Carleman estimates for semi-discrete parabolic operators with a discontinuous diffusion coefficient and application to controllability

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Thuy N.T. Nguyen*

Abstract

In the discrete setting of one-dimensional finite-differences we prove a Carleman estimate for a semi-discretization of the parabolic operator $\partial_t - \partial_x(c\partial_x)$ where the diffusion coefficient c has a jump. As a consequence of this Carleman estimate, we deduce consistent null-controllability results for classes of semi-linear parabolic equations.

1 Introduction and settings

Let Ω, ω be connected non-empty open interval of $\mathbb{R}$ with $\omega \Subset \Omega$. We consider the following parabolic problem in $(0,T)\times\Omega$, with $T>0$,

$$\partial_t y - \partial_x(c\partial_x y) = \mathbf{1}_\omega v \ \text{ in } (0,T)\times\Omega,\ y|_{\partial\Omega}=0, \text{ and } y|_{t=0}=y_0, \qquad (1.1)$$

where the diffusion coefficient $c = c(x) > 0$.

System (1.1) is said to be null controllable from $y_0 \in L^2(\Omega)$ in time T if there exists $v \in L^2((0,T)\times\Omega)$, such that $y(T)=0$.

In the continuous framework, we refer to [FI96] and [LR95] who proved such a controllability result by means of a global/local Carleman observability estimates in the case the diffusion coefficient c is smooth. The authors of [BDL07] produced this controllability result in the case of a discontinuous coefficient in the one-dimensional case later extended to arbitrary dimension by [LR10]. Additionally, a result of controllability in the case of a coefficient with bounded variation (BV) was shown in [FCZ02, L07].

The authors of [LZ98] show that uniform controllability holds in the one-dimensional case with constant diffusion coefficient c and for a constant step size finite-difference scheme. Here, "uniform" is meant with respect to the discretization parameter h. The situation becomes more complex in

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higher dimension. In fact, a counter-example to null-controllability due to O. Kavian is provided in [Zua06] for a finite-difference discretization scheme for the heat equation in a square.

In recent works, by means of discrete Carleman estimate, the authors of [BHL10a], [BHL10b] and [BL12] obtained weak observability inequalities in the case of a smooth diffusion coefficient $c(x)$. Such observability estimates are charaterized by an additional term that vanishes exponentially fast. Morever, also with a constant diffusion coffiencient c, under the assumption that the discretized semigroup is uniformly analytic and that the degree of unboundedness of control operator is lower than $1/2$, a uniform observability property of semi-discrete approximations for System (1.1) is achieved in L^2 [LT06]. Besides that, such a result continues to hold even with the condition that the degree of unboundedness of control operator is greater than $1/2$ [N12].

In the case of a non-smooth coefficient, our aim is to investigate the uniform controllability of System (1.1) after discretization. It is well known that controllability and observability are dual aspects of the same problem. We shall therefore focus on uniform observability which is shown to hold when the observability constant of the finite dimensional approximation systems does not depend on the step-size h.

In the present paper we prove a Carleman estimate for system (1.1) in the case of:

- the heat equation in one space dimension;
- a piecewise $\mathbf{C}^1$ coefficient c with jumps at a finite number of points in Ω;
- a finite-difference discretization in space.

The main idea of the proof is combination of the derivation of a discrete Carleman estimate as in [BHL10a, BL12] and tecniques of [BDL07] for operators with discontinuous coefficients in the one-dimensional case. A similar question in n-dimensional case, $n \geq 2$, remains open, to our knowledge.

When considering a discontinuous coefficient c the parabolic problem (1.1) can be understood as a transmission problem. For instance, assume that c exhibits a jump at $a \in \Omega$. Then we write

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) = \mathbf{1}_\omega v & \text{in } (0,T) \times \big((0,a) \cup (a,1)\big), \\ c\partial_x y|_{a^+} = c\partial_x y|_{a^-}, \qquad y|_{a^+} = y|_{a^-}, \\ y|_{\partial\Omega} = 0, \qquad \text{and} \qquad y|_{t=0} = y_0. \end{cases}$$

The second line is thus a transmission condition implying the continuity of the solution and of the flux at $x = a$.

When one gives a finite-difference version of this transmission problem, a similar condition can be given for the continuity of the solution. Yet, for

the flux, it is only achieved up to a consistent term. In what follows, in the finite-difference approximation, we shall in fact write

$$\begin{cases} y(a^-) = y(a^+) = y_{n+1}, \\ (c_d Dy)_{n+\frac{3}{2}} - (c_d Dy)_{n+\frac{1}{2}} = h\big(\bar{D}(c_d Dy)\big)_{n+1}, \end{cases}$$

(the discrete notation will be given below). Note that the flux condition converges to the continuous one if $h \to 0$, h being the discretization parameter. This difference between the continuous and the discrete case will be the source of several technical points.

An important point in the proof of Carleman estimate is the construction of a suitable weight function ψ whose gradient does not vanish in the complement of the observation region. The weight function is chosen smooth in the case of a smooth diffusion coefficient $c(x)$. In general, the technique to construct such a function is based on Morse functions (see some details in [FI96]). In one space dimension, this construction is in fact straightforward. In the case of a discontinuous diffusion coefficient, authors of [BDL07] introduced an *ad hoc* transmission condition on the weight function: its derivative exhibits jumps at the singular points of the coefficient. In this paper, we construct a weight function based on these techniques in the one-dimentional discrete case.

From the semi-discrete Carleman we obtain, we give an observability inequality for semi-discrete parabolic problems with potential. As compared to the result in continuous case [BDL07] the observability estimate we state here is weak because of an additional term that describes the obstruction to the null-controllability. This term is exponentially small in agreement with the results obtained in [BHL10a, BHL10b] in the smooth coefficient case. A precise statement is given in Section 6.

Finally, the observability inequality allows one to obtain controllability results for semi-discrete parabolic with semi-linear terms. In continuous case, this was achieved in [BDL07]. Taking advantage of one-dimensional situation, the results we state are uniform with respect to the discretization parameter h (see Section 6).

1.1 Discrete settings

We restrict our analysis to one dimension in space. Let us consider the operator formally defined by $\mathcal{A} = -\partial_x(c\partial_x)$ on the open interval $\Omega = (0, L) \subset \mathbb{R}$. We let $a' \in \Omega$ and set $\Omega_1 := (0, a')$ and $\Omega_2 := (a', L)$. The diffusion coefficient c is assumed to be piecewise regular such that

$$0 < c_{min} \leq c \leq c_{max} \tag{1.2}$$

$$c = \begin{cases} c_0 \text{ in } \Omega_1, \\ c_1 \text{ in } \Omega_2, \end{cases}$$

with $c_i \in C^1(\overline{\Omega_i}), i = 1, 2$.

The domain of $\mathcal{A}$ is $D(\mathcal{A}) = \left\{u \in H_0^1(\Omega); \quad c\partial_x u \in H^1(\Omega)\right\}$.

Let $T > 0$. We shall use the following notation $\Omega' = \Omega_1 \cup \Omega_2$, $Q = (0,T) \times \Omega$, $Q' = (0,T) \times \Omega'$, $Q_i = (0,T) \times \Omega_i$, $i = 1, 2$, $\Gamma = \{0, L\}$, and $\Sigma = (0,T) \times \Gamma$. We also set $S = \{a'\}$. We consider the following parabolic problem

$$\begin{cases} \partial_t y + \mathcal{A}y = f \text{ in } Q', \\ y(0,x) = y_0(x) \text{ in } \Omega \quad . \end{cases}$$

(real valued coefficient and solution), for $y_0 \in L^2(\Omega)$ and $f \in L^2(Q)$, with the following transmission conditions at a'

$$(TC) \begin{cases} y(a'^-) = y(a'^+), \\ c(a'^-)\partial_x y(a'^-) = c(a'^+)\partial_x y(a'^+). \end{cases}$$

Now, we introduce finite-difference approximations of the operator $\mathcal{A}$. Let $0 = x'_0 < x'_1 < \ldots < \underline{x'_{n+1} = a'} < \ldots < x'_{n+m+1} < x'_{n+m+2} = L$. We refer to this discretization as to the primal mesh $\mathfrak{M} := (x'_i)_{1 \le i \le n+m+1}$. We set $|\mathfrak{M}| := n + m + 1$. We set $h'_{i+\frac{1}{2}} = x'_{i+1} - x'_i$ and $x'_{i+\frac{1}{2}} = (x'_{i+1} + x'_i)/2$, $i = 0, \ldots, n+m+1$, and $h' = \max_{0 \le i \le n+m+1} h'_{i+\frac{1}{2}}$. We call $\overline{\mathfrak{M}} := (x'_{i+\frac{1}{2}})_{0 \le i \le n+m+1}$ the dual mesh and set $h'_i = x'_{i+\frac{1}{2}} - x'_{i-\frac{1}{2}} = (h'_{i+\frac{1}{2}} + h'_{i-\frac{1}{2}})/2$, $i = 0, \ldots, n+m+1$.

In this paper, we shall address to some families of non uniform meshes, that will be precisely defined in Section 1.2.

We introduce the following notation

$$[\rho_1 \star]_a = \rho_1(a^+) - \rho_1(a^-), \tag{1.3}$$

$$[\star \rho_2]_a = \rho_2(n + \frac{3}{2}) - \rho_2(n + \frac{1}{2}), \tag{1.4}$$

$$[\rho_1 \star \rho_2]_a = \rho_1(a^+)\rho_2(n + \frac{3}{2}) - \rho_1(a^-)\rho_2(n + \frac{1}{2}). \tag{1.5}$$

We follow some notation of [BHL10a] for discrete functions in the one-dimensional case. We denote by $\mathbb{C}^{\mathfrak{M}}$ and $\mathbb{C}^{\overline{\mathfrak{M}}}$ the sets of discrete functions defined on $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ respectively. If $u \in \mathbb{C}^{\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}}$), we denote by u_i (resp. $u_{i+\frac{1}{2}}$) its value corresponding to x'_i (resp. $x'_{i+\frac{1}{2}}$). For $u \in \mathbb{C}^{\mathfrak{M}}$ we define

$$u^{\mathfrak{M}} = \sum_{i=1}^{n+m+1} 1|_{[x'_{i-\frac{1}{2}}, x'_{i+\frac{1}{2}}]} u_i \in L^\infty(\Omega).$$

And for $u \in \mathbb{C}^{\mathfrak{M}}$ we define $\int_\Omega u := \int_\Omega u^{\mathfrak{M}}(x)dx = \sum_{i=1}^{n+m+1} h'_i u_i$.

For $u \in \mathbb{C}^{\overline{\mathfrak{M}}}$ we define

$$u^{\overline{\mathfrak{M}}} = \sum_{i=0}^{n+m+1} 1|_{[x'_i, x'_{i+1}]} u_{i+\frac{1}{2}}.$$

As above, for $u \in \mathbb{C}^{\overline{\mathfrak{M}}}$, we define $\int_\Omega u := \int_\Omega u^{\overline{\mathfrak{M}}}(x)dx = \sum_{i=0}^{n+m+1} h'_{i+\frac{1}{2}} u_{i+\frac{1}{2}}$. In particular we define the following L^2 inner product on $\mathbb{C}^{\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}}$)

$$(u,v)_{L^2} = \int_\Omega u^{\mathfrak{M}}(x) v^{\mathfrak{M}}(x) dx, \text{ resp. } (u,v)_{L^2} = \int_\Omega u^{\overline{\mathfrak{M}}}(x) v^{\overline{\mathfrak{M}}}(x) dx.$$

For some $u \in \mathbb{C}^{\mathfrak{M}}$, we shall need to associate boundary conditions $u^{\partial\mathfrak{M}} = \{u_0, u_{n+m+2}\}$. The set of such extended discrete functions is denoted by $\mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$. Homogeneous Dirichlet boundary conditions then consist in the choice $u_0 = u_{n+m+2} = 0$, in short $u^{\partial\mathfrak{M}} = 0$. We can define translation operators $\tau^\pm$, a difference operator D and an averaging operator as the map $\mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}} \to \mathbb{C}^{\overline{\mathfrak{M}}}$ given by

$$\begin{aligned}
&(\tau^+ u)_{i+\frac{1}{2}} := u_{i+1}, \qquad (\tau^- u)_{i+\frac{1}{2}} := u_i, \qquad i = 0, \dots n+m+1, \\
&(Du)_{i+\frac{1}{2}} := \frac{1}{h'_{i+\frac{1}{2}}} (\tau^+ u - \tau^- u)_{i+\frac{1}{2}}, \qquad \tilde{u} := \frac{1}{2}(\tau^+ + \tau^-)u.
\end{aligned}$$

We also define, on the dual mesh, translation operators $\tau^\pm$, a difference operator $\bar{D}$ and an averaging operator as the map $\mathbb{C}^{\overline{\mathfrak{M}}} \to \mathbb{C}^{\mathfrak{M}}$ given by

$$\begin{aligned}
&(\tau^+ u)_i := u_{i+\frac{1}{2}}, \qquad (\tau^- u)_i := u_{i-\frac{1}{2}}, \qquad i = 1, \dots n+m+1, \\
&(\bar{D}u)_i := \frac{1}{h'_i} (\tau^+ u - \tau^- u)_i, \qquad \bar{u} := \frac{1}{2}(\tau^+ + \tau^-)u.
\end{aligned}$$

1.2 Families of non-uniform meshes

In this paper, we address non-uniform meshes that are obtained as the smooth image of an uniform grid.

More precisely, let $\Omega_0 =]0,1[$ and let $\vartheta : \mathbb{R} \to \mathbb{R}$ be an increasing map such that

$$\vartheta(\Omega_0) = \Omega, \quad \vartheta \in \mathcal{C}^\infty, \quad \inf \vartheta' > 0 \text{ and } \quad \vartheta(a) = a' \tag{1.6}$$

with a to be kept fixed in what follows and chosen such that $a \in (0,1) \cap \mathbb{Q}$, i.e $a = \frac{p}{q}$ with $p, q \in \mathbb{N}^*$. Clearly, we have $q > p$. We impose the function ϑ to be affine on $[a-\delta, a+\delta]$ $\vartheta|_{[a-\delta,a+\delta]}$ (for some $\delta > 0$).

Given $r \in \mathbb{N}^*$ and set $m = (q-p)r$ and $n = pr$. The parameter r is used to refine the mesh when increased. Set $a = x_{n+1} = x_{pr+1}$. The interval $\overline{\Omega_{01}} = [0, a]$ is then discretized with $n = pr$ interior grid points (excluding 0

and a). The interval $\overline{\Omega_{02}} = [a, 1]$ is discretized with $m = (q-p)r$ exterior grid points (excluding a and 1). Let $\mathfrak{M}_0 = (ih)_{1\le i\le n+m+1}$ with $h = \frac{1}{n+m+2}$ be uniform mesh of Ω_0 and $\overline{\mathfrak{M}_0}$ be the associated dual mesh. We define a non-uniform mesh $\mathfrak{M}$ of Ω as image of $\mathfrak{M}_0$ by the map ϑ, settings

$$\begin{aligned} &x'_i = \vartheta(ih), \quad \forall i \in \{0, ..., n\} \cup \{n+2, ..., n+m+2\} \\ &x'_{n+1} := a' = \vartheta(a). \end{aligned} \tag{1.7}$$

The dual mesh $\overline{\mathfrak{M}}$ and the general notation are then those of the previous section.

1.3 Main results

With the notation we have introduced, a consistent finite-difference approximation of $\mathcal{A}u$ with homogeneous boundary condition is

$$\mathcal{A}^{\mathfrak{M}} u = -\bar{D}(c_d D u)$$

for $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ satisfying $u|_{\partial\Omega} = u^{\partial\mathfrak{M}} = 0$. We have

$$(\mathcal{A}^{\mathfrak{M}} u)_i = -\frac{c_d(x_{i+\frac{1}{2}})\frac{u_{i+1}-u_i}{h_{i+\frac{1}{2}}} - c_d(x_{i-\frac{1}{2}})\frac{u_i-u_{i-1}}{h_{i-\frac{1}{2}}}}{h_i}, \quad i = 1, .., n+m+1.$$

For a suitable weight function φ (to be defined below), the announced semi-discrete Carleman estimate for the operator $\mathcal{P}^{\mathfrak{M}} = -\partial_t + \mathcal{A}^{\mathfrak{M}}$ with a discontinuous diffusion coefficient c, for the non-uniform meshes we consider, is of the form

$$\begin{aligned} \tau^{-1} \left\| \theta^{-\frac{1}{2}} e^{\tau\theta\varphi} \partial_t u \right\|^2_{L^2(Q)} + \tau \left\| \theta^{\frac{1}{2}} e^{\tau\theta\varphi} D u \right\|^2_{L^2(Q)} + \tau^3 \left\| \theta^{\frac{3}{2}} e^{\tau\theta\varphi} u \right\|^2_{L^2(Q)} \\ \le C_{\lambda,\mathfrak{K}} \left(\left\| e^{\tau\theta\varphi} P^{\mathfrak{M}} u \right\|^2_{L^2(Q)} + \tau^3 \left\| \theta^{\frac{3}{2}} e^{\tau\theta\varphi} u \right\|^2_{L^2((0,T)\times\omega)} \right. \\ \left. + h^{-2} \left| e^{\tau\theta\varphi} u|_{t=0} \right|^2_{L^2(\Omega)} + h^{-2} \left| e^{\tau\theta\varphi} u|_{t=T} \right|^2_{L^2(\Omega)} \right), \end{aligned} \tag{1.8}$$

for properly chosen functions $\theta = \theta(t)$ and $\varphi = \varphi(x)$, for all $\tau \ge \tau_0(T + T^2)$, $0 < h \le h_0$ and $\tau h(\alpha T)^{-1} \le \epsilon_0$, $0 < \alpha < T$ and for all $u \in C^\infty(0, T; \mathbb{C}^{\mathfrak{M}})$ satisfying the discrete transmission conditions, where τ_0, h_0, ϵ_0 only depend on the data. We refer to Theorem 4.1 (uniform mesh) and Theorem 5.6 (non uniform mesh) below for a precise result. The proof of this estimate will be first carried out for piecewise uniform meshes, and then adapted to the case of the non-uniform meshes we introduced in Section 1.2.

From the semi-discrete Carleman estimate we obtain allows we deduce following weak observability estimate

$$|q(0)|_{L^2(\Omega)} \leq C_{obs} \|q\|^2_{L^2((0,T)\times\omega)} + e^{-\frac{C}{h}} |q(T)|^2_{L^2(\Omega)},$$

for any q solution to the adjoint system

$$\partial_t q + \mathcal{A}^{\mathfrak{M}} q + aq = 0, \qquad q|_{\partial\Omega} = 0.$$

A precise statement is given in Section 6.

Moreover, from the weak observability estimate given above we obtain a controllability result for the linear operator $P^{\mathfrak{M}}$. This result can be extended to classes of semi-linear equations

$$\big(\partial_t + \mathcal{A}^{\mathfrak{M}}\big)y + G(y) = \mathbf{1}_\omega v, \quad y \in (0,T) \qquad y|_{\partial\Omega} = 0, \quad y(0) = y_0,$$

with $G(x) = xg(x)$, where $g \in L^\infty(\mathbb{R})$ and

$$|g(x)| \leq K \ln^r(e + |x|), \quad x \in \mathbb{R}, \qquad \text{with} \qquad 0 \leq r < \frac{3}{2}.$$

We shall state controllability results with a control that satisfies

$$\|v\|_{L^2(Q)} \leq C\,|y_0|\,.$$

Thanks to one space dimension the size of the control function is uniform with respect to the discretization parameter h.

1.4 Sketch of proof of the Carleman estimate

The main idea of the proof lays in the combination of the derivation of a discrete Carleman estimate as in [BHL10a, BL12] and techniques used in [BDL07] to achieve such estimates for operators with discontinuous coefficients in the one-dimensional case.

We set $v = e^{-s\varphi}u$ yielding $e^{s\varphi}Pe^{-s\varphi}v = e^{s\varphi}f_1$ in Q_0' if $Pu = f_1$

We obtain $g = Av + Bv$ in Q_0', with A and iB 'essentially' selfadjoint.

We write $\|g\|^2_{L^2} = \|Av\|^2_{L^2} + \|Bv\|^2_{L^2} + 2(Av, Bv)_{L^2}$ and the main part of the proof is dedicated to computing the inner product $(Av, Bv)_{L^2(Q_0')}$, involving (discrete) integration by parts.

We proceed with these computations separately in each domain Ω_{01}, Ω_{02}. As in [BL12] we obtain terms involving boundary points $x = 0$ and $x = 1$ such as $v(0)$,$v(1)$,$\partial_t v(0)$,$\partial_t v(1)$, $(Dv)_{n+m+\frac{1}{2}}$, $(Dv)_{n+m+\frac{3}{2}}$. In our case we obtain additional terms involving the jump point a such as $v(a), \partial_t v(a)$, $\tilde{v}_{n+\frac{1}{2}}, \tilde{v}_{n+\frac{3}{2}}$, $(Dv)_{n+\frac{1}{2}}$, $(Dv)_{n+\frac{3}{2}}$. Main difficulties of our work come from dealing with these new terms. To reduce the number of terms to control, we find relations among connecting these various values at jump point allowing to focus our computations on terms only involving $v(a), \partial_t v(a)$ and $(Dv)_{n+\frac{1}{2}}$.

