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About Pillai and Fermat-Catalan conjectures

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Abstract

(MSC=11D04) More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all? Mihailescu's solution utilizes computation on machines, we propose here not really a proof as it is entended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation.

(Keywords : Diophantine equations, Catalan equation ; Algebraic resolution)

Introduction

Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers strictly greater than 1, $X > 1$ and $Y > 1$, for which with exponents strictly greater than 1, $p > 1$ and $q > 1$,

$$Y^p = X^q + 1$$

but for $(X, Y, p, q) = (2, 3, 2, 3)$. We can verify that

$$3^2 = 2^3 + 1$$

Euler has proved that the equation $X^3 + 1 = Y^2$ has this only solution. We propose in this study a general solution. The particular cases already solved concern $p = 2$, solved by Ko Chao in 1965, and $q = 3$ which has been solved in 2002. The case $q = 2$ has been solved by Lebesgue in 1850. We solve here the equation for the general case.

The resolution

Let Catalan equation

$$Y^p = X^q + 1$$

We pose

$$c = \frac{X^p - 1}{Y^{\frac{p}{2}}}; \quad c' = \frac{7 - X^p}{Y^{\frac{p}{2}}}$$

Thus

$$Y^{\frac{p}{2}} = \frac{6}{c+c'}$$

$$X^p = cY^{\frac{p}{2}} + 1 = \frac{6c}{c+c'} + 1 = \frac{7c+c'}{c+c'}$$

And

$$X^q = Y^p - 1 = \frac{36 - (c+c')^2}{(c+c')^2}$$

We have $X^p > 7$ hence $c' < 0$ and $c > c'$ and $Y^{\frac{p}{2}} \geq 3$ thus $0 < c+c' \leq 2$. We have to discuss two cases $c^2 > 1$ and $c^2 \leq 1$

$c^2 > 1$ We have

$$X^q - X^{2p} = \frac{36 - (c+c')^2 - (7c+c')^2}{(c+c')^2}$$

$$< \frac{36 - 36c^2}{(c+c')^2} < 0$$

We deduce

$$q+1 \leq 2p$$

Also

$$2c^2 X^q - X^{2p} = \frac{72c^2 - 2c^2(c+c')^2 - (7c+c')^2}{(c+c')^2}$$

$$> \frac{64c^2 - (7c+c')^2}{(c+c')^2} = \frac{(c-c')(15c+c')}{(c+c')^2} > 0$$

And

$$c^2 X^q - X^{2p} = \frac{36c^2 - (c+c')^2 - (7c+c')^2}{(c+c')^2}$$

$$< \frac{36c^2 - (7c+c')^2}{(c+c')^2} = \frac{(c'-c)(13c+c')}{(c+c')^2} < 0$$

But

$$2c^2 X^{2p-1} \geq 2c^2 X^q > X^{2p} \Rightarrow 2c^2 > X$$

But

$$2c^2 > 1 \Rightarrow 2c^2 = \frac{n+2}{n+1} = X$$

If

$$2n+1 > 0 \Rightarrow 2c^2 = 1 + \frac{1}{n+1} < 3 < X$$

Thus $2n+1 < 0$ and

$$n^2 > \frac{1}{4} \Rightarrow \frac{17-n^2}{n^2} < 67 < X$$

$$\Rightarrow n^2(1+X) - 2n(3-X) - 14 - X > n^2(1+X) + X - 17 > n^2(X+1) - 17 > 0$$

$$\Rightarrow -X(n+1)^2 < n^2 - 6n - 14$$

$$\begin{aligned}
&\Rightarrow X > \frac{n^2 - 6n - 14}{-n^2 - 2n - 1} \\
&\Rightarrow 2nX + 3X - 6n - 12 > 2X + 2 - Xn^2 - n^2 \\
&\quad \Rightarrow \frac{X - n}{X + 1} > \frac{2 - n^2}{2n + 3} \\
&\quad \Rightarrow \frac{n^2 - 2}{(2n + 3)(n + 1)} > \frac{n - X}{(n + 1)(X + 1)} \\
&\quad \Rightarrow \frac{3n + 4}{2n + 3} - \frac{n + 2}{n + 1} > \frac{X + 2}{X + 1} - \frac{n + 2}{n + 1} \\
&\quad \Rightarrow \frac{3n + 4}{2n + 3} = \frac{2c^2 + 2}{2c^2 + 1} > \frac{X + 2}{X + 1} \\
&\Rightarrow X > 2c^2 \Rightarrow 2c^2 > X > 2c^2 \Rightarrow X = 2c^2
\end{aligned}$$

