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## About Pillai and Fermat-Catalan conjectures

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### Abstract

More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all ? Mihailescu's solution utilizes computation on machines, we propose here not really a proof as it is entended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation. We generalize our approach to Pillai and Fermat-Catalan equations and discuss the solutions. (Keywords : Diophantine equations, Catalan, Fermat-Catalan, Pillai, Conjectures, Proofs, Algebraic resolution).

### Introduction

Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers stricly greater than 1,  $X > 1$  and  $Y > 1$ , for which with exponents strictly greater than 1,  $p > 1$  and  $q > 1$ ,  $Y^p = X^q + 1$  but for  $(X, Y, p, q) = (2, 3, 2, 3)$ . We can verify that  $3^2 = 2^3 + 1$ . Euler has proved that the equation  $Y^2 = X^3 + 1$  has this only solution. We propose in this study a general solution. The particular cases already solved concern  $p=2$ , solved by Ko Chao in 1965, and  $q=3$  which has been solved in 2002. The case  $q=2$  has been solved by Lebesgue in 1850. We solve here the equation and prove that Pillai and Fermat-Catalan equations are related to this problem.

### The proof

Catalan equation is  $Y^p = X^q + 1$  we pose  $c = \frac{X^p - 1}{Y^{\frac{p}{2}}}$  and  $c' = \frac{7 - X^p}{Y^{\frac{p}{2}}}$  thus  $Y^{\frac{p}{2}} = \frac{6}{c + c'}$  and

$$X^p = cY^{\frac{p}{2}} + 1 = \frac{7c + c'}{c + c'} \text{ and } X^q = Y^p - 1 = \frac{36 - (c + c')^2}{(c + c')^2} \text{ but } X^p > 7 \Rightarrow c' < 0 \text{ and } c > c' \text{ and}$$

$Y^{\frac{p}{2}} \geq 3 \Rightarrow 0 < c + c' \leq 2$ . We will prove now that  $q + 1 = 2p$  and will discuss two cases :  
 $c^2 > 1$  and  $c^2 \leq 1$ .

1)  $c^2 > 1$

We have  $X^q - X^{2p} = \frac{36 - (c + c')^2 - (7c + c')^2}{(c + c')^2} < 36 \frac{1 - c^2}{(c + c')^2} < 0$  we deduce  $q + 1 \leq 2p$

$$2c^2 X^q - X^{2p} = \frac{72c^2 - 2c^2(c + c')^2 - (7c + c')^2}{(c + c')^2} > \frac{64c^2 - (7c + c')^2}{(c + c')^2} > 0$$

$$c^2 X^q - X^{2p} = \frac{36c^2 - c^2(c + c')^2 - (7c + c')^2}{(c + c')^2} < \frac{36c^2 - c^2(c + c')^2 - 36c^2}{(c + c')^2} < 0$$

But  $2c^2 X^{2p-1} \geq 2c^2 X^q > X^{2p} \Rightarrow 2c^2 > X$  let  $2c^2 = \frac{n+2}{n+1}$  and  $X = \frac{m+2}{m+1}$  we want to prove that

$X > 2c^2$ . It is evident that  $n < 0$  and  $m < 0$  ( $2X - 1 - X - 1 = \frac{-m}{m+1} = X - 2 > 0$ ). We have

$$u = \frac{2}{3}$$

$$v = 1$$

$$\Rightarrow \frac{8}{3} n \left( \left( \frac{2}{3} - v \right) n + 4v - \frac{2}{3} \right) < 0$$

$$\Rightarrow 4u(u-v)n^2 + 4un(u-2+4v-2u+2) + (u-2)^2 + 8u(u-1)$$

$$= 4u(u-v)n^2 + 4u(4v-u)n + 9u^2 - 12u + 4 < 0$$

$$\Rightarrow (2un+u-2)^2 - 4u(vn^2 + (4v-2u+2)n + 2-2u) < 0$$

$$\Rightarrow \frac{u}{(X-1)^2} + \frac{2un+u-2}{X-1} + vn^2 + (4v-2u+2)n + 2-2u > 0$$

$$\Rightarrow u(m+1)^2 + n(vn+2u(m+1)-2u+4v+2) + (u-2)(m+1) + 2-2u$$

$$= u(m+1)^2 + n(vn+2um+4v+2) + (u-2)m - u > 0$$

$$\Rightarrow (um+vn)(m-1) + (2n+2)(um+vn) + 2(u-1)m + 2(v+1)n$$

$$= u(m+1)^2 + n(2vn+2um+4v+2) + 2(u-1)m - u > 0$$

$$\Rightarrow (um+vn)(m-1) + 2(n+1)(um+vn) > -2(u-1)m - 2(v+1)n$$

$$\Rightarrow (um+vn)(m-1+2n+2) = (um+vn)(m+2n+1) > -2(u-1)m - 2(v+1)n$$

$$\Rightarrow (um+vn) \left( \frac{m}{2} + n + \frac{1}{2} \right) + (u-1)m + (v+1)n > 0$$