Those relations are stated in **Lemma 3.17**. In the limit $h \to 0$ they give back the transmission conditions for the function $v = e^{-s\varphi}u$ used crucial way in [BDL07]. The idea of this technique comes from a similar technique shown in continuos case by [BDL07].

The discrete setting could allow computation on the whole Ω. Yet such computation would yield constant that would depend on discrete derivatives of the diffusions coefficient, yielding non-uniformity with respect to the discretization parameter h. This explains why we resort to working on both Ω_0 and Ω_1 separately and deal with the interface terms that appear. As in [BDL07] the weight function is chosen to obtain positive contributions for these terms.

Sketch of proof Theorem

1. We compute the inner product (Av, Bv) in a series of terms and collect them together in an estimate (see **Lemma 4.4**–**Lemma 4.12**). In that estimate, we need to tackle two parts: volume integrals, integrals involving boundary points and the jump point. Volume integrals and boundary terms are dealt with similar to [BL12]. Terms at the jump point require special case.

2. Treatment of terms the jump point

   - Terms at jump point involving $\partial_t v$: when treating the term Y_{13} we obtain a positive integral of $(\partial_t v(a))^2$ in the LHS of the estimate as shown in **Lemma 4.15**. We keep this term in the LHS of the estimate.
   - Other terms: We collect together the terms at the jump point that already exist in the continuous case. As in [BDL07] we obtain a quadratic form because of the choice of the weight function (jump of its slope). This allows us to obtain positive two integrals involving $v^2(a)$, $(Dv)^2_{n+\frac{1}{2}}$ in the LHS of our estimate (see **Lemma 4.14**).
   - The remaining terms at the jump point are placed in the RHS of estimate. After that, we apply Young's inequality to them (as shown in **Lemma 4.16**) and they then can be absorded by the positive integrals involving $v^2(a)$, $(Dv)^2_{n+\frac{1}{2}}$, $(\partial_t v(a))^2$ in the LHS of estimate as described above.

1.5 Outline

In section 2, we construct the weight functions to be used in the Carleman estimate. In section 3 we have gathered some preliminary discrete calculus results and we present how transmission conditions can be expressed

in the discretization scheme. Section 4 is devoted to the proof the semi-discrete parabolic Carleman estimate in the case of a discontinuous diffusion cofficient for piecewise uniform meshes in the one-dimensional case. To ease the reading, a large number of proofs of intermediate estimates have been provided in Appendix A. This result is then extended to non-uniform meshes in Section 5. Finally, in Section 6, as consequences of the Carleman estimate, we present the weak observability estimate and associated some controllability results.

2 Weight functions

We shall first introduce a particular type of weight functions, which are constructed through the following lemma.

We enlarge the open intervals Ω_1, Ω_2 to large open sets $\tilde{\Omega}_1, \tilde{\Omega}_2$.

Lemma 2.1. *Let $\tilde{\Omega}_1$, $\tilde{\Omega}_2$ be a smooth open and connected neighborhoods of intervals $\overline{\Omega}_1$, $\overline{\Omega}_2$ of $\mathbb{R}$ and let $\omega \subset \Omega_2$ be a non-empty open set. Then, there exists a function $\psi \in C(\bar{\Omega})$ such that*

$$\psi(x) = \begin{cases} \psi_1 \text{ in } \overline{\Omega}_1, \\ \psi_2 \text{ in } \overline{\Omega}_2, \end{cases}$$

with $\psi_i \in C^\infty(\overline{\tilde{\Omega}_i}), i = 1, 2$, $\psi > 0$ in $\Omega, \psi = 0$ on $\Gamma, \psi_2' \neq 0$ in $\overline{\Omega}_2 \setminus \omega$, $\psi_1' \neq 0$ in $\overline{\Omega}_1$ and the function ψ satisfies the following trace properties, for some $\alpha_0 > 0$,

$$(\boldsymbol{A}u, u) \geq \alpha_0 |u|^2 \quad u \in \mathbb{R}^2,$$

with the matrix $\boldsymbol{A}$ defined by

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

with

$$\begin{aligned} a_{11} &= [\psi' \star]_{a'}, \\ a_{22} &= [c\psi' \star]_{a'}^2 (\psi')(a'^+) + [c^2 (\psi')^3 \star]_{a'}, \\ a_{12} &= a_{21} = [c\psi' \star]_{a'} (\psi')(a'^+), \end{aligned}$$

(see the notation (1.3) - (1.5) introduced in Section 1.1).

Remark 2.2. *Here we choose a weight function that yields an observation in the region $\omega \subset \Omega_2$ in the Carleman estimate of Section 4. This choice is of course arbitrary.*

Proof. We refer to Lemma 1.1 in [BDL07] for a similar proof. □

Choosing a function ψ, as in the previous lemma, for $\lambda > 0$ and $K > \|\psi\|_\infty$, we define the following weight functions

$$\begin{aligned} &\varphi(x) = e^{\lambda\psi(x)} - e^{\lambda K} < 0, \quad \phi(x) = e^{\lambda\psi(x)}, \\ &r(t,x) = e^{s(t)\varphi(x)}, \quad \rho(t,x) = (r(t,x))^{-1}, \end{aligned} \tag{2.1}$$

with

$$s(t) = \tau\theta(t), \quad \tau > 0, \quad \theta(t) = ((t+\alpha)(T+\alpha-t))^{-1},$$

for $0 < \alpha < T$.

We have

$$\max_{[0,T]}\theta = \theta(0) = \theta(T) = \alpha^{-1}(T+\alpha)^{-1}, \tag{2.2}$$

and $\min_{[0,T]}\theta \geq T^{-2}$. We note that

$$\partial_t\theta = (2t-T)\theta^2. \tag{2.3}$$

For the Carleman estimate and the observation/control results we choose here to treat the case of an distributed-observation in $\omega \subset \Omega$. The weight function is of the form $r = e^{s\varphi}$ with $\varphi = e^{\lambda\psi}$, with ψ fulfilling the following assumption. Construction of such a weight function is classical (see e.g [FI96]).

Assumption 2.3. *Let $\omega \subset \Omega$ be an open set. Let $\tilde{\Omega}$ be a smooth open and connected neighborhood of $\bar{\Omega}$ in R. The function $\psi = \psi(x)$ is in $\boldsymbol{C}^p\big(\overline{\tilde{\Omega}}, R\big)$, p sufficiently large, and satisfies, for some $c > 0$,*

$$\begin{aligned} &\psi > 0 \text{ in } \quad \tilde{\Omega}, \qquad &&|\nabla\psi| \geq c \text{ in } \quad \tilde{\Omega}\backslash\omega_0, \\ &\partial_n\psi(x) \leq -c < 0, \qquad &&\partial_x^2\psi(x) \leq 0 \text{ in } \quad V_{\partial\Omega}. \end{aligned}$$

where $V_{\partial\Omega}$ is a sufficiently small neighborhood of $\partial\Omega$ in $\tilde{\Omega}$, in which the outward unit normal n to Ω is extended from $\partial\Omega$.

3 Some preliminary discrete calculus results for uniform meshes

Here, to prepare for Section 4, we only consider constant-step discretizations, i.e., $h_{i+\frac{1}{2}} = h$, $i = 0, \ldots, n+m+1$.

We use here the following notation: $\Omega_0 = (0,1)$, $\Omega_{01} = (0,a)$, $\Omega_{02} = (a,1)$, $\Omega_0' = \Omega_{01} \cup \Omega_{02}$, $Q_0 = (0,T)\times\Omega_0$, $Q_0' = (0,T)\times\Omega_0'$, $Q_{0i} = (0,T)\times\Omega_{0i}$ with $i = 1,2$ and $\partial\Omega_0 = \{0,1\}$.

This section aims to provide calculus rules for discrete operators such as $D_i, \bar{D}_i$ and also to provide estimates for the successive applications of such operators on the weight functions. To avoid cumbersome notation we

introduce the following continuous difference and averaging operators on continuous functions. For a function f defined on Ω_0 we set

$$\tau^+ f(x) := f(x + h/2), \qquad \tau^- f(x) := f(x - h/2),$$
$$\mathtt{D} f(x) := (\tau^+ - \tau^-) f(x)/h, \qquad \hat{f}(x) = (\tau^+ + \tau^-) f(x)/2.$$

Remark 3.1. *To iterate averaging symbols we shall sometimes write $Af = \hat{f}$, and thus $A^2 f = \hat{\hat{f}}$.*

3.1 Discrete calculus formulae

We present calculus results for finite-difference operators that were defined in the introductory section. Proofs can be found in Appendix of [BHL10a] in the one-dimension case.

Lemma 3.2. *Let the functions f_1 and f_2 be continuously defined in a neighborhood of $\bar{\Omega}$. We have:*

$$\mathtt{D}(f_1 f_2) = \mathtt{D}(f_1)\hat{f}_2 + \hat{f}_1 \mathtt{D}(f_2).$$

Note that the immediate translation of the proposition to discrete functions $f_1, f_2 \in \mathbb{C}^{\mathfrak{M}}$ and $g_1, g_2 \in \mathbb{C}^{\overline{\mathfrak{M}}}$ is

$$D(f_1 f_2) = D(f_1)\tilde{f}_2 + \tilde{f}_1 D(f_2), \;\; \bar{D}(g_1 g_2) = \bar{D}(g_1)\bar{g_2} + \bar{g_1}\bar{D}(g_2).$$

Lemma 3.3. *Let the functions f_1 and f_2 be continuously defined in a neighborhood of $\bar{\Omega}$. We have:*

$$\widehat{f_1 f_2} = \hat{f}_1 \hat{f}_2 + \frac{h^2}{4}\mathtt{D}(f_1)\mathtt{D}(f_2).$$

Note that the immediate translation of the proposition to discrete functions $f_1, f_2 \in \mathbb{C}^{\mathfrak{M}}$ and $g_1, g_2 \in \mathbb{C}^{\overline{\mathfrak{M}}}$ is

$$\widetilde{f_1 f_2} = \tilde{f}_1 \tilde{f}_2 + \frac{h^2}{4} D(f_1) D(f_2), \;\; \overline{g_1 g_2} = \bar{g_1}\bar{g_2} + \frac{h^2}{4}\bar{D}(g_1)\bar{D}(g_2).$$

Some of the following properties can be extended in such a manner to discrete functions. We shall not always write it explicitly.

Averaging a function twice gives the following formula.

Lemma 3.4. *Let the function f be continuously defined over $\mathbb{R}$. We then have*

$$A^2 f := \hat{\hat{f}} = f + \frac{h^2}{4}\mathtt{DD} f.$$

The following proposition covers discrete integrations by parts and related formula.

Proposition 3.5. *Let $f \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ and $g \in \mathbb{C}^{\overline{\mathfrak{M}}}$. We have the following formulae:*

$$\int_{\Omega_0} f(\bar{D}g) = -\int_{\Omega_0} (Df)g + f_{n+m+2}g_{n+m+\frac{3}{2}} - f_0 g_{\frac{1}{2}},$$
$$\int_{\Omega_0} f\bar{g} = \int_{\Omega_0} \tilde{f}g - \frac{h}{2}f_{n+m+2}g_{n+m+\frac{3}{2}} - \frac{h}{2}f_0 g_{\frac{1}{2}}.$$

Lemma 3.6. *Let f be a smooth function defined in a neighborhood of $\bar{\Omega}$. We have*

$$\tau^{\pm}f = f \pm \frac{h}{2}\int_0^T \partial_x f(. \pm \sigma h/2)d\sigma,$$
$$A^j f = f + C_j h^2 \int_{-1}^1 (1-|\sigma|)\partial_x^2 f(. + l_j\sigma h)d\sigma,$$
$$D^j f = \partial_x^j f + C'_j h^2 \int_{-1}^1 (1-|\sigma|)^{j+1}\partial_x^{j+2} f(. + l_j\sigma h)d\sigma,\ j = 1,2,\ l_1 = \frac{1}{2},\ l_2 = 1.$$

3.2 Calculus results related to the weight functions

We now present some technical lemmata related to discrete operators performed on the Carleman weight functions that is of the form $e^{s\varphi}$, $\varphi = e^{\lambda\psi} - e^{\lambda K}$, where ψ satisfies the properties listed in Section 2 in the domain Ω_0. For concision, we set $r(t,x) = e^{s(t)\varphi(x)}$ and $\rho = r^{-1}$, with $s(t) = \tau\theta(t)$. From Section 2, we have $\psi_{|\Omega_{01}} = \psi_{1|\Omega_{01}}$, $\psi_{|\Omega_{01}} = \psi_{2|\Omega_{01}}$ where $\psi_i \in \mathbb{C}^2(\overline{\tilde{\Omega}}_{0i})$. Then $\rho = e^{-s\varphi}$ can be replaced by

$$\begin{array}{llll} \rho_1 = e^{-s\varphi_1} \text{ with} & \varphi_1 = e^{\lambda\psi_1} - e^{\lambda K} \text{ in domain} & \Omega_{01} \\ \rho_1 = e^{-s\varphi_2} \text{ with} & \varphi_2 = e^{\lambda\psi_2} - e^{\lambda K} \text{ in domain} & \Omega_{02} \end{array}$$

And $r = \rho^{-1}$ is also replaced by

$$\begin{array}{ll} r_1 = \rho_1^{-1} \text{ in domain} & \Omega_{01} \\ r_2 = \rho_2^{-1} \text{ in domain} & \Omega_{02} \end{array}$$

The positive parameters τ and h will be large and small respectively and we are particularly interested in the dependence on τ, h and λ in the following basic estimates in each domain Ω_{01}, Ω_{02}.

We assume $\tau \geq 1$ and $\lambda \geq 1$.

Lemma 3.7. *Let $\alpha,\ \beta \in \mathbb{N}$, i=1,2. We have*

$$\begin{aligned} \partial_x^\beta(r_i\partial_x^\alpha\rho_i) &= \alpha^\beta(-s\phi_i)^\alpha\lambda^{\alpha+\beta}(\bigtriangledown\psi_i)^{\alpha+\beta} \\ &+ \alpha\beta(s\phi_i)^\alpha\lambda^{\alpha+\beta-1}\mathcal{O}_\lambda(1) + s^{\alpha-1}\alpha(\alpha-1)\mathcal{O}_\lambda(1) = \mathcal{O}_\lambda(s^\alpha). \end{aligned}$$

Let $\sigma \in [-1,1]$, we have

$$\partial_x^\beta\big(r_i(t,.)(\partial_x^\alpha \rho_i)(t,.+\sigma h)\big) = \mathcal{O}_\lambda(s^\alpha(1+(sh)^\beta))e^{\mathcal{O}_\lambda(sh)}.$$

Provided $0 < \tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$ we have $\partial_x^\beta\big(r_i(t,.)(\partial^\alpha \rho_i)(t,.+\sigma h)\big) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|})$. The same expressions hold with r and ρ interchanged and with s changed into -s.

A proof is given in [BHL10a, proof of Lemma 3.7] in the time independent case. Additionally, we provide a result below to the time-dependent case whose proof is refered to [BL12, proof of Lemma 2.8]. Note that the condition $0 < \tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$ implies that $s(t)h \leq \mathfrak{K}$ for all $t \in [0,T]$.

Lemma 3.8. *Let $\alpha \in \mathbb{N}$, i=1,2. We have*

$$\partial_t(r_i\partial_x^\alpha \rho_i) = s^\alpha T\theta \mathcal{O}_\lambda(1).$$

With Leibniz formula we have the following estimates

Corollary 3.9. *Let $\alpha, \beta, \delta \in \mathbb{N}$, i=1,2. We have*

$$\begin{aligned}\partial_x^\delta(r_i^2(\partial_x^\alpha \rho_i)\partial_x^\beta \rho_i) &= (\alpha+\beta)^\delta(-s\phi_i)^{\alpha+\beta}\lambda^{\alpha+\beta+\delta}(\bigtriangledown\psi_i)^{\alpha+\beta+\delta}\\ &+ \delta(\alpha+\beta)(s\phi_i)^{\alpha+\beta}\lambda^{\alpha+\beta+\delta-1}\mathcal{O}(1)\\ &+ s^{\alpha+\beta-1}(\alpha(\alpha-1)+\beta(\beta-1))\mathcal{O}_\lambda(1) = \mathcal{O}_\lambda(s^{\alpha+\beta}).\end{aligned}$$

The proofs of the following properties can be found in Appendix A of [BHL10a].

Proposition 3.10. *Let $\alpha \in \mathbb{N}$, i=1,2. Provided $0 < \tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, we have*

$$\begin{aligned}&r_i\tau^\pm\partial_x^\alpha\rho_i = r_i\partial_x^\alpha\rho_i + s^\alpha\mathcal{O}_{\lambda,\mathfrak{K}}(sh) = s^\alpha\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\ &r_iA^k\partial_x^\alpha\rho_i = r_i\partial_x^\alpha\rho_i + s^\alpha\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = s^\alpha\mathcal{O}_{\lambda,\mathfrak{K}}(1),\ k=0,1,2,\\ &r_iA^kD\rho_i = r_i\partial_x\rho_i + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = s\mathcal{O}_{\lambda,\mathfrak{K}}(1),\ k=0,1,\\ &r_iD^2\rho_i = r_i\partial_x^2\rho_i + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1).\end{aligned}$$

The same estimates hold with ρ_i and r_i interchanged.

Lemma 3.11. *Let $\alpha, \beta \in \mathbb{N}$ and $k = 1,2; j = 1,2; i = 1,2$. Provided $0 < \tau h(\max_{[0,T]}\theta) \leq \mathfrak{K}$, we have*

$$\begin{aligned}&D^k(\partial_x^\beta(r_i\partial_x^\alpha\rho_i)) = \partial_x^{k+\beta}(r_i\partial_x^\alpha\rho_i) + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^\alpha),\\ &A^j\partial_x^\beta(r_i\partial_x^\alpha\rho_i) = \partial^\beta(r_i\partial_x^\alpha\rho_i) + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^\alpha).\end{aligned}$$

Let $\sigma \in [-1,1]$, we have $D^k\partial^\beta(r_i(t,.)\partial^\alpha\rho_i(t,.+\sigma h)) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|})$. The same estimates hold with r_i and ρ_i interchanged.

Lemma 3.12. *Let $\alpha,\ \beta,\delta \in \mathbb{N}$ and $k=1,2; j=1,2; i=1,2$. Provided $0<\tau h(max_{[0,T]}\theta)\leq \mathfrak{K}$, we have*

$$A^j\partial_x^\delta(r_i^2(\partial_x^\alpha\rho_i)\partial_x^\beta\rho_i)=\partial_x^\delta(r_i^2(\partial_x^\alpha\rho_i)\partial_x^\beta\rho_i)+h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^{\alpha+\beta})=\mathcal{O}_{\lambda,\mathfrak{K}}(s^{\alpha+\beta}),$$

$$D^k\partial_x^\delta(r_i^2(\partial_x^\alpha\rho_i)\partial_x^\beta\rho_i) \ = \ \partial_x^{k+\delta}(r_i^2(\partial_x^{\alpha+\beta}\rho_i)+h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^{\alpha+\beta})=\mathcal{O}_{\lambda,\mathfrak{K}}(s^{\alpha+\beta}).$$

Let $\sigma,\sigma'\in[-1,1]$. We have

$$\begin{aligned}&A^j\partial^\delta(r_i(t,.)^2(\partial^\alpha\rho_i(t,.+\sigma h))\partial^\beta\rho_i(y,.+\sigma' h))=\mathcal{O}_{\lambda,\mathfrak{K}}(s^{\alpha+\beta}),\\&D^k\partial^\delta(r_i(t,.)^2(\partial^\alpha\rho_i(t,.+\sigma h))\partial^\beta\rho_i(t,.+\sigma' h))=\mathcal{O}_{\lambda,\mathfrak{K}}(s^{\alpha+\beta}).\end{aligned}$$

The same estimates hold with r_i and ρ_i interchanged.

Proposition 3.13. *Let $\alpha\in\mathbb{N}$ and $k=0,1,2; j=0,1,2; i=1,2$. Provided $0<sh\leq\mathfrak{K}$, we have*

$$\begin{aligned}&D^kA^j\partial_x^\alpha(r_i\widehat{D\rho_i})=\partial_x^{k+\alpha}(r_i\partial_x\rho_i)+s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2=s\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\&D^k(r_iD^2\rho_i)=\partial_x^k(r_i\partial^2\rho_i)+s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2=s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\&r_iA^2\rho_i=1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,\ D^k(r_iA^2\rho_i)=\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2.\end{aligned}$$

The same estimates hold with r_i and ρ_i interchanged.

Proposition 3.14. *Provided $0<\tau h(max_{[0,T]}\theta)\leq\mathfrak{K}$ and σ is bounded, we have*

$$\begin{aligned}&\partial_t(r_i(.,x)(\partial^\alpha\rho_i)(.,x+\sigma h))=Ts^\alpha\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\&\partial_t(r_iA^2\rho_i)=T(sh)^2\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\&\partial_t(r_iD^2\rho_i)=Ts^2\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(1).\end{aligned}$$

The same estimates hold with r_i and ρ_i interchanged.