Another proof. There exists $u > 0$ for which

$$\begin{aligned}
n < -\frac{1}{2^u} &\Rightarrow 2c^2 - X = \frac{n + 2}{n + 1} - X < \frac{-\frac{1}{2^u} + 2}{n + 1} - X \\
&= \frac{2^{u+1} - 1}{2^u(n + 1)} - X < \frac{(2^{u+1} - 1)X}{2^{u+1}(n + 1)} - X \\
&= X \left(\frac{-2^{u+1}n - 1}{2^{u+1}(n + 1)} \right)
\end{aligned}$$

$\forall u > 0$ Thus u for which $-\frac{1}{2^{u+1}} > n > \frac{1-2^u}{(u-1)2^{u+1}}$ then

$$\begin{aligned}
u(2c^2 - X) &< \frac{X}{n + 1} \left(\frac{-4n - 1}{4} + \frac{-8n - 1}{8} + \dots + \frac{-2^{u+1}n - 1}{2^{u+1}} \right) \\
&= \frac{X}{n + 1} \left(\frac{-2^{u+1}n - 2^{u-1} - 2^{u+1}n - 2^{u-2} - \dots - 2^{u+1}n - 1}{2^{u+1}} \right) \\
&= \frac{X}{n + 1} \left(\frac{-(u-1)2^{u+1}n - 2^u + 1}{2^{u+1}} \right) < 0
\end{aligned}$$

With $-(u-1)2^{u+1}n < 2^u - 1$. Thus

$$2c^2 \geq X \geq 2c^2 \Rightarrow 2c^2 = X$$

And

$$X^{2p} < 2c^2 X^q = X^{q+1} \leq X^{2p} \Rightarrow q + 1 = 2p$$

The equation becomes

$$Y^p = X^{2p-1} + 1$$

And also

$$\begin{aligned}
(2c^2 - X)Y^p = 0 &= 2X^{2p} - 4X^p + 2 - X^{2p} - X = X^{2p} - 4X^p + 2 - X \\
&\Rightarrow \frac{-2}{X} = X^{2p-1} - 2X^{p-1} - 1 \in \mathbb{N} \\
&\Rightarrow X = 2 \Rightarrow 2^{2p-1} - 2^p = 0 \Rightarrow 2^{p-2} = 1 \\
&\Rightarrow p = 2 \Rightarrow q = 2p - 1 = 3 \Rightarrow Y = \pm 3
\end{aligned}$$

$$c^2 \leq 1$$

$$\begin{aligned} 2X^q - X^{2p} &= \frac{72 - 2(c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &> \frac{64 - 49c^2}{(c + c')^2} > 0 \end{aligned}$$

Thus

$$X^q \geq X^{2p-1}$$

Or

$$q + 1 \geq 2p$$

Also

$$\begin{aligned} 2c^2 X^q - X^{2p} &= \frac{72c^2 - 2c^2(c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &> \frac{64c^2 - (7c + c')^2}{(c + c')^2} = \frac{(c - c')(15c + c')}{(c + c')^2} > 0 \end{aligned}$$

And

$$\begin{aligned} c^2 X^q - X^{2p} &= \frac{36c^2 - (c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &< \frac{36c^2 - (7c + c')^2}{(c + c')^2} = \frac{(c' - c)(13c + c')}{(c + c')^2} < 0 \end{aligned}$$

If

$$2n + 1 > 0 \Rightarrow n > -\frac{1}{2} \Rightarrow 2c^{-2} = 1 + \frac{1}{n+1} < 3 < X$$

If

$$2n + 1 > 0 \Rightarrow 2c^{-2} = 1 + \frac{1}{n+1} < 3 < X$$

Thus $2n + 1 < 0$ and

$$\begin{aligned} n^2 &> \frac{1}{4} \Rightarrow \frac{17 - n^2}{n^2} < 67 < X \\ \Rightarrow n^2(1 + X) - 2n(3 - X) - 14 - X &> n^2(1 + X) + X - 17 > n^2(X + 1) - 17 > 0 \\ \Rightarrow -X(n + 1)^2 &< n^2 - 6n - 14 \\ \Rightarrow X &> \frac{n^2 - 6n - 14}{-n^2 - 2n - 1} \\ \Rightarrow 2nX + 3X - 6n - 12 &> 2X + 2 - Xn^2 - n^2 \\ \Rightarrow \frac{X - n}{X + 1} &> \frac{2 - n^2}{2n + 3} \\ \Rightarrow \frac{n^2 - 2}{(2n + 3)(n + 1)} &> \frac{n - X}{(n + 1)(X + 1)} \\ \Rightarrow \frac{3n + 4}{2n + 3} - \frac{n + 2}{n + 1} &> \frac{X + 2}{X + 1} - \frac{n + 2}{n + 1} \\ \Rightarrow \frac{3n + 4}{2n + 3} = \frac{2c^{-2} + 2}{2c^{-2} + 1} &> \frac{X + 2}{X + 1} \\ \Rightarrow X > 2c^{-2} \Rightarrow 2c^{-2} > X > 2c^{-2} &\Rightarrow X = 2c^{-2} \end{aligned}$$

