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About Pillai and Fermat-Catalan conjectures

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Abstract

(MSC=11D04) More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all? Mihailescu's solution utilizes computation on machines, we propose here not really a proof as it is intended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation. We generalize our approach to Pillai and Fermat-Catalan equations and discuss the solutions.

(Keywords : Diophantine equations, Catalan, Fermat-Catalan, Pillai, Conjectures, Proofs, Algebraic resolution)

Introduction

Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers strictly greater than 1, $X > 1$ and $Y > 1$, for which with exponents strictly greater than 1, $p > 1$ and $q > 1$,

$$Y^p = X^q + 1$$

but for $(X, Y, p, q) = (2, 3, 2, 3)$. We can verify that

$$3^2 = 2^3 + 1$$

Euler has proved that the equation $X^3 + 1 = Y^2$ has this only solution. We propose in this study a general solution. The particular cases already solved concern $p = 2$, solved by Ko Chao in 1965, and $q = 3$ which has been solved in 2002. The case $q = 2$ has been solved by Lebesgue in 1850. We solve here the equation and prove that Pillai and Fermat-Catalan equations are related to this problem.

The resolution

Let Catalan equation

$$Y^p = X^q + 1$$

We pose

$$c = \frac{X^p - 1}{Y^{\frac{p}{2}}}; \quad c' = \frac{7 - X^p}{Y^{\frac{p}{2}}}$$

Thus

$$Y^{\frac{p}{2}} = \frac{6}{c + c'}$$

$$X^p = cY^{\frac{p}{2}} + 1 = \frac{6c}{c + c'} + 1 = \frac{7c + c'}{c + c'}$$

And

$$X^q = Y^p - 1 = \frac{36 - (c + c')^2}{(c + c')^2}$$

We have $X^p > 7$ hence $c' < 0$ and $c > c'$ and $Y^{\frac{p}{2}} \geq 3$ thus $0 < c + c' \leq 2$. We will prove now that $q + 1 = 2p$. We have to discuss two cases $c^2 > 1$ and $c^2 \leq 1$

$c^2 > 1$ We have

$$\begin{aligned} X^q - X^{2p} &= \frac{36 - (c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &< \frac{36 - 36c^2}{(c + c')^2} < 0 \end{aligned}$$

We deduce

$$q + 1 \leq 2p$$

Also

$$\begin{aligned} 2c^2 X^q - X^{2p} &= \frac{72c^2 - 2c^2(c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &> \frac{64c^2 - (7c + c')^2}{(c + c')^2} = \frac{(c - c')(15c + c')}{(c + c')^2} > 0 \end{aligned}$$

And

$$\begin{aligned} c^2 X^q - X^{2p} &= \frac{36c^2 - (c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &< \frac{36c^2 - (7c + c')^2}{(c + c')^2} = \frac{(c' - c)(13c + c')}{(c + c')^2} < 0 \end{aligned}$$

But

$$2c^2 X^{2p-1} \geq 2c^2 X^q > X^{2p} \Rightarrow 2c^2 > X$$

Let

$$4c^2 = \frac{n + 2}{n + 1}, \quad X = \frac{m + 2}{m + 1}$$

We want to prove that $X > 2c^2$. It is evident that $n < 0$ and $m < 0$ ($2X - 1 - X - 1 = X - 2 = \frac{-m}{m+1} > 0$). But let us study the sign of

$$f(n) = 16n^3 + 48n^2 + 23n + 9, \quad n \in] -1, 0[$$

$$f'(n) = 48n^2 + 96n + 23 = 48\left(n + \frac{48 + \sqrt{1200}}{48}\right)\left(n + \frac{48 - \sqrt{1200}}{48}\right)$$