Proposition 3.15. *Let $\alpha,\ \beta\in\mathbb{N}$ and $k=0,1,2; j=0,1,2; i=1,2$, provided $0<sh\leq\mathfrak{K}$, we have*

$$A^jD^k\partial^\beta(r_i^2(\partial^\alpha)\widehat{D\rho_i}) \ = \ \partial_x^{k+\beta}(r_i^2(\partial^\alpha\rho)\partial\rho_i)+s^{\alpha+1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2=s^{\alpha+1}\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$

$$A^jD^k\partial^\beta(r_i^2(\partial^\alpha)A^2\rho_i) \ = \ \partial_x^{k+\beta}(r_i(\partial^\alpha\rho_i))+s^\alpha\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)=s^\alpha\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$

$$A^jD^k\partial^\beta(r_i^2(\partial^\alpha)D^2\rho_i) \ = \ \partial_x^{k+\beta}(r_i^2(\partial^\alpha\rho)\partial^2\rho_i)+s^{\alpha+2}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2=s^{\alpha+2}\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$

and we have

$$\begin{aligned}&A^jD^k\partial^\alpha(r_i^2\widehat{D\rho_i}D^2\rho_{0i})=\partial_x^{k+\alpha}(r_i^2(\partial\rho_i)\partial^2\rho_i)+s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2=s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\&A^jD^k\partial^\alpha(r_i^2\widehat{D\rho_i}A^2\rho_i)=\partial_x^{k+\alpha}(r_i\partial\rho_i)+s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2=s\mathcal{O}_{\lambda,\mathfrak{K}}(1).\end{aligned}$$

3.3 Transmission conditions

We consider here discrete version of the transmission conditions (TC) at the point a. For $u \in \mathbb{C}^{\mathfrak{M}}$ we set $f := \bar{D}(c_d Du)$ we then have

$$\begin{cases} u(a^-) = u(a^+) = u_{n+1}, \\ (c_d Du)_{n+\frac{3}{2}} - (c_d Du)_{n+\frac{1}{2}} = hf_{n+1} \end{cases} .$$

Remark 3.16. *These transmission conditions provide the continuity for u and the discrete flux at the singular point of coefficient up to a consistent factor.*

From these conditions, we obtain the following lemma whose proof is given in Appendix A

Lemma 3.17. *For the parameter λ chosen sufficiently large and sh sufficiently small and with $u = \rho v$ we have*

$$[\star c_d Dv]_a = (c_d Dv)_{n+\frac{3}{2}} - (c_d Dv)_{n+\frac{1}{2}} = J_1 v_{n+1} + J_2 (c_d Dv)_{n+\frac{1}{2}} + J_3 h(rf)_{n+1} \quad (3.1)$$

where

$$J_1 = \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)\lambda s[\star c\phi\psi']_a + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh),$$
$$J_2 = \mathcal{O}_{\lambda,\mathfrak{K}}(sh), \qquad J_3 = \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big).$$

Furthermore, we have

$$\partial_t J_1 = sT\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh),$$
$$\partial_t J_2 = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \qquad \partial_t J_3 = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh).$$

For simplicity, (3.1) *can be written in form*

$$[\star c_d Dv]_a = \lambda s[\star c\phi\psi']_a v_{n+1} + r_0, \quad (3.2)$$

where $r_0 = \lambda s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1} + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{1}{2}} + h\Big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\Big)(rf)_{n+1}$.

4 Carleman estimate for uniform meshes

In this section, we prove a Carleman estimate in case of picewise uniform meshes, i.e, constant-step discretizations in each subinterval $(0, a)$ and $(a, 1)$. The case of non-uniform meshes is treated in Section 5.

We let $\omega_0 \subset \Omega_{02}$ be a nonempty open subset. We set the operator $P^{\mathfrak{M}}$ to be $\mathcal{P}^{\mathfrak{M}} = -\partial_t + \mathcal{A}^{\mathfrak{M}} = -\partial_t - \bar{D}(c_d D)$, continuous in the variable $t \in (0, T)$ with $T > 0$, and discrete in the variable $x \in \Omega_0$.

The Carleman weight function is of the form $r = e^{s\varphi}$ with $\varphi = e^{\lambda\psi} - e^{\lambda K}$ where ψ satisfies the properties listed in Section 2 in the domain Ω_0. Here, to treat the semi-discrete case, we use the enlarged neighborhoods $\tilde{\Omega}_{01}$, $\tilde{\Omega}_{02}$ of Ω_{01}, Ω_{02} as introduced in Lemma 2.1. This allows one to apply multiple

discrete operators such as D and A on the weight functions. In particular, we take ψ such that $\partial_x\psi \geq 0$ in V_0 and $\partial_x\psi \leq 0$ in V_1 where V_0 and V_1 are neighborhoods of 0 and 1 respectively. This then yields on $\partial\Omega_0$

$$(r\overline{D\rho})_0 \leq 0, \quad (r\overline{D\rho})_{n+m+2} \geq 0 \tag{4.1}$$

Theorem 4.1. *Let $\omega_0 \subset \Omega_{02}$ be a non-empty open set and we set $f := \bar{D}(c_d Du)$. For the parameter $\lambda > 1$ sufficiently large, there exists C, $\tau_0 \geq 1$, $h_0 > 0$, $\epsilon_0 > 0$, depending on ω_0 so that the following estimate holds*

$$\begin{aligned}\tau^{-1}\left\|\theta^{-\frac{1}{2}}e^{\tau\theta\varphi}\partial_t u\right\|^2_{L^2(Q_0)} + \tau\left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi}Du\right\|^2_{L^2(Q_0)} + \tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi}u\right\|^2_{L^2(Q_0)}\\ \leq C_{\lambda,\mathfrak{K}}\left(\left\|e^{\tau\theta\varphi}P^{\mathfrak{M}}u\right\|^2_{L^2(Q_0)} + \tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi}u\right\|^2_{L^2((0,T)\times\omega_0)}\right.\\ \left. + h^{-2}\left|e^{\tau\theta\varphi}u|_{t=0}\right|^2_{L^2(\Omega_0)} + h^{-2}\left|e^{\tau\theta\varphi}u|_{t=T}\right|^2_{L^2(\Omega_0)}\right)\end{aligned} \tag{4.2}$$

for all $\tau \geq \tau_0(T+T^2)$, $0 < h \leq h_0$ and $\tau h(\alpha T)^{-1} \leq \epsilon_0$ and for all $u \in C^\infty(0,T;\mathbb{C}^{\mathfrak{M}})$ satisfying $u|_{\partial\Omega_0} = 0$.

Remark 4.2. *Observation was chosen in Ω_{02} here. This is an arbitrary choice (see Remark 2.2).*

Proof. We set $f_1 := -P^{\mathfrak{M}} = \partial_t u + \bar{D}(c_d Du)$ and $f = \bar{D}(c_d Du)$. At first, we shall work with the function $v = ru$, i.e., $u = \rho v$, that satisfies

$$r\Big(\partial_t(\rho v) + \bar{D}\big(c_d D(\rho v)\big)\Big) = rf_1 \text{ in } Q_0'. \tag{4.3}$$

We have

$$r\partial_t(\rho v) = \partial_t v + r(\partial_t\rho)v = \partial_t v - \tau(\partial_t\theta)\varphi v.$$

We write: $g = Av + Bv$,
where $Av = A_1v + A_2v + A_3v$, $Bv = B_1v + B_2v + B_3v$ with

$$\begin{aligned}&A_1v = r\overline{\rho}\bar{D}(c_dDv),\ A_2v = cr(\bar{D}D\rho)\overline{\tilde{v}},\ A_3v = -\tau(\partial_t\theta)\varphi v,\\ &B_1v = 2cr\overline{D\rho}\ \overline{Dv},\ B_2v = -2sc\phi''v,\ B_3v = \partial_t v,\\ &g = rf_1 - \frac{h}{4}r\overline{D\rho}(\bar{D}c_d)(\tau^+Dv - \tau^-Dv) - \frac{h^2}{4}(\bar{D}c_d)r(\bar{D}D\rho)\overline{Dv}\\ &\qquad -h\mathcal{O}(1)r\overline{D\rho}\ \overline{Dv} - \big(r(\bar{D}c_d)\ \overline{D\rho} + h\mathcal{O}(1)r(\bar{D}D\rho)\big)\overline{\tilde{v}} - 2sc(\phi'')v,\end{aligned}$$

as derived in [BL12].

Equation (4.3) now reads $Av + Bv = g$ and we write

$$\|Av\|^2_{L^2(Q_0')} + \|Bv\|^2_{L^2(Q_0')} + 2(Av, Bv)_{L^2(Q_0')} = \|g\|^2_{L^2(Q_0')}. \tag{4.4}$$

First we need an estimation of $\|g\|^2_{L^2(Q_0')}$. The proof can be adapted from [BHL10a].

Lemma 4.3. *For $\tau h(\max_{[0,T]}\theta) \leq \mathfrak{K}$ we have*

$$\|g\|^2_{L^2(Q'_0)} \leq C_{\lambda,\mathfrak{K}}(\|rf_1\|^2_{L^2(Q'_0)}) + \|sv\|^2_{L^2(Q'_0)} + h^2\,\|sDv\|^2_{L^2(Q'_0)}\,. \qquad (4.5)$$

Most of the remaining of the proof will be dedicated to computing the inner product $(Av, Bv)_{L^2(Q'_0)}$. Developing the inner-product $(Av, Bv)_{L^2(Q'_0)}$, we set $I_{ij} = (A_i v, B_j v)_{L^2(Q'_0)}$. The proofs of the following lemmata are provided in Appendix A.

Lemma 4.4 (Estimate of I_{11})**.** *For $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$ we have*

$$I_{11} \geq -\int_{Q'} s\lambda^2(c^2\phi(\psi')^2)_d(Dv)^2 - X_{11} + Y_{11},$$

where $X_{11} = \int_{Q'_0} \nu_{11}(Dv)^2$ with ν_{11} of the form $s\lambda\phi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ and

$$\begin{aligned}
Y_{11} &= Y_{11}^{(1)} + Y_{11}^{(2,1)} + Y_{11}^{(2,2)},\\
Y_{11}^{(1)} &= \int_0^T \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)(c\bar{c}_d)(1)(r\overline{D\rho})(1)(Dv)^2_{n+m+\frac{3}{2}}\\
&\quad - \int_0^T \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)(c\bar{c}_d)(0)(r\overline{D\rho})(0)(Dv)^2_{\frac{1}{2}},\\
Y_{11}^{(2,1)} &= \int_0^T s\lambda\phi(a)\bar{c}_d(a)\Big((c\psi')(a^+)(Dv)^2_{n+\frac{3}{2}} - (c\psi')(a^-)(Dv)^2_{n+\frac{1}{2}}\Big),\\
Y_{11}^{(2,2)} &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2_{n+\frac{1}{2}} - \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2_{n+\frac{3}{2}}.
\end{aligned}$$

Lemma 4.5 (Estimate of I_{12})**.** *For $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, the term I_{12} is of the following form*

$$I_{12} = 2\int_{Q'_0} s\lambda^2(c^2\phi(\psi')^2)_d(Dv)^2 - X_{12} + Y_{12},$$

with

$$\begin{aligned}
Y_{12} &= \int_0^T s\lambda^2\phi(a)v(a)[c(\psi')^2 \star c_d Dv]_a\\
&+ \int_0^T \delta_{12}v(a)(cDv)_{n+\frac{3}{2}} + \int_0^T \bar{\delta}_{12}v(a)(cDv)_{n+\frac{1}{2}},
\end{aligned}$$

where $\delta_{12}, \bar{\delta}_{12}$ are of the form $s\big(\lambda\phi(a)\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\big)$ and

$$X_{12} = \int_{Q'_0} \nu_{12}(Dv)^2 + \int_{Q'_0} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\tilde{v}Dv,$$

where

$$\nu_{12} = s\lambda\phi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(h + (sh)^2).$$

Lemma 4.6 (Estimate of I_{13})**.** *There exists $\epsilon_1(\lambda) > 0$ such that, for $0 < \tau h(max_{[0,T]}\theta) \leq \epsilon_1(\lambda)$, the term I_{13} can be estimated from below in following way:*

$$I_{13} \geq -\int_{\Omega_0'} \mathcal{C}_{\lambda,\mathfrak{K}}(1)(Dv(T))^2 - X_{13} + Y_{13}.$$

with

$$X_{13} = \int_{Q_0'} \big(s(sh) + T(sh)^2\theta\big)\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2 + \int_{Q_0} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v)^2,$$

$$Y_{13} = -\int_0^T r\bar{\tilde{\rho}}(a^+)\partial_t v(a)(c_d Dv)_{n+\frac{3}{2}} + \int_0^T r\bar{\tilde{\rho}}(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}}.$$

Lemma 4.7 (Estimate of I_{21})**.** *For $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, the term I_{21} can be estimated as*

$$I_{21} \geq 3\int_{Q_0'} \lambda^4 s^3\phi^3 c^2(\psi')^4 v^2 - X_{21} + Y_{21},$$

with

$$X_{21} = \int_{Q_0'} \mu_{21} v^2 + \int_{Q_0'} \nu_{21}(Dv)^2,$$

where

$$\mu_{21} = (s\lambda\phi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2, \quad \nu_{21} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,$$

and

$$\begin{aligned}
&Y_{21} = Y_{21}^{(1,1)} + Y_{21}^{(1,21)} + Y_{21}^{(1,22)} + Y_{21}^{(2)},\\
&Y_{21}^{(1,1)} = \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(r\overline{D\rho})(1)(Dv)^2_{n+m+\frac{3}{2}} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(r\overline{D\rho})(0)(Dv)^2_{\frac{1}{2}},\\
&Y_{21}^{(1,21)} = \int_0^T s^3\lambda^3\phi^3(a)[c^2(\psi')^3 \star (\tilde{v})^2]_a,\\
&Y_{21}^{(1,22)} = \int_0^T \big(s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\big)\big((\tilde{v})^2_{n+\frac{1}{2}} + (\tilde{v})^2_{n+\frac{3}{2}}\big),\\
&Y_{21}^{(2)} = \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2(a).
\end{aligned}$$

Lemma 4.8 (Estimate of I_{22})**.** *For $sh \leq \mathfrak{K}$, we have*

$$I_{22} = -2\int_{Q_0'} c^2 s^3\lambda^4\phi^3(\psi')^4 v^2 - X_{22} + Y_{22},$$

with

$$
\begin{aligned}
&Y_{22} = Y_{22}^{(1)} + Y_{22}^{(2)},\\
&Y_{22}^{(1)} = \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h^2}{2}(Dv)_{n+\frac{1}{2}} + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h^2}{2}(Dv)_{n+\frac{3}{2}},\\
&Y_{22}^{(2)} = \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v^2(a),
\end{aligned}
$$

and

$$
X_{22} = \int_{Q_0'} \mu_{22}v^2 + \int_{Q_0'} \nu_{22}(Dv)^2,
$$

where

$$
\mu_{22} = (s\lambda\phi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2, \quad \nu_{22} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2.
$$

Lemma 4.9 (Estimate of I_{23})**.** *For $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, the term I_{23} can be estimated from below in the following way*

$$
I_{23} \geq \int_{\Omega_0'} s^2\big(\mathcal{O}_{\lambda,\mathfrak{K}}(1)v_{|t=0}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}(1)v_{|t=T}^2\big) - X_{23} + Y_{23},
$$

with

$$
\begin{aligned}
X_{23} \;=\; & \int_{Q_0} Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{Q_0} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v)^2\\
& + \int_{Q_0'} (sh)^2s\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&Y_{23} = Y_{23}^{(1)} + Y_{23}^{(2)} + Y_{23}^{(3)},\\
&Y_{23}^{(1)} = \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)\frac{h}{2}(\tilde{v}_{n+\frac{1}{2}}) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)\frac{h}{2}(\tilde{v}_{n+\frac{3}{2}}),\\
&Y_{23}^{(2)} = \int_0^T sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2(a),\\
&Y_{23}^{(3)} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v^2(a)|_{t=0}^{t=T}.
\end{aligned}
$$

Lemma 4.10 (Estimate of I_{31})**.** *For $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, we have*

$$
I_{31} = -X_{31} + Y_{31},
$$

with

$$
X_{31} = \int_{Q_0'} T\theta s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{Q_0'} T\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2,
$$

and

$$
\begin{aligned}
&Y_{31} = Y_{31}^{(1)} + Y_{31}^{(2)},\\
&Y_{31}^{(1)} = \int_0^T T\theta s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) \frac{h}{2} (Dv)_{n+\frac{1}{2}} + \int_0^T T\theta s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) \frac{h}{2} (Dv)_{n+\frac{3}{2}},\\
&Y_{31}^{(2)} = \int_0^T T\theta s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2(a).
\end{aligned}
$$

Lemma 4.11 (Estimate of I_{32})**.** *[BL12] For $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, the term I_{32} can be estimated from below in the following way*

$$
I_{32} = -X_{32} = \int_{Q_0'} T s^2 \theta \mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2.
$$

Lemma 4.12 (Estimate of I_{33})**.** *[BL12, proof of Lemma 3.9] For $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, the term I_{33} can be estimated from below in the following way*

$$
I_{33} \geq -X_{33} = \frac{1}{2}\tau \int_{Q_0'} c' \varphi (\partial_t^2 \theta) v^2.
$$

Continuation of the proof of Theorem 4.1. Collecting the terms we have obtained in the previous lemmata, from (4.4) and (4.5) for $0 < \tau h(max_{[0,T]}\theta) \leq \epsilon_1(\lambda)$ we find

$$
\begin{aligned}
&\|Av\|^2_{L^2(Q_0')} + \|Bv\|^2_{L^2(Q_0')} + 2\int_{Q_0'} s\lambda^2 (c^2\phi(\psi')^2)_d (Dv)^2 + 2\int_{Q_0'} c^2 s^3 \lambda^4 \phi^3 (\psi')^4 v^2\\
&\quad +2\big(Y_{11}^{(1)} + Y_{21}^{(1,1)}\big) + 2\big(Y_{11}^{(2,1)} + Y_{21}^{(1,21)}\big) + 2Y_{13}\\
\leq\ & C_{\lambda,\mathfrak{K}}\Big(\|rf_1\|^2_{L^2(Q_0')} + \int_{\Omega_0'} s^2 \big(v^2_{|_{t=0}} + v^2_{|_{t=T}}\big) + \int_{\Omega_0'} (Dv(T))^2 \Big) + 2X + 2\underline{Y},
\end{aligned}
$$

with

$$
\begin{aligned}
&\underline{Y} = -\Big(Y_{11}^{(2,2)} + Y_{12} + Y_{21}^{(1,22)} + Y_{21}^{(2)} + Y_{22} + Y_{23} + Y_{31}\Big),\\
&X = X_{11} + X_{12} + X_{13} + X_{21} + X_{22} + X_{23} + X_{31} + X_{32} + X_{33}\\
&\qquad + C_{\lambda,\mathfrak{K}}\big(\|sv\|^2_{L^2(Q_0')} + h^2 \|sDv\|^2_{L^2(Q_0')} \big).
\end{aligned}
$$

With the following lemma, we may in fact ignore the term $Y_{11}^{(1)} + Y_{21}^{(1,1)}$ in the previous inequality.

Lemma 4.13. *For all λ there exists $0 < \epsilon_2(\lambda) < \epsilon_1(\lambda)$ such that for $0 < \tau h(max_{[0,T]}\theta) \leq \epsilon_2(\lambda)$ we have $Y_{11}^{(1)} + Y_{21}^{(1,1)} \geq 0$.*

Recalling that $\nabla\psi \geq C > 0$ in $\Omega\backslash\omega_0$ we may thus write

$$\begin{aligned}
&\|Av\|^2_{L^2(Q_0')} + \|Bv\|^2_{L^2(Q_0')} + \int_{Q_0'} s(Dv)^2 + \int_{Q_0'} s^3v^2 \\
&\qquad +2\big(Y_{11}^{(2,1)} + Y_{21}^{(1,21)}\big) + 2Y_{13} \\
\leq\ & C_{\lambda,\mathfrak{K}}\left(\|rf_1\|^2_{L^2(Q_0')} + 2\int_0^T\int_{\omega_0} s(Dv)^2 + 2\int_0^T\int_{\omega_0} s^3v^2\right. \\
&\left. + \int_{\Omega_0'} s^2(v^2_{|_{t=0}} + v^2_{|_{t=T}}) + \int_{\Omega_0'}(Dv(T))^2\right) + 2X + 2\underline{Y}. \qquad (4.6)
\end{aligned}$$

Lemma 4.14. *With the function ψ satisfing the properties of Lemma 2.1 and for $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, we have*

$$Y_{11}^{(2,1)} + Y_{21}^{(1,21)} \geq C\alpha_0\int_0^T s\lambda\phi(a)(c_dDv)^2_{n+\frac{1}{2}} + C\alpha_0\int_0^T s^3\lambda^3\phi^3(a)v^2_{n+1} + \mu_1 + \mu_r,$$

with α_0 as given in Lemma 2.1 and where

$$\begin{aligned}
\mu_r \ &= \ \int_0^T s\mathcal{O}_\lambda(1)r_0^2 + \int_0^T s^2\mathcal{O}_\lambda(1)r_0v_{n+1} + \int_0^T s\mathcal{O}_\lambda(1)r_0(c_dDv)_{n+\frac{1}{2}} \\
&+ \ \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2_{n+1} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_dDv)_{n+\frac{1}{2}} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)r_0v_{n+1},
\end{aligned}$$

with r_0 as given in Lemma 3.17 and

$$\mu_1 = \mu_1^{(1)} + \mu_1^{(2)},$$

where

$$\begin{aligned}
\mu_1^{(1)} &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)^2_{n+\frac{3}{2}} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)^2_{n+\frac{1}{2}}, \\
\mu_1^{(2)} &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{3}{2}}v_{n+1} + \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}v_{n+1}.
\end{aligned}$$

For a proof see Appendix A.

