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On the optimality of the ideal right-angled 24-cell

ALEXANDER KOLPAKOV

We prove that among four-dimensional ideal right-angled hyperbolic polytopes the 24-cell is of minimal volume and of minimal facet number. As a corollary, a dimension bound for ideal right-angled hyperbolic polytopes is obtained.

Key words: Coxeter polytope, right-angled Coxeter group.

MSC (2010) 20F55; 52B11, 51M20

1 Introduction

In this work, we consider the 24-cell $\mathscr{C}$, that is a four-dimensional regular ideal hyperbolic polytope with Schläfli symbol $\{3,4,3\}$ with all dihedral angles right. The polytope $\mathscr{C}$ has 24 octahedral facets, 96 triangular faces, 96 edges and 24 cubical vertex figures. We shall show that it provides a solution to the following problems in the class of ideal right-angled polytopes in $\mathbb{H}^4$:

I: Find a polytope of minimal volume,

II: Find a polytope of minimal facet number.

Since Coxeter's work [3], the 24-cell is known for its nice combinatorial and geometric structure in the Euclidean sense. We demonstrate that it possesses optimal properties **I** and **II** in the hyperbolic setting. Question **I** is closely related to the volume spectrum of four-dimensional hyperbolic manifolds [10], question **II** is new and is both of combinatorial and geometric nature. Furthermore, using the results of [7, 8], we obtain a new dimension bound for ideal right-angled hyperbolic polytopes. The case of right-angled hyperbolic polytopes with both proper and ideal vertices was considered before in [6, 9].

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2 Preliminaries

Let $\mathscr{P}$ be *a polytope* in the hyperbolic n-dimensional space $\mathbb{H}^n$, that means $\mathscr{P} = \bigcap_{i\in I} H_i^-$, $|I| < \infty$, with finitely many half-spaces $H_i^- \subset \mathbb{H}^n$ enclosed by the respective hyperplanes H_i. In particular, $\mathscr{P}$ is convex. The part of the polytope $\mathscr{P}$ contained in such a hyperplane is called *a facet* and has dimension $n-1$. We define the low-dimensional faces of $\mathscr{P}$ by induction as non-empty intersections of the higher-dimensional ones. Let us denote the set of k-dimensional faces by $\Omega_k(\mathscr{P})$, $0 \leq k < n$, $\Omega_n(\mathscr{P}) := \mathscr{P}$. We call the k-dimensional faces of $\mathscr{P}$ for $k = 0$ and 1, its *vertices* and *edges*, respectively. Let f_k denote the number of k-dimensional faces and let $\mathbf{f}(\mathscr{P}) = (\mathrm{f}_0, \cdots, \mathrm{f}_{n-1})$ be *the face vector* of the polytope $\mathscr{P}$.

The faces of a given n-polytope $\mathscr{P}$ form a lattice $\Omega(\mathscr{P}) := \bigcup_{k=0}^{n} \Omega_k(\mathscr{P})$ ordered by the set-theoretical face inclusion. Call two polytopes $\mathscr{P}_1$ and $\mathscr{P}_2$ combinatorially isomorphic if their face lattices $\Omega(\mathscr{P}_1)$ and $\Omega(\mathscr{P}_2)$ are isomorphic. Call $\mathscr{P} \subset \mathbb{H}^n$ *a regular* hyperbolic polytope if it is combinatorially isomorphic to a regular n-dimensional Euclidean polytope and all the dihedral angles of $\mathscr{P}$ in its co-dimension two faces are equal. Recall that there are infinitely many regular polygons. Dimension three provides five Platonic solids. There exist six regular four-dimensional polytopes. Starting from dimension five, there are only three combinatorial types of convex regular polytopes (see [3, Table I]).

A polytope is called *non-obtuse* if all the dihedral angles in its co-dimension two faces are not greater than $\pi/2$. A polytope is *right-angled* if all these dihedral angles equal $\pi/2$. Call a hyperbolic polytope $\mathscr{P}$ *ideal* if all its vertices are ideal points of $\mathbb{H}^n$, i.e. all of them belong to $\partial\overline{\mathbb{H}^n}$. A polytope $\mathscr{P} \subset \mathbb{H}^n$ is *simple* if each vertex belongs to n facets only, and $\mathscr{P}$ is called *simple at edges* if each edge belongs to $n-1$ facets only. An infinitesimal link (w.r.t. the Euclidean metric on $\overline{\mathbb{H}^n}$) of a vertex of a polytope is called its *vertex figure* (w.r.t. the given vertex). Every vertex figure of a compact non-obtuse hyperbolic polytope is a combinatorial simplex of co-dimension one [11, p. 108, Theorem 1.8]. Every vertex figure of an ideal right-angled hyperbolic polytope is a combinatorial cube of co-dimension one [6, Proposition 1]. Thus, a compact non-obtuse hyperbolic polytope is simple and an ideal right-angled hyperbolic polytope is simple at edges.

The 24-cell considered as a regular polytope has the Schläfli symbol $\{3,4,3\}$, octahedral facets $\{3,4\}$ and cubical vertex figures $\{4,3\}$. Note that this is the only regular four-dimensional polytope having each vertex figure a cube [3, Table I] and thus realisable as an ideal right-angled hyperbolic one. We denote it by $\mathscr{C}$ and call *the hyperbolic* 24*-cell*.

3 The 24-cell and volume minimality

Lemma 1 (Combinatorial identities) *Let $\mathscr{P} \subset \mathbb{H}^4$ be an ideal right-angled polytope with face vector $\mathbf{f}(\mathscr{P}) = (\mathrm{f}_0, \mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3)$. Then the following combinatorial identities hold.*

$$\mathrm{f}_0 - \mathrm{f}_1 + \mathrm{f}_2 - \mathrm{f}_3 = 0, \tag{1}$$

$$\mathrm{f}_1 = 4\,\mathrm{f}_0, \tag{2}$$

$$12\,\mathrm{f}_0 = \sum_{F \in \Omega_2(\mathscr{P})} \mathrm{f}_0(F). \tag{3}$$

Proof We list the proofs of (1)-(3) below in the respective order.

(1) This is Euler's identity. Since $\mathscr{P}$ is a convex four-dimensional polytope, its surface $\partial\mathscr{P}$ is homeomorphic to $\mathbb{S}^3$. Hence, for the Euler characteristic of $\partial\mathscr{P}$, we have $\mathrm{f}_0 - \mathrm{f}_1 + \mathrm{f}_2 - \mathrm{f}_3 =: \chi(\partial\mathscr{P}) = \chi(\mathbb{S}^3) = 0$.

(2) Let $v \in \Omega_0(\mathscr{P})$ be a vertex. Then each vertex figure $\mathscr{P}_v$ is a cube, since $\mathscr{P}$ is an ideal right-angled polytope [6, Proposition 1]. The vertices of $\mathscr{P}_v$ correspond to the edges of $\mathscr{P}$ emanating from a given vertex $v \in \Omega_0(\mathscr{P})$. This means that eight edges are adjacent at v. On the other hand, each edge has two vertices. Thus, we obtain $2\,\mathrm{f}_1 = 8\,\mathrm{f}_0$ and (2) follows.

