  $S_j \cap B(x, 4a)$ , so can define a function  $F_{j,x}$  from  $H = \pi_x(S_j \cap B(x, 4a))$  to  $P_x^\perp$ , by the relation  $F_{j,x}(\pi_x(y)) = \pi_x^\perp(y)$  for  $y \in S_j \cap B(x, 4a)$ . In addition, (2.7) says that  $F_{j,x}$  is  $A'_j \varepsilon$ -Lipschitz on  $H$ , and  $S_j \cap B(x, 4a)$  is its graph.

We then use the Whitney extension theorem to extend  $F_{j,x}$  into an  $A_j \varepsilon$ -Lipschitz function defined on  $P_x$ , and we get (2.6). Since we allow ourselves to take  $A_j$  larger than  $A'_j$ , we don't need to use Kirzbraun's theorem and the construction of the extension is simpler.

Let us now check that (2.7) holds for  $j = 0$ . Recall that we took  $S_0 = X$ . Let  $y, z \in S_0 \cap B(x, 4a) = X \cap B(x, 4a)$  be given. Then

$$(2.8) \quad |\pi_x^\perp(y) - \pi_x^\perp(x)| = \text{dist}(y, P_x) \leq \varepsilon$$

because  $P_x$  goes through  $x$ , because  $y \in X \subset E$ , and by (2.4) (also see the definition (1.2)). Similarly,  $|\pi_x^\perp(z) - \pi_x^\perp(x)| \leq \varepsilon$ . If  $y = z$ , the inequality in (2.7) is obvious. Otherwise,  $|y - z| \geq a$  by definition of  $X$ , and

$$(2.9) \quad |\pi_x^\perp(y) - \pi_x^\perp(z)| \leq 2\varepsilon \leq 2a^{-1}\varepsilon|y - z|;$$

then  $|\pi_x(y) - \pi_x(z)| \geq |y - z| - |\pi_x^\perp(y) - \pi_x^\perp(z)| \geq |y - z|/2$  and (2.7) holds with  $A'_0 = 4a^{-1} = 128$ . So there are Lipschitz functions  $F_{0,x}$  such that (2.6) holds for  $j = 0$ .

Next assume that  $0 \leq j < N$  and that we constructed  $S_j$  and the  $F_{j,x}$  with the property (2.6). We take

$$(2.10) \quad S_{j+1} = S_j \cup \bigcup_{w \in X_{j+1}} \mathcal{G}(F_{j,w}) \cap B(w, 3a).$$

Let us now prove (2.7) for  $j + 1$ . Let  $x \in X$  and  $y, z \in S_{j+1} \cap B(x, 4a)$  be given; we need to prove that

$$(2.11) \quad |\pi_x^\perp(y) - \pi_x^\perp(z)| \leq A'_{j+1} \varepsilon |\pi_x(y) - \pi_x(z)|.$$

If  $y, z$  both lie on  $S_j$ , (2.11) simply follows from (2.7) for  $j$ . So we may assume that one of the two points (say  $y$  for definiteness) lies in  $\mathcal{G}(F_{j,w}) \cap B(w, 3a)$  for some  $w \in X_{j+1}$ .

**Case 1.** We first assume that  $z \in \mathcal{G}(F_{j,w})$ . Then

$$(2.12) \quad |\pi_w^\perp(y) - \pi_w^\perp(z)| \leq A_j \varepsilon |\pi_w(y) - \pi_w(z)|,$$

just because  $F_{j,w}$  is  $A_j \varepsilon$ -Lipschitz. This is not exactly (2.11), because we project on different planes, but the difference will be small. Indeed,  $|w - x| \leq |w - y| + |y - x| \leq 7a < 1/2$ , so (2.5) says that

$$(2.13) \quad d_{x,1/4}(P_x, P_w) \leq 10\varepsilon.$$

The reader will not be surprised to learn that (2.11) follows from (2.12) and (2.13), but let us check this anyway. Denote by  $\tilde{P}_x$  and  $\tilde{P}_w$  the vector planes parallel to  $P_x$  and  $P_w$ , and by  $\tilde{\pi}_x$  (respectively  $\tilde{\pi}_w$ ) the orthogonal projection on  $\tilde{P}_x$  (respectively  $\tilde{P}_w$ ); we want to check that

$$(2.14) \quad \|\tilde{\pi}_x - \tilde{\pi}_w\| \leq 100\varepsilon.$$

Let  $v \in \mathbb{R}^n$  be given; we want to estimate  $|\tilde{\pi}_x(v) - \tilde{\pi}_w(v)|$ , and we may as well assume that  $v$  is a unit vector. We start with the case when  $v \in \tilde{P}_x$ , and we first check that