Lemma 4.15. *With $0 < \epsilon_3(\lambda) < \epsilon_2(\lambda)$ sufficiently small we obtain*

$$\begin{aligned}
Y_{13} &\geq \int_0^T \mathcal{C}_{\lambda,\mathfrak{K}}h(\partial_tv(a))^2 + \int_0^T \big(sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + T^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)v^2(a) \\
+ \quad & s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)|_{t=0}^{t=T} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_tv(a)(c_dDv)_{n+\frac{1}{2}} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_tv(a)h(rf_1)_{n+1}.
\end{aligned}$$

where $C_{\lambda,\mathfrak{K}}$ is positive constant whose value depends on λ and sh.

For a proof see Appendix A.

If we choose $\lambda_2 \geq \lambda_1$ sufficiently large, then for $\lambda = \lambda_2$ (fixed for the rest of the proof) and $0 < \tau h(\max_{[0,T]}\theta) \leq \epsilon_3$, from (4.6) and Lemma 4.14 and Lemma 4.15, we can thus achieve the following inequality

$$
\begin{aligned}
& \|Av\|^2_{L^2(Q_0')} + \|Bv\|^2_{L^2(Q_0')} + \int_{Q_0'} s\,|Dv|^2\,dt + \int_{Q_0'} s^3 v^2 dt \\
+ \quad & C\alpha_0 \int_0^T s(c_d Dv)^2_{n+\frac{1}{2}} + C\alpha_0 \int_0^T s^3 v^2(a) + \int_0^T C_{\lambda,\mathfrak{K}} h(\partial_t v(a))^2 \\
\leq \quad & C_{\lambda,\mathfrak{K}} \left(\|rf_1\|^2_{L^2(Q_0')} + 2\int_0^T \int_{\omega_0} s(Dv)^2 + 2\int_0^T \int_{\omega_0} s^3 v^2 \right. \\
& \left. + \int_{\Omega_0'} s^2 (v^2_{|_{t=0}} + v^2_{|_{t=T}}) + \int_{\Omega_0'} (Dv(T))^2 + s v^2(a)|_{t=0}^{t=T} \right) \\
+ \quad & \int_0^T \big(sT\theta \mathcal{O}_{\lambda,\mathfrak{K}}(sh) + T^2\theta^2 \mathcal{O}_{\lambda,\mathfrak{K}}(sh) \big) v^2(a) + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh) \partial_t v(a) (c_d Dv)_{n+\frac{1}{2}} \\
+ \quad & \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1) \partial_t v(a) h (rf_1)_{n+1} + 2\underline{X} + 2\underline{Y} + 2\underline{Z} \qquad , \qquad (4.7)
\end{aligned}
$$

where $\underline{Z} = \mu_r + \mu_1$ with μ_r and μ_1 are given as in Lemma 4.14 and where

$$
\begin{aligned}
\underline{X} \quad = \quad & \int_{Q_0'} \bar{\mu} v^2 \quad + \quad \int_{Q_0'} \bar{\nu} (Dv)^2 \\
+ \quad & X_{12} + X_{13} + X_{23} + X_{31} + X_{32} + X_{33},
\end{aligned}
$$

with $\bar{\mu} = s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ and $\bar{\nu}$ of the form $s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$.

By using the Young's inequality, we estimate in turn all the terms of $\underline{Y}$, $\underline{Z}$ and the two terms at the RHS of (4.7) through the following Lemma whose proof can be found in Appendix A

Lemma 4.16. *For $sh \leq \mathfrak{K}$, we have*

$$
\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1) \partial_t v(a) h (rf_1)_{n+1} \leq \epsilon \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1) h (\partial_t v(a))^2 + C_\epsilon \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1) h (rf_1)^2_{n+1},
$$

$$
\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh) \partial_t v(a) (c_d Dv)_{n+\frac{1}{2}} \leq \epsilon \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1) h (\partial_t v(a))^2 + C_\epsilon \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) (c_d Dv)^2_{n+\frac{1}{2}}.
$$

$$
\left| Y_{11}^{(2,2)} \right| \leq \int_0^T \alpha_{11} v^2_{n+1} + \int_0^T \beta_{11} h (\partial_t v)^2_{n+1} + \int_0^T \gamma_{11} (c_d Dv)^2_{n+\frac{1}{2}} + \int_0^T \eta_{11} h (rf_1)^2_{n+1},
$$

$$
\begin{aligned}
\alpha_{11} = \left(s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 + sT^2\theta^2 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4 \right) \qquad & \beta_{11} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3, \\
\gamma_{11} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 \qquad & \eta_{11} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3.
\end{aligned}
$$

$$
|Y_{12}| \leq \int_0^T \alpha_{12} v^2_{n+1} + \int_0^T \beta_{12} h (\partial_t v)^2_{n+1} + \int_0^T \gamma_{12} (c_d Dv)^2_{n+\frac{1}{2}} + \int_0^T \eta_{12} h (rf_1)^2_{n+1},
$$

$$\alpha_{12} = \left(s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\right) \qquad \beta_{12} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh),$$
$$\gamma_{12} = \mathcal{O}_{\lambda,\mathfrak{K}}(1) \qquad \eta_{12} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh).$$

$$\left|Y_{21}^{(1,22)}\right| \leq \int_0^T \alpha_{21} v_{n+1}^2 + \int_0^T \beta_{21} h(\partial_t v)_{n+1}^2 + \int_0^T \gamma_{21}(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_{21} h(rf_1)_{n+1}^2,$$

$$\alpha_{21} = \left(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4\right) \qquad \beta_{21} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3,$$
$$\gamma_{21} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 \qquad \eta_{21} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3.$$

$$\left|Y_{22}^{(1)}\right| \leq \int_0^T \alpha_{22} v_{n+1}^2 + \int_0^T \beta_{22} h(\partial_t v)_{n+1}^2 + \int_0^T \gamma_{22}(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_{22} h(rf_1)_{n+1}^2,$$

$$\alpha_{22} = \left(s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4\right) \qquad \beta_{22} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3,$$
$$\gamma_{22} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 \qquad \eta_{22} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3.$$

$$\left|Y_{23}^{(1)}\right| \leq \int_0^T \alpha_{23} v_{n+1}^2 + \int_0^T \beta_{23} h(\partial_t v(a))^2 + \int_0^T \gamma_{23}(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_{22} h(rf_1)_{n+1}^2,$$

$$\alpha_{23} = \left(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3\right) \qquad \beta_{23} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh),$$
$$\gamma_{23} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 \qquad \eta_{22} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2.$$

$$\left|Y_{31}^{(1)}\right| \leq \int_0^T \alpha_{31} v_{n+1}^2 + \int_0^T \beta_{23} h(\partial_t v)_{n+1}^2 + \int_0^T \gamma_{23}(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_{22} h(rf_1)_{n+1}^2,$$

$$\alpha_{31} = \left(s^2T\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\right) \qquad \beta_{23} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh),$$
$$\gamma_{23} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) \qquad \eta_{22} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh).$$

$$\mu_1 \leq \int_0^T \alpha_1 v_{n+1}^2 + \int_0^T \beta_1 h(\partial_t v)_{n+1}^2 + \int_0^T \gamma_1(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_1 h(rf_1)_{n+1}^2,$$

$$\alpha_1 = \left(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3\right) \qquad \beta_1 = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,$$
$$\gamma_1 = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) \qquad \eta_1 = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2.$$

$$\mu_r \leq \int_0^T \alpha_r v_{n+1}^2 + \int_0^T \beta_r h(\partial_t v)_{n+1}^2 + \int_0^T \gamma_r(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_r h(rf_1)_{n+1}^2,$$

$$\alpha_r = \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2\mathcal{O}_{\epsilon,\lambda,\mathfrak{K}}(sh)^2 + \epsilon s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)\Big) \qquad \beta_r = \mathcal{O}_{\epsilon,\lambda,\mathfrak{K}}(sh),$$
$$\gamma_r = \Big(s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + \epsilon s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\Big) \qquad \eta_r = \mathcal{O}_{\epsilon,\lambda,\mathfrak{K}}(sh).$$

Futhermore, we can estimate the term in X_{12} as follows

$$\begin{aligned}\int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\tilde{v}Dv &\leq \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)(\tilde{v})^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2 \\ &\leq \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\overline{|v|}^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2 \\ &= \int_{Q_0} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2,\end{aligned}$$

by Lemma 3.3 and as $\int_{\Omega_0'} \mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\overline{|v|}^2 = \int_{\Omega_0} \mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2$.

Observe that

$$1 \leq T^2\theta \text{ and } \left|\partial_t^2\theta\right| \leq CT^2\theta^3.$$

We can now choose ϵ_4 and h_0 sufficiently small, with $0 < \epsilon_4 \leq \epsilon_3(\lambda_2)$, $0 < h_0 \leq h_1(\lambda_2)$, and $\tau_2 \geq 1$ sufficiently large, such that for $\tau \geq \tau_2(T+T^2)$, $0 < h \leq h_0$, and $\tau h(max_{[0,T]}\theta) \leq \epsilon_4$, from (4.7) and Lemma 4.16 we get

$$\begin{aligned}&\|Av\|^2_{L^2(Q_0')} + \|Bv\|^2_{L^2(Q_0')} + \int_{Q_0'} s\,|Dv|^2 + \int_{Q_0'} s^3v^2 \\ +\;& C\alpha_0\int_0^T s(c_dDv)^2_{n+\frac{1}{2}} + C\alpha_0\int_0^T s^3v_{n+1}^2 + C_{\lambda,\mathfrak{K}}\int_0^T h(\partial_t v(a))^2 \\ \leq\;& C_{\lambda,\mathfrak{K}}\Bigg(\|rf_1\|^2_{L^2(Q_0')} + \int_0^T\int_{\omega_0} s(Dv)^2 + \int_0^T\int_{\omega_0} s^3v^2 \\ &+h^{-2}\Big(\int_{\Omega_0'} v^2_{|_{t=0}} + \int_{\Omega_0'} v^2_{|_{t=T}}\Big) + sv^2(a)|_{t=0}^{t=T}\Bigg) + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)h(rf_1)^2_{n+1} \\ +\;& \int_{Q_0} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{Q_0} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v)^2 + \int_{Q_0} s^2T\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2.\end{aligned} \tag{4.8}$$

where we used that $(Dv)^2 \leq Ch^{-2}((\tau^+v)^2+(\tau^-v)^2)$ and the last three terms whose integral taken on domain Q_0 come from the term in X_{12}, X_{13} and X_{23} respectively.

As $\tau \geq \tau_2(T+T^2)$ then $s \geq \tau_2 > 0$ and furthermore we observe that

$$\begin{aligned}\left\|s^{-\frac{1}{2}}\partial_t v\right\|^2_{L^2(Q_0')} &\leq \mathcal{C}_{\lambda,\mathfrak{K}}\Big(\left\|s^{-\frac{1}{2}}Bv\right\|^2_{L^2(Q_0')} + \left\|s^{\frac{1}{2}}v\right\|^2_{L^2(Q_0')} + \left\|s^{\frac{1}{2}}Dv\right\|^2_{L^2(Q_0')}\Big) \\ &\leq \mathcal{C}_{\lambda,\tau,\mathfrak{K}}\Big(\|Bv\|^2_{L^2(Q_0')} + \left\|s^{\frac{3}{2}}v\right\|^2_{L^2(Q_0')} + \left\|s^{\frac{1}{2}}Dv\right\|^2_{L^2(Q_0')}\Big).\end{aligned}$$

We then add the following terms $\int_0^T hs^3v_{n+1}^2$ and $\int_0^T hs^{-1}(\partial_t v(a))^2$ on both the right hand side and the left hand side of (4.8). This allows us to change the domain of integration from Q_0' to Q_0 for the discrete integrals on the primal mesh.

No additional term is required for discrete integrals on the dual mesh. For sh sufficiently small and $s \geq 1$ sufficiently large, these terms at the right hand side are then absorbed by the terms at the left hand side. More precisely, with $0 < \epsilon_0 \leq \epsilon_4$ sufficiently small and for $\tau \geq \tau_2(T + T^2)$, $0 < h \leq h_0$, and $0 < \tau h(max_{[0,T]}\theta) \leq \epsilon_0$ we thus obtain

$$\begin{aligned} & \left\| s^{-\frac{1}{2}} \partial_t v \right\|^2_{L^2(Q_0)} \quad + \quad \int_{Q_0} s(Dv)^2 \quad + \quad \int_{Q_0} s^3 v^2 \\ \leq \; & C_{\lambda,\mathfrak{K}} \left(\|rf_1\|^2_{L^2(Q_0)} + \int_0^T \int_{\omega_0} s(Dv)^2 + \int_0^T \int_{\omega_0} s^3 v^2 \right. \\ & \left. + h^{-2} \Big(\int_{\Omega_0} v^2_{|t=0} + \int_{\Omega_0} v^2_{|t=T} \Big) + s\mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2(a)|_{t=0}^{t=T} \right). \end{aligned} \tag{4.9}$$

Now we shall estimate the term $s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)_{|t=T}$. We have

$$\left\| v_{|t=T} \right\|^2_{L^2(\Omega_0)} = \sum_{j=1}^{n+m+1} h v^2_{j|t=T} \geq h \left\| v_{|t=T} \right\|^2_{L^\infty(\Omega_0)}.$$

It follows that, as sh is bounded

$$\begin{aligned} \left| s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)_{|t=T} \right| \leq C_{\lambda,\mathfrak{K}}\; s \left\| v_{|t=T} \right\|^2_{L^\infty(\Omega_0)} \quad & \leq \quad C_{\lambda,\mathfrak{K}}\; sh^{-1} \left\| v_{|t=T} \right\|^2_{L^2(\Omega_0)} \\ & \leq \quad C_{\lambda,\mathfrak{K}}\; h^{-2} \left\| v_{|t=T} \right\|^2_{L^2(\Omega_0)}. \end{aligned}$$

Similarly, we treat the term $s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)_{|t=0}$ as

$$\left| s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)_{|t=0} \right| \leq C_{\lambda,\mathfrak{K}}\; h^{-2} \left\| v_{|t=0} \right\|^2_{L^2(\Omega_0)}.$$

Therefore, (4.9) can be written as

$$\begin{aligned} & \left\| s^{-\frac{1}{2}} \partial_t v \right\|^2_{L^2(Q_0)} \quad + \quad \left\| s^{\frac{1}{2}} Dv \right\|^2_{L^2(Q_0)} + \quad \left\| s^{\frac{3}{2}} v \right\|^2_{L^2(Q_0)} \\ \leq \; & C_{\lambda,\mathfrak{K}} \left(\|rf_1\|^2_{L^2(Q_0)} + \left\| s^{\frac{1}{2}} Dv \right\|^2_{L^2\left((0,T)\times\omega_0\right)} + \left\| s^{\frac{3}{2}} v \right\|^2_{L^2\left((0,T)\times\omega_0\right)} \right. \\ & \left. + h^{-2} \Big(\int_{\Omega_0} v^2_{|t=0} + \int_{\Omega_0} v^2_{|t=T} \Big) \right). \end{aligned}$$

We next remove the volume norm $\left\| s^{\frac{1}{2}} Dv \right\|^2_{L^2((0,T)\times\omega_0)}$ by proceeding as in the proof of Theorem 4.1 in [BHL10a] we thus write

$$\begin{aligned} & \tau^{-1} \left\| \theta^{-\frac{1}{2}} e^{\tau\theta\varphi} \partial_t u \right\|^2_{L^2(Q_0)} + \tau \left\| \theta^{\frac{1}{2}} e^{\tau\theta\varphi} Du \right\|^2_{L^2(Q_0)} + \tau^3 \left\| \theta^{\frac{3}{2}} e^{\tau\theta\varphi} u \right\|^2_{L^2(Q_0)} \\ & \leq C_{\lambda,\mathfrak{K}} \left(\left\| e^{\tau\theta\varphi} P^{\mathfrak{M}} u \right\|^2_{L^2(Q_0)} + \tau^3 \left\| \theta^{\frac{3}{2}} e^{\tau\theta\varphi} u \right\|^2_{L^2((0,T)\times\omega_0)} \right. \\ & \qquad \left. + h^{-2} \left| e^{\tau\theta\varphi} u|_{t=0} \right|^2_{L^2(\Omega_0)} + h^{-2} \left| e^{\tau\theta\varphi} u|_{t=T} \right|^2_{L^2(\Omega_0)} \right), \end{aligned} \tag{4.10}$$

As we have $\max_{[0,T]}\theta \leq \frac{1}{T\alpha}$, we see that a sufficient condition for $\tau h\big(\max_{[0,T]}\theta\big) \leq \epsilon_0$ then becomes $\tau h(T\alpha)^{-1} \leq \epsilon_0$. To finish the proof, we need to express all the terms in the estimate above in terms of the original function u. We can proceed exactly as in the end of proof of Theorem 4.1 in [BHL10a].

5 Carleman estimates for regular non uniform meshes

In this section we focus on extending the above result to the class of non piecewise uniform meshes introduced in Section 1.2. We choose a function ϑ satisfying (1.6) and further $\vartheta|_{[a-\delta,a+\delta]}$ is chosen affine (for some $\delta > 0$ to remain fixed in the sequel). The way we proceed here is similar to what is done in [BHL10a]. In this framework, we shall prove a non-uniform Carleman estimate for the parabolic operator $\mathcal{P}^{\mathfrak{M}} = -\partial_t + \mathcal{A}^{\mathfrak{M}}$ on the mesh $\mathfrak{M}$ by using the result on uniform meshes of Section 4.

By using first-order Taylor formulae we obtain the following result.

Lemma 5.1. *Let us define $\zeta \in \mathbb{R}^{\overline{\mathfrak{M}}}$ and $\overline{\zeta} \in \mathbb{R}^{\mathfrak{M}}$ as follows*

$$\zeta_{i+\frac{1}{2}} = \frac{h'_{i+\frac{1}{2}}}{h}, i \in \{0,\dots,n+m+1\} \qquad \overline{\zeta_i} = \frac{h'_i}{h}, i \in \{1,\dots,n+m+1\}$$

These two discrete functions are connected to the geometry of the primal and dual meshes $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ and we have

$$0 < \inf_{\Omega_0}\vartheta' \leq \zeta_{i+\frac{1}{2}} \leq \sup_{\Omega_0}\vartheta', \qquad \forall i \in 0,\dots,n+m+1$$
$$0 < \inf_{\Omega_0}\vartheta' \leq \overline{\zeta}_i \leq \sup_{\Omega_0}\vartheta', \qquad \forall i \in 1,\dots,n+m+1$$
$$\big|\bar{D}\zeta\big|_{L_\infty(\Omega)} \leq \frac{\|\vartheta''\|_{L_\infty}}{\inf_{\Omega_0}\vartheta'}, \qquad \big|D\bar{\zeta}\big|_{L_\infty(\Omega)} \leq \frac{\|\vartheta''\|_{L_\infty}}{\inf_{\Omega_0}\vartheta'}.$$

We introduce some notation. To any $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$, we associate the discrete function denoted by $\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u \in \mathbb{C}^{\mathfrak{M}_0\cup\partial\mathfrak{M}_0}$ defined on the uniform mesh $\mathfrak{M}_0$ which takes the same values as u at the corresponding nodes. More precisely, if $u = \sum_{i=1}^{n+m+1}\mathbf{1}_{[x'_{i-\frac{1}{2}},x'_{i+\frac{1}{2}}]}u_i$, we let

$$\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u = \sum_{i=1}^{n+m+1}\mathbf{1}_{[(i-\frac{1}{2})h,(i+\frac{1}{2})h]}u_i$$

and $(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u)_0 = u_0$, $(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u)_{n+m+2} = u_{n+m+2}$. Similarly, for $u \in \mathbb{C}^{\overline{\mathfrak{M}}}$, $u = \sum_{i=1}^{n+m+1}\mathbf{1}_{[x'_i,x'_{i+1}]}u_{i+\frac{1}{2}}$, we set

$$\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}u = \sum_{i=0}^{n+m+1}\mathbf{1}_{[ih,(i+1)h]}u_{i+\frac{1}{2}}.$$

The operators $\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}$ and $\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}$ are invertible and we denote by $\mathcal{Q}^{\mathfrak{M}}_{\mathfrak{M}_0}$ and $\mathcal{Q}^{\overline{\mathfrak{M}}}_{\overline{\mathfrak{M}_0}}$ their respective inverses. We give commutation properties between these operators and discrete-difference operators through the following Lemmata whose proofs can be found in [BHL10a].