(3) The edges of the vertex figure $\mathscr{P}_v$ correspond to the two-dimensional faces of $\mathscr{P}$ meeting v. Thus, twelve two-dimensional faces meet at each vertex. Hence, if we sum up the number of vertices $\mathrm{f}_0(F)$ over all the two-dimensional faces $F \in \Omega_2(\mathscr{P})$, we count each vertex of $\mathscr{P}$ twelve times. Then the desired formula follows and the lemma is proven. □

Lemma 2 (Volume formula) *Let $\mathscr{P} \subset \mathbb{H}^4$ be an ideal right-angled polytope with face vector $\mathbf{f}(\mathscr{P}) = (\mathrm{f}_0, \mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3)$. Then its volume equals*

$$\mathrm{Vol}\,\mathscr{P} = \frac{\mathrm{f}_0 - \mathrm{f}_3 + 4}{3}\,\pi^2.$$

Proof Let $\mathscr{G}$ be the group generated by reflections in the supporting hyperplanes of the facets of $\mathscr{P}$. The group $\mathscr{G}$ is a Coxeter group acting discretely on $\mathbb{H}^4$ with Poincaré polytope $\mathscr{P}$. Hence $\operatorname{Vol}\mathscr{P} = \operatorname{coVol}\mathscr{G}$. By [12, Corollary 4.2], we obtain

$$\operatorname{coVol}\mathscr{G} = \frac{\pi^2}{3}\left(\kappa(\mathscr{P}) - 2\sum_{F\in\Omega_2(\mathscr{P})}\frac{\mathrm{f}_0(F)-2}{|Stab(F)|} + 4\sum_{v\in\Omega_0(\mathscr{P})}\frac{1}{|Stab(v)|}\right), \tag{4}$$

where $\kappa(\mathscr{P}) := 4 - 2(\mathrm{f}_0 + \mathrm{f}_2) + \sum_{F\in\Omega_2(\mathscr{P})}\mathrm{f}_0(F)$. Here $Stab(\star) \leqslant \mathscr{G}$ means the stabilizer of $\star$, where $\star$ is a two-dimensional face F or a vertex v of $\mathscr{P}$.

For each $F \in \Omega_2(\mathscr{P})$ the group $Stab(F)$ is generated by two reflections in the facets meeting at F. Since $\mathscr{P}$ is a right-angled polytope, the group $Stab(F)$ is generated by reflections in two orthogonal mirrors. Thus, $Stab(F) \simeq D_2$, the dihedral group of order four. Each vertex $v \in \Omega_0(\mathscr{P})$ is ideal, i.e. belongs to $\partial\overline{\mathbb{H}^4}$, so the stabilizer $Stab(v)$ is a Coxeter group generated by reflection in the faces of a Euclidean cube. Hence $Stab(v)$ is infinite.

We obtain the desired volume formula by substituting the values $|Stab(F)| = 4$, $|Stab(v)| = \infty$ into (4) and by applying the formulas of Lemma 1 for the computation. □

The hyperbolic 24-cell $\mathscr{C}$ has $\mathrm{f}_0 = \mathrm{f}_3 = 24$, $\mathrm{f}_1 = \mathrm{f}_2 = 96$, see [3, Table I, (ii)]. Hence, by the lemma above, its volume equals $4\pi^2/3$.

Theorem 1 (Minimal volume) *A four-dimensional ideal right-angled hyperbolic polytope of minimal volume is $\mathscr{C}$, up to an isometry.*

Proof Let us consider an ideal right-angled hyperbolic polytope $\mathscr{P} \subset \mathbb{H}^4$. Let $\mathrm{f}_2(k)$ denote the number of its two-dimensional k-gonal faces, $k \geq 3$, which are ideal hyperbolic polygons. Then

$$\mathrm{f}_2 = \mathrm{f}_2(3) + \cdots + \mathrm{f}_2(N),$$

where $N = \max_{F\in\Omega_2(\mathscr{P})}\mathrm{f}_0(F) \geq 3$. By Lemma 1, formula (3), we obtain

$$12\,\mathrm{f}_0 = \sum_{F\in\Omega_2(\mathscr{P})}\mathrm{f}_0(F) = 3\,\mathrm{f}_2(3) + \cdots + N\,\mathrm{f}_2(N).$$

By using Lemma 1, formulas (1)-(2), one subsequently computes

$$\mathrm{f}_0 - \mathrm{f}_3 = 4\mathrm{f}_0 - \mathrm{f}_2 = \frac{1}{3}\sum_{k=4}^{N}(k-3)\mathrm{f}_2(k) \geq 0. \tag{5}$$

Then, by Lemma 2,

$$\operatorname{Vol}\mathscr{P} \geq \frac{4}{3}\pi^2 = \operatorname{Vol}\mathscr{C}.$$

If $\operatorname{Vol}\mathscr{P}$ equals the volume of $\mathscr{C}$, one immediately has $\mathrm{f}_2(k) = 0$ for all $k \geq 4$ by (5). This means that all the two-dimensional faces of $\mathscr{P}$ are triangles. Consider a facet $P \in \Omega_3(\mathscr{P})$. Observe that $P \subset \mathbb{H}^3$ is an ideal right-angled polyhedron which has only triangular faces. Then P is a combinatorial octahedron and it is isometric to the right-angled hyperbolic octahedron by [1, Theorem 3] and [2, Theorem 2]. Hence, all the facets of $\mathscr{P}$ are ideal right-angled octahedra. So the polytope $\mathscr{P}$ is combinatorially isomorphic to a regular four-dimensional Euclidean polytope with octahedral facets only, that is, the 24-cell by [3, Table I, (ii)]. Thus $\mathscr{P}$ is isometric to $\mathscr{C}$ by Andreev's theorem [1, Theorem 3]. □

4 The 24-cell and facet number minimality

Theorem 2 (Minimal facet number) *The facet number of a four-dimensional ideal right-angled hyperbolic polytope $\mathscr{P}$ satisfies the inequality $\mathrm{f}_3(\mathscr{P}) \geq \mathrm{f}_3(\mathscr{C}) = 24$. Any four-dimensional ideal right-angled hyperbolic polytope $\mathscr{P}$ with $\mathrm{f}_3(\mathscr{P}) = 24$ is isometric to the hyperbolic 24-cell $\mathscr{C}$.*

The proof will be based on Proposition 1 and Lemma 3 below. Their proofs will be given in Section 3.3.

3.1 Three-dimensional ideal right-angled hyperbolic polyhedra with few faces. Let $\mathscr{A}_k \subset \mathbb{H}^3$, $k \geq 3$, be an ideal right-angled antiprism depicted in Fig. 1. In the figure, the leftmost and the rightmost edges are identified, so that the surface of the polyhedron is partitioned into top and bottom k-gonal faces and $2k$ triangular faces in the annulus between them. Such an antiprism exists for every $k \geq 3$ and it is unique up to an isometry due to [1, Theorem 3], [2, Theorem 2].

Antiprisms $\mathscr{A}_k$ will later play the rôle of possible facets for a four-dimensional ideal right-angled hyperbolic polytope in the proof of Theorem 2.

Proposition 1 (Antiprism's optimality) *A three-dimensional ideal right-angled hyperbolic polyhedron of minimal facet number, which has at least one k-gonal face, $k \geq 3$, is isometric to the antiprism $\mathscr{A}_k$ with $\mathrm{f}_2(\mathscr{A}_k) = 2k + 2$.*

Proof Let $\mathscr{P} \subset \mathbb{H}^3$ be an ideal right-angled polyhedron. Let $F \in \Omega_2(\mathscr{P})$ be a k-gonal face, $k \geq 3$. For each edge $e \in \Omega_1(F)$ there is exactly one further face adjacent to F along e. For each vertex v, being four-valent by Andreev's theorem [2, Theorem 2], there exists a face intersecting F at v only. Moreover, all the faces mentioned above are different from each other, so that we have $\mathrm{f}_2(\mathscr{P}) \geq 2k+1$. Observe that these faces can not constitute yet a polyhedron. Indeed, consider F as a "bottom" face of $\mathscr{P}$. Then the new faces we have added make a surface wrapping around the interior of $\mathscr{P}$ along the edges of F. Since all vertices are four-valent, at least one additional "top" face is required to close up the polyhedron. Hence $\mathrm{f}_2(\mathscr{P}) \geq 2k+2$. The antiprism $\mathscr{A}_k$ satisfies

$$\mathrm{f}_2(\mathscr{A}_k) = 2k+2 \tag{6}$$

and so has minimal facet number.