$$\begin{aligned}
f(0) &= 9 > 0, & f(-1) &= 18 > 0, & f\left(\frac{\sqrt{1200} - 48}{48}\right) &= 5.97 > 0 \\
&\Rightarrow f(n) = 16n^3 + 48n^2 + 23n + 9 > 0 \\
&\Rightarrow 16n(n+1)(n+2) > 9(n-1) \\
&\Rightarrow 16(n+1)^4 - 16(n+1)^2 > 9(n^2 - 1) \\
&\Rightarrow (4(n+1)^2 - 3)^2 > 9n^2 - 8(n+1)^2 = n^2 - 16n - 8 \\
&\Rightarrow X - 1 > 1 > \frac{3 + \sqrt{n^2 - 16n - 8}}{4(n+1)^2} \\
&\quad > \frac{-3n + \sqrt{n^2 - 16n - 8}}{4(n+1)^2} \\
&\Rightarrow 16(n+1)^2(X-1)^4 + 8(X-1)^2 + 24n(X-1)^3 \\
&= 8(X-1)^2(2(n+1)^2(X-1)^2 + 1 + 3n(X-1)) > 0 \\
&\quad \Rightarrow 2(n+1)^2(X-1)^2 + 3n(X-1) + 1 > 0 \\
&\quad \Rightarrow (2n^2 + 4n + 2)(X-1)^2 + 3n(X-1) + 1 > 0 \\
&\Rightarrow \frac{1}{(X-1)^2} + \frac{3n}{(X-1)} + 2n^2 + 4n + 2 = (m+1)^2 + 3n(m+1) + 2n^2 + 4n + 2 > 0 \\
&\quad \Rightarrow (m+1)^2 + n(2n + 3m + 7) + 2 > 0 \\
&\Rightarrow (m+n+1)(m-1) + (2n+2)(m+n+1) + 4n + 2 = (m+1)^2 + n(2n + 3m + 7) + 2 > 0 \\
&\quad \Rightarrow (m+n+1)(m-1) + 2(n+1)(m+n+1) > -4n - 2 \\
&\Rightarrow (m+n+1)(m-1 + 2n + 2) = (m+n+1)(m+2n+1) > -4n - 2 \\
&\quad \Rightarrow (m+n+1)\left(\frac{m}{2} + n + \frac{1}{2}\right) + 2n + 1 > 0
\end{aligned}$$

But

$$\begin{aligned}
m+1+n &= \frac{2-X}{X-1} + \frac{2-4c^2}{4c^2-1} + 1 = \frac{1}{4c^2-1} + \frac{2-X}{X-1} \\
&= \frac{2X-1+8c^2-2-4c^2X}{X-1} = \frac{2X(1-c^2)+2c^2(4-X)-3}{X-1} < 0 \\
&\Rightarrow (2n+1)(m+1) < 0 < 2(n+1) \Rightarrow -1-m < -1 + \frac{1}{-2n-1} = \frac{2n+2}{-2n-1} \\
&\quad \Rightarrow -m < \frac{1}{-2n-1} \Rightarrow m(2n+1) < 1 \\
&\Rightarrow m+2mn-1 < 0 \Rightarrow 2m+2n+2mn < m+2n+1 \\
&\Rightarrow (m+n+1)(m+n+mn) + 2n+1 > (m+n+1)\left(\frac{m}{2} + n + \frac{1}{2}\right) + 2n+1 > 0 \\
&\quad \Rightarrow m(n+m+mn) + n(m+n+mn+2) + 2(m+n+mn+1) \\
&> m(n+m+mn) + n(m+n+mn) + (m+n+mn+1) + 2n = (m+n+1)(m+n+mn) + 2n+1 > 0 \\
&\quad \Rightarrow m(m+1)(n+1) - m + n(m+1)(n+1) + n + 2(m+1)(n+1) > 0
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \frac{(m+1)^2(n+1) + n+1 + (n+1)^2(m+1) - (m+1)}{(m+1)(n+1)} > 0 \\
&\Rightarrow \frac{(m+1)^2+1}{m+1} + \frac{(n+1)^2-1}{n+1} > 0 \\
&\Rightarrow m+1 + \frac{1}{m+1} + n+1 - \frac{1}{n+1} = m+1 + X - 1 + n+1 - 4c^2 + 1 > 0 \\
&\Rightarrow 2 + X - 4c^2 > -m - n > 0 \Rightarrow 2X > 2 + X > 4c^2 > 2X \\
&\Rightarrow X^{q+1} > 2c^2 X^q > X^{2p} \geq X^{q+1} \Rightarrow q+1 = 2p
\end{aligned}$$

Thus

$$\begin{aligned}
&2c^2 > X > 2c^2 \Rightarrow X = 2c^2 \\
&\Rightarrow 2c^2 Y^p = 2(X^p - 1)^2 = XY^p = X^{q+1} + X \\
&\Rightarrow \frac{2}{X} = X^q + 1 - 2X^{2p-1} + 4X^{p-1} \in \mathbb{N} \\
&\Rightarrow X = 2 \Rightarrow -2^{2p-1} + 2^{p+1} = 0 \Rightarrow q = 2p - 1 = 3 \\
&\Rightarrow Y = \pm 3
\end{aligned}$$