Lemma 5.2. *[BHL10a, see the proof of Lemma 5.2]*

1. *For any $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ and any $v \in \mathbb{C}^{\overline{\mathfrak{M}}}$, we have*

$$D(\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}u) = \mathcal{Q}_{\overline{\mathfrak{M}}}^{\overline{\mathfrak{M}_0}}(\zeta Du), \qquad \bar{D}\mathcal{Q}_{\overline{\mathfrak{M}}}^{\overline{\mathfrak{M}_0}}v = \mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}(\overline{\zeta}\ \bar{D}v)$$

2. *For any $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ we have*

$$\bar{D}(c_dDu) = (\overline{\zeta})^{-1}\mathcal{Q}_{\mathfrak{M}_0}^{\mathfrak{M}}\Big(\bar{D}\big((\mathcal{Q}_{\overline{\mathfrak{M}}}^{\overline{\mathfrak{M}_0}}\frac{c_d}{\zeta})D(\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}u)\big)\Big).$$

Lemma 5.3. *[BHL10a, see proof of Lemma 5.3]*
For any $u \in \mathbb{C}^{\mathfrak{M}}$ and any $v \in \mathbb{C}^{\overline{\mathfrak{M}}}$, we have

$$(\sup_{\Omega_0}\vartheta')^{-1}\,|u|^2_{L^2(\Omega)} \le \Big|\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}u\Big|^2_{L^2(\Omega_0)} \le (\inf_{\Omega_0}\vartheta')^{-1}\,|u|^2_{L^2(\Omega)}$$
$$(\sup_{\Omega_0}\vartheta')^{-1}\,|v|^2_{L^2(\Omega)} \le \Big|\mathcal{Q}_{\overline{\mathfrak{M}}}^{\overline{\mathfrak{M}_0}}u\Big|^2_{L^2(\Omega_0)} \le (\inf_{\Omega_0}\vartheta')^{-1}\,|v|^2_{L^2(\Omega)}$$

Futhermore, the same inequalities hold by replacing Ω by ω and Ω_0 by ω_0, respectively.

For any continuous function f defined on $\overline{\Omega}$ (resp. on $\overline{\Omega_0}$) we denote by $\Pi_{\mathfrak{M}}f = (f(x_i'))_{0\le i\le n+m+2} \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ the sampling of f on $\mathfrak{M}$ (resp. $\Pi_{\mathfrak{M}_0}f = (f(ih))_{0\le i\le n+m+2} \in \mathbb{C}^{\mathfrak{M}_0\cup\partial\mathfrak{M}_0}$ the sampling of f on $\mathfrak{M}_0$).

Lemma 5.4. *[BHL10a, see the proof of Lemma 5.4]*
Let f be a continuous function defined on Ω

$$\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}\Pi_{\mathfrak{M}}f = \Pi_{\mathfrak{M}_0}(f\circ\vartheta).$$

In particular, for $u \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ we have

$$\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}\Big((\Pi_{\mathfrak{M}}f)u\Big) = \Pi_{\mathfrak{M}_0}(f\circ\vartheta)(\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}u).$$

Moreover, by making use of Taylor formulae we get the following result

Lemma 5.5. *With ζ defined as in Lemma 5.1 we have*

$$\big\|\bar{D}D\nu\big\|_\infty < \infty, \qquad \big\|\bar{D}\tilde{\nu}\big\|_\infty < \infty, \qquad 0 < \|\nu\|_\infty, \|\tilde{\nu}\|_\infty < \infty$$

where $\nu := \frac{1}{\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}\overline{\zeta}}$.

Proof. From the definition of ζ, $\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}$ and D acting on $\mathbb{C}^{\mathfrak{M}_0}$, $\bar{D}$ acting on $\mathbb{C}^{\overline{\mathfrak{M}_0}}$ we have

$$\begin{aligned}\Big(\bar{D}D(\frac{1}{\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}\overline{\zeta}})\Big)_i &:= (\bar{D}D\nu)_i \\ &= \frac{\nu_{i+1}-2\nu_i+\nu_{i-1}}{h^2} \\ &= \frac{1}{h}\frac{(h_i'-h_{i+1}')h_{i-1}' - (h_{i-1}'-h_i')h_{i+1}'}{h_{i-1}'h_i'h_{i+1}'}. \qquad (5.1)\end{aligned}$$

We find

$$h'_i = x'_{i+\frac{1}{2}} - x'_{i-\frac{1}{2}} = \frac{x'_{i+1} - x'_{i-1}}{2} = \frac{\vartheta\big((i+1)h\big) - \vartheta\big((i-1)h\big)}{2} = \frac{\vartheta_{i+1} - \vartheta_{i-1}}{2},$$
$$h'_{i+1} = \frac{\vartheta\big((i+2)h\big) - \vartheta\big(ih\big)}{2} = \frac{\vartheta_{i+2} - \vartheta_i}{2},$$
$$h'_{i-1} = \frac{\vartheta(ih) - \vartheta\big((i-2)h\big)}{2} = \frac{\vartheta_i - \vartheta_{i-2}}{2}.$$

By using Taylor formulae we write

$$\begin{aligned}
\vartheta_{i+2} &= \vartheta_i + (2h)\vartheta'_i + \frac{(2h)^2}{2}\vartheta''_i + \frac{(2h)^3}{6}\vartheta'''_i + \frac{(2h)^4}{24}\vartheta^{(4)}_i + \mathcal{O}(h^5),\\
\vartheta_{i-2} &= \vartheta_i - (2h)\vartheta'_i + \frac{(2h)^2}{2}\vartheta''_i - \frac{(2h)^3}{6}\vartheta'''_i + \frac{(2h)^4}{24}\vartheta^{(4)}_i + \mathcal{O}(h^5),\\
\vartheta_{i+1} &= \vartheta_i + h\vartheta'_i + \frac{h^2}{2}\vartheta''_i + \frac{h^3}{6}\vartheta'''_i + \frac{(h)^4}{24}\vartheta^{(4)}_i + \mathcal{O}(h^5),\\
\vartheta_{i-1} &= \vartheta_i - h\vartheta'_i + \frac{h^2}{2}\vartheta''_i - \frac{h^3}{6}\vartheta'''_i + \frac{(h)^4}{24}\vartheta^{(4)}_i + \mathcal{O}(h^5).
\end{aligned}$$

Thus we have

$$\begin{aligned}
h'_i &= 2h\vartheta'_i + \frac{2h^3}{6}\vartheta'''_i + \mathcal{O}(h^5),\\
h'_{i+1} &= 2h\vartheta'_i + \frac{(2h)^2}{2}\vartheta''_i + \frac{(2h)^3}{6}\vartheta''' + \frac{(2h)^4}{24}\vartheta^{(4)} + \mathcal{O}(h^5),\\
h'_{i-1} &= 2h\vartheta'_i - \frac{(2h)^2}{2}\vartheta''_i + \frac{(2h)^3}{6}\vartheta''' - \frac{(2h)^4}{24}\vartheta^{(4)} + \mathcal{O}(h^5).
\end{aligned}$$

From (5.1) we obtain

$$\bar{D}D(\frac{1}{\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}\overline{\zeta}})_i = \frac{N}{D},$$

where

$$\begin{aligned}
N &= (h'_i - h'_{i+1})h'_{i-1} - (h'_{i-1} - h'_i)h'_{i+1}\\
&= \Big(-\frac{(2h)^2}{2}\vartheta''_i - h^3\vartheta'''_i - \frac{(2h)^4}{24}\vartheta^{(4)}_i + \mathcal{O}(h^5)\Big)\big((2h)\vartheta'_i - \frac{(2h)^2}{2}\vartheta''_i + \mathcal{O}(h^3)\big)\\
&- \Big(-\frac{(2h)^2}{2}\vartheta''_i - h^3\vartheta'''_i - \frac{(2h)^4}{24}\vartheta^{(4)}_i + \mathcal{O}(h^5)\Big)\big((2h)\vartheta'_i + \frac{(2h)^2}{2}\vartheta''_i + \mathcal{O}(h^3)\big)\\
&= \frac{(2h)^4}{2}(\vartheta''_i)^2 + \mathcal{O}(h^5),
\end{aligned}$$

and

$$D = h \times h'_{i-1} \times h'_i \times h'_{i+1} = (2h)^4(\vartheta'_i)^3 + \mathcal{O}(h^5).$$

Thus, we have

$$\left|\bar{D}D(\frac{1}{\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}\overline{\zeta}})_i\right| \lesssim (\inf\vartheta')^{-3} < \infty,$$

which proves the first result. Next, we proceed with the second result in the same manner as above. We have

$$(\bar{D}\tilde{\nu})_i = \frac{\tilde{\nu}_{i+\frac{1}{2}} - \tilde{\nu}_{i+\frac{1}{2}}}{h} = \frac{\nu_{i+1} - \nu_{i-1}}{2h} = \frac{1}{2h}\big(\frac{h}{h'_{i+1}} - \frac{h}{h'_{i-1}}\big) = \frac{h'_{i-1} - h'_{i+1}}{h'_{i+1}h'_{i-1}}.$$

By using the computations of h'_{i-1}, h'_{i+1} above we find

$$\left|(\bar{D}\tilde{\nu})_i\right| = \left|\frac{-(2h)^2\vartheta'' + \mathcal{O}(h^4)}{(2h)^2\vartheta'^2 + \mathcal{O}(h^3)}\right| \lesssim \frac{\|\vartheta''\|_\infty}{(\inf\vartheta')^2} < \infty,$$

which yields the second result.

Moreover, with the properties of ζ shown as in Lemma 5.1 we can assert

$$0 < \|\nu\|_\infty, \|\tilde{\nu}\|_\infty < \infty.$$

□

From Lemmata 5.2 – 5.4 we thus obtain the following discrete Carleman estimate for the operator $P^{\mathfrak{M}} = -\partial_t - D(c_d D.)$ on the mesh $\mathfrak{M}$.

Theorem 5.6. *Let $\omega \subset \Omega_2$ be a non-empty open set and we set $f := \bar{D}(c_d Du)$. For the parameter $\lambda > 1$ sufficiently large, there exists C, $\tau_0 \geq 1$, $h_0 > 0$, $\epsilon_0 > 0$, depending on ω such that for any mesh $\mathfrak{M}$ obtained from ϑ by (1.6) – (1.7), we have*

$$\begin{aligned}
&\tau^{-1}\left\|\theta^{-\frac{1}{2}}e^{\tau\theta\varphi}\partial_t u\right\|^2_{L^2(Q)} + \tau\left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi}Du\right\|^2_{L^2(Q)} + \tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi}u\right\|^2_{L^2(Q)}\\
&\leq C_{\lambda,\mathfrak{K}}\left(\left\|e^{\tau\theta\varphi}P^{\mathfrak{M}}u\right\|^2_{L^2(Q)} + \tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi}u\right\|^2_{L^2((0,T)\times\omega)}\right.\\
&\qquad\left. + h^{-2}\left|e^{\tau\theta\varphi}u|_{t=0}\right|^2_{L^2(\Omega)} + h^{-2}\left|e^{\tau\theta\varphi}u|_{t=T}\right|^2_{L^2(\Omega)}\right), \quad (5.2)
\end{aligned}$$

for all $\tau \geq \tau_0(T+T^2)$, $0 < h \leq h_0$ and $\tau h(\alpha T)^{-1} \leq \epsilon_0$ and for all $u \in C^\infty(0,T;\mathbb{C}^{\mathfrak{M}})$ satisfying $u|_{\partial\Omega} = 0$.

Proof. We set $w = \mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}} u$ defined on the uniform mesh $\mathfrak{M}_0$. By using Lemma 5.2 we have

$$\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}(\bar{\zeta}P^{\mathfrak{M}}u) = -(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}\bar{\zeta})\partial_t w - \bar{D}\Big((\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}\frac{c_d}{\zeta})Dw\Big). \quad (5.3)$$

We observe that the right-hand side of (5.3) is a semi-discrete parabolic operator of the form $P^{\mathfrak{M}_0} = \xi'(-\partial_t - \frac{1}{\xi'}D(\xi_d D.))$ applied to w, where

$$\xi' = \mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}\bar{\zeta}, \quad \xi_d = \mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}\frac{c_d}{\zeta}. \quad (5.4)$$

We set $\nu := \frac{1}{\xi'} = \frac{1}{\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}\bar{\zeta}}$ and we find

$$\bar{\tilde{\nu}} = \nu + h^2\bar{D}D\nu = \nu + h^2\mathcal{O}(1),$$

by using Lemma 3.3 and Lemma 5.5.

Thus, the operator $P^{\mathfrak{M}_0}$ can be written in form as

$$P^{\mathfrak{M}_0}w = \xi'\big(-\partial_t w - \bar{\tilde{\nu}}\bar{D}(\xi_d Dw) + h^2\mathcal{O}(1)\bar{D}(\xi_d Dw)\big).$$

Moreover, using Lemma 3.2 we have

$$\bar{\tilde{\nu}}\bar{D}(\xi_d Dw) = \bar{D}(\tilde{\nu}\xi_d Dw) - \bar{D}(\tilde{\nu})\overline{\xi_d Dw}.$$

We set $P_0^{\mathfrak{M}_0} w := -\partial_t w - \bar{D}(\tilde{\nu}\xi_d Dw) = -\partial_t w - \bar{D}(b_d Dw)$ with $b_d = \tilde{\nu}\xi_d$. From the properties of $\tilde{\nu}$ and ξ_d it follows that

$$0 < b_{min} \leq b \leq b_{max} \quad \text{and} \quad \left\|\bar{D}(b_d)\right\|_\infty < \infty.$$

First, we shall obtain a Carleman estimate for $P_0^{\mathfrak{M}_0}$. Then we shall deduce a Carleman estimate for the operator

$$P^{\mathfrak{M}_0} w = \xi'\big(P_0^{\mathfrak{M}_0} w + \bar{D}(\tilde{\nu})\overline{\xi_d Dw} + h^2\mathcal{O}(1)\bar{D}(\xi_d Dw)\big) \tag{5.5}$$

Now, we consider the function $\psi \circ \vartheta : (t,x) \mapsto \psi\big(t, \vartheta(x)\big)$. By using the properties listed in Lemma 2.1 and (1.6), we shall see that $\psi \circ \vartheta$ is a suitable weight function associated to the control domain $\omega_0 = \vartheta^{-1}(w)$ in Ω_0, i.e., that $\psi \circ \vartheta$ satisfies Lemma 2.1 for the domaims Ω_0 and ω_0.

The important property to checking is the trace property. The remaining properties are left to the reader. We set

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

with

$$\begin{aligned} b_{11} &= [(\psi\circ\vartheta)'\star]_a \\ b_{22} &= [b(\psi\circ\vartheta)'\star]_a^2(\psi\circ\vartheta)'(a^+) + [b^2(\psi\circ\vartheta)'^3\star]_a \\ b_{12} = b_{21} &= [b(\psi\circ\vartheta)'\star]_a(\psi\circ\vartheta)'(a^+) \end{aligned}$$

where $b = \frac{1}{\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}\bar{\zeta}}\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}\frac{c}{\zeta} = \frac{c\circ\vartheta}{\bar{\zeta}^2}$. Morever, we have $\vartheta'_+(a) = \vartheta'_-(a)$ and $\bar{\zeta}_{n+1} = \vartheta'_{n+1}$ (recall that $\vartheta|_{[a-\delta,a+\delta]}$ is an affine function). It follows that

$$\begin{aligned} b_{11} &= [\psi'(\vartheta)\star]_a\vartheta'(a) = [\psi'\star]_{a'}\vartheta'(a), \\ b_{22} &= [\frac{c\circ\vartheta}{\bar{\zeta}^2}\psi'(\vartheta)\vartheta'\star]_a^2\psi'(\vartheta)(a^+)\vartheta'(a^+) + [(\frac{c\circ\vartheta}{\bar{\zeta}^2})^2\big(\psi'(\vartheta)\vartheta'\big)^3\star]_a \\ &= [c\psi'\star]_{a'}\psi'(a^+)\frac{1}{\vartheta'(a)} + [c^2(\psi')^3\star]_{a'}\frac{1}{\vartheta'(a)}, \\ b_{12} = b_{21} &= [\frac{c\circ\vartheta}{\bar{\zeta}^2}\psi'(\vartheta)\vartheta'\star]_a\psi'(\vartheta(a^+))\vartheta'(a^+) \\ &= [c\psi'\star]_{a'}\psi'(a'^+). \end{aligned}$$

We can see that $(Bw,w) = (Aw,w) \geq \alpha_0 \|w\|^2$. This means that $\psi\circ\vartheta$ satisfies the trace property.

Through Theorem 4.1, we obtained a discrete uniform Carleman estimate for $P_0^{\mathfrak{M}_0}$ and the Carleman weight function is of the form $r_0 = e^{s\varphi_0}$, with $\varphi_0 = \varphi\circ\vartheta = e^{\lambda\psi_0} - e^{\lambda K}$ where $\psi_0 = \psi\circ\vartheta$ on the uniform mesh $\mathfrak{M}_0$. We can deduce the same result on the non-uniform mesh $\mathfrak{M}$. Namely, through (4.2) we see that the following estimate holds

$$\begin{aligned} &\tau^{-1}\left\|\theta^{-\frac{1}{2}}e^{\tau\theta\varphi_0}\partial_t w\right\|^2_{L^2(Q_0)} + \tau\left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi_0}Dw\right\|^2_{L^2(Q_0)} + \tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi_0}w\right\|^2_{L^2(Q_0)} \\ &\leq C\left(\left\|e^{\tau\theta\varphi_0}P_0^{\mathfrak{M}_0}w\right\|^2_{L^2(Q_0)} + \tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi_0}w\right\|^2_{L^2((0,T)\times\omega_0)}\right. \\ &\qquad \left. + h^{-2}\left|e^{\tau\theta\varphi_0}w|_{t=0}\right|^2_{L^2(\Omega_0)} + h^{-2}\left|e^{\tau\theta\varphi_0}w|_{t=T}\right|^2_{L^2(\Omega_0)}\right), \end{aligned} \tag{5.6}$$

and the constant C is uniform in h for τ sufficiently large and with $\tau h(\alpha T)^{-1} \leq \epsilon_0$, for ϵ_0 sufficiently small. Note that, setting $\tilde{\epsilon}_0 = (\inf_{\Omega_0} \vartheta')\epsilon_0$, we see that the condition $\tau h'(\alpha T)^{-1} \leq \tilde{\epsilon}_0$ on the size of the non-uniform mesh $\mathfrak{M}$ implies the condition $\tau h(\alpha T)^{-1} \leq \epsilon_0$ for the uniform mesh $\mathfrak{M}_0$.

From (5.5) – (5.6) we deduce the following Carleman estimate for $P^{\mathfrak{M}_0}$

$$\begin{aligned}
&\tau^{-1} \left\| \theta^{-\frac{1}{2}} e^{\tau\theta\varphi_0} \partial_t w \right\|^2_{L^2(Q_0)} + \tau \left\| \theta^{\frac{1}{2}} e^{\tau\theta\varphi_0} Dw \right\|^2_{L^2(Q_0)} + \tau^3 \left\| \theta^{\frac{3}{2}} e^{\tau\theta\varphi_0} w \right\|^2_{L^2(Q_0)} \\
&\leq C \Big(\left\| e^{\tau\theta\varphi_0} P^{\mathfrak{M}_0} w \right\|^2_{L^2(Q_0)} + \left\| e^{\tau\theta\varphi_0} \bar{D}(\tilde{\nu}) \overline{\xi_d Dw} \right\|^2_{L^2(Q_0)} + h^4 \left\| e^{\tau\theta\varphi_0} \bar{D}(\xi_d Dw) \right\|^2_{L^2(Q_0)} \\
&+\tau^3 \left\| \theta^{\frac{3}{2}} e^{\tau\theta\varphi_0} w \right\|^2_{L^2((0,T)\times\omega_0)} + h^{-2} \left| e^{\tau\theta\varphi_0} w|_{t=0} \right|^2_{L^2(\Omega_0)} + h^{-2} \left| e^{\tau\theta\varphi_0} w|_{t=T} \right|^2_{L^2(\Omega_0)} \Big).
\end{aligned} \tag{5.7}$$

Now, by using Lemma 5.5 we estimate $\left\| e^{\tau\theta\varphi_0} \bar{D}(\tilde{\nu}) \overline{\xi_d Dw} \right\|^2_{L^2(Q_0)}$ in the RHS of (5.7) as

$$\left\| e^{\tau\theta\varphi_0} \bar{D}(\tilde{\nu}) \overline{\xi_d Dw} \right\|^2_{L^2(Q_0)} \leq C \left\| e^{s\varphi_0} \overline{\xi_d Dw} \right\|^2_{L^2(Q_0)}.$$

We see that

$$\overline{\xi_d Dw} = \frac{1}{2} \Big(\bar{\tau}_+(\xi_d Dw) + \bar{\tau}_-(\xi_d Dw) \Big).$$

Hence we find

$$\begin{aligned}
&\left\| e^{s\varphi_0} \overline{\xi_d Dw} \right\|^2_{L^2(Q_0)} \\
\leq\ & C \Big(\|e^{s\varphi_0} \bar{\tau}_+(\xi_d Dw)\|^2_{L^2(Q_0)} + \|e^{s\varphi_0} \bar{\tau}_-(\xi_d Dw)\|^2_{L^2(Q_0)} \Big) \\
\leq\ & C \Big(\left\| e^{s\varphi_0} \bar{\tau}_+ \big(\xi_d D(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}} u) \big) \right\|^2_{L^2(Q_0)} + \left\| e^{s\varphi_0} \bar{\tau}_- \big(\xi_d D(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}} u) \big) \right\|^2_{L^2(Q_0)} \Big) \\
\leq\ & C \Big(\left\| e^{s\varphi_0} \bar{\tau}_+ \big(\xi_d \mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}} (\zeta Du) \big) \right\|^2_{L^2(Q_0)} + \left\| e^{s\varphi_0} \bar{\tau}_- \big(\xi_d \mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}} (\zeta Du) \big) \right\|^2_{L^2(Q_0)} \Big) \\
\leq\ & C \Big(\left\| e^{s\varphi_0} \bar{\tau}_+ \mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}} (c_d Du) \right\|^2_{L^2(Q_0)} + \left\| e^{s\varphi_0} \bar{\tau}_- \mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}} (c_d Du) \right\|^2_{L^2(Q_0)} \Big) \\
\leq\ & C \Big(\left\| \mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}} \left(e^{s\varphi} \bar{\tau}_+ (c_d Du) \right) \right\|^2_{L^2(Q_0)} + \left\| \mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}} \left(e^{s\varphi} \bar{\tau}_- (c_d Du) \right) \right\|^2_{L^2(Q_0)} \Big) \\
\leq\ & C(\inf \vartheta')^{-1} \Big(\|e^{s\varphi} \bar{\tau}_+ (c_d Du)\|^2_{L^2(Q)} + \|e^{s\varphi} \bar{\tau}_- (c_d Du)\|^2_{L^2(Q)} \Big),
\end{aligned}$$

by using (5.4) and Lemmata 5.2 – 5.4.