It remains to show that a polyhedron $\mathscr{P}$ with $\mathrm{f}_2(\mathscr{P}) = \mathrm{f}_2(\mathscr{A}_k)$ is in fact isometric to $\mathscr{A}_k$. Since $\mathscr{P}$ has four-valent vertices, $2\mathrm{f}_1(\mathscr{P}) = 4\mathrm{f}_0(\mathscr{P})$. From this equality and Euler's identity $\mathrm{f}_0(\mathscr{P}) - \mathrm{f}_1(\mathscr{P}) + \mathrm{f}_2(\mathscr{P}) = 2$ we obtain that

$$\mathrm{f}_2(\mathscr{P}) = \mathrm{f}_0(\mathscr{P}) + 2. \tag{7}$$

Consider the faces adjacent to the k-gon F along its edges. We shall prove that no pair of them can have a common vertex $v \notin \Omega_0(\mathscr{P})$. By supposing the contrary, let us denote two such faces F_i, $i = 1,2$, and let them intersect at v. Observe that F_i, $i = 1,2$, are adjacent to F along two disjoint edges e_1 and e_2. In fact, if e_1 intersects e_2 in a vertex $u \in \Omega_0(F)$, then since $\mathscr{P}$ has convex faces we obtain two geodesic segments joining v to u. One of them belongs to F_1 and the other belongs to F_2. This is impossible, unless the considered segments are represented by a common edge e of F_i, $i = 1,2$, adjacent to both v and u. But then the vertex u has only three adjacent edges: e_1, e_2 and e. This is a contradiction to u having valency four. Now if F_1 and F_2 share an edge e such that $v \in \Omega_0(\mathscr{P})$, then condition $(\mathfrak{m}_2)$ of Andreev's theorem [2, Theorem 2] does not hold as depicted in Fig. 2. If F_1 and F_2 share only the vertex v, then condition $(\mathfrak{m}_5)$ of [2, Theorem 2] is not satisfied as depicted in Fig. 3.

Suppose that a face F' adjacent to the k-gon $F \in \Omega_2(\mathscr{P})$ along an edge is not triangular. Then F' has $\mathrm{f}_0(F')$ vertices, and two among them are already counted in $\mathrm{f}_0(F)$. Hence we have at least

$$\sum_{\substack{F' \text{ adjacent to} \\ F \text{ along an edge}}} \big(\mathrm{f}_0(F') - 2\big) \geq (k-1) + 2 = k+1$$

additional vertices, since $f_0(F') \geq 3$ for each F' among k faces adjacent to F and at least one such face F' has $f_0(F') \geq 4$. Thus $f_0(\mathscr{P}) \geq 2k+1$, and by (7) the estimate $f_2(\mathscr{P}) \geq 2k+3$ follows. Equality (6) implies $f_2(\mathscr{P}) > f_2(\mathscr{A}_k)$ and we arrive at a contradiction. Hence all the faces adjacent to F along its edges are triangular.

Consider the faces of $\mathscr{P}$ adjacent to the k-gon $F \in \Omega_2(\mathscr{P})$ only at its vertices. Suppose that one of them, say F', is not triangular. Then we have

$$\sum_{\substack{F' \text{ adjacent to} \\ F \text{ at a vertex}}} f_1(F') \geq 3(k-1)+4 = 3k+1$$

additional edges. But then $f_1(\mathscr{P}) \geq 4k+1$ and we arrive at a contradiction. Indeed, in this case $f_1(\mathscr{P}) > f_1(\mathscr{A}_k) = 4k$, while a polyhedron of minimal facet number has all f_i minimal, $i = 0, 1, 2$, as follows from equation (7) and the remarks before.

Hence we have a k-gonal face F, $k \geq 3$, together with $2k$ triangular side faces adjacent to it along the edges and at the vertices. By adding another one k-gonal face we close up the polyhedron $\mathscr{P}$, while its vertex number remains unchanged. Observe that there is no other way to finish this construction without increasing the vertex number.

Thus, an ideal right-angled polyhedron $\mathscr{P} \subset \mathbb{H}^3$ having minimal face number, which contains at least one k-gon, is combinatorially isomorphic to $\mathscr{A}_k$. By [1, Theorem 3] and [2, Theorem 2], the polyhedron $\mathscr{P}$ is isometric to $\mathscr{A}_k$. □

Note (to Proposition 1) The classification of *polygonal maps* on the two-dimensional sphere given in [5] provides another argument to show the uniqueness of antiprism stated above. Namely, [5, Theorem 1] says that $\mathscr{P}$ has in fact not less than two k-gonal faces. Hence $f_2(\mathscr{P}) = 2k+2$ if and only if $\mathscr{P}$ has exactly two k-gonal faces and $2k$ triangular faces. Polygonal maps of this kind are classified by [5, Theorem 2]. Among them only the map isomorphic to the one-skeleton of $\mathscr{A}_k$ satisfies Steiniz's theorem [13, Chapter 4]. Thus, the polyhedron $\mathscr{P}$ is combinatorially isomorphic to $\mathscr{A}_k$.*

3.2 Combinatorial constraints on facet adjacency. Let $F_1, \ldots, F_m$ be an ordered sequence of facets of a given hyperbolic polytope $\mathscr{P} \subset \mathbb{H}^4$ such that each facet is adjacent only to the previous and the following ones either through a co-dimension two face or through an ideal vertex, while the last facet F_m is adjacent only to the first facet F_1 (through a co-dimension two face or through an ideal vertex, as before) and

* the author is grateful to Michel Deza for indicating the very recent paper [5].

no three of them share a lower-dimensional face. Call the sequence $F_1, \ldots, F_m$ *a* (k,ℓ) *circuit*, $k+\ell=m$, if it comprises k co-dimension two faces and ℓ ideal vertices shared by the facets. We complete the analysis carried out in [9] in the following way.

Lemma 3 (Adjacency constraints) *Let $\mathscr{P} \subset \mathbb{H}^4$ be an ideal right-angled polytope. Then $\mathscr{P}$ contains no $(3,0)$, $(4,0)$ and $(2,1)$ circuits.*

Proof By [9, Proposition 4.1] there are no $(3,0)$ and $(2,1)$ circuits. Suppose on the contrary that there exists a $(4,0)$ circuit formed by the facets $F_k \in \Omega_3(\mathscr{P})$, $k=1,2,3,4$. Let e_k, $k=1,2,3,4$, denote the outer unit vector normal to the support hyperplane of F_k. Consider the Gram matrix of these vectors w.r.t. the Lorentzian form $\langle\cdot,\cdot\rangle_{4,1}$:

$$G = \left(\langle e_i, e_j\rangle\right)_{i,j=1}^4 = \begin{pmatrix} 1 & 0 & -\cosh\rho_{13} & 0 \\ 0 & 1 & 0 & -\cosh\rho_{24} \\ -\cosh\rho_{13} & 0 & 1 & 0 \\ 0 & -\cosh\rho_{24} & 0 & 1 \end{pmatrix},$$

where $\rho_{ij} > 0$ is the length of the common perpendicular between two disjoint support hyperplanes for F_i and F_j respectively. The eigenvalues of G are $\{1 \pm \cosh\rho_{13}, 1 \pm \cosh\rho_{24}\}$, that means two of them are strictly negative and two are strictly positive. Thus, we arrive at a contradiction with the signature of a Lorentzian form. □

3.3 Proof of Theorem 2. Let $\mathscr{P} \subset \mathbb{H}^4$ be an ideal right-angled polytope. Let $P \in \Omega_3(\mathscr{P})$ be a facet. For every two-face $F \in \Omega_2(P)$ there exists a corresponding facet $P' \in \Omega_3(\mathscr{P})$, $P' \neq P$, such that P and P' share the face F. Since each vertex figure of $\mathscr{P}$ is a cube, there exists a respective facet $P'' \in \Omega_3(\mathscr{P})$ for every vertex $v \in \Omega_0(P)$. The vertex figure is depicted in Fig. 4, where the grey bottom face of the cube corresponds to P and the top face corresponds to P''. These new facets P' and P'' together with P are pairwise different. In order to show this we use the following convexity argument.