We want to prove now that $c^2 = 1$ with another matter and solve the equation. We know that for all $k > 1$, $kc^2 > 1$. Let for all $k > 1$

$$kc^2 = 1 + \frac{1}{n+1} > 1$$

If

$$\begin{aligned}
n &> \frac{1-10^u}{10^u} \Rightarrow kc^2 < 10^u + 1 \\
k &= 10^u + 1 \Rightarrow 1 > c^2 > 1 \Rightarrow c^2 = 1
\end{aligned}$$

Else for u enough great

$$n < \frac{1-10^u}{10^u} \Rightarrow -1 < n < \lim_{u \rightarrow \infty} \left(\frac{1-10^u}{10^u} \right) = -1$$

Impossible! It means that $c = 1$ and we have $2c^2 = 2 \geq X$, and

$$\Rightarrow X = 2c^2 = 2$$

And

$$X^{2p} < 2c^2 X^q = X^{q+1} \leq X^{2p} \Rightarrow q+1 = 2p$$

The equation becomes

$$Y^p = 2^{2p-1} + 1$$

And also

$$\begin{aligned}
(c^2 - 1)Y^p X^{-p-1} = 0 &= X^{p-1} - 1 - X^{p-2} = 2^{p-1} - 1 - 2^{p-2} \Rightarrow 2^{p-2} = 1 \\
&\Rightarrow p = 2 \Rightarrow q = 2p - 1 = 3 \Rightarrow Y = \pm 3
\end{aligned}$$

$$c^2 \leq 1$$

$$\begin{aligned} 2X^q - X^{2p} &= \frac{72 - 2(c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &> \frac{64 - 49c^2}{(c + c')^2} > 0 \end{aligned}$$

Thus

$$X^q \geq X^{2p-1}$$

Or

$$q + 1 \geq 2p$$

Also

$$\begin{aligned} 2c^2 X^q - X^{2p} &= \frac{72c^2 - 2c^2(c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &> \frac{64c^2 - (7c + c')^2}{(c + c')^2} = \frac{(c - c')(15c + c')}{(c + c')^2} > 0 \end{aligned}$$

And

$$\begin{aligned} c^2 X^q - X^{2p} &= \frac{36c^2 - (c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &< \frac{36c^2 - (7c + c')^2}{(c + c')^2} = \frac{(c' - c)(13c + c')}{(c + c')^2} < 0 \end{aligned}$$

Let

$$4c^{-2} = \frac{n + 2}{n + 1}, \quad X = \frac{m + 2}{m + 1}$$

It is evident that $n < 0$ and $m < 0$ ($2X - 1 - X - 1 = X - 2 = \frac{-m}{m+1} > 0$). And

$$4c^{-2} = 1 + \frac{1}{n + 1} > 3 \Rightarrow n < -\frac{1}{2}$$

But let us study the sign of

$$f(n) = 16n^3 + 48n^2 + 23n + 9, \quad n \in] -1, 0[$$

$$f'(n) = 48n^2 + 96n + 23 = 48\left(n + \frac{48 + \sqrt{1200}}{48}\right)\left(n + \frac{48 - \sqrt{1200}}{48}\right)$$

$$f(0) = 9 > 0, \quad f(-1) = 18 > 0, \quad f\left(\frac{\sqrt{1200} - 48}{48}\right) = 5.97 > 0$$