We treat $\|e^{s\varphi} \bar{\tau}_+ (c_d Du)\|_{L^2(Q)}$ (the term $\|e^{s\varphi} \bar{\tau}_- (c_d Du)\|_{L^2(Q)}$ can be treated similarly). We find

$$\|e^{s\varphi} \bar{\tau}_+ (c_d Du)\|_{L^2(Q)} = \|r \bar{\tau}_+ (c_d Du)\|_{L^2(Q)} \leq \|(\tau_- r)(c_d Du)\|_{L^2(Q)} \leq C \|(\tau_- r) Du\|_{L^2(Q)}. \tag{5.8}$$

We have $\tau_- r = r(\rho \tau_- r) = r\big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)$ (due to Proposition 3.10). From that we can write

$$\left\| e^{\tau\theta\varphi_0} \bar{D}(\tilde{\nu}) \overline{\xi_d Dw} \right\|^2_{L^2(Q_0)} \leq C(\inf \vartheta')^{-1} \|e^{s\varphi} Du\|_{L^2(Q)},$$

which allows one to absorb by the term at the LHS of the Carleman estimate by choosing τ sufficiently large.

Next, we estimate $h^4 \left\|e^{\tau\theta\varphi_0}\bar{D}(\xi_d Dw)\right\|^2_{L^2(Q_0)}$ in the RHS of (5.7) as

$$\begin{aligned}
& h^4 \left\|e^{\tau\theta\varphi_0}\bar{D}(\xi_d Dw)\right\|^2_{L^2(Q_0)} \\
= \;& h^2 \left\|e^{s\varphi_0}\bar{\tau}_+(\xi_d Dw) - e^{s\varphi_0}\bar{\tau}_-(\xi_d Dw)\right\|^2_{L^2(Q_0)} \\
\leq \;& Ch^2\Big(\|e^{s\varphi_0}\bar{\tau}_+(\xi_d Dw)\|^2_{L^2(Q_0)} + \|e^{s\varphi_0}\bar{\tau}_-(\xi_d Dw)\|^2_{L^2(Q_0)}\Big) \\
\leq \;& Ch^2\Big(\left\|e^{s\varphi_0}\bar{\tau}_+\big(\xi_d D(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u)\big)\right\|^2_{L^2(Q_0)} + \left\|e^{s\varphi_0}\bar{\tau}_-\big(\xi_d D(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u)\big)\right\|^2_{L^2(Q_0)}\Big) \\
\leq \;& Ch^2\Big(\left\|e^{s\varphi_0}\bar{\tau}_+\big(\xi_d \mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(\zeta Du)\big)\right\|^2_{L^2(Q_0)} + \left\|e^{s\varphi_0}\bar{\tau}_-\big(\xi_d \mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(\zeta Du)\big)\right\|^2_{L^2(Q_0)}\Big) \\
\leq \;& Ch^2\Big(\left\|e^{s\varphi_0}\bar{\tau}_+\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(c_d Du)\right\|^2_{L^2(Q_0)} + \left\|e^{s\varphi_0}\bar{\tau}_-\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(c_d Du)\right\|^2_{L^2(Q_0)}\Big) \\
\leq \;& Ch^2\Big(\left\|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}\Big(e^{s\varphi}\bar{\tau}_+(c_d Du)\Big)\right\|^2_{L^2(Q_0)} + \left\|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}\Big(e^{s\varphi}\bar{\tau}_-(c_d Du)\Big)\right\|^2_{L^2(Q_0)}\Big) \\
\leq \;& Ch^2(\inf\vartheta')^{-1}\Big(\|e^{s\varphi}\bar{\tau}_+(c_d Du)\|^2_{L^2(Q)} + \|e^{s\varphi}\bar{\tau}_-(c_d Du)\|^2_{L^2(Q)}\Big),
\end{aligned}$$

by using (5.4) and Lemmata 5.2 – 5.4. We proceed with an estimate as in (5.8). We thus obtain

$$h^4 \left\|e^{\tau\theta\varphi_0}\bar{D}(\xi_d Dw)\right\|^2_{L^2(Q_0)} \leq Ch^2(\inf\vartheta')^{-1}\left\|e^{s\varphi}Du\right\|_{L^2(Q)},$$

which allows one to absorb by the term in the LHS of Carleman estimate by choosing τ sufficiently large.

Futhermore, by using the previous Lemmata 5.1 – 5.4 and considering each term in (5.7) separately, we see that we have the following estimates

- For the first term in LHS of (5.7)

$$\begin{aligned}
\left\|\theta^{-\frac{1}{2}}e^{\tau\theta\varphi_0}\partial_t w\right\|^2_{L^2(Q_0)} &= \left\|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}(\theta^{-\frac{1}{2}}e^{\tau\theta\varphi}\partial_t u)\right\|^2_{L^2(Q_0)} \\
&\geq (\sup_{\Omega_0}\vartheta')^{-1}\left\|\theta^{-\frac{1}{2}}e^{\tau\theta\varphi}\partial_t u\right\|^2_{L^2(Q)},
\end{aligned}$$

  and a similar inequality holds for $\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi_0}w\right\|^2_{L^2(Q_0)}$.

- For the second term of LHS of (5.7) we use Lemma 5.2 and Lemma 5.3 as follows

$$\begin{aligned}
\left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi_0}Dw\right\|^2_{L^2(Q_0)} &= \left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi_0}D(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u)\right\|^2_{L^2(Q_0)} = \left\|\theta^{\frac{1}{2}}\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(e^{\tau\theta\varphi})\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(\zeta Du)\right\|^2_{L^2(Q_0)} \\
&= \left\|\theta^{\frac{1}{2}}\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(\zeta e^{\tau\theta\varphi}Du)\right\|^2_{L^2(Q_0)} \gtrsim \left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi}Du\right\|^2_{L^2(Q)}.
\end{aligned}$$

- By using (5.3) and Lemma 5.3 we have

$$\begin{aligned}
\left\|e^{\tau\theta\varphi_0}P^{\mathfrak{M}_0}w\right\|^2_{L^2(Q_0)} &= \left\|e^{\tau\theta\varphi_0}\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}(\bar{\zeta}P^{\mathfrak{M}}u)\right\|^2_{L^2(Q_0)} = \left\|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}(e^{\tau\theta\varphi}\bar{\zeta}P^{\mathfrak{M}}u)\right\|^2_{L^2(Q_0)} \\
&\lesssim \left\|e^{\tau\theta\varphi}\bar{\zeta}P^{\mathfrak{M}}u\right\|^2_{L^2(Q)} \lesssim \left\|e^{\tau\theta\varphi}P^{\mathfrak{M}}u\right\|^2_{L^2(Q)}.
\end{aligned}$$

- For the third term of RHS of (5.7)

$$\begin{aligned}\left|e^{\tau\theta\varphi_0}w|_{t=0}\right|^2_{L^2(\Omega_0)} &= \left|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}(e^{\tau\theta\varphi}w|_{t=0})\right|^2_{L^2(\Omega_0)}\\ &\lesssim \left|e^{\tau\theta\varphi}w|_{t=0}\right|^2_{L^2(\Omega)}\end{aligned}$$

and a similar inequality holds for $\left|e^{\tau\theta\varphi_0}w|_{t=T}\right|^2_{L^2(\Omega_0)}$, $\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi_0}w\right\|^2_{L^2((0,T)\times\omega_0)}$.

- Finally, since $\vartheta(\omega_0)=\omega$ we have

$$\begin{aligned}\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi_0}Du\right\|^2_{L^2((0,T)\times\omega_0)} &= \left\|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}(\theta^{\frac{3}{2}}e^{\tau\theta\varphi}Du)\right\|^2_{L^2((0,T)\times\omega_0)}\\ &\lesssim \left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi}Du\right\|^2_{L^2((0,T)\times\omega}\end{aligned}$$

The proof is complete. □

6 Controllability results

The Carleman estimate proved in the previous Section allows to give observability estimate that yields results of controllability to the trajectories for classes of semi-linear heat equations.

6.1 The linear case

We consider the following semi-discrete parabolic problem with potential

$$\partial_t y + \mathcal{A}^{\mathfrak{M}} y + ay = \mathbf{1}_\omega v, \quad t\in(0,T) \qquad y|_{\partial\Omega}=0 \tag{6.1}$$

The adjoint system associated with the controlled system with potential (6.1) is given by

$$-\partial_t q + \mathcal{A}^{\mathfrak{M}} y + ay = 0, \quad t\in(0,T) \qquad q|_{\partial\Omega}=0 \tag{6.2}$$

We assume that a piecewise C^1 diffusion coefficient c satisfies (1.2) and $\Omega=(0,1)$. From Carleman estimate (4.2) we obtain a following observability estimate.

Proposition 6.1. *There exists positive constants C_0, C_1 and C_2 such that for all $T>0$ and all potential fucntion a, under the condition $h\le\min(h_0,h_1)$ with*

$$h_1 = C_0\big(1+\frac{1}{T}+\|a\|_\infty^{\frac{2}{3}}\big)^{-1}$$

any solution of (6.2) *satisfies*

$$|q(0)|_{L^2(\Omega)} \le C_{obs}\,\|q\|^2_{L^2((0,T)\times\omega)} + e^{-\frac{C_1}{h}+T\|a\|_\infty}\,|q(T)|^2_{L^2(\Omega)}\,, \tag{6.3}$$

with $C_{obs}=e^{C_2\left(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{\frac{2}{3}}\right)}$.

Remark 6.2. *In comparision the observability inequality in continuous case which performed in [BDL07], we find that the observability inequality obtained here is weak since there is an additional term depending upon h at right-hand-side of inequality* (6.3).

From the result of Proposition 6.1 we deduce the following controllability result for system (6.1).

Proposition 6.3. *There exists positive constants C_1, C_2, C_3 and for $T > 0$ a map $L_{T,a} : \mathbb{R}^{\mathfrak{M}} \to L^2(0,T;\mathbb{R}^{\mathfrak{M}})$ such that if $h \leq \min(h_0, h_2)$ with*

$$h_1 = C_0\big(1 + \frac{1}{T} + T\,\|a\|_\infty + \|a\|_\infty^{\frac{2}{3}}\big)^{-1}$$

for all initial data $y_0 \in \mathbb{R}^{\mathfrak{M}}$, there exists a semi-discrete control function v given by $v = L_a(y_0)$ such that the solution to (6.1) satisfies

$$|y(T)|_{L^2(\Omega)} \leq C_0 e^{-C_2/h}\,|y_0|_{L^2(\Omega)}$$

and

$$\|v\|_{L^2(Q)} \leq C_0\,|y_0|_{L^2(\Omega)}$$

, with $C_0 = e^{C_3\left(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{\frac{2}{3}}\right)}$.

Note that the final state is of size $e^{-C/h}\,|y_0|_{L^2(\Omega)}$. The proof of these proposition are given in [BL12].

6.2 The semilinear case

We consider the following semilinear semi-discrete control problem

$$\big(\partial_t + \mathcal{A}^{\mathfrak{M}}\big)y + G(y) = \mathbf{1}_\omega v, \quad y \in (0,T) \qquad y|_{\partial\Omega} = 0, \quad y(0) = y_0 \tag{6.4}$$

where $\omega \subset \Omega$. The function $G : \mathbb{R} \to \mathbb{R}$ is assumed of the form

$$G(x) = xg(x), \qquad x \in \mathbb{R}, \tag{6.5}$$

with g Lipschitz continuous. Here, we consider the function g in two cases: $g \in L^\infty(\mathbb{R})$ and the more general case as

$$|g(x)| \leq K \ln^r(e + |x|), \quad x \in \mathbb{R}, \qquad \text{with} \qquad 0 \leq r < \frac{3}{2} \tag{6.6}$$

The results of semi-discrete parabolic with potential above allows one to obtain controllability results for parabolic equation with semi-linear terms whose proofs are given in [BL12]

Theorem 6.4. *We assume that $g \in L^\infty(\mathbb{R})$ and c satisfies (1.2). There exists positive constants C_0, C_1 such that for all $T > 0$ and h chosen sufficiently small, for all initial data $y_0 \in \mathbb{R}^{\mathfrak{M}}$, there exists a semi-discrete control function v with*

$$\|v\|_{L^2(Q)} \leq C\,|y_0|_{L^2(\Omega)}$$

such that the solution to the semi-linear parabolic equation (6.4) satisfies

$$|y(T)|_{L^2(\Omega)} \leq Ce^{-C_0/h}\,|y_0|_{L^2(\Omega)}$$

with $C_0 = e^{C_1\left(1+\frac{1}{T}+T\|g\|_\infty+\|g\|_\infty^{\frac{2}{3}}\right)}$.

Theorem 6.5. *Let $\Omega = (0,1)$, c satisfy (1.2) and G satisfy (6.5) - (6.6). There exists C_0 such that, for $T > 0$ and $M > 0$, there exists positive constants C, h_0 such that for $0 < h \le h_0$ and for all initial data $y_0 \in \mathbb{R}^{\mathfrak{M}}$ satisfying $|y_0|_{L^2(\Omega)} \le M$ there exists a semi-discrete control function v such that the solution to the semi-linear parabolic equation*

$$\big(\partial_t - \bar{D}cD\big)y + G(y) = \mathbf{1}_\omega v, \quad y \in (0,T) \qquad y|_{\partial\Omega} = 0, \quad y(0) = y_0 \tag{6.7}$$

satisfies

$$|y(T)|_{L^2(\Omega)} \le Ce^{-C_0/h}\,|y_0|_{L^2(\Omega)}$$

where $C = C(T, M)$.

Observe that the constants are uniform with respect to discretization parameter h.

A Proofs of Lemma 3.17 and intermediate results in Section 4

A.1 Proof of Lemma 3.17

We have

$$(c_dDu)_{n+\frac{3}{2}} - (c_dDu)_{n+\frac{1}{2}} = hf_{n+1}.$$

As $Du = \tilde{\rho}Dv + D\rho\tilde{v}$ we obtain

$$\begin{aligned} &r_{n+1}\Big(\tilde{\rho}_{n+\frac{3}{2}}(cDv)_{n+\frac{3}{2}} - \tilde{\rho}_{n+\frac{1}{2}}(cDv)_{n+\frac{1}{2}} + (D\rho)_{n+\frac{3}{2}}(c\tilde{v})_{n+\frac{3}{2}} - (D\rho)_{n+\frac{1}{2}}(c\tilde{v})_{n+\frac{1}{2}}\Big) \\ &= h(rf)_{n+1}. \end{aligned} \tag{A.1}$$

We write

$$r_{n+1}\tilde{\rho}_{n+\frac{3}{2}} = \frac{r_{n+1}\rho_{n+1} + r_{n+1}\rho_{n+2}}{2} = \frac{1 + (((\tau^+)^2\rho)r)_{n+1}}{2} := K_{11},$$

$$\begin{aligned} r_{n+1}(c_dD\rho)_{n+\frac{3}{2}} &= (r\tau^+\rho)_{n+1}(c_drD\rho)_{n+\frac{3}{2}} \\ &= (r\tau^+\rho)_{n+1}\Big((c_dr\partial\rho)_{n+\frac{3}{2}} + (c_drD\rho)_{n+\frac{3}{2}} - (c_dr\partial\rho)_{n+\frac{3}{2}}\Big) \\ &= K_{21}\Big((c_dr\partial\rho)_{n+\frac{3}{2}} + K_{22}\Big), \end{aligned}$$

where $K_{21} = (r\tau^+\rho)_{n+1}$ and $K_{22} = (c_drD\rho)_{n+\frac{3}{2}} - (c_dr\partial\rho)_{n+\frac{3}{2}}$.

Similarly,

$$r_{n+1}\tilde{\rho}_{n+\frac{1}{2}} = \frac{r_{n+1}\rho_{n+1} + r_{n+1}\rho_n}{2} = \frac{1 + (((\tau^-)^2\rho)r)_{n+1}}{2} := K_{31},$$

$$\begin{aligned} r_{n+1}(c_dD\rho)_{n+\frac{1}{2}} &= (r\tau^-\rho)_{n+1}(c_drD\rho)_{n+\frac{1}{2}} \\ &= (r\tau^-\rho)_{n+1}\Big((c_dr\partial\rho)_{n+\frac{1}{2}} + (c_drD\rho)_{n+\frac{1}{2}} - (c_dr\partial\rho)_{n+\frac{1}{2}}\Big) \\ &:= K_{41}\Big((c_dr\partial\rho)_{n+\frac{1}{2}} + K_{42}\Big), \end{aligned}$$

where $K_{41} = (r\tau^- \rho)_{n+1}$ and $(c_d r D\rho)_{n+\frac{1}{2}} - (c_d r \partial\rho)_{n+\frac{1}{2}}$.