$$(2.15) \quad \text{dist}(v, \tilde{P}_w) \leq 25\varepsilon.$$

Observe that both  $x$  and  $x + v/5$  lie in  $P_x \cap B(x, 1/4)$ , so by (2.13) we can find  $x'$  and  $x''$  in  $P_w$  such that  $|x' - x| \leq 10\varepsilon/4$  and  $|x'' - x - v/5| \leq 10\varepsilon/4$ ; then  $v' = 5(x'' - x')$  lies in  $\tilde{P}_w$ , and  $|v' - v| = 5|(x'' - x - v/5) + (x - x')| \leq 25\varepsilon$ , as needed for (2.15).

Because of (2.15), we get that  $|\tilde{\pi}_x(v) - \tilde{\pi}_w(v)| = |v - \tilde{\pi}_w(v)| = \text{dist}(v, \tilde{P}_w) \leq 25\varepsilon$  for every unit vector  $v \in \tilde{P}_x$ .

Now suppose  $v$  is a unit vector in  $\tilde{P}_x^\perp$ . First set  $\xi = \tilde{P}_w(v)$ , and check that

$$(2.16) \quad \text{dist}(\xi, \tilde{P}_x) \leq 25\varepsilon.$$

Pick  $x' \in P_w$  such that  $|x' - x| \leq 10\varepsilon/4$  (use (2.13) as before), and, since  $x' + \xi/5 \in P_w \cap B(x, 1/4)$ , use (2.13) to find  $y \in P_x$  such that  $|y - x' - \xi/5| \leq 10\varepsilon/4$ . Recall that  $x \in P_x$ ; then  $5(y - x) \in \tilde{P}_x$ , and  $|5(y - x) - \xi| \leq 5[|y - x' - \xi/5| + |x' - x|] \leq 25\varepsilon$ , which proves (2.16). Thus

$$(2.17) \quad |\xi - \tilde{\pi}_x(\xi)| = \text{dist}(\xi, \tilde{P}_x) \leq 25\varepsilon.$$

Also, for each unit vector  $e \in \tilde{P}_x$ , use (2.15) to find  $e' \in \tilde{P}_w$ , with  $|e' - e| \leq 25\varepsilon$ , and observe that

$$(2.18) \quad \begin{aligned} |\langle e, \xi \rangle| &\leq |\langle e', \xi \rangle| + |e' - e||\xi| \leq |\langle e', \xi \rangle| + 25\varepsilon = |\langle e', v \rangle| + 25\varepsilon \\ &\leq |\langle e, v \rangle| + |e' - e| + 25\varepsilon \leq |\langle e, v \rangle| + 50\varepsilon = 50\varepsilon \end{aligned}$$

because  $\xi - v \perp e'$  by definition of  $\xi = \tilde{\pi}_w(v)$ , and  $v \in \tilde{P}_x^\perp$ . Hence  $|\tilde{\pi}_x(\xi)| \leq 50\varepsilon$  and, adding up with (2.17),  $|\xi| \leq 75\varepsilon$ . Then  $|\tilde{\pi}_x(v) - \tilde{\pi}_w(v)| = |\tilde{\pi}_w(v)| = |\xi| \leq 75\varepsilon$  for every unit vector  $v \in \tilde{P}_x^\perp$ . Since we also have  $|\tilde{\pi}_x(v) - \tilde{\pi}_w(v)| \leq 25\varepsilon$  for every unit vector  $v \in \tilde{P}_x$ , we get that  $|\tilde{\pi}_x(v) - \tilde{\pi}_w(v)| \leq 100\varepsilon|v|$  for every  $v \in \mathbb{R}^n$ . That is, (2.14) holds.

Return to our  $y$  and  $z \in S_{j+1} \cap B(x, 4a)$ , recall that  $\pi_x^\perp$  denotes the orthogonal projection on the vector space  $\tilde{P}_x^\perp$ , define  $\pi_w^\perp$  similarly, and observe that

$$(2.19) \quad \begin{aligned} |(\pi_x^\perp(y) - \pi_x^\perp(z)) - (\pi_w^\perp(y) - \pi_w^\perp(z))| &= |\pi_x^\perp(y - z) - \pi_w^\perp(y - z)| \\ &= |\tilde{\pi}_x(y - z) - \tilde{\pi}_w(y - z)| \\ &\leq \|\tilde{\pi}_x - \tilde{\pi}_w\| |y - z| \leq 100\varepsilon |y - z| \end{aligned}$$



because  $\pi_x^\perp + \tilde{\pi}_x = I = \pi_w^\perp + \tilde{\pi}_w$ . Now (2.11) follows from (2.12) and (2.13), as soon as we take  $A'_{j+1} \geq A_j + 100$ . This completes our proof of (2.11) when  $z \in \mathcal{G}(F_{j,w})$ .