$$\Rightarrow f(n) = 16n^3 + 48n^2 + 23n + 9 > 0$$

$$\Rightarrow 16n(n + 1)(n + 2) > 9(n - 1)$$

$$\Rightarrow 16(n + 1)^4 - 16(n + 1)^2 > 9(n^2 - 1)$$

$$\Rightarrow (4(n + 1)^2 - 3)^2 > 9n^2 - 8(n + 1)^2 = n^2 - 16n - 8$$

$$\Rightarrow X - 1 > 1 > \frac{3 + \sqrt{n^2 - 16n - 8}}{4(n + 1)^2}$$

$$> \frac{-3n + \sqrt{n^2 - 16n - 8}}{4(n + 1)^2}$$

$$\begin{aligned}
&\Rightarrow 16(n+1)^2(X-1)^4 + 8(X-1)^2 + 24n(X-1)^3 \\
&= 8(X-1)^2(2(n+1)^2(X-1)^2 + 1 + 3n(X-1)) > 0 \\
&\Rightarrow 2(n+1)^2(X-1)^2 + 3n(X-1) + 1 > 0 \\
&\Rightarrow (2n^2 + 4n + 2)(X-1)^2 + 3n(X-1) + 1 > 0 \\
\Rightarrow \frac{1}{(X-1)^2} + \frac{3n}{(X-1)} + 2n^2 + 4n + 2 &= (m+1)^2 + 3n(m+1) + 2n^2 + 4n + 2 > 0 \\
&\Rightarrow (m+1)^2 + n(2n+3m+7) + 2 > 0 \\
\Rightarrow (m+n+1)(m-1) + (2n+2)(m+n+1) + 4n + 2 &= (m+1)^2 + n(2n+3m+7) + 2 > 0 \\
&\Rightarrow (m+n+1)(m-1) + 2(n+1)(m+n+1) > -4n-2 \\
\Rightarrow (m+n+1)(m-1+2n+2) &= (m+n+1)(m+2n+1) > -4n-2 \\
&\Rightarrow (m+n+1)\left(\frac{m}{2} + n + \frac{1}{2}\right) + 2n + 1 > 0
\end{aligned}$$

But

$$\begin{aligned}
m+1+n &= \frac{2-X}{X-1} + \frac{2-4c^{-2}}{4c^{-2}-1} + 1 = \frac{1}{4c^{-2}-1} + \frac{2-X}{X-1} \\
&= \frac{2X-1+8c^{-2}-2-4c^{-2}X}{X-1} = \frac{2X(1-c^{-2})+2c^{-2}(4-X)-3}{X-1} < 0 \\
\Rightarrow (2n+1)(m+1) < 0 < 2(n+1) &\Rightarrow -1-m < -1 + \frac{1}{-2n-1} = \frac{2n+2}{-2n-1} \\
&\Rightarrow -m < \frac{1}{-2n-1} \Rightarrow m(2n+1) < 1 \\
&\Rightarrow m+2mn-1 < 0 \Rightarrow 2m+2n+2mn < m+2n+1 \\
\Rightarrow (m+n+1)(m+n+mn) + 2n+1 &> (m+n+1)\left(\frac{m}{2} + n + \frac{1}{2}\right) + 2n+1 > 0 \\
&\Rightarrow m(n+m+mn) + n(m+n+mn+2) + 2(m+n+mn+1) \\
> m(n+m+mn) + n(m+n+mn) + (m+n+mn+1) + 2n &= (m+n+1)(m+n+mn) + 2n+1 > 0 \\
&\Rightarrow m(m+1)(n+1) - m + n(m+1)(n+1) + n + 2(m+1)(n+1) > 0 \\
&\Rightarrow \frac{(m+1)^2(n+1) + n+1 + (n+1)^2(m+1) - (m+1)}{(m+1)(n+1)} > 0 \\
&\Rightarrow \frac{(m+1)^2+1}{m+1} + \frac{(n+1)^2-1}{n+1} > 0 \\
\Rightarrow m+1 + \frac{1}{m+1} + n+1 - \frac{1}{n+1} &= m+1 + X-1 + n+1 - 4c^2 + 1 > 0 \\
&\Rightarrow 2+X-4c^{-2} > -m-n > 0 \Rightarrow 2X > 2+X > 4c^{-2} \\
&\Rightarrow X^{2p} > c^2 X^q > 2X^{q-1} \geq 2X^{2p-2} \Rightarrow q+1 = 2p
\end{aligned}$$

But

$$\begin{aligned}
0 > (c^2-1)Y^p &= X^{2p} - 2X^p - X^q = X^{2p} - 2X^p - X^{2p-1} > X^{2p-1} - 2X^p > 0 \\
&\Rightarrow (c^2-1)Y^p = 0 = X^p(X^{p-1}-2) \Rightarrow (X, p, q) = (2, 2, 3) \Rightarrow Y = \pm 3
\end{aligned}$$

We want to prove with another matter that $c^{-2} = 1$, we have for all $k > 1$

$$kc^{-2} = \frac{n+2}{n+1}$$

If

$$n > \frac{1-10^u}{10^u} \Rightarrow kc^{-2} < 10^u + 1$$

$$k = 10^u + 1 \Rightarrow 1 < c^{-2} < 1 \Rightarrow c^2 = 1$$

Else

$$-1 < n < \lim_{u \rightarrow \infty} \left(\frac{1-10^u}{10^u} \right) = -1$$

Impossible! It means that $c^2 = 1$ and we have the Catalan solutions. The only solution of Catalan equation is then

$$(Y, X, p, q) = (\pm 3, 2, 2, 3)$$

Generalization to Pillai and Fermat-Catalan equation

The Pillai equation is $Y^p = X^q + a$ let us pose $a = 1_a 1_a$ thus

$$Y^p = X^q + 1_a 1_a$$

and Fermat-Catalan equation is $Y^p = X^q + Z^a$. Let $Z^a = 1_a 1_a$. Thus

$$Y^p = X^q + 1_a 1_a$$

We pose

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}}, \quad c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}}$$