Additionally,

$$
\begin{aligned}
(\tilde{v})_{n+\frac{1}{2}} &= v_{n+1} + \frac{v_n - v_{n+1}}{2} = v_{n+1} + \mathcal{O}(h)(Dv)_{n+\frac{1}{2}}, \\
(\tilde{v})_{n+\frac{3}{2}} &= v_{n+1} + \frac{v_{n+2} - v_{n+1}}{2} = v_{n+1} + \mathcal{O}(h)(Dv)_{n+\frac{3}{2}}.
\end{aligned}
$$

From (A.1) we thus write

$$
\begin{aligned}
& K_{11}(c_d Dv)_{n+\frac{3}{2}} - K_{31}(c_d Dv)_{n+\frac{1}{2}} \\
+ \quad & K_{21}\Big((cr\partial\rho)_{n+\frac{3}{2}} + K_{22}\Big)\Big(v_{n+1} + \mathcal{O}(h)(Dv)_{n+\frac{3}{2}}\Big) \\
- \quad & K_{41}\Big((cr\partial\rho)_{n+\frac{1}{2}} + K_{42}\Big)\Big(v_{n+1} + \mathcal{O}(h)(Dv)_{n+\frac{1}{2}}\Big) \\
= \quad & h(rf)_{n+1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& K_{11}\Big((c_d Dv)_{n+\frac{3}{2}} - (c_d Dv)_{n+\frac{1}{2}}\Big) + (K_{11} - K_{31})(c_d Dv)_{n+\frac{1}{2}} \\
+ \quad & K_{21}[\star cr\partial\rho]_a v_{n+1} + (K_{21} - K_{41})(cr\partial\rho)_{n+\frac{1}{2}} v_{n+1} + (K_{21}K_{22} - K_{41}K_{42})v_{n+1} \\
+ \quad & K_{21}\Big((cr\partial\rho)_{n+\frac{3}{2}} + K_{22}\Big)\mathcal{O}(h)(Dv)_{n+\frac{3}{2}} + K_{41}\Big((cr\partial\rho)_{n+\frac{1}{2}} + K_{42}\Big)\mathcal{O}(h)(Dv)_{n+\frac{1}{2}} \\
= \quad & h(rf)_{n+1}.
\end{aligned}
$$

Moreover, as $r\partial\rho = -\lambda s\phi\partial\psi = s\mathcal{O}_\lambda(1)$ we have

$$
\begin{aligned}
& K_{11}\Big((c_d Dv)_{n+\frac{3}{2}} - (c_d Dv)_{n+\frac{1}{2}}\Big) + (K_{11} - K_{31})(c_d Dv)_{n+\frac{1}{2}} \\
= \quad & K_{21}\lambda s[\star c\phi\partial\psi]_a v_{n+1} - Kv_{n+1} + \Big(K_{21}\mathcal{O}_\lambda(sh) + K_{21}K_{22}\mathcal{O}(h)\Big)\Big((c_d Dv)_{n+\frac{3}{2}} - (c_d Dv)_{n+\frac{1}{2}}\Big) \\
+ \quad & \Big(K_{21}\mathcal{O}_\lambda(sh) + K_{21}K_{22}\mathcal{O}(h) + K_{41}\mathcal{O}_\lambda(sh) + K_{41}K_{42}\mathcal{O}(h)\Big)(c_d Dv)_{n+\frac{1}{2}} \\
+ \quad & h(rf)_{n+1},
\end{aligned}
$$

where

$$
\begin{aligned}
K &= (K_{21} - K_{41})(cr\partial\rho)_{n+\frac{1}{2}} + K_{21}K_{22} - K_{41}K_{42} \\
&= (K_{21} - K_{41})s\mathcal{O}_\lambda(1) + K_{21}K_{22} - K_{41}K_{42}.
\end{aligned}
$$

From that, we can write

$$
\begin{aligned}
& L\Big((c_d Dv)_{n+\frac{3}{2}} - (c_d Dv)_{n+\frac{1}{2}}\Big) \\
= \quad & K_{21}\lambda s[\star c\phi\partial\psi]_a v_{n+1} - Kv_{n+1} + H(c_d Dv)_{n+\frac{1}{2}} + h(rf)_{n+1},
\end{aligned}
$$

where

$$
\begin{aligned}
L &= K_{11} - K_{21}\mathcal{O}_\lambda(sh) - K_{21}K_{22}\mathcal{O}(h), \\
K &= (K_{21} - K_{41})s\mathcal{O}_\lambda(1) + K_{21}K_{22} - K_{41}K_{42}, \\
H &= K_{21}\mathcal{O}_\lambda(sh) + K_{21}K_{22}\mathcal{O}(h) + K_{41}\mathcal{O}_\lambda(sh) + K_{41}K_{42}\mathcal{O}(h) - K_{11} + K_{31}.
\end{aligned}
$$

As $L = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh) \neq 0$ (see below) then we read

$$\begin{aligned}
&(c_d Dv)_{n+\frac{3}{2}} - (c_d Dv)_{n+\frac{1}{2}} \\
= \;& \Big(L^{-1}K_{21}\lambda s[\star c\phi\partial\psi]_a - L^{-1}K\Big)v_{n+1} + L^{-1}H(c_d Dv)_{n+\frac{1}{2}} + L^{-1}h(rf)_{n+1}.
\end{aligned}$$

We set

$$\begin{aligned}
&J_1 = L^{-1}K_{21}\lambda s[\star c\phi\partial\psi]_a - L^{-1}K, \\
&J_2 = L^{-1}H, \qquad\qquad J_3 = L^{-1}.
\end{aligned}$$

We thus have

$$(c_d Dv)_{n+\frac{3}{2}} - (c_d Dv)_{n+\frac{1}{2}} = J_1 v_{n+1} + J_2(c_d Dv)_{n+\frac{1}{2}} + J_3 h(rf)_{n+1} \tag{A.2}$$

By using Proposition 3.10 we find

$$\begin{aligned}
K_{11} &= \frac{1 + (((\tau^+)^2\rho)r)_{n+1}}{2} = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
K_{31} &= \frac{1 + (((\tau^-)^2\rho)r)_{n+1}}{2} = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
K_{21} &= (r\tau^+\rho)_{n+1} = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
K_{41} &= (r\tau^-\rho)_{n+1} = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
K_{22} &= (c_d rD\rho)_{n+\frac{3}{2}} - (c_d r\partial\rho)_{n+\frac{3}{2}} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2, \\
K_{42} &= (c_d rD\rho)_{n+\frac{1}{2}} - (c_d r\partial\rho)_{n+\frac{1}{2}} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2.
\end{aligned}$$

From that we estimate

$$\begin{aligned}
K &= (K_{21} - K_{41})s\mathcal{O}_\lambda(1) + K_{21}K_{22} - K_{41}K_{42} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
H &= K_{21}\mathcal{O}_\lambda(sh) + K_{21}K_{22}\mathcal{O}(h) + K_{41}\mathcal{O}_\lambda(sh) + K_{41}K_{42}\mathcal{O}(h) - K_{11} + K_{31} \\
&= \mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
L &= K_{11} - K_{21}\mathcal{O}_\lambda(sh) + K_{21}K_{22}\mathcal{O}(h) = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh).
\end{aligned}$$

For sh sufficiently small we have $L^{-1} = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ and then we obtain

$$\begin{aligned}
J_1 &= L^{-1}K_{21}\lambda s[\star c\phi\partial\psi]_a - L^{-1}K \\
&= \Big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\Big)\lambda s[\star c\phi\partial\psi]_a + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
J_2 &= L^{-1}H = \mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
J_3 &= L^{-1} = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh).
\end{aligned}$$

By using Proposition 3.14, Lemma 3.8 and Lemma 3.6 yield

$$\begin{aligned}
\partial_t K_{11} &= \partial_t\Big((((\tau^+)^2\rho)r)_{n+1}\Big) = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
\partial_t K_{31} &= \partial_t\Big((((\tau^-)^2\rho)r)_{n+1}\Big) = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
\partial_t K_{21} &= \partial_t(r\tau^+\rho)_{n+1} = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
\partial_t K_{41} &= \partial_t(r\tau^-\rho)_{n+1} = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
\partial_t K_{22} &= \partial_t\Big((c_d rD\rho)_{n+\frac{3}{2}} - (c_d r\partial\rho)_{n+\frac{3}{2}}\Big) = sT\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2, \\
\partial_t K_{42} &= \partial_t\Big((c_d rD\rho)_{n+\frac{1}{2}} - (c_d r\partial\rho)_{n+\frac{1}{2}}\Big) = sT\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,
\end{aligned}$$

which give

$$\begin{aligned}\partial_t L^{-1} = -\frac{\partial_t L}{L^2} &= \Big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\Big)\,(\partial_t K_{11}+\partial_t K_{21}\mathcal{O}_\lambda(sh)+K_{21}(\partial_t s)\mathcal{O}_\lambda(h)\\ &\qquad +\partial_t K_{21}K_{22}\mathcal{O}(h)+\partial_t K_{22}K_{21}\mathcal{O}(h))\\ &= T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\end{aligned}$$

where sh sufficiently small and

$$\begin{aligned}&\partial_t H = \partial_t K_{21}\mathcal{O}_\lambda(sh)+K_{21}(\partial_t s)\mathcal{O}_\lambda(h)+\partial_t K_{21}K_{22}\mathcal{O}(h)+K_{21}\partial_t K_{22}\mathcal{O}(h)\\ +\;&\partial_t K_{41}\mathcal{O}_\lambda(sh)+K_{41}(\partial_t s)\mathcal{O}_\lambda(h)+\partial_t K_{41}K_{42}\mathcal{O}(h)+K_{41}\partial_t K_{42}\mathcal{O}(h)-\partial_t K_{11}+\partial_t K_{31}\\ =\;& T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh).\end{aligned}$$

It follows that we have

$$\begin{aligned}&\partial_t J_1 = sT\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\ &\partial_t J_2 = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \qquad\qquad \partial_t J_3 = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh).\end{aligned}$$

Furthermore, we can write (A.2) in the simple form

$$\begin{aligned}&(c_d Dv)_{n+\frac{3}{2}}-(c_d Dv)_{n+\frac{1}{2}}\\ =\;&\lambda s[\star c\phi\partial\psi]_a v_{n+1}+\lambda s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}\\ +\;&\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{1}{2}}+\Big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\Big)h(rf)_{n+1},\end{aligned}$$

which yields the conclusion.

A.2 Proof of Lemma 4.4

By using Lemma 3.2 in each domain Ω_{01}, Ω_{02} , we have

$$\begin{aligned}I_{11} &= 2\int_{Q_0'} cr^2\bar{\tilde{\rho}}\;\overline{D\rho}\bar{D}(c_d Dv)\overline{Dv}\\ &= 2\int_{Q_{01}} cr^2\bar{\tilde{\rho}}\;\overline{D\rho}\bar{D}(c_d Dv)\overline{Dv}+2\int_{Q_{02}} cr^2\bar{\tilde{\rho}}\;\overline{D\rho}\bar{D}(c_d Dv)\overline{Dv}\\ &= 2\int_{Q_{01}} cr^2\bar{\tilde{\rho}}\;\overline{D\rho}\bar{c}_d\bar{D}(Dv)\overline{Dv}+2\int_{Q_{01}} cr^2\bar{\tilde{\rho}}\;\overline{D\rho}(\bar{D}c_d)(\overline{Dv})^2\\ &+ 2\int_{Q_{02}} cr^2\bar{\tilde{\rho}}\;\overline{D\rho}\bar{c}_d\bar{D}(Dv)\overline{Dv}+2\int_{Q_{02}} cr^2\bar{\tilde{\rho}}\;\overline{D\rho}(\bar{D}c_d)(\overline{Dv})^2\\ &= \sum_{i=1}^{2}\int_{Q_{0i}} cr^2\bar{\tilde{\rho}}\;\overline{D\rho}\bar{c}_d\bar{D}(Dv)^2+2\sum_{i=1}^{2}\int_{Q_{0i}} cr^2\bar{\tilde{\rho}}\;\overline{D\rho}(\bar{D}c_d)(\overline{Dv})^2.\end{aligned}$$

We then apply a discrete integration by parts (Proposition 3.5) in each domain Ω_{01}, Ω_{02} with $\partial\Omega_{01}=\{0,a\}$ and $\partial\Omega_{02}=\{a,1\}$ for the first two terms and we

obtain

$$\begin{aligned} I_{11} &= -\sum_{i=1}^{2}\int_{Q_{0i}} D(c\bar{c}_d r^2\bar{\tilde{\rho}}\ \overline{D\rho})(Dv)^2 + 2\sum_{i=1}^{2}\int_{Q_{0i}} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}(\bar{D}c_d)(\overline{Dv})^2 \\ &+ \int_0^T (c\bar{c}_d r^2\bar{\tilde{\rho}}\ \overline{D\rho})(1)(Dv)^2_{n+m+\frac{3}{2}} - \int_0^T (c\bar{c}_d r^2\bar{\tilde{\rho}}\ \overline{D\rho})(a^+)(Dv)^2_{n+\frac{3}{2}} \\ &+ \int_0^T (c\bar{c}_d r^2\bar{\tilde{\rho}}\ \overline{D\rho})(a^-)(Dv)^2_{n+\frac{1}{2}} - \int_0^T (c\bar{c}_d r^2\bar{\tilde{\rho}}\ \overline{D\rho})(0)(Dv)^2_{\frac{1}{2}} \\ &= -\sum_{i=1}^{2}\int_{Q_{0i}} D(c\bar{c}_d r^2\bar{\tilde{\rho}}\ \overline{D\rho})(Dv)^2 + 2\sum_{i=1}^{2}\int_{Q_{0i}} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}(\bar{D}c_d)(\overline{Dv})^2 + Y_{11}. \end{aligned}$$

where

$$\begin{aligned} Y_{11} &= Y_{11}^{(1)} + Y_{11}^{(2)} \\ Y_{11}^{(1)} &= \int_0^T (c\bar{c}_d r^2\bar{\tilde{\rho}}\ \overline{D\rho})(1)(Dv)^2_{n+m+\frac{3}{2}} - \int_0^T (c\bar{c}_d r^2\bar{\tilde{\rho}}\ \overline{D\rho})(a^+)(Dv)^2_{n+\frac{3}{2}} \\ Y_{11}^{(2)} &= \int_0^T (c\bar{c}_d r^2\bar{\tilde{\rho}}\ \overline{D\rho})(a^-)(Dv)^2_{n+\frac{1}{2}} - \int_0^T (c\bar{c}_d r^2\bar{\tilde{\rho}}\ \overline{D\rho})(0)(Dv)^2_{\frac{1}{2}} \end{aligned}$$

Lemma A.1. *(see Lemma B.3 in [BHL10a]) Provided $sh \le \mathfrak{K}$ we have*

$$\begin{aligned} &D(c_i\bar{c}_{di}r_i^2\bar{\tilde{\rho}}_i\ \overline{D\rho_i}) = -s\lambda^2(c_i^2\phi_i(\psi_i')^2)_d + s\lambda\phi_{di}\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\ &c_i r_i^2\bar{\tilde{\rho}}_i\overline{D\rho_i}(\bar{D}c_{di}) = s\lambda\phi_i\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2), \\ &r_i^2\bar{\tilde{\rho}}_i\overline{D\rho_i} = r_i\partial\rho_i + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = -s\lambda\phi_i\psi_i' + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2), \\ &r_i^2\bar{\tilde{\rho}}_i\overline{D\rho_i} = r_i\bar{\tilde{\rho}}_i r_i\overline{D\rho_i} = (1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh))r_i\overline{D\rho_i}. \end{aligned}$$

Moreover, by Lemma 3.3 and Proposition 3.5 in each domain Ω_{01}, Ω_{02} we obtain

$$\int_{\Omega_0'} s\lambda\phi(\overline{Dv})^2 \le \int_{\Omega_0'} s\lambda\phi\overline{(Dv)^2} = \int_{\Omega_0'} s\lambda\tilde{\phi}(Dv)^2 - \frac{h}{2}\sum_{i=1}^{2} BT_i \le \int_{\Omega_0'} s\lambda\tilde{\phi}(Dv)^2$$

since

$$\begin{aligned} BT_1 &= s\lambda\phi(a)(Dv)^2_{n+\frac{1}{2}} + s\lambda\phi(a)(Dv)^2_{\frac{1}{2}} \ge 0 \\ BT_2 &= s\lambda\phi(1)(Dv)^2_{n+m+\frac{3}{2}} + s\lambda\phi(a)(Dv)^2_{n+\frac{3}{2}} \ge 0 \end{aligned}$$

and $\tilde{\phi} = \phi + h^2\mathcal{O}_\lambda(1)$ then we can write

$$\int_{\Omega_{0i}} s\lambda\phi(\overline{Dv})^2 \le \int_{\Omega_{0i}} s\lambda\phi(Dv)^2 + \int_{\Omega_{0i}} s\lambda h^2\mathcal{O}_\lambda(1)(Dv)^2$$

Similarly, we have

$$\left|\int_{\Omega_{0i}} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\overline{Dv})^2\right| \le \int_{\Omega'} s\left|\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\right|\overline{(Dv)^2} \le \int_{\Omega'} s\left|\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\right|(Dv)^2.$$

Thus

$$I_{11} \ge -\int_{Q'} s\lambda^2(c^2\phi(\psi')^2)_d(Dv)^2 - X_{11} + Y_{11},$$

where $X_{11} = \int_{Q_0'} \nu_{11}(Dv)^2$ with ν_{11} of the form $s\lambda\phi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ and

$$
\begin{aligned}
Y_{11} &= Y_{11}^{(1)} + Y_{11}^{(2,1)} + Y_{11}^{(2,2)},\\
Y_{11}^{(1)} &= \int_0^T \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)(c\bar{c}_d)(1)(r\overline{D\rho})(1)(Dv)^2_{n+m+\frac{3}{2}}\\
&\quad - \int_0^T \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)(c\bar{c}_d)(0)(r\overline{D\rho})(0)(Dv)^2_{\frac{1}{2}},\\
Y_{11}^{(2,1)} &= \int_0^T s\lambda\phi(a)\bar{c}_d(a)\Big(- (c\psi')(a^-)(Dv)^2_{n+\frac{1}{2}} + (c\psi')(a^+)(Dv)^2_{n+\frac{3}{2}}\Big),\\
Y_{11}^{(2,2)} &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2_{n+\frac{1}{2}} - \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2_{n+\frac{3}{2}}.
\end{aligned}
$$

A.3 Proof of Lemma 4.5

We set $q = r\bar{\tilde{\rho}}c\phi^{''}$. By using a discrete integrations by parts (Proposition 3.5) and Lemma 3.2 in each domain Ω_{01}, Ω_{02} we have

$$
\begin{aligned}
I_{12} &= -2\sum_{i=1}^{2}\int_{Q_{0i}} sqv\bar{D}(c_dDv)\\
&= 2\sum_{i=1}^{2}\int_{Q_{0i}} s\tilde{q}c_d(Dv)^2 + 2\sum_{i=1}^{2}\int_{Q_{0i}} sDqc_d\tilde{v}Dv\\
&- \int_0^T sq(a^-)v(a)(c_dDv)_{n+\frac{1}{2}} + \int_0^T sq(a^+)v(a)(c_dDv)_{n+\frac{3}{2}}\\
&= 2\sum_{i=1}^{2}\int_{Q_{0i}} s\tilde{q}c_d(Dv)^2 + 2\sum_{i=1}^{2}\int_{Q_{0i}} sDqc_d\tilde{v}Dvdt + Y_{12},
\end{aligned}
$$

since $v|_{\partial\Omega_0} = 0$ and with $\partial\Omega_{01} = \{0, a\}$, $\partial\Omega_{02} = \{a, 1\}$.

Lemma A.2. *(see the proof as given in Lemma 4.4 of [BHL10a]) Let $i = 1, 2$. Provided $sh \le \mathfrak{K}$ we have*

$$
\begin{aligned}
&\phi_i'' = \lambda^2\phi_i(\psi_i')^2 + \lambda\phi_i\mathcal{O}(1),\\
&q_i = r_i\bar{\tilde{\rho}}_i c\phi_i^{''} = \lambda^2 c\phi_i(\psi_i')^2 + \lambda\phi_i\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,\\
&\tilde{q}_i = \lambda^2(c\phi_i(\psi_i')^2)_d + \lambda\phi_i\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2 + h),\\
&Dq_i = D(r_i\bar{\tilde{\rho}}_i)\,\widetilde{c\phi_i^{''}} + \widetilde{(r_i\bar{\tilde{\rho}}_i)}\,D(c\phi_i^{''}) = \mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}
$$

Note that the proof and the use of Lemma A.2 are carried out in each domain Ω_{01}, Ω_{02} independently.

It follows that

$$
I_{12} = 2\sum_{i=1}^{2}\int_{Q_{0i}} s\lambda^2(c^2\phi(\psi')^2)_d(Dv)^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \nu_{12}(Dv)^2 + \sum_{i=1}^{2}\int_{Q_{0i}} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\tilde{v}Dv + Y_{12},
$$

Then

$$I_{12} = 2\int_{Q_0'} s\lambda^2(c^2\phi(\psi')^2)_d(Dv)^2 - X_{12} + Y_{12},$$

with

$$\begin{aligned} Y_{12} &= \int_0^T s\lambda^2\phi(a)v(a)[c(\psi')^2 \star c_d Dv]_a \\ &+ \int_0^T \delta_{12}v(a)(cDv)_{n+\frac{3}{2}} + \bar{\delta}_{12}v(a)(cDv)_{n+\frac{1}{2}}, \end{aligned}$$

where $\delta_{12}, \bar{\delta}_{12}$ are of form $s\big(\lambda\phi(a)\mathcal{O}(1) + \mathcal{O}_{\lambda,\Re}(sh)^2\big)$ and

$$X_{12} = \int_{Q_0'} \nu_{12}(Dv)^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\Re}(1)\tilde{v}Dv,$$

where

$$\nu_{12} = s\lambda\phi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\Re}(h + (sh)^2).$$

A.4 Proof of Lemma 4.6

We carry out a discrete integration by parts (Proposition 3.5) in each domain Ω_{01}, Ω_{02} with $\partial\Omega_{01} = \{0, a\}$ and $\partial\Omega_{02} = \{a, 1\}$ as follows

$$\begin{aligned} I_{13} &= \int_{Q_{01}} r\bar{\tilde{\rho}}\bar{D}(c_d Dv)\partial_t v + \int_{Q_{02}} r\bar{\tilde{\rho}}\bar{D}(c_d Dv)\partial_t v \\ &= -\int_{Q_{01}} D(r\bar{\tilde{\rho}}\partial_t v)c_d Dv - \int_{Q_{02}} D(r\bar{\tilde{\rho}}\partial_t v)c_d Dv \\ &+ \int_0^T (r\bar{\tilde{\rho}})(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} - \int_0^T (r\bar{\tilde{\rho}})(0)\partial_t v(0)(c_d Dv)_{\frac{1}{2}} \\ &+ \int_0^T (r\bar{\tilde{\rho}})(1)\partial_t v(1)(c_d Dv)_{n+m+\frac{3}{2}} - \int_0^T (r\bar{\tilde{\rho}})(a^+)\partial_t v(a)(c_d Dv)_{n+\frac{3}{2}} \\ &= -\int_{Q_{01}} D(r\bar{\tilde{\rho}}\partial_t v)c_d Dv - \int_{Q_{02}} D(r\bar{\tilde{\rho}}\partial_t v)c_d Dv \\ &+ \int_0^T (r\bar{\tilde{\rho}})(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} - \int_0^T (r\bar{\tilde{\rho}})(a^+)\partial_t v(a)(c_d Dv)_{n+\frac{3}{2}} \\ &= \underbrace{-\sum_{i=1}^2 \int_{Q_{0i}} D(r\bar{\tilde{\rho}})\partial_t\tilde{v}c_d Dv}_{\bar{Q}_1} \underbrace{-\sum_{i=1}^2 \int_{Q_{0i}} \overline{r\tilde{\rho}}\,(\partial_t Dv)c_d Dv}_{\bar{Q}_2} + Y_{13}, \end{aligned}$$

by Lemma 3.2 and with

$$Y_{13} = \int_0^T (r\bar{\tilde{\rho}})(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} - \int_0^T (r\bar{\tilde{\rho}})(a^+)\partial_t v(a)(c_d Dv)_{n+\frac{3}{2}},$$

as $v|_{\partial\Omega_0} = 0$.