But

$$m+1+n = \frac{2-X}{X-1} + \frac{2-2c^2}{2c^2-1} + 1 = \frac{1}{2c^2-1} + \frac{2-X}{X-1}$$

$$= \frac{2X-1+4c^2-2-2c^2X}{(2c^2-1)(X-1)} = \frac{X(2-c^2)+c^2(4-X)-3}{(2c^2-1)(X-1)} < 0$$

$$\Rightarrow (2n+1)(m+1) < 0 < 2(n+1) \Rightarrow -1-m < -1 + \frac{1}{-2n-1} = \frac{2n+2}{-2n-1}$$

$$\Rightarrow -m < \frac{1}{-2n-1} \Rightarrow m(2n+1) < 1$$

$$\Rightarrow m+2mn-1 < 0 \Rightarrow 2m+2n+2mn < m+2n+1$$

$$\Rightarrow (um+vn)(m+n+mn) + (u-1)m + (v+1)n > 0$$

$$\Rightarrow m(um+un+umn+u-1) + n(vm+vn+vmn+v+1) = (um+vn)(m+n+mn) + (u-1)m + (v+1)n > 0$$

$$\Rightarrow um(m+1)(n+1) - m + vn(m+1)(n+1) + n > 0$$

$$\Rightarrow \frac{u(m+1)^2(n+1) + n + 1 + v(n+1)^2(m+1) - (m+1)}{(m+1)(n+1)} > u+v$$

$$\Rightarrow \frac{u(m+1)^2+1}{m+1} + \frac{v(n+1)^2-1}{n+1} > u+v$$

$$\Rightarrow um+u-u + \frac{1}{m+1} + vn+v-v - \frac{1}{n+1} = um+X-2+vn-2c^2+2 > 0$$

$$\begin{aligned} &\Rightarrow X - 2c^2 > -um - vn > 0 \\ &\Rightarrow X^{q+1} > 2c^2 X^q > X^{2p} \geq X^{q+1} \Rightarrow q+1=2p \\ \text{Thus } &2c^2 > X > 2c^2 \Rightarrow X = 2c^2 \\ &\Rightarrow 2c^2 Y^p = 2(X^p - 1)^2 = X^{q+1} + X \\ &\Rightarrow \frac{2}{X} = X^q + 1 - 2X^{2p-1} + 4X^{p-1} \in N \Rightarrow X = 2 \\ &\Rightarrow 2^{p+1} - 2^{2p-1} = 0 \Rightarrow p = 2 \Rightarrow q = 2p - 1 = 3 \Rightarrow Y = \pm 3 \end{aligned}$$

Now let

$$\begin{aligned} &2) \quad c^{-2} > 1 \\ &2X^q - X^{2p} = \frac{72 - 2(c+c')^2 - (7c+c')^2}{(c+c')^2} > \frac{64 - (7c+c')^2}{(c+c')^2} > 0 \\ &\Rightarrow X^{2p-q-1} < \frac{2}{X} < 1 \Rightarrow 2p \leq q+1 \end{aligned}$$

$$\text{We have } 2c^2 X^q - X^{2p} = \frac{72c^2 - 2c^2(c+c')^2 - (7c+c')^2}{(c+c')^2} > \frac{64c^2 - (7c+c')^2}{(c+c')^2} > 0$$

$$c^2 X^q - X^{2p} = \frac{36c^2 - c^2(c+c')^2 - (7c+c')^2}{(c+c')^2} < \frac{36c^2 - c^2(c+c')^2 - 36c^2}{(c+c')^2} < 0$$