(*Convexity argument*) First, observe that no facet of a convex polytope can meet another one at two different two-faces. Now suppose that $P' \in \Omega_3(\mathscr{P})$ is a facet adjacent to P at a face $F \in \Omega_2(P)$ and a single vertex $v \in \Omega_0(P)$ not in F. The facets P and P' have non-intersecting interiors, but the geodesic going through a given point of F to v belongs to both of them by the convexity of $\mathscr{P}$. So we arrive at a contradiction.

The same contradiction arises if we suppose that there is a facet $P' \in \Omega_3(\mathscr{P})$ adjacent to P at two distinct vertices $v, v' \in \Omega_0(P)$. In this case we consider the geodesic in P going through v to v'.

By the convexity argument above, the facet number of $\mathscr{P}$ has the lower bound

$$\mathrm{f}_3(\mathscr{P}) \geq \mathrm{f}_2(P) + \mathrm{f}_0(P) + 1,$$

or, by means of equality (7),

$$\mathrm{f}_3(\mathscr{P}) \geq 2\,\mathrm{f}_2(P) - 1. \tag{8}$$

Observe that the hyperbolic 24-cell $\mathscr{C}$ has only triangle two-faces. Suppose that $\mathscr{P}$ has at least one k-gonal face $F \in \Omega_2(\mathscr{P})$ with $k \geq 4$. We shall show that the estimate $\mathrm{f}_3(\mathscr{P}) \geq 25$ holds, by considering several cases as follows.

A) Suppose that $\mathscr{P}$ has a k-gonal two-dimensional face with $k \geq 6$. Then, by (8) and Proposition 1, we have

$$\mathrm{f}_3(\mathscr{P}) \geq 2\,\mathrm{f}_2(\mathscr{A}_k) - 1 = 2(2k+2) - 1 \geq 27.$$

Thus $\mathscr{P}$ is not of minimal facet number.

B) Suppose that $\mathscr{P}$ has a pentagonal two-dimensional face F contained in a facet $P \in \Omega_3(\mathscr{P})$. Suppose P is not isometric to $\mathscr{A}_5$. This assumption implies $\mathrm{f}_2(P) > 12$. Then (8) grants $\mathrm{f}_3(\mathscr{P}) \geq 25$.

C) Suppose that all the facets of $\mathscr{P}$ containing a pentagonal two-face are isometric to $\mathscr{A}_5$. Let P_0 be one of them. Then it has two neighbouring facets P_k, $k = 1,2$ both isometric to $\mathscr{A}_5$. Now we count the facets adjacent to P_k, $k = 0,1,2$ in Fig. 5, where P_0 is coloured grey. Observe that two-faces in Fig. 5 sharing an edge are marked with the same number and belong to a common facet, since $\mathscr{P}$ is simple at edges. However, the two-faces marked with different numbers, correspond to different adjacent facets. Suppose on the contrary that there are two faces $F \in \Omega_2(P_i)$, $F' \in \Omega_2(P_j)$, $i,j \in \{0,1,2\}$, marked with distinct numbers and a facet $P' \in \Omega_3(\mathscr{P})$ such that P' is adjacent to P_i at F and to P_j at F' and consider the following cases.

C.1) If $i = j$, we arrive at a contradiction by the convexity argument above.

C.2) If $i = 0$, $j \in \{1,2\}$, then there exists a unique geodesic joining a point p of F to a point p' of F'. Observe in Fig. 5, that the point p' may be chosen so that $p' \in F' \cap P_0$. Then the geodesic between p and p' intersects both the interior of P' and the interior of P_0. Again, we use the convexity argument and arrive at a contradiction.

C.3) Let $i = 1$, $j = 2$. Then if there exist a face $\tilde{F} \in \Omega_2(P_0)$, $\tilde{F} \cap F \neq \emptyset$, and a face $\tilde{F}' \in \Omega_2(P_0)$, $\tilde{F}' \cap F' \neq \emptyset$, we reduce our argument to case **C.1** by considering a geodesic segment joining a point of $\tilde{F} \cap F$ to a point of $\tilde{F}' \cap F'$.

The only case when no such two faces $\tilde{F}$ and $\tilde{F}'$ exist is if F has number 21 and F' has number 22 in Fig. 5. Then the $(4,0)$-circuit $P_0P_1P'P_2$ appears, in contrary to Lemma 3.

Thus, one has 22 new facets adjacent to P_k, $k=0,1,2$. Together with P_k themselves, $k=0,1,2$, they provide $\mathrm{f}_3(\mathscr{P}) \geq 25$.

D) By now, cases **A**, **B** and **C** imply that if an ideal right-angled hyperbolic polytope $\mathscr{P} \subset \mathbb{H}^4$ has at least one k-gonal face with $k \geq 5$, then $\mathrm{f}_3(\mathscr{P}) \geq 25$. Suppose that $\Omega_2(\mathscr{P})$ contains only triangles and quadrilaterals.

By Andreev's theorem [2], each facet $P \in \Omega_3(\mathscr{P})$ has only four-valent vertices. By assumption, P has only triangular and quadrilateral faces. Combinatorial polyhedra of this type are introduced in [4] as *octahedrites* and the list of those possessing up to 17 vertices is given. Note that in view of (8) we may consider octahedrites that have not more than twelve faces or, by equality (7) from Proposition 1, ten vertices. In Fig. 6, 7 we depict only those realisable as ideal right-angled hyperbolic polyhedra with eight, nine and ten vertices. The ideal right-angled octahedron has six vertices and completes the list. By considering each of the polyhedra in Fig. 6 and Fig. 7 as a possible facet $P \in \Omega_3(\mathscr{P})$, we shall derive the estimate $\mathrm{f}_3(\mathscr{P}) \geq 25$.

D.1) Let $P_0 \in \Omega_3(\mathscr{P})$ be the hyperbolic octahedrite with ten vertices depicted in Fig. 8. Consider the facets of $\mathscr{P}$ adjacent to P_0 at its faces. One has $\mathrm{f}_2(P_0) = 12$, and hence $\mathrm{f}_3(\mathscr{P}) \geq 12$. Consider the faces coloured grey in Fig. 8: the front face is called F_1 and the back face, called F_2, is indicated by the grey arrow.

The facets $P_1, P_2 \in \Omega_3(\mathscr{P})$ adjacent to P_0 at F_1 and F_2, respectively, contain quadrilaterals among their faces. By Proposition 1, it follows that $\mathrm{f}_2(P_i) \geq \mathrm{f}_2(\mathscr{A}_4) = 10$, $i=1,2$. We shall count all new facets P' brought by face adjacency to P_i, $i=1,2$.

Observe that no P', which does not share an edge with P_0, can be adjacent simultaneously to P_i and P_j, $i,j \in \{1,2\}$, at two-faces, since otherwise the $(4,0)$ circuit $P_1P_0P_2P'$ appears in contrary to Lemma 3.

Each facet P' that shares an edge with F_k, $k=1,2$, is already counted as adjacent to P_0. The facets P_1 and P_2 are already counted as well, by the same reason. Then the total number of new facets coming together with P_1 and P_2 is at least $\sum_{i=1}^{2} \mathrm{f}_2(P_i) - \sum_{i=1}^{2} \mathrm{f}_1(F_i) - 2 \geq 2 \cdot 10 - 2 \cdot 4 - 2 = 10$. This implies the estimate $\mathrm{f}_3(\mathscr{P}) \geq 12 + 10 = 22$.