**Case 2.** We still need to check (2.11) when  $z \notin \mathcal{G}(F_{j,w})$ . Notice that  $z \in \mathcal{G}(F_{j,w'}) \cap B(w', 3a)$  for some other  $w' \in X_{j+1}$  is impossible, because we would get that

$$(2.20) \quad |w - w'| \leq |w - y| + |y - x| + |x - z| + |z - w'| \leq 3a + 4a + 4a + 3a < 16a,$$

which is forbidden by (2.3). By (2.10) and because  $z \notin \mathcal{G}(F_{j,w})$ ,  $z \in S_j$ . In addition, by (2.6) and because  $z \notin \mathcal{G}(F_{j,w})$ ,  $z$  lies out of  $B(w, 4a)$ , and so  $|z - y| \geq |z - w| - |w - y| \geq 4a - |w - y| \geq a$  (recall from the line below (2.11) that  $y \in \mathcal{G}(F_{j,w}) \cap B(w, 3a)$ ), and (2.11) will follow as soon as we prove that

$$(2.21) \quad |\pi_x^\perp(y) - \pi_x^\perp(z)| \leq C\varepsilon$$

for some  $C$  that depends on  $n$  and  $A'_j$ , because then  $|\pi_x(y) - \pi_x(z)| \geq a/2$ . First observe that

$$(2.22) \quad |\pi_x^\perp(x) - \pi_x^\perp(z)| \leq 4aA'_j\varepsilon$$

by (2.7) and because we just checked that  $z \in S_j \cap B(x, 4a)$ . We are left with

$$(2.23) \quad \begin{aligned} |\pi_x^\perp(y) - \pi_x^\perp(x)| &\leq |\pi_x^\perp(y) - \pi_x^\perp(w)| + |\pi_x^\perp(w) - \pi_x^\perp(x)| \\ &= |\pi_x^\perp(y) - \pi_x^\perp(w)| + \text{dist}(w, P_x) \\ &\leq |\pi_x^\perp(y) - \pi_x^\perp(w)| + \varepsilon. \end{aligned}$$

by (2.4) for  $x$ , and because  $|w - x| \leq |w - y| + |y - x| \leq 7a < 1/2$ . Let us again use (2.14) (its proof is valid also in case 2; in fact neither the statement nor the proof involves  $z$ ); this yields

$$(2.24) \quad \begin{aligned} |\pi_x^\perp(y) - \pi_x^\perp(w)| &= |\pi_x^\perp(y - w)| \leq |\pi_w^\perp(y - w)| + |y - w| \|\pi_x^\perp - \pi_w^\perp\| \\ &= |\pi_w^\perp(y - w)| + |y - w| \|\pi_x - \pi_w\| \\ &\leq |\pi_w^\perp(y - w)| + 100\varepsilon|y - w| \end{aligned}$$

because  $\pi_x^\perp$  is a linear projection and  $\pi_x^\perp + \tilde{\pi}_x = I = \pi_w^\perp + \tilde{\pi}_w$ . Now recall that  $y \in \mathcal{G}(F_{j,w}) \cap B(w, 3a)$  (see below (2.11)), and so does  $w$  (because  $w \in X = S_0 \subset S_j$ , and by (2.6)); then

$$(2.25) \quad \begin{aligned} |\pi_w^\perp(y - w)| &= |\pi_w^\perp(y) - \pi_w^\perp(w)| = |F_{j,w}(\pi_w(y)) - F_{j,w}(\pi_w(w))| \\ &\leq A_j\varepsilon|\pi_w(y) - \pi_w(w)| \leq A_j\varepsilon|y - w| \leq A_j\varepsilon \end{aligned}$$

because  $F_{j,w}$  is  $A_j\varepsilon$ -Lipschitz. Altogether,  $|\pi_x^\perp(y) - \pi_x^\perp(z)| \leq 4aA'_j\varepsilon + \varepsilon + 300a\varepsilon + A_j\varepsilon$ , by (2.22)-(2.25), which proves (2.21) and then (2.11) in our second case.

This completes our proof of (2.7) for  $j+1$ . Thus we have the local Lipschitz description of  $S_{j+1}$ , with (2.6), and this completes our construction of sets  $S_j$  by induction.