Thus

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'}$$

$$X^p = cY^{\frac{p}{2}} + 1_a = \frac{6_a c}{c + c'} + 1_a = \frac{7_a c + 1_a c'}{c + c'}$$

And

$$X^q = Y^p - 1_a 1_a = \frac{36_a 1_a - 1_a 1_a (c + c')^2}{(c + c')^2}$$

We have $X^p > 7_a$ hence $c' < 0$ and $c > c'$ and $Y^{\frac{p}{2}} \geq 3_a$ thus $0 < c + c' \leq 2$ Two cases $c^2 > 1$ and $c^2 \leq 1$

$c^2 > 1$ We have

$$\begin{aligned} X^q - X^{2p} &= \frac{36_a 1_a - 1_a 1_a (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2} \\ &< \frac{36_a 1_a - 36_a 1_a c^2}{(c + c')^2} < 0 \end{aligned}$$

We deduce

$$q + 1 \leq 2p$$

Also

$$\begin{aligned} 2c^2 X^q - X^{2p} &= \frac{72_a 1_a c^2 - 2_a 1_a c^2 (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2} \\ &> \frac{64_a 1_a c^2 - (7_a c + 1_a c')^2}{(c + c')^2} = 1_a 1_a \frac{(c - c')(15c + c')}{(c + c')^2} > 0 \end{aligned}$$

And

$$\begin{aligned} c^2 X^q - X^{2p} &= \frac{36_a 1_a c^2 - 1_a 1_a (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2} \\ &< \frac{36_a 1_a c^2 - (7_a c + 1_a c')^2}{(c + c')^2} = 1_a 1_a \frac{(c' - c)(13c + c')}{(c + c')^2} < 0 \end{aligned}$$

But

$$2c^2 X^{2p-1} \geq 2c^2 X^q > X^{2p} \Rightarrow 2c^2 > X$$

Let

$$4c^2 = \frac{n+2}{n+1}, \quad X = \frac{m+2}{m+1}$$

It is evident that $n < 0$ and $m < 0$ ($2X - 1 - X - 1 = X - 2 = \frac{-m}{m+1} > 0$). But let us study the sign of

$$f(n) = 16n^3 + 48n^2 + 23n + 9, \quad n \in]-1, 0[$$

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$$\Rightarrow (4(n+1)^2 - 3)^2 > 9n^2 - 8(n+1)^2 = n^2 - 16n - 8$$

$$\Rightarrow X - 1 > 1 > \frac{3 + \sqrt{n^2 - 16n - 8}}{4(n+1)^2}$$

$$> \frac{-3n + \sqrt{n^2 - 16n - 8}}{4(n+1)^2}$$

$$\Rightarrow 16(n+1)^2(X-1)^4 + 8(X-1)^2 + 24n(X-1)^3$$

$$= 8(X-1)^2(2(n+1)^2(X-1)^2 + 1 + 3n(X-1)) > 0$$

$$\Rightarrow 2(n+1)^2(X-1)^2 + 3n(X-1) + 1 > 0$$

$$\Rightarrow (2n^2 + 4n + 2)(X-1)^2 + 3n(X-1) + 1 > 0$$

$$\Rightarrow \frac{1}{(X-1)^2} + \frac{3n}{(X-1)} + 2n^2 + 4n + 2 = (m+1)^2 + 3n(m+1) + 2n^2 + 4n + 2 > 0$$

$$\begin{aligned}
&\Rightarrow (m+1)^2 + n(2n+3m+7) + 2 > 0 \\
\Rightarrow (m+n+1)(m-1) + (2n+2)(m+n+1) + 4n+2 &= (m+1)^2 + n(2n+3m+7) + 2 > 0 \\
&\Rightarrow (m+n+1)(m-1) + 2(n+1)(m+n+1) > -4n-2 \\
\Rightarrow (m+n+1)(m-1+2n+2) &= (m+n+1)(m+2n+1) > -4n-2 \\
&\Rightarrow (m+n+1)\left(\frac{m}{2} + n + \frac{1}{2}\right) + 2n+1 > 0
\end{aligned}$$