Consider the facets P_i, $i = 3,4$, adjacent to P_0 only at the corresponding circumscribed grey vertices v_i, $i = 3,4$, in Fig. 8. Then consider the case if P' is adjacent to P_j, $j \in \{1,2\}$ at a two-face $F' \in \Omega_2(P_j)$. If there exist a face $\tilde{F}' \in \Omega_2(P_0)$ such that $F' \cap \tilde{F}' \neq \emptyset$, then choose a point $p \in F' \cap \tilde{F}'$ and use the convexity argument again for the geodesic going through p to v_i. If $F' \cap \tilde{F}' = \emptyset$, then the $(2,1)$ circuit P_0P_1P' appears in contrary to Lemma 3. Adding up two new facets gives $\mathrm{f}_3(\mathscr{P}) \geq 24$. Finally, we count P_0 itself and arrive at the estimate $\mathrm{f}_3(\mathscr{P}) \geq 25$.

D.2) Let $P_0 \in \Omega_3(\mathscr{P})$ be the hyperbolic octahedrite with nine vertices and eleven faces depicted on the right in Fig. 6. Consider the facets adjacent to P_0 at its two-dimensional faces. By counting them, we have $\mathrm{f}_3(\mathscr{P}) \geq \mathrm{f}_2(P_0) = 11$.

Consider the facet P_1 adjacent to the triangle face F_1 of P_0 coloured grey in the center of Fig. 9. By Proposition 1, we have $\mathrm{f}_2(P_1) \geq \mathrm{f}_2(\mathscr{A}_3) = 8$. By excluding already counted facets adjacent to P_0 like in case **D.1**, the facet P_1 brings new $\mathrm{f}_2(P_1) - \mathrm{f}_1(F_1) - 1 \geq 8 - 3 - 1 = 4$ ones by face adjacency. Then $\mathrm{f}_3(\mathscr{P}) \geq 15$. The visible part of the facet P_2 adjacent to P_0 at its back face F_2 is coloured grey in Fig. 9. Again, we have $\mathrm{f}_2(P_2) \geq \mathrm{f}_2(\mathscr{A}_3) = 8$. By counting new facets adjacent to P_2 at faces, it brings another $\mathrm{f}_2(P_2) - \mathrm{f}_1(F_2) - 1 \geq 8 - 3 - 1 = 4$ new ones. Hence $\mathrm{f}_3(\mathscr{P}) \geq 19$.

The facets $\widehat{P}_k$, $k = 3,4,5$, adjacent to P_0 only at the circumscribed hollow vertices v_k, $k = 3,4,5$, in Fig. 9 are different from the already counted ones either by the convexity argument or by Lemma 3, which forbids $(2,1)$ circuits, c.f. the argument of case **D.1**. Thus $\mathrm{f}_3(\mathscr{P}) \geq 22$.

Let $\widehat{P}_k$, $k = 6,7,8$, be the facets of $\mathscr{P}$ adjacent to P_2 only at the respective circumscribed grey vertices v_k, $k = 6,7,8$ in Fig. 9. Let the faces of P_1 and P_2, that contain a single circumscribed hollow or grey vertex, be F_k, $k = 3,\ldots,8$. Finally, let $P(k)$, $k = 6,7,8$, denote the facets adjacent to P_2 at F_k, $k = 6,7,8$, respectively.

By the convexity argument or by Lemma 3, similar to **D.1**, the facets $\widehat{P}_i$, $i = 6,7,8$ can not coincide with the already counted ones, except for $\widehat{P}_j$, $j = 3,4,5$ and the facets adjacent only to P_1.

First consider the case when a facet from $\widehat{P}_i$, $i \in \{6,7,8\}$, coincides with $\widehat{P}_j$, $j \in \{3,4,5\}$. Then

1) either $\widehat{P}_i = \widehat{P}_j$ is such that $(i,j) \neq (7,3)$, $(6,4)$ and $(8,5)$, so the $(2,1)$ circuit $\widehat{P}_jP(i)P_0$ appears;

2) or $\widehat{P}_i = \widehat{P}_j$ has $(i,j) = (7,3),(6,4)$, or $(8,5)$, and contains therefore a part of the geodesic going from v_i to v_j by convexity. Since the edge shared by F_i and F_j belongs to three facets P_0, P_2 and $P(i)$, then $P(i)$ is adjacent to P_0 at F_j and to P_2 at F_i. Hence $P(i)$ contains the vertices v_i, v_j and the geodesic segment between them as well. Since $P(i)$ and $\widehat{P}_i$ have non-intersecting interiors, the two following cases are only possible.

2.1) The geodesic segment v_iv_j belongs to a triangle face of $P(i)$: then v_iv_j is an edge. Observe that the face F_j of $P(i)$ is always a triangle, as in Fig. 9, while the face F_i is either a triangle or a quadrilateral. Then the edges of F_i, F_j and the edge v_iv_j constitute a sub-graph in the one-skeleton of $P(i)$. The possible sub-graphs τ and σ depending on the vertex number of F_i are depicted in Fig. 10. The graph τ is the one-skeleton of a tetrahedron. The graph σ is the one-skeleton of a square pyramid without one vertical edge. By assumption, the facet $P(i)$ is an octahedrite with not more than ten vertices. Such octahedrites are depicted in Fig. 6-7, and none of them contains in its one-skeleton a sub-graph combinatorially isomorphic to τ or σ.

The case when $P(i)$ is an octahedron still remains. Clearly, its one-skeleton does not contain a sub-graph combinatorially isomorphic to τ. However, it contains a sub-graph isomorphic to σ. The only possible sub-graph embedding of σ into the one-skeleton of an octahedron, up to a symmetry, is given in Fig. 11 on the left. But then the face F_i of P_2 correspond to the interior domain F in $P(i)$ coloured grey in Fig. 11 on the right. Thus, we arrive at a contradiction with the convexity of facets.

2.2) The geodesic segment v_iv_j belongs to a quadrilateral face of $P(i)$. The general picture of this case is given in Fig. 12. Again two sub-graphs ν and ω arise, as depicted in Fig. 13. Such sub-graphs appear at most for the octahedrites as given in Fig. 6-7. Observe, that none of them contains in its one-skeleton a sub-graph isomorphic to ν.

All possible embeddings of ω into the one-skeleton of each considered octahedrite are given, up to a symmetry, in Fig. 14-21. Since the edges e and e' belong to a single face as in Fig. 12, we arrive at a contradiction, since there is no embedding of ω with this property.

Finally, consider the case when a facet from $\widehat{P}_i$, $i \in \{6,7,8\}$, coincides with a facet P' adjacent only to P_1 at a two-face. Then the $(4,0)$ circuit $P_0P_1P'P(i)$ arises, in contrary to Lemma 3.

So the facets $\widehat{P}_k$, $k = 6,7,8$, are different from the already counted ones. Adding them up, we obtain $\mathrm{f}_3(\mathscr{P}) \geq 22 + 3 = 25$.

D.3) Let $P_0 \in \Omega_3(\mathscr{P})$ be the hyperbolic octahedrite with eight vertices depicted on the left in Fig. 6. Observe that this polyhedron is combinatorially isomorphic to $\mathscr{A}_4$, and hence isometric to it by Andreev's theorem [2]. Moreover, we suppose that all facets of $\mathscr{P}$ are isometric to $\mathscr{A}_4$, since other possible facet types are already considered in **D.1** and **D.2**.