Let us now check that for  $1 \leq j \leq N$  and  $x \in X_j$ ,

$$(2.26) \quad S_N \cap B(x, 2a) = \mathcal{G}(F_{j-1,x}) \cap B(x, 2a).$$

Set  $\mathcal{G} = \mathcal{G}(F_{j-1,x})$ . By (2.10),  $\mathcal{G} \cap B(x, 3a) \subset S_j \subset S_N$  so we get a first inclusion. Notice also that since the Lipschitz constant is small and  $\mathcal{G}$  goes through  $x$ ,

$$(2.27) \quad \pi_x(\mathcal{G} \cap B(x, 3a)) \text{ contains } P_x \cap B(x, 2a).$$

Now let  $z \in S_N \cap B(x, 2a)$  be given. Then  $\pi_x(z) \in P_x \cap B(x, 2a)$ ; by (2.27) we can find  $w \in \mathcal{G} \cap B(x, 3a)$  such that  $\pi_x(w) = \pi_x(z)$ . Then  $w \in S_N \cap B(x, 3a)$ . But (2.6) says that  $S_N \cap B(x, 4a)$  is contained in a Lipschitz graph, so  $\pi_x$  is injective on  $S_N \cap B(x, 4a)$ , hence  $w = z$  and  $z \in \mathcal{G}$ . So  $S_N \cap B(x, 2a) \subset \mathcal{G}$ , as needed for (2.26).

Now we check the properties (1.4) and (1.5) for  $S_N$ . First let  $y \in E$  be given. By (2.1), we can find  $x \in X$  such that  $|x - y| \leq a$ . Let  $j$  be such that  $x \in X_j$ , and set  $w = \pi_x(y)$  and  $z = w + F_{j-1,x}(w) \in \mathcal{G}(F_{j-1,x})$ . We know from (2.26) that  $z \in S_N$ , and so

$$(2.28) \quad \begin{aligned} \text{dist}(y, S_N) &\leq |y - z| = |\pi_x^\perp(y) - \pi_x^\perp(z)| = |\pi_x^\perp(y) - F_{j-1,x}(w)| \\ &\leq |\pi_x^\perp(y) - \pi_x^\perp(x)| + |\pi_x^\perp(x) - F_{j-1,x}(w)| \\ &= \text{dist}(y, P_x) + |\pi_x^\perp(x) - F_{j-1,x}(w)| \end{aligned}$$

because  $\pi_x(y) = \pi_x(z) = w$  and then  $x \in P_x$ . But  $x \in X \subset S_N$ , so  $x \in \mathcal{G}(F_{j-1,x})$  by (2.26) and hence  $F_{j-1,x}(x) = \pi_x^\perp(x)$ . Thus

$$(2.29) \quad |\pi_x^\perp(x) - F_{j-1,x}(w)| = |F_{j-1,x}(x) - F_{j-1,x}(w)| \leq A_{j-1}\varepsilon|x - w| \leq C\varepsilon$$

because  $F_{j-1,x}$  is  $A_{j-1}\varepsilon$ -Lipschitz. Since  $\text{dist}(y, P_x) \leq \varepsilon$  by (2.4), we deduce from (2.28) that  $\text{dist}(y, S_N) \leq C\varepsilon$ . So (1.4) holds.

For (1.5), let  $y \in S_N$  be given, and let  $j(y)$  be the first index such that  $y \in S_j$ ; if  $j(y) = 0$ ,  $y$  lies in  $X \subset E$ , and  $\text{dist}(y, E) = 0$ . Otherwise, (2.10) says that  $y \in \mathcal{G}(F_{j-1,x}) \cap B(x, 3a)$  for some  $x \in X_j$ . Set  $w = \pi_x(y)$ ; then  $w \in P_x \cap B(x, 3a)$  and hence  $\text{dist}(w, E) \leq \varepsilon$  by (2.4). Now

$$(2.30) \quad \begin{aligned} \text{dist}(y, E) &\leq |y - w| + \text{dist}(w, E) \leq |y - w| + \varepsilon = |\pi_x^\perp(y - w)| = |\pi_x^\perp(y - x)| \\ &= |F_{j-1,x}(w) - F_{j-1,x}(x)| + \varepsilon \leq A_{j-1}\varepsilon|x - w| + \varepsilon \leq C\varepsilon, \end{aligned}$$

where we used the facts that  $w = \pi_x(y)$ , that  $\pi_x^\perp(w - x) = 0$  because  $x$  and  $w$  lie in  $P_x$ , that  $y = w + F_{j-1,x}(w)$  and  $x = x + F_{j-1,x}(w)$  (both points lie in  $\mathcal{G}(F_{j-1,x})$ ; also see the definition (1.7)), and that  $F_{j-1,x}$  is  $A_{j-1}\varepsilon$ -Lipschitz. This proves (1.5) for  $S_N$ .