Another proof. There exists $u > 0$ for which

$$\begin{aligned} n < -\frac{1}{2^u} &\Rightarrow 2c^{-2} - X = \frac{n+2}{n+1} - X < \frac{-\frac{1}{2^u} + 2}{n+1} - X \\ &= \frac{2^{u+1} - 1}{2^u(n+1)} - X < \frac{(2^{u+1} - 1)X}{2^{u+1}(n+1)} - X \\ &= X \left(\frac{-2^{u+1}n - 1}{2^{u+1}(n+1)} \right) \end{aligned}$$

$\forall u > 0$ Thus u for which $-\frac{1}{2^{u+1}} > n > \frac{1-2^u}{(u-1)2^{u+1}}$ then

$$\begin{aligned} u(2c^{-2} - X) &< \frac{X}{n+1} \left(\frac{-4n-1}{4} + \frac{-8n-1}{8} + \dots + \frac{-2^{u+1}n-1}{2^{u+1}} \right) \\ &= \frac{X}{n+1} \left(\frac{-2^{u+1}n - 2^{u-1} - 2^{u+1}n - 2^{u-2} - \dots - 2^{u+1}n - 1}{2^{u+1}} \right) \\ &= \frac{X}{n+1} \left(\frac{-(u-1)2^{u+1}n - 2^u + 1}{2^{u+1}} \right) < 0 \end{aligned}$$

With $-(u-1)2^{u+1}n < 2^u - 1$. And

$$X^{2p} > c^2 X^q > 2X^{q-1} \geq X^{2p-2}$$

It means that

$$q + 1 = 2p$$

The equation becomes

$$Y^p = X^{2p-1} + 1$$

And

$$\begin{aligned} 0 &\geq (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q = X^{2p} - 2X^p - X^{2p-1} \geq 0 \\ &\Rightarrow (c^2 - 1)Y^p = 0 = X^{2p} - 2X^p - X^{2p-1} \\ &\Rightarrow \frac{2}{X} = X^{2p} - 2X^p + 1 - 2X^{2p-2} \in \mathbb{N} \\ &\Rightarrow X = 2 \Rightarrow 2^{2p} - 2^{p+1} - 2^{2p-1} = 0 \Rightarrow 2^{p-1} = 1 + 2^{p-2} \\ &\Rightarrow p = 2 \Rightarrow q = 2p - 1 = 3 \Rightarrow Y = \pm 3 \end{aligned}$$

The only solution of Catalan equation is then

$$(Y, X, p, q) = (\pm 3, 2, 2, 3)$$

Generalization to Pillai and Fermat-Catalan equation

The Pillai equation is $Y^p = X^q + a$ let us pose $a = 1_a 1_a$ thus

$$Y^p = X^q + 1_a 1_a$$

and Fermat-Catalan equation is $Y^p = X^q + Y^a$. Let $Y^a = 1_a 1_a$. Thus

$$Y^p = X^q + 1_a 1_a$$

We pose

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}}, \quad c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}}$$

Thus

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'}$$

$$X^p = cY^{\frac{p}{2}} + 1_a = \frac{6_a c}{c + c'} + 1_a = \frac{7_a c + 1_a c'}{c + c'}$$

And

$$X^q = Y^p - 1_a 1_a = \frac{36_a 1_a - 1_a 1_a (c + c')^2}{(c + c')^2}$$

We have $X^p > 7_a$ hence $c' < 0$ and $c > c'$ and $Y^{\frac{p}{2}} \geq 3_a$ thus $0 < c + c' \leq 2$ Two cases $c^2 > 1$ and $c^2 \leq 1$

$c^2 > 1$ We have

$$X^q - X^{2p} = \frac{36_a 1_a - 1_a 1_a (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2}$$

$$< \frac{36_a 1_a - 36_a 1_a c^2}{(c + c')^2} < 0$$

We deduce

$$q + 1 \leq 2p$$

Also

$$2c^2 X^q - X^{2p} = \frac{72_a 1_a c^2 - 2_a 1_a c^2 (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2}$$

$$> \frac{64_a 1_a c^2 - (7_a c + 1_a c')^2}{(c + c')^2} = 1_a 1_a \frac{(c - c')(15c + c')}{(c + c')^2} > 0$$