Consider the facets P_k, $k = 1, 2$, adjacent to the front and the back quadrilateral faces of P_0. The facets P_i, $i = 0, 1, 2$, are depicted together in Fig. 22, where P_0 is coloured grey. We count the facets adjacent to P_i, $i = 1, 2, 3$, at faces in Fig. 22. Observe that different numbers on the faces shown in Fig. 22 correspond to distinct facets of $\mathscr{P}$ adjacent to them. The counting arguments are completely analogous to those of **C**. Hence, we obtain the estimate $\mathrm{f}_3(\mathscr{P}) \geq 18$. By taking into account the facets P_i, $i = 1, 2, 3$, themselves, it becomes $\mathrm{f}_3(\mathscr{P}) \geq 21$.

Consider the facets $\widehat{P}_i$, $i = 1, 2, 3, 4$, adjacent to P_2 only at its circumscribed vertices v_i, $i = 1, 2, 3, 4$ in Fig. 22. By analogy with the proof in **D.2**, the $\widehat{P}_i$'s are different from the already counted ones. Thus, we add four new facets and obtain $\mathrm{f}_3(\mathscr{P}) \geq 25$.

Hence, a polytope $\mathscr{P}$ with $\mathrm{f}_3(\mathscr{P}) = 24$ has only octahedral facets and, by the argument from Theorem 1, is isometric to the hyperbolic 24-cell. □

5 A dimension bound for ideal right-angled hyperbolic polytopes

Given a combinatorial n-dimensional polytope $\mathscr{P}$, define the average number of ℓ-dimensional faces over k-dimensional ones as

$$\mathrm{f}_k^\ell(\mathscr{P}) = \frac{1}{\mathrm{f}_k(\mathscr{P})} \sum_{P \in \Omega_k(\mathscr{P})} \mathrm{f}_\ell(P).$$

The Nikulin-Khovanskiĭ inequality [7, 8] applied to the polytope $\mathscr{P}$ which is simple at edges, states that

$$\mathrm{f}_k^\ell(\mathscr{P}) < \binom{n-\ell}{n-k} \frac{\binom{\lfloor \frac{n}{2} \rfloor}{\ell} + \binom{\lfloor \frac{n+1}{2} \rfloor}{\ell}}{\binom{\lfloor \frac{n}{2} \rfloor}{k} + \binom{\lfloor \frac{n+1}{2} \rfloor}{k}}, \tag{9}$$

where $\lfloor \circ \rfloor$ means the floor function.

Corollary (of Theorem 2) *There are no ideal right-angled hyperbolic polytopes in $\mathbb{H}^n$, if $n \geq 7$.*

Proof Suppose that $\mathscr{P} \subset \mathbb{H}^n$ is an ideal right-angled hyperbolic polytope, $n \geq 4$. Since we have $f_4^3(\mathscr{P}) \geq 24$ by Theorem 2, then (9) implies $n \leq 5$ for n odd and $n \leq 6$ for n even. □

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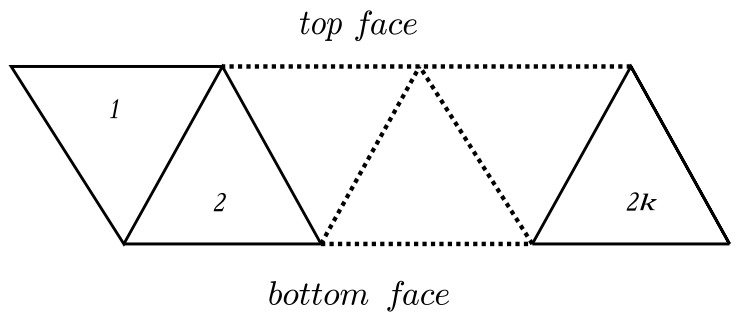

Figure 1: Antiprism $\mathscr{A}_k$, $k \geq 3$.

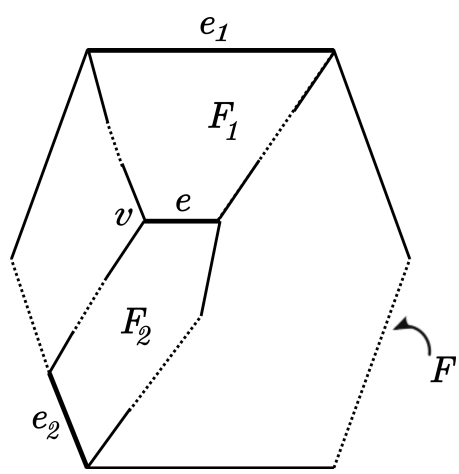

Figure 2: Three-circuit deprecated by Andreev's theorem consists of the faces F, F_1 and F_2

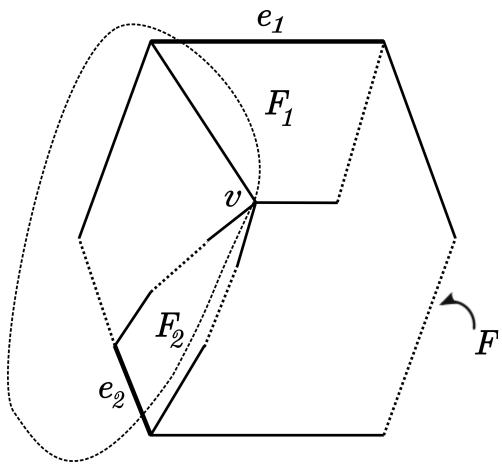

Figure 3: The circuit deprecated by Andreev's theorem is indicated by the dashed line

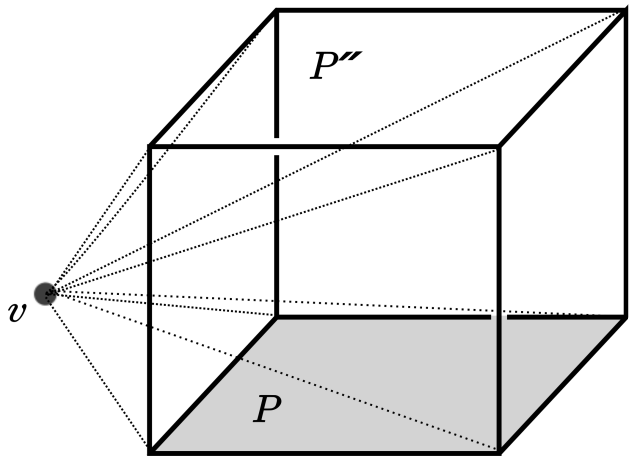

Figure 4: The vertex figure $\mathscr{P}_v$

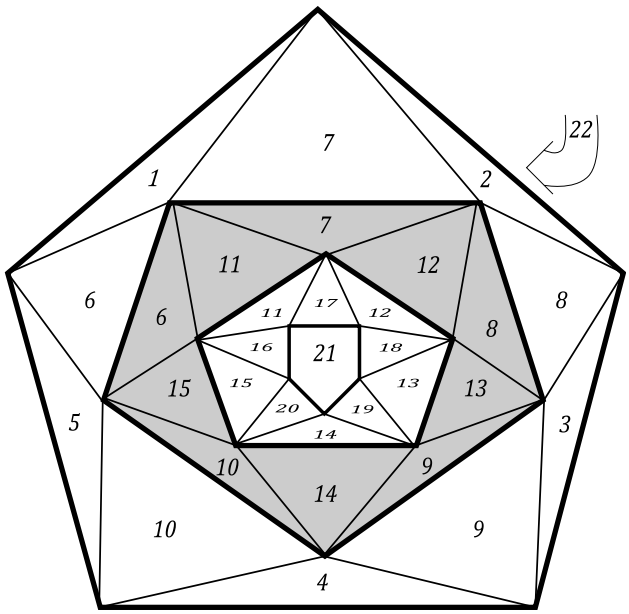

Figure 5: Three facets of $\mathscr{P}$ isometric to $\mathscr{A}_5$ and their neighbours