But

$$\begin{aligned}
m+1+n &= \frac{2-X}{X-1} + \frac{2-4c^2}{4c^2-1} + 1 = \frac{1}{4c^2-1} + \frac{2-X}{X-1} \\
&= \frac{2X-1+8c^2-2-4c^2X}{X-1} = \frac{2X(1-c^2)+2c^2(4-X)-3}{X-1} < 0 \\
\Rightarrow (2n+1)(m+1) < 0 < 2(n+1) &\Rightarrow -1-m < -1 + \frac{1}{-2n-1} = \frac{2n+2}{-2n-1} \\
&\Rightarrow -m < \frac{1}{-2n-1} \Rightarrow m(2n+1) < 1 \\
&\Rightarrow m+2mn-1 < 0 \Rightarrow 2m+2n+2mn < m+2n+1 \\
\Rightarrow (m+n+1)(m+n+mn) + 2n+1 &> (m+n+1)\left(\frac{m}{2} + n + \frac{1}{2}\right) + 2n+1 > 0 \\
&\Rightarrow m(n+m+mn) + n(m+n+mn+2) + 2(m+n+mn+1) \\
> m(n+m+mn) + n(m+n+mn) + (m+n+mn+1) + 2n &= (m+n+1)(m+n+mn) + 2n+1 > 0 \\
&\Rightarrow m(m+1)(n+1) - m + n(m+1)(n+1) + n + 2(m+1)(n+1) > 0 \\
&\Rightarrow \frac{(m+1)^2(n+1) + n+1 + (n+1)^2(m+1) - (m+1)}{(m+1)(n+1)} > 0 \\
&\Rightarrow \frac{(m+1)^2+1}{m+1} + \frac{(n+1)^2-1}{n+1} > 0 \\
\Rightarrow m+1 + \frac{1}{m+1} + n+1 - \frac{1}{n+1} &= m+1 + X-1 + n+1 - 4c^2 + 1 > 0 \\
&\Rightarrow 2+X-4c^2 > -m-n > 0 \Rightarrow 2X > 2+X > 4c^2 > 2X \\
&\Rightarrow X^{q+1} > 2c^2 X^q > X^{2p} \geq X^{q+1} \Rightarrow q+1 = 2p
\end{aligned}$$

We want to prove with another matter that $c^2 = 1$, we have for all $k > 1$, $kc^2 > 1$, let

$$kc^2 = \frac{n+2}{n+1}$$

If

$$\begin{aligned}
n &> \frac{1-10^u}{10^u} \Rightarrow kc^{-2} < 10^u + 1 \\
k &= 10^u + 1 \Rightarrow 1 < c^{-2} < 1 \Rightarrow c^2 = 1
\end{aligned}$$

Else

$$-1 < n < \lim_{u \rightarrow \infty} \left(\frac{1-10^u}{10^u}\right) = -1$$

Impossible! It means that $c^2 = 1$

$$\Rightarrow 2c^2 = 2 \geq X \Rightarrow X = 2c^2 = 2$$

And

$$X^{2p} < 2c^2 X^q = X^{q+1} \leq X^{2p} \Rightarrow q + 1 = 2p$$

The equation becomes

$$Y^p = X^{2p-1} + 1_a 1_a$$

And also

$$(2c^2 - X)Y^p = 0 = 2X^{2p} - 4_a X^p + 2_a 1_a - X^{2p} - 1_a 1_a X$$

$$\Rightarrow \frac{2_a 1_a}{X} = -X^{2p-1} + 4_a X^{p-1} + 1_a 1_a \in \mathbb{N}$$

$$\Rightarrow X \leq 2_a 1_a$$

The equation has a finite number of solutions for a fixed $1_a 1_a$, then for Pillai equation and we will discuss below Fermat-Catalan.

$$c^2 \leq 1$$

$$2X^q - X^{2p} = \frac{72_a 1_a - 2_a 1_a (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2}$$

$$> 1_a 1_a \frac{64 - 49c^2}{(c + c')^2} > 0$$

Thus

$$X^q \geq X^{2p-1}$$

Or

$$q + 1 \geq 2p$$

Let

$$4c^{-2} = \frac{n+2}{n+1}, \quad X = \frac{m+2}{m+1}$$

It is evident that $n < 0$ and $m < 0$ ($2X - 1 - X - 1 = X - 2 = \frac{-m}{m+1} > 0$). But let us study the sign of

$$f(n) = 16n^3 + 48n^2 + 23n + 9, \quad n \in] -1, 0[$$

$$f'(n) = 48n^2 + 96n + 23 = 48\left(n + \frac{48 + \sqrt{1200}}{48}\right)\left(n + \frac{48 - \sqrt{1200}}{48}\right)$$