With the same calculus than for  $c^2 > 1$ , we prove that

$$X > 2c^{-2} \Rightarrow c^2 X > 2 \Rightarrow X^{2p} > c^2 X^q > 2X^{q-1}$$

$$\Rightarrow \frac{X^{2p+1}}{2} > X^q \geq X^{2p-1}$$

For Catalan,  $q$  and  $2p$  do not have the same parity thus  $q+1=2p$ , for the other equations there is also  $q=2p$ . But for Catalan

$$\begin{aligned} &0 > (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q \\ &= X^{2p} - 2X^p - X^{2p-1} \geq X^{2p-1} - 2X^p \geq 0 \\ &\Rightarrow (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q = 0 \\ &= X^{2p} - 2X^p - X^{2p-1} = 0 \Rightarrow \frac{2}{X} = X^{p-1} - X^{p-2} \in N \\ &\Rightarrow (X, Y, p, q) = (2, \pm 3, 2, 3) \end{aligned}$$

### Generalization to Pillai and Fermat-Catlan equations

We have for Pillai

$$\begin{aligned} &Y^p = X^q + a \text{ and for Fermat-Catalan } Y^p = X^q + Z^c \text{ we pose } Y^p = X^q + 1_a 1_a \text{ and } c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}} \\ &c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}}, Y^{\frac{p}{2}} = \frac{6_a}{c+c'}, X^p = cY^{\frac{p}{2}} + 1_a = \frac{7_a c + 1_a c'}{c+c'}, X^q = Y^p - 1_a 1_a = 1_a 1_a \frac{36 - (c+c')^2}{(c+c')^2}, \end{aligned}$$

with the same equalities than for Catalan, we discuss two cases

$$\begin{aligned} &1) \quad c^2 > 1 \\ &X^q - X^{2p} = \frac{36_a 1_a - 1_a 1_a (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2} < 36_a 1_a \frac{1-c^2}{(c+c')^2} < 0 \\ &\Rightarrow q+1 \leq 2p \end{aligned}$$

$$2c^2 X^q - X^{2p} = \frac{72_a 1_a c^2 - 2_a 1_a c^2 (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2} > \frac{64_a 1_a c^2 - (7_a c + 1_a c')^2}{(c+c')^2} > 0$$

$$c^2 X^q - X^{2p} = \frac{36_a 1_a c^2 - 1_a 1_a c^2 (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2} < \frac{36_a 1_a c^2 - (7_a c + 1_a c')^2}{(c+c')^2} < 0$$

By the same way  $2c^2 X^{2p-1} \geq 2c^2 X^q > X^{2p} \Rightarrow 2c^2 > X$  and we prove then that  $X = 2c^2 \Rightarrow q+1 = 2p$  for Pillai it is enough to conclude that there are a finite number of solutions because

$$\Rightarrow 2c^2 Y^p = 2(X^p - 1_a)^2 = X^{q+1} + 1_a 1_a X$$

$$\Rightarrow \frac{2_a 1_a}{X} = X^q + 1_a 1_a - 2X^{2p-1} + 4_a X^{p-1} \in N \Rightarrow X \leq 2_a 1_a$$

And for Fermat-Catlan the solutions are  $q+1 = 2p$

$$2) \quad c^{-2} > 1$$

The proof is the same than for Catalan, we prove that  $q+1 = 2p$  or  $q=2p$

$$\Rightarrow (c^2 - 1)Y^p = X^{2p} - 2_a X^p - X^q = 0$$

$$= X^{2p} - 2_a X^p - X^{2p-1} = 0 \Rightarrow \frac{2_a}{X} = X^{p-1} - X^{p-2} \in N$$

And there is a finite number of solutions !

### Discussion of the solutions of Fermat-Catalan equation

We have the following solutions and explanations

$$1 + 2^3 = 3^2$$

When we have an equation like  $Y^p = X^q - 1_a 1_a$  our approach is available but not for

$$Y^p = -X^q + 1_a 1_a$$

and why there are the solution  $q+1=2p$  in one of the following equations and not in the other (for which  $q+2=2p$ ) :  $7^2 = 3^4 - 2^5$  and  $13^2 = -7^3 + 2^9$  but for the same equations, we do not have  $Y^p = X^q + 1_a 1_a$

because  $1_a 1_a$  is negative and can not be a square. But for  $71^2 = 17^3 + 2^7$  our approach is available, we have  $p=2, X=17, q=3, Y=71$  and all the conditions are satisfied, but for the same equation and  $p=2, X=2, q=7, Y=71$  we do not have 4 greater than  $7_a$ , that is why  $q+1$  is not equal to  $2p$ . Also for the other solutions !

### Conclusion

Catalan equation implies two other equations, and an original solution of Catalan equation exists. It seems that many problems of number theory can be solved like this. How ? We showed an example.

### The bibliography

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