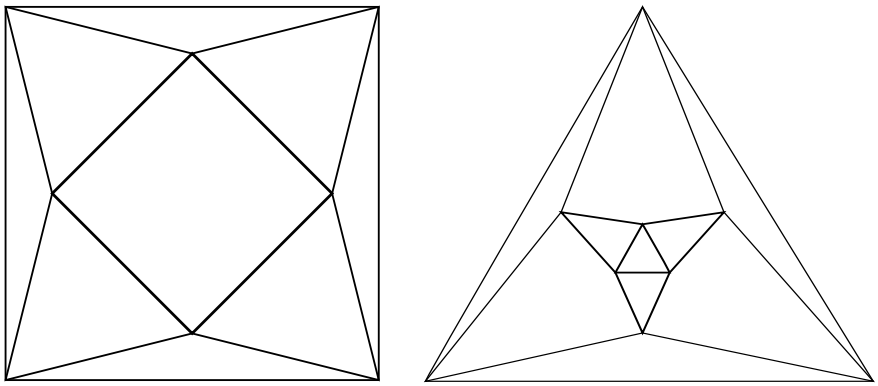

Figure 6: Hyperbolic octahedrites with 8 (left) and 9 (right) vertices

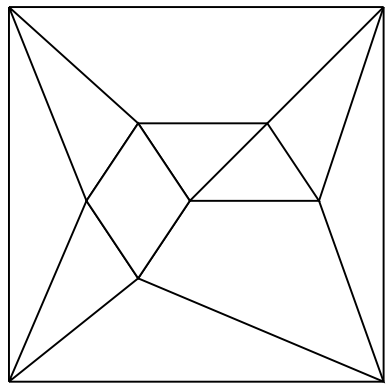

Figure 7: Hyperbolic octahedrite with 10 vertices

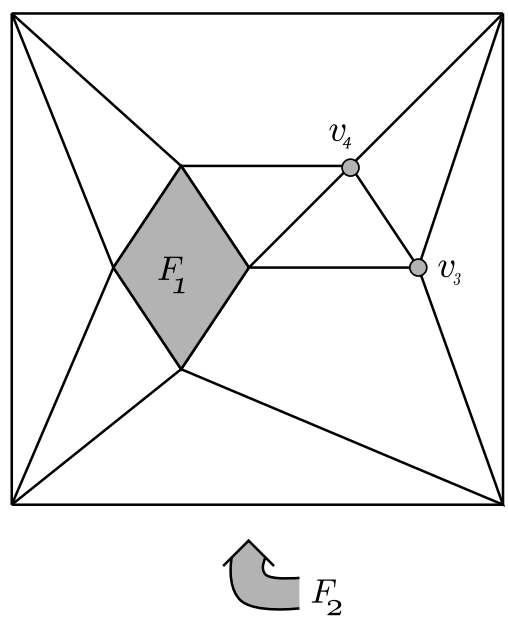

Figure 8: Hyperbolic octahedrite with 10 vertices as a facet of $\mathscr{P}$ and its neighbours

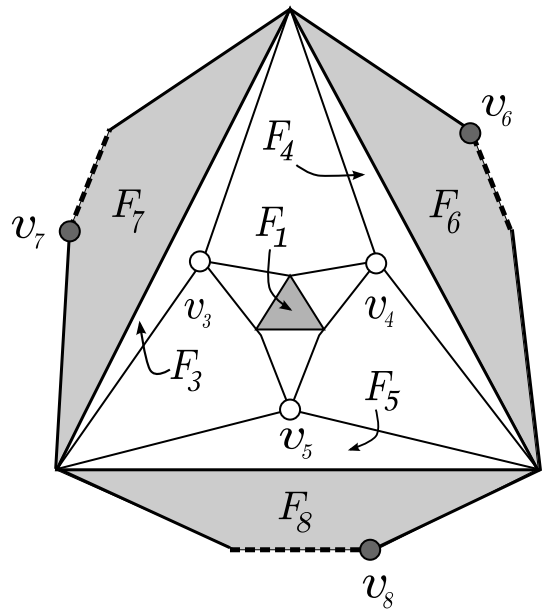

Figure 9: Hyperbolic octahedrite with 9 vertices as a facet of $\mathscr{P}$ and its neighbours (omitted edges are dotted)

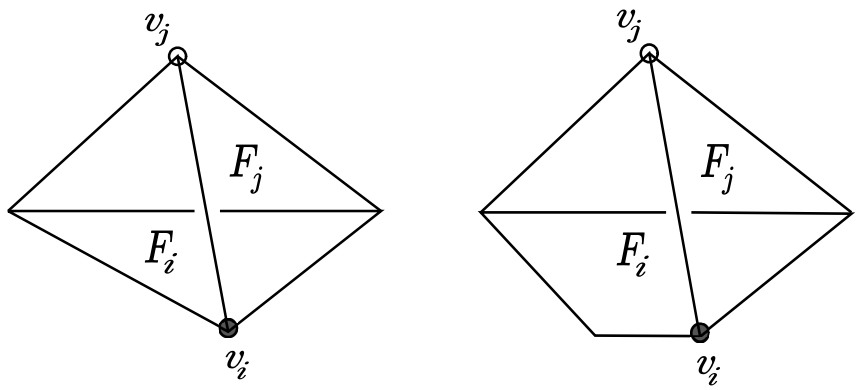

Figure 10: Sub-graphs τ (on the left) and σ (on the right)

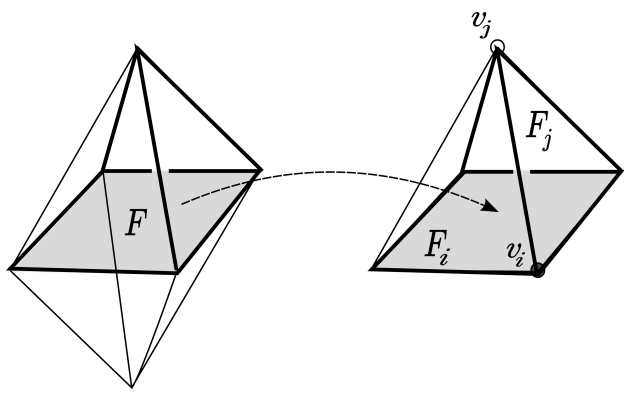

Figure 11: Sub-graph σ in an octahedron (on the left) and in the facet $P(i)$ (on the right)

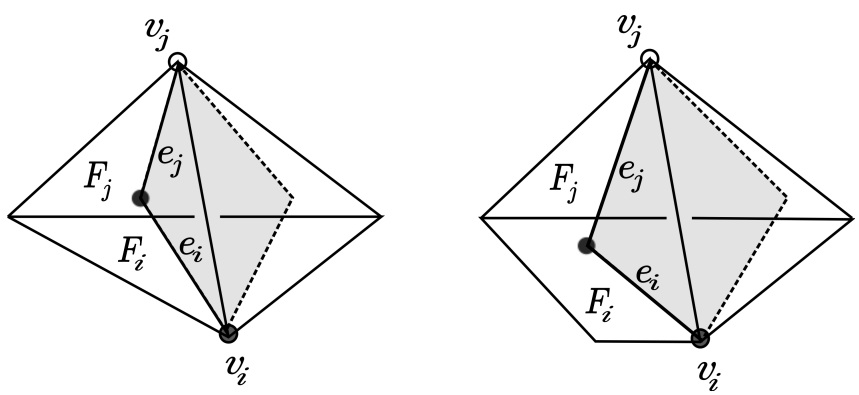

Figure 12: The segment $v_i v_j$ belongs to a quadrilateral face

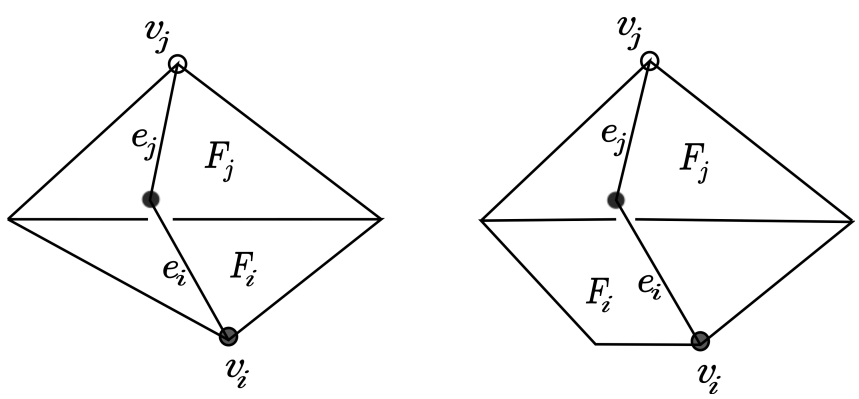

Figure 13: Sub-graphs ν (on the left) and ω (on the right)