And

$$c^2 X^q - X^{2p} = \frac{36_a 1_a c^2 - 1_a 1_a (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2}$$

$$< \frac{36_a 1_a c^2 - (7_a c + 1_a c')^2}{(c + c')^2} = 1_a 1_a \frac{(c' - c)(13c + c')}{(c + c')^2} < 0$$

But

$$2c^2 X^{2p-1} \geq 2c^2 X^q > X^{2p} \Rightarrow 2c^2 > X$$

If

$$2n + 1 > 0 \Rightarrow 2c^2 = 1 + \frac{1}{n+1} < 3 < X$$

Thus $2n + 1 < 0$ and

$$n^2 > \frac{1}{4} \Rightarrow \frac{17 - n^2}{n^2} < 67 < X$$

$$\Rightarrow n^2(1 + X) - 2n(3 - X) - 14 - X > n^2(1 + X) + X - 17 > n^2(X + 1) - 17 > 0$$

$$\Rightarrow -X(n + 1)^2 < n^2 - 6n - 14$$

$$\begin{aligned}
&\Rightarrow X > \frac{n^2 - 6n - 14}{-n^2 - 2n - 1} \\
&\Rightarrow 2nX + 3X - 6n - 12 > 2X + 2 - Xn^2 - n^2 \\
&\quad \Rightarrow \frac{X - n}{X + 1} > \frac{2 - n^2}{2n + 3} \\
&\Rightarrow \frac{n^2 - 2}{(2n + 3)(n + 1)} > \frac{n - X}{(n + 1)(X + 1)} \\
&\Rightarrow \frac{3n + 4}{2n + 3} - \frac{n + 2}{n + 1} > \frac{X + 2}{X + 1} - \frac{n + 2}{n + 1} \\
&\quad \Rightarrow \frac{3n + 4}{2n + 3} = \frac{2c^2 + 2}{2c^2 + 1} > \frac{X + 2}{X + 1} \\
&\Rightarrow X > 2c^2 \Rightarrow 2c^2 > X > 2c^2 \Rightarrow X = 2c^2
\end{aligned}$$

Another proof. If

$$2n + 1 > 0 \Rightarrow n > -\frac{1}{2} \Rightarrow 2c^2 = 1 + \frac{1}{n + 1} < 3 < X$$

There exists $u > 0$ for which

$$\begin{aligned}
n < -\frac{1}{2^u} &\Rightarrow 2c^2 - X = \frac{n + 2}{n + 1} - X < \frac{-\frac{1}{2^u} + 2}{n + 1} - X \\
&= \frac{2^{u+1} - 1}{2^u(n + 1)} - X < \frac{(2^{u+1} - 1)X}{2^{u+1}(n + 1)} - X \\
&= X \left(\frac{-2^{u+1}n - 1}{2^{u+1}(n + 1)} \right)
\end{aligned}$$

$\forall u > 0$ Thus u for which $-\frac{1}{2^{u+1}} > n > \frac{1-2^u}{(u-1)2^{u+1}}$ then

$$\begin{aligned}
u(2c^2 - X) &< \frac{X}{n + 1} \left(\frac{-4n - 1}{4} + \frac{-8n - 1}{8} + \dots + \frac{-2^{u+1}n - 1}{2^{u+1}} \right) \\
&= \frac{X}{n + 1} \left(\frac{-2^{u+1}n - 2^{u-1} - 2^{u+1}n - 2^{u-2} - \dots - 2^{u+1}n - 1}{2^{u+1}} \right) \\
&= \frac{X}{n + 1} \left(\frac{-(u - 1)2^{u+1}n - 2^u + 1}{2^{u+1}} \right) < 0
\end{aligned}$$

With $-(u - 1)2^{u+1}n < 2^u - 1$. Thus

$$2c^2 \geq X \geq 2c^2 \Rightarrow 2c^2 = X$$

And

$$X^{2p} < 2c^2 X^q = X^{q+1} \leq X^{2p} \Rightarrow q + 1 = 2p$$

The equation becomes

$$Y^p = X^{2p-1} + 1_a 1_a$$

And also

$$\begin{aligned}
(2c^2 - X)Y^p = 0 &= 2X^{2p} - 4_a X^p + 2_a 1_a - X^{2p} - 1_a 1_a X \\
&\Rightarrow \frac{-2_a 1_a}{X} = X^{2p-1} - 4_a X^{p-1} - 1_a 1_a \in \mathbb{N} \\
&\Rightarrow X \leq 2_a 1_a
\end{aligned}$$

The equation has a finite number of solutions!