$$f(0) = 9 > 0, \quad f(-1) = 18 > 0, \quad f\left(\frac{\sqrt{1200} - 48}{48}\right) = 5.97 > 0$$

$$\Rightarrow f(n) = 16n^3 + 48n^2 + 23n + 9 > 0$$

$$\Rightarrow 16n(n+1)(n+2) > 9(n-1)$$

$$\Rightarrow 16(n+1)^4 - 16(n+1)^2 > 9(n^2 - 1)$$

$$\Rightarrow (4(n+1)^2 - 3)^2 > 9n^2 - 8(n+1)^2 = n^2 - 16n - 8$$

$$\begin{aligned}
&\Rightarrow X - 1 > 1 > \frac{3 + \sqrt{n^2 - 16n - 8}}{4(n+1)^2} \\
&> \frac{-3n + \sqrt{n^2 - 16n - 8}}{4(n+1)^2} \\
&\Rightarrow 16(n+1)^2(X-1)^4 + 8(X-1)^2 + 24n(X-1)^3 \\
&= 8(X-1)^2(2(n+1)^2(X-1)^2 + 1 + 3n(X-1)) > 0 \\
&\Rightarrow 2(n+1)^2(X-1)^2 + 3n(X-1) + 1 > 0 \\
&\Rightarrow (2n^2 + 4n + 2)(X-1)^2 + 3n(X-1) + 1 > 0 \\
\Rightarrow \frac{1}{(X-1)^2} + \frac{3n}{(X-1)} + 2n^2 + 4n + 2 &= (m+1)^2 + 3n(m+1) + 2n^2 + 4n + 2 > 0 \\
&\Rightarrow (m+1)^2 + n(2n+3m+7) + 2 > 0 \\
\Rightarrow (m+n+1)(m-1) + (2n+2)(m+n+1) + 4n+2 &= (m+1)^2 + n(2n+3m+7) + 2 > 0 \\
&\Rightarrow (m+n+1)(m-1) + 2(n+1)(m+n+1) > -4n-2 \\
\Rightarrow (m+n+1)(m-1+2n+2) = (m+n+1)(m+2n+1) &> -4n-2 \\
&\Rightarrow (m+n+1)\left(\frac{m}{2} + n + \frac{1}{2}\right) + 2n+1 > 0
\end{aligned}$$

But

$$\begin{aligned}
m+1+n &= \frac{2-X}{X-1} + \frac{2-4c^{-2}}{4c^{-2}-1} + 1 = \frac{1}{4c^{-2}-1} + \frac{2-X}{X-1} \\
&= \frac{2X-1+8c^{-2}-2-4c^{-2}X}{X-1} = \frac{2X(1-c^{-2})+2c^{-2}(4-X)-3}{X-1} < 0 \\
\Rightarrow (2n+1)(m+1) < 0 < 2(n+1) &\Rightarrow -1-m < -1 + \frac{1}{-2n-1} = \frac{2n+2}{-2n-1} \\
&\Rightarrow -m < \frac{1}{-2n-1} \Rightarrow m(2n+1) < 1 \\
&\Rightarrow m+2mn-1 < 0 \Rightarrow 2m+2n+2mn < m+2n+1 \\
\Rightarrow (m+n+1)(m+n+mn) + 2n+1 &> (m+n+1)\left(\frac{m}{2} + n + \frac{1}{2}\right) + 2n+1 > 0 \\
&\Rightarrow m(n+m+mn) + n(m+n+mn+2) + 2(m+n+mn+1) \\
> m(n+m+mn) + n(m+n+mn) + (m+n+mn+1) + 2n &= (m+n+1)(m+n+mn) + 2n+1 > 0 \\
&\Rightarrow m(m+1)(n+1) - m + n(m+1)(n+1) + n + 2(m+1)(n+1) > 0 \\
&\Rightarrow \frac{(m+1)^2(n+1) + n+1 + (n+1)^2(m+1) - (m+1)}{(m+1)(n+1)} > 0 \\
&\Rightarrow \frac{(m+1)^2+1}{m+1} + \frac{(n+1)^2-1}{n+1} > 0 \\
\Rightarrow m+1 + \frac{1}{m+1} + n+1 - \frac{1}{n+1} &= m+1 + X-1 + n+1 - 4c^2 + 1 > 0 \\
&\Rightarrow 2+X-4c^{-2} > -m-n > 0 \Rightarrow 2X > 2+X > 4c^{-2} \\
&\Rightarrow X^{q+1} > 2c^{-2}X^q > X^{2p} \geq X^{q+1} \Rightarrow q+1 = 2p
\end{aligned}$$