The local Lipschitz description (1.7)-(1.8) for  $S_N$  (but with no extra smoothness yet) will easily follow from (2.26). Indeed, if  $y \in S_N$ , (1.5) and (2.1) imply that  $\text{dist}(y, X) \leq a + \varepsilon$ ; we choose  $j \in [1, N]$  and  $x \in X_j$  so that  $|x - y| \leq a + \varepsilon$  and take  $P_y = P_x$  and  $F_y = F_{j-1,x}$ ; then (1.8) follows from (2.26) because  $B(y, 10^{-2}) \subset B(x, 2a)$ . (Recall that  $a = \frac{1}{32}$ .) We also get (1.9) for  $k = 1$  (i.e.,  $F_y = F_{j-1,x}$  is  $C\varepsilon$ -Lipschitz).

Our proof is not complete yet, because we want our smooth surface  $\Sigma_0$  to satisfy (1.9) for all  $k$ . If we were only interested in a finite number of derivatives, we could modify the argument above, use Whitney extensions with functions of class  $C^k$ , and conclude as before. But since we decided to require an infinite number of derivatives, the simplest seems to start from  $S_N$  and smooth it out. Notice that if we do not move it by more than  $C\varepsilon$ , (1.4) and (1.5) stay true (with larger constants), so the reader should not expect trouble here.

Let us rapidly describe the argument. The details would be standard and boring, so we skip them. We shall use the same sets  $X_j$ ,  $1 \leq j \leq N$ , as before, and define sets  $T_0 = S_N, T_1, \dots$ , and finally  $\Sigma_0 = T_N$ , by induction. For  $j = 1, \dots, N$ , we obtain  $T_j$  from  $T_{j-1}$  by smoothing  $T_j$  in the balls  $B(x, \frac{3a}{2})$ ,  $x \in X_j$ . Since these balls are far from each other, we can do this independently in all these balls. We start from a description of  $T_{j-1}$  as a Lipschitz graph near  $B(x, \frac{3a}{2})$  (which we get from an induction assumption), use a convolution with a smooth function with a support of small diameter to smooth  $T_{j-1}$  in  $B(x, \frac{3a}{2})$ , interpolate smoothly in a small ring near  $\partial B(x, \frac{3a}{2})$ , and get the next  $T_j$ . Each time we get a Lipschitz graph with a slightly worse constant, but this does not matter. The smoothness that has been produced in the previous stages is preserved too, and at the end,  $\Sigma_0 = T_N$  is the desired smooth set. This completes our proof of Theorem 1.10.  $\square$

**Remark 2.31.** In [DS], one also obtains parameterizations of sets that contain  $E$ , assuming that for  $x \in E$  and  $0 < r \leq 1$ , we can find a  $d$ -dimensional affine plane  $P(x, r)$  through  $x$  such that

$$(2.32) \quad \text{dist}(y, P(x, r)) \leq \varepsilon r \quad \text{for } y \in E \cap B(x, r),$$

and also that the  $P(x, r)$  vary slowly. Here we can do something similar, suppose that for  $x \in E$ , we can find a  $d$ -plane  $P_x$  through  $x$ , such that

$$(2.33) \quad \text{dist}(y, P_x) \leq \varepsilon \quad \text{for } y \in E \cap B(x, 1),$$

and such that (2.5) holds. Then our construction gives a set  $S_N$ , with a Lipschitz description near  $E$ , as in (2.26) included. We also get (1.4) as before, but of course not (1.5). But slightly more unpleasantly,  $S_N$  has a boundary; that is, we only get (2.26) for balls centered on  $X \subset E$ , but not necessarily for balls centered on points of  $S_N$  far from  $E$ . For instance, the parts near  $E$  may be far from each other and then  $S_N$  stops somewhere in between. It is possible that we can connect all the boundary pieces of  $S_N$ , and get a larger set  $\Sigma_0$  which is smooth and without boundary, but this could be unpleasant to do. The author did not find any obvious topological obstruction either.

**Remark 2.34.** When  $E$  is unbounded, we can try to prove a variant of Theorem 1.10, where the radius  $r_0$  depends slowly on the location of the balls. We shall not give any detail.

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