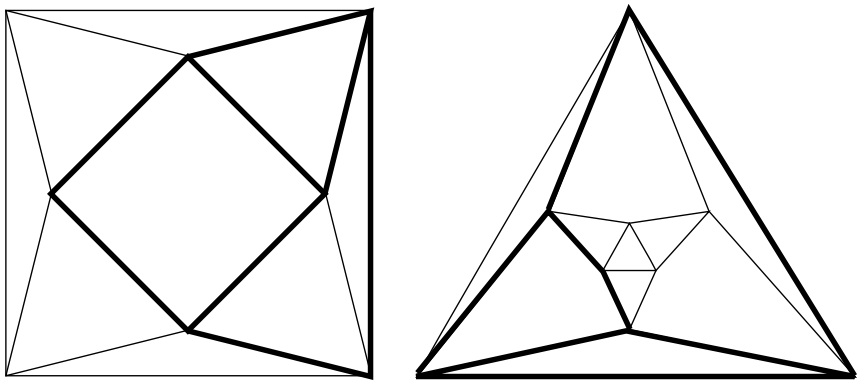

Figure 14: Embeddings of the graph ν into octahedrite facets with 8 (left) and 9 (right) vertices

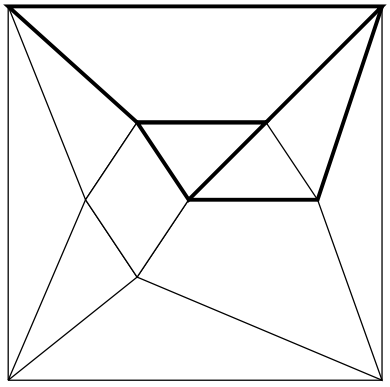

Figure 15: An embedding of the graph ω into the octahedrite facet with 10 vertices

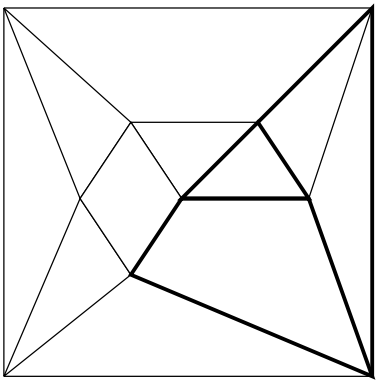

Figure 16: An embedding of the graph ω into the octahedrite facet with 10 vertices

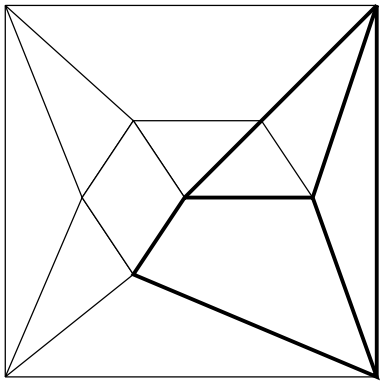

Figure 17: An embedding of the graph ω into the octahedrite facet with 10 vertices

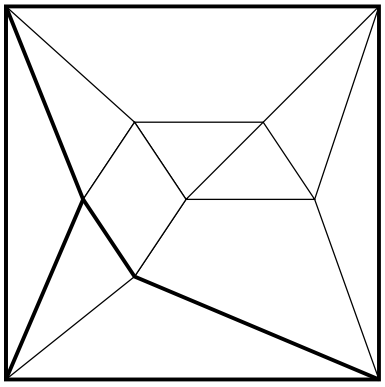

Figure 18: An embedding of the graph ω into the octahedrite facet with 10 vertices

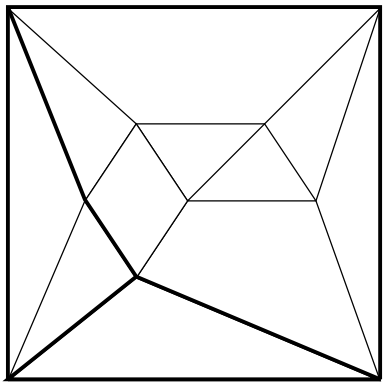

Figure 19: An embedding of the graph ω into the octahedrite facet with 10 vertices

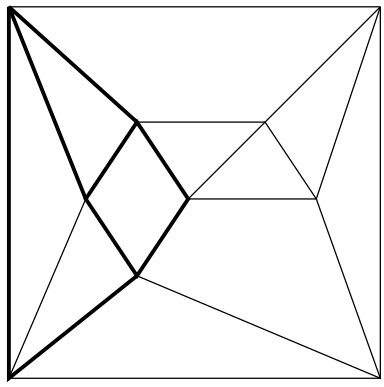

Figure 20: An embedding of the graph ω into the octahedrite facet with 10 vertices

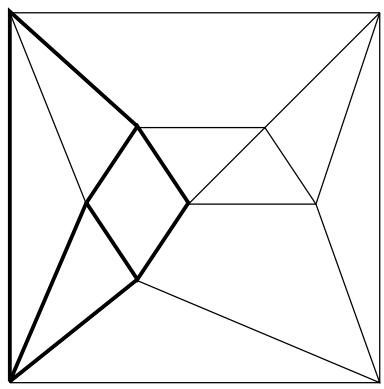

Figure 21: An embedding of the graph ω into the octahedrite facet with 10 vertices

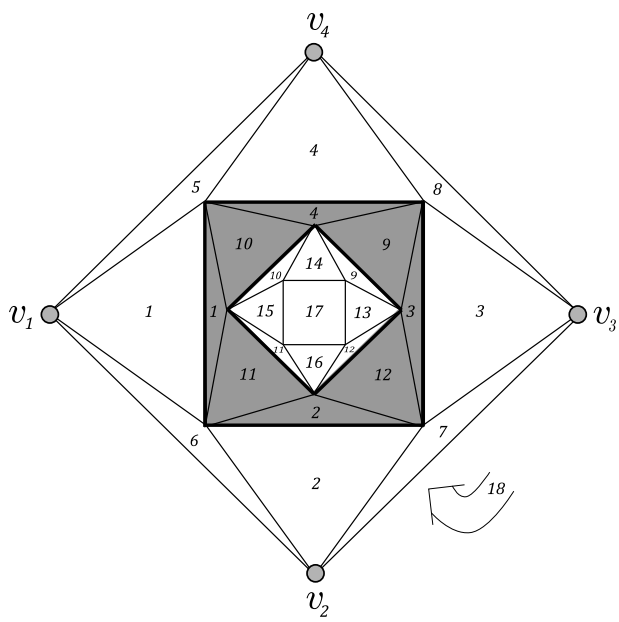

Figure 22: Hyperbolic octahedrite with 8 vertices as a facet of $\mathscr{P}$ and its neighbours