$$c^2 \leq 1$$

$$\begin{aligned} 2X^q - X^{2p} &= \frac{72a1_a - 2a1_a(c+c')^2 - (7ac + 1ac')^2}{(c+c')^2} \\ &> 1a1_a \frac{64 - 49c^2}{(c+c')^2} > 0 \end{aligned}$$

Thus

$$X^q \geq X^{2p-1}$$

Or

$$q + 1 \geq 2p$$

If

$$2n + 1 > 0 \Rightarrow 2c^{-2} = 1 + \frac{1}{n+1} < 3 < X$$

Thus $2n + 1 < 0$ and

$$n^2 > \frac{1}{4} \Rightarrow \frac{17 - n^2}{n^2} < 67 < X$$

$$\Rightarrow n^2(1 + X) - 2n(3 - X) - 14 - X > n^2(1 + X) + X - 17 > n^2(X + 1) - 17 > 0$$

$$\Rightarrow -X(n + 1)^2 < n^2 - 6n - 14$$

$$\Rightarrow X > \frac{n^2 - 6n - 14}{-n^2 - 2n - 1}$$

$$\Rightarrow 2nX + 3X - 6n - 12 > 2X + 2 - Xn^2 - n^2$$

$$\Rightarrow \frac{X - n}{X + 1} > \frac{2 - n^2}{2n + 3}$$

$$\Rightarrow \frac{n^2 - 2}{(2n + 3)(n + 1)} > \frac{n - X}{(n + 1)(X + 1)}$$

$$\Rightarrow \frac{3n + 4}{2n + 3} - \frac{n + 2}{n + 1} > \frac{X + 2}{X + 1} - \frac{n + 2}{n + 1}$$

$$\Rightarrow \frac{3n + 4}{2n + 3} = \frac{2c^{-2} + 2}{2c^{-2} + 1} > \frac{X + 2}{X + 1}$$

$$\Rightarrow X > 2c^{-2} \Rightarrow 2c^{-2} > X > 2c^{-2} \Rightarrow X = 2c^{-2}$$

Another proof. If

$$2n + 1 > 0 \Rightarrow n > -\frac{1}{2} \Rightarrow 2c^{-2} = 1 + \frac{1}{n+1} < 3 < X$$

There exists $u > 0$ for which

$$n < -\frac{1}{2^u} \Rightarrow 2c^{-2} - X = \frac{n + 2}{n + 1} - X < \frac{-\frac{1}{2^u} + 2}{n + 1} - X$$

$$= \frac{2^{u+1} - 1}{2^u(n + 1)} - X < \frac{(2^{u+1} - 1)X}{2^{u+1}(n + 1)} - X$$

$$= X \left(\frac{-2^{u+1}n - 1}{2^{u+1}(n + 1)} \right)$$

$\forall u > 0$ Thus u for which $-\frac{1}{2^{u+1}} > n > \frac{1-2^u}{(u-1)2^{u+1}}$ then

$$\begin{aligned} u(2c^{-2} - X) &< \frac{X}{n+1} \left(\frac{-4n-1}{4} + \frac{-8n-1}{8} + \dots + \frac{-2^{u+1}n-1}{2^{u+1}} \right) \\ &= \frac{X}{n+1} \left(\frac{-2^{u+1}n - 2^{u-1} - 2^{u+1}n - 2^{u-2} - \dots - 2^{u+1}n - 1}{2^{u+1}} \right) \\ &= \frac{X}{n+1} \left(\frac{-(u-1)2^{u+1}n - 2^u + 1}{2^{u+1}} \right) < 0 \end{aligned}$$

With $-(u-1)2^{u+1}n < 2^u - 1$. And

$$X^{2p} > c^2 X^q > 2X^{q-1} \geq X^{2p-2}$$

It means that

$$q + 1 = 2p$$

The equation becomes

$$Y^p = X^{2p-1} + 1$$

And

$$\begin{aligned} 0 &\geq (c^2 - 1)Y^p = X^{2p} - 2_a X^p - X^q = X^{2p} - 2_a X^p - X^{2p-1} \geq 0 \\ &\Rightarrow (c^2 - 1)Y^p = 0 = X^{2p} - 2_a X^p - X^{2p-1} \\ &\Rightarrow \frac{2_a 1_a}{X} \in \mathbb{N} \end{aligned}$$

The equation has a finite number of solutions! Both Pillai and Fermat-Catalan equations have a finite number of solutions!

Conclusion

Catalan equation implies a second one, and an original solution of Catalan equation exists. We have generalized the approach to both Pillai and Fermat-Catalan equations and proved that they have a finite number of solutions!

Références

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