We want to prove with another matter that $c^{-2} = 1$, we have for all $k > 1$

$$kc^{-2} = \frac{n+2}{n+1} > 1$$

If

$$n > \frac{1-10^u}{10^u} \Rightarrow kc^{-2} < 10^u + 1$$

$$k = 10^u + 1 \Rightarrow 1 < c^{-2} < 1 \Rightarrow c^2 = 1$$

Else

$$-1 < n < \lim_{u \rightarrow \infty} \left(\frac{1-10^u}{10^u} \right) = -1$$

Impossible! It means that $c^2 = 1$ and

$$X^{2p} > c^2 X^q > 2X^{q-1} \geq X^{2p-2}$$

It means that

$$q + 1 = 2p$$

The equation becomes

$$Y^p = X^{2p-1} + 1$$

And

$$0 \geq (c^2 - 1)Y^p = X^{2p} - 2_a X^p - X^q = X^{2p} - 2_a X^p - X^{2p-1} \geq 0$$

$$\Rightarrow (c^2 - 1)Y^p = 0 = X^{2p} - 2_a X^p - X^{2p-1}$$

$$\Rightarrow \frac{2_a 1_a}{X} \in \mathbb{N}$$

The equation has a finite number of solutions for Pillai! For Fermat-Catalan equation which is

$$Y^p = X^q + Z^a$$

We have the following solutions and explanations

$$1 + 2^3 = 3^1$$

When we have an equation like $Y^p = X^q - 1_a 1_a$ our approach is available but not for $Y^p = 1_a 1_a - X^q$, it is why we must pose here $c^2(1_a 1_a - X^q) = 1_a 1_a + 2X^p - 2X^{2p}$ and also $c'^2(1_a 1_a - X^q) = 49_a 1_a + 2X^p - X^{2p}$ and why there are the solution $q + 1 = 2p$ in one of the following equations and not in the other (for which $q + 3 = 2p$): $7^2 = 3^4 - 2^5$ and $13^2 = -7^3 + 2^9$, but for the same equations, we do not have $Y^p = X^q + 1_a 1_a$ because $1_a 1_a$ is negative and can not be a square. But for $71^2 = 17^3 + 2^7$, our approach is available, we have $p = 2$, $X = 17$, $q = 3$, $Y = 71$, $1_a 1_a = 2^7$ and all the conditions are satisfied, we have $q = 2p - 1$, but for the same equation and $p = 2$, $X = 2$, $q = 7$, $Y = 71$, $1_a 1_a = 17^3$ we do not have $X^p = 2^2 = 4$ greater than 7_a , that is why $q + 1$ is not equal to $2p$. Also $3^5 + 11^4 = 122^2$, $p = 2$, $X = 3$, $q = 5$, $Y = 122$, $1_a 1_a = 11^4$ (also $X = 11$, $q = 4$, $1_a 1_a = 3^5$), we do not have $X^2 = 9$ greater than 7_a that is why $q + 1$ is not equal to $2p$. And for $1549304^2 = 15613^3 - 33^8$, we have $q + 1 = 2p = 3$. And for $2213459^2 = 65^7 - 1414^3$ and for $15312283^2 = 113^7 - 9262^3$, we have $q + 1 = 2p$. But

for $21062938^2 = 76271^3 + 17^7$, we have all the conditions for $p = 2$, $X = 76271$, $q = 3$, $Y = 21062938$, $1_a 1_a = 1$, and also for $30042907^2 = 96222^3 + 43^8$, we have all the conditions for $p = 2$, $X = 96222$, $q = 3$, $Y = 30042907$, $1_a 1_a = 43^8$ but not for $p = 2$, $X = 43$, $q = 8$, $1_a 1_a = 96222^3$ as 43^2 is not greater than 7_a . Thus Fermat-Catalan equation is equivalent to $Y^p = X^{2p-1} + Z^{2p-1}$ if the conditions stated in our approach are satisfied, and the equation does not have solutions for $p > 2$. It has only a finite number of solutions in all cases!

Conclusion

Catalan equation implies a second one, and an original solution of Catalan equation exists. We have generalized the approach to both Pillai and Fermat-Catalan equations and proved that they have a finite number of solutions!

Références

- [1] P. MIHAILESCU A class number free criterion for catalan's conjecture, *Journal of Number theory*, **99** (2003).