By applying Proposition 3.13 in each domain Ω_{01}, Ω_{02} we find

$$\begin{aligned} &D(r_i\bar{\tilde{\rho}}_i) = \mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\ &r_i\bar{\tilde{\rho}}_i = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = \mathcal{O}_{\lambda,\mathfrak{K}}(1). \end{aligned}$$

On the one hand, we have

$$\begin{aligned} \left|\bar{Q}_1\right| &\leq \sum_{i=1}^{2}\int_{Q_{0i}} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t\tilde{v})^2 + \sum_{i=1}^{2}\int_{Q_{0i}} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(Dv)^2\\ &\leq \sum_{i=1}^{2}\int_{Q_{0i}} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\,\widetilde{(\partial_t v)^2} + \sum_{i=1}^{2}\int_{Q_{0i}} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(Dv)^2\\ &= \int_{Q_0} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v)^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(Dv)^2, \end{aligned}$$

by $(\partial_t\tilde{v})^2 \leq \widetilde{(\partial_t\tilde{v})^2}$ in each domain Ω_{01}, Ω_{02} and $\sum_{i=1}^{2}\int_{\Omega_{0i}}\mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{(\partial_t\tilde{v})^2} = \int_{\Omega_0}\mathcal{O}_{\lambda,\mathfrak{K}}(1)(\partial_t\tilde{v})^2$.

On the other hand, by an integrations by parts w.r.t t we write as

$$\begin{aligned} \bar{Q}_2 &= -\frac{1}{2}\sum_{i=1}^{2}\int_{Q_{0i}} \widetilde{r\bar{\tilde{\rho}}}\, c_d.\partial_t(Dv)^2\\ &= \frac{1}{2}\sum_{i=1}^{2}\int_{Q_{0i}} \partial_t(\widetilde{r\bar{\tilde{\rho}}})c_d(Dv)^2 - \frac{1}{2}\sum_{i=1}^{2}\int_{\Omega_{0i}} \widetilde{r\bar{\tilde{\rho}}}\, c_d.(Dv)^2|_{t=0}^{t=T}. \end{aligned}$$

We observe that for $sh \leq \epsilon_1(\lambda)$ with $\epsilon_1(\lambda)$ sufficiently small we have $r\bar{\tilde{\rho}} > 0$ by Proposition 3.13. The sign of the term at $t = T$ and $t = 0$ are thus prescribed. Furthermore, Proposition 3.14 leads to $\partial_t(r_i\bar{\tilde{\rho}}_i) = T(sh)^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)$, so that, for $sh \leq \mathfrak{K}$ we obtain

$$\bar{Q}_2 \geq \sum_{i=1}^{2}\int_{Q_{0i}} T(sh)^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2 - \mathcal{C}_{\lambda,\mathfrak{K}}(1)\sum_{i=1}^{2}\int_{\Omega_{0i}}(Dv(T))^2.$$

Thus,

$$I_{13} \geq -\int_{\Omega_0'}\mathcal{C}_{\lambda,\mathfrak{K}}(1)(Dv(T))^2 - X_{13} + Y_{13}.$$

with

$$\begin{aligned} X_{13} &= \int_{Q_0'}\big(s(sh) + T(sh)^2\theta\big)\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2 + \int_{Q_0} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v)^2.\\ Y_{13} &= \int_0^T (r\bar{\tilde{\rho}})(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} - \int_0^T (r\bar{\tilde{\rho}})(a^+)\partial_t v(a)(c_d Dv)_{n+\frac{3}{2}} \end{aligned}$$

A.5 Proof of Lemma 4.7

We set $q = c^2 r^2 (\bar{D}D\rho)\overline{D\rho}$. Observing that $\overline{Dv} = \bar{D}\tilde{v}$ we get

$$\begin{aligned}
I_{21} &= 2\int_{Q_{01}} \underbrace{c^2 r^2 (\bar{D}D\rho)\overline{D\rho}}_{q}\,\bar{\tilde{v}}\;\overline{Dv} + 2\int_{Q_{02}} \underbrace{c^2 r^2 (\bar{D}D\rho)\overline{D\rho}}_{q}\,\bar{\tilde{v}}\;\overline{Dv} \\
&= \int_{Q_{01}} q\bar{D}(\tilde{v})^2 + \int_{Q_{02}} q\bar{D}(\tilde{v})^2 \\
&= -\int_{Q_{01}} Dq(\tilde{v})^2 - \int_{Q_{02}} Dq(\tilde{v})^2 \\
&+ \int_0^T q(a^-)(\tilde{v})^2_{n+\frac{1}{2}} - \int_0^T q(0)(\tilde{v})^2_{\frac{1}{2}} \\
&+ \int_0^T q(1)(\tilde{v})^2_{n+m+\frac{3}{2}} - \int_0^T q(a^+)(\tilde{v})^2_{n+\frac{3}{2}} \\
&= -\sum_{i=1}^{2}\int_{Q_{0i}} Dq\;\overline{v^2} + \sum_{i=1}^{2}\frac{h^2}{4}\int_{Q_{0i}} (Dq)(Dv)^2 + Y_{21}^{(1)} \\
&= -\sum_{i=1}^{2}\int_{Q_{0i}} \overline{Dq}(v)^2 + \sum_{i=1}^{2}\frac{h^2}{4}\int_{Q_{0i}} (Dq)(Dv)^2 + Y_{21}^{(1)} + Y_{21}^{(2)},
\end{aligned}$$

by means of Proposition 3.5, Lemma 3.2, Lemma 3.3 in each domain Ω_{01}, Ω_{02} independently and where

$$\begin{aligned}
&Y_{21} = Y_{21}^{(1)} + Y_{21}^{(2)} = Y_{21}^{(1,1)} + Y_{21}^{(1,2)} + Y_{21}^{(2)}, \\
&Y_{21}^{(1,1)} = \int_0^T q(1)(\tilde{v})^2_{n+m+\frac{3}{2}} - \int_0^T q(0)(\tilde{v})^2_{\frac{1}{2}}, \\
&Y_{21}^{(1,2)} = \int_0^T q(a^-)(\tilde{v})^2_{n+\frac{1}{2}} - \int_0^T q(a^+)(\tilde{v})^2_{n+\frac{3}{2}}, \\
&Y_{21}^{(2)} = -\frac{h}{2}\int_0^T v^2(a)(Dq)_{n+\frac{1}{2}} - \frac{h}{2}\int_0^T v^2(0)(Dq)_{\frac{1}{2}} \\
&-\frac{h}{2}\int_0^T v^2(1)(Dq)_{n+m+\frac{3}{2}} - \frac{h}{2}\int_0^T v^2(a)(Dq)_{n+\frac{3}{2}} \\
&= -\frac{h}{2}\int_0^T v^2(a)(Dq)_{n+\frac{1}{2}} - \frac{h}{2}\int_0^T v^2(a)(Dq)_{n+\frac{3}{2}},
\end{aligned}$$

as $v|_{\partial\Omega_0} = 0$.

We note that $\tilde{v}_{\frac{1}{2}} = \frac{h}{2}(Dv)_{\frac{1}{2}}$, $\tilde{v}_{n+m+\frac{3}{2}} = -\frac{h}{2}(Dv)_{n+m+\frac{3}{2}}$. On the one hand, by Proposition 3.10 we have $q = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)r\overline{D\rho}$ in each domain Ω_{01}, Ω_{02}. It follows that

$$\begin{aligned}
Y_{21}^{(1,1)} &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(r\overline{D\rho})(1)(\tilde{v})^2_{n+m+\frac{3}{2}} + \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(r\overline{D\rho})(0)(\tilde{v})^2_{\frac{1}{2}} \\
&= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(r\overline{D\rho})(1)(Dv)^2_{n+m+\frac{3}{2}} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(r\overline{D\rho})(0)(Dv)^2_{\frac{1}{2}}.
\end{aligned}$$

On the other hand, by Proposition 3.15, Corollary 3.9 we have $q = -c^2(s\phi\lambda)^3(\psi')^3 + s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_{\lambda,\Re}(sh)^2$ in each domain Ω_{01}, Ω_{02}. We thus obtain

$$\begin{aligned} Y_{21}^{(1,2)} &= \int_0^T \big(s\lambda\phi(a)\big)^3\Big(-(c^2\psi'^3)(a^-)(\tilde{v})^2_{n+\frac{1}{2}} + (c^2\psi'^3)(a^+)(\tilde{v})^2_{n+\frac{3}{2}}\Big) \\ &\quad + \int_0^T \Big(s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_{\lambda,\Re}(sh)^2\Big)\Big((\tilde{v})^2_{n+\frac{1}{2}} - (\tilde{v})^2_{n+\frac{3}{2}}\Big) \\ &= Y_{21}^{(1,21)} + Y_{21}^{(1,22)}, \end{aligned}$$

where

$$Y_{21}^{(1,21)} = \int_0^T s^3\lambda^3\phi^3(a)[c^2(\psi')^3 \star \tilde{v}^2]_a.$$

Lemma A.3. *(see Lemma B.8 in [BHL10a]) Provided $sh \leq \Re$ we have*

$$\begin{aligned} &Dq_i = s^3\mathcal{O}_{\lambda,\Re}(1), \\ &\overline{Dq_i} = -3s^3\lambda^4\phi_i^3c^2(\psi_i')^4 + (s\lambda\phi_i)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\Re}(1) + s^3\mathcal{O}_{\lambda,\Re}(sh)^2. \end{aligned}$$

Note that the proof and the use of Lemma A.3 are done in each domain Ω_{01}, Ω_{02} separately.

We then obtain

$$\begin{aligned} Y_{21}^{(2)} &= -\frac{h}{2}\int_0^T v^2(a)(Dq)_{n+\frac{1}{2}} - \frac{h}{2}\int_0^T v^2(a)(Dq)_{n+\frac{3}{2}} \\ &= \int_0^T s^2\mathcal{O}_{\lambda,\Re}(sh)v^2(a). \end{aligned}$$

We thus write I_{21}

$$I_{21} \geq 3\int_{Q_0'} \lambda^4 s^3\phi^3c^2(\psi')^4(v)^2 - \int_{Q_0'} \mu_{21}(v)^2 - \int_{Q_0'} \nu_{21}(Dv)^2 + Y_{21},$$

where

$$\begin{aligned} &\mu_{21} = (s\lambda\phi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\Re}(1) + s^3\mathcal{O}_{\lambda,\Re}(sh)^2, \quad \nu_{21} = s\mathcal{O}_{\lambda,\Re}(sh)^2, \\ &Y_{21} = Y_{21}^{(1,1)} + Y_{21}^{(1,21)} + Y_{21}^{(1,22)} + Y_{21}^{(2)}. \end{aligned}$$

A.6 Proof of Lemma 4.8

We set $q = c^2r(\bar{D}D\rho)\phi''$ and by Lemma 3.4 we have $\overline{\overline{v}} = v + h^2\bar{D}Dv/4$ in each domain Ω_{01}, Ω_{02}. It follows that

$$\begin{aligned} I_{22} &= -2\int_{Q_{01}} sq\bar{\overline{v}}v - 2\int_{Q_{02}} sq\bar{\overline{v}}v \\ &= -2\int_{Q_{01}} sqv^2 - \int_{Q_{01}} \frac{sh^2}{2}q(\bar{D}Dv)v \\ &\quad -2\int_{Q_{02}} sqv^2 - \int_{Q_{02}} \frac{sh^2}{2}q(\bar{D}Dv)v. \end{aligned}$$

Applying a discrete integration by parts (Proposition 3.5) and Lemma 3.2 in each domain Ω_{01}, Ω_{02} yield

$$\begin{aligned}
I_{22} &= -2\sum_{i=1}^{2}\int_{Q_{0i}} sqv^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{2} D(qv)Dv + Y_{22}^{(1)} \\
&= -2\sum_{i=1}^{2}\int_{Q_{0i}} sqv^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{2}\tilde{q}(Dv)^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{2} D(q)\tilde{v}Dv + Y_{22}^{(1)} \\
&= -2\sum_{i=1}^{2}\int_{Q_{0i}} sqv^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{2}\tilde{q}(Dv)^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{4} D(q)D(v^2) + Y_{22}^{(1)} \\
&= -2\sum_{i=1}^{2}\int_{Q_{0i}} sqv^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{2}\tilde{q}(Dv)^2 - \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{4} \bar{D}Dqv^2 + Y_{22}^{(1)} + Y_{22}^{(2)},
\end{aligned}$$

where

$$\begin{aligned}
Y_{22}^{(1)} &= -\int_0^T \frac{sh^2}{2} q(a^-)v(a)(Dv)_{n+\frac{1}{2}} + \int_0^T \frac{sh^2}{2} q(a^+)v(a)(Dv)_{n+\frac{3}{2}}, \\
Y_{22}^{(2)} &= \int_0^T \frac{sh^2}{4} v^2(a)(Dq)_{n+\frac{1}{2}} - \int_0^T \frac{sh^2}{4} v^2(a)(Dq)_{n+\frac{3}{2}},
\end{aligned}$$

as $v|_{\partial\Omega_0} = 0$.

In each domain Ω_{01}, Ω_{02}, we have $\phi^{''} = \mathcal{O}_\lambda(1)$ and from Proposition 3.13 we have $q = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)$ and $Dq = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)$. We thus obtain

$$\begin{aligned}
Y_{22}^{(1)} &= \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h^2}{2}(Dv)_{n+\frac{1}{2}} + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h^2}{2}(Dv)_{n+\frac{3}{2}}, \\
Y_{22}^{(2)} &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 v^2(a).
\end{aligned}$$

Lemma A.4. *(see Lemma B.9 and Lemma B.10 in [BHL10a]) Provided $sh \leq \mathfrak{K}$ we have*

$$\begin{aligned}
&c^2 r_i \bar{D}D\rho_i = c^2(r_i\partial^2\rho_i + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2) = c^2(s\lambda\phi_i)^2(\psi_i')^2 + s\mathcal{O}_\lambda(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2, \\
&h^2\bar{D}Dq_i = s(sh)\mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}$$

Note that the proof and use of above Lemma A.4 are done in each domain Ω_{01}, Ω_{02} separately.

Futhermore, we have $\phi'' = \lambda^2(\psi')^2\phi + \lambda\phi\mathcal{O}(1)$ in each domain Ω_{01}, Ω_{02}. It follows that

$$\begin{aligned}
sq_i &= s\Big(c^2(s\lambda\phi_i)^2(\psi_i')^2 + s\mathcal{O}_\lambda(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\Big)\Big(\lambda^2(\psi_i')^2\phi_i + \lambda\phi_i\mathcal{O}(1)\Big) \\
&= c^2 s^3\lambda^4(\psi_i')^4\phi_i^3 + s^3\lambda^3\phi_i^3\mathcal{O}(1) + s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,
\end{aligned}$$

in each domain Ω_{01}, Ω_{02}.

We thus write I_{22} as

$$I_{22} = -2\int_{Q_0'} c^2 s^3\lambda^4\phi^3(\psi')^4 v^2 + \int_{Q_0'} \mu_{22}v^2 + \int_{Q_0'} \nu_{22}(Dv)^2 + Y_{22},$$

where

$$\mu_{22} = (s\lambda\phi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\Re}(1) + s^3\mathcal{O}_{\lambda,\Re}(sh)^2, \quad \nu_{22} = s\mathcal{O}_{\lambda,\Re}(sh)^2,$$
$$Y_{22} = Y_{22}^{(1)} + Y_{22}^{(2)}.$$

A.7 Proof of Lemma 4.9

By means of a discrete integration by parts (Proposition 3.5) in each domain Ω_{01}, Ω_{02}, we obtain

$$\begin{aligned}
I_{23} &= \sum_{i=1}^{2}\int_{Q_{0i}} cr(\bar{D}D\rho)\bar{\tilde{v}}\partial_t v \\
&= \sum_{i=1}^{2}\int_{Q_{0i}} \widetilde{cr(\bar{D}D\rho)\partial_t v}\ \tilde{v} \\
&- \frac{h}{2}\int_0^T (cr(\bar{D}D\rho))(0)\partial_t v(0)\tilde{v}_{\frac{1}{2}} - \frac{h}{2}\int_0^T (cr(\bar{D}D\rho))(a^-)\partial_t v(a)\tilde{v}_{n+\frac{1}{2}} \\
&- \frac{h}{2}\int_0^T (cr(\bar{D}D\rho))(a^+)\partial_t v(a)\tilde{v}_{n+\frac{3}{2}} - \frac{h}{2}\int_0^T (cr(\bar{D}D\rho))(1)\partial_t v(1)\tilde{v}_{n+m+\frac{3}{2}} \\
&= \bar{Q}_1 + \bar{Q}_2 + Y_{23}^{(1)},
\end{aligned}$$

by Lemma 3.3 and where

$$\bar{Q}_1 = \sum_{i=1}^{2}\int_{Q_{0i}} \widetilde{(cr(\bar{D}D\rho))}\ \partial_t\tilde{v}\tilde{v},$$
$$\bar{Q}_2 = \sum_{i=1}^{2}\frac{h^2}{4}\int_{Q_{0i}} D(cr\bar{D}D\rho)(D\partial_t v)\tilde{v},$$
$$Y_{23}^{(1)} = -\frac{h}{2}\int_0^T (cr(\bar{D}D\rho))(a^-)\partial_t v(a)\tilde{v}_{n+\frac{1}{2}} - \frac{h}{2}\int_0^T (cr(\bar{D}D\rho))(a^+)\partial_t v(a)\tilde{v}_{n+\frac{3}{2}}$$

as $\partial_t v|_{\partial\Omega_0} = 0$.

With an integrations by parts w.r.t t we have

$$\bar{Q}_1 = \frac{-1}{2}\sum_{i=1}^{2}\int_{Q_{0i}} \partial_t\widetilde{(cr\bar{D}D\rho)}(\tilde{v})^2 + \frac{1}{2}\sum_{i=1}^{2}\int_{\Omega_{0i}} \widetilde{(cr(\bar{D}D\rho))}\ (\tilde{v})^2|_{t=0}^{t=T}.$$

By means of Proposition 3.13 and Lemma 3.7 in each domain Ω_{01}, Ω_{02} we get

$$\widetilde{cr_i(\bar{D}D\rho_i)} = s^2\mathcal{O}_{\lambda,\Re}(1),$$
$$r_i\bar{D}D\rho_i = s^2\mathcal{O}_{\lambda,\Re}(1),$$

and we further have

Lemma A.5. *(see Lemma A.1 in [BL12])*

$$\partial_t\widetilde{(cr_i\bar{D}D\rho_i)} = Ts^2\theta\mathcal{O}_{\lambda,\Re}(1).$$

Note that the proof and use of Lemma A.5 are done in each domain Ω_{01}, Ω_{02} separately.

It follows that

$$\bar{Q}_1 = \sum_{i=1}^{2} \int_{Q_{0i}} Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{v^2} + \sum_{i=1}^{2} \int_{\Omega_{0i}} s^2\big(\mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{v^2}_{|t=0} + \mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{v^2}_{|t=T}\,\big)$$

as $|\tilde{v}|^2 \leq \widetilde{|v|^2}$ in each domain Ω_{01}, Ω_{02}.

Moreover, we observe that $\sum_{i=1}^{2}\int_{\Omega_{0i}} \mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{v^2} = \int_{\Omega_0} \mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2$. Then,

$$\bar{Q}_1 = \int_{Q_0} Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{\Omega_0} s^2\big(\mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{v^2}_{|t=0} + \mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{v^2}_{|t=T}\,\big). \tag{A.3}$$

We have

$$Y_{23}^{(1)} = \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)\frac{h}{2}(\tilde{v}_{n+\frac{1}{2}}) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)\frac{h}{2}(\tilde{v}_{n+\frac{3}{2}}).$$

By an integration by parts w.r.t t and Lemma 3.2 in each domain Ω_{01}, Ω_{02} we find

$$\bar{Q}_2 = \underbrace{-\sum_{i=1}^{2}\frac{h^2}{4}\int_{Q_{0i}} \partial_t(D(cr\bar{D}D\rho)\tilde{v})Dv}_{\bar{Q}_2^1} + \underbrace{\sum_{i=1}^{2}\frac{h^2}{8}\int_{\Omega_{0i}} D(cr\bar{D}D\rho)D(v)^2|_{t=0}^{t=T}}_{\bar{Q}_2^2}.$$

By means of Lemma 3.2 and a discrete intergration by parts in space (Proposition 3.5) in each domain Ω_{01}, Ω_{02} we see that

$$\begin{aligned}
\bar{Q}_2^1 &= \sum_{i=1}^{2}\frac{h^2}{8}\int_{Q_{0i}} \partial_t(\bar{D}D(cr\bar{D}D\rho))v^2 - \sum_{i=1}^{2}\frac{h^2}{4}\int_{Q_{0i}} D(cr\bar{D}D\rho)(\partial_t\tilde{v})Dv \\
&\quad -\frac{h^2}{8}\int_0^T v^2(a)\partial_t(D(cr\bar{D}D\rho))_{n+\frac{1}{2}} + \frac{h^2}{8}\int_0^T v^2(a)\partial_t(D(cr\bar{D}D\rho))_{n+\frac{3}{2}} \\
&= \sum_{i=1}^{2}\frac{h^2}{8}\int_{Q_{0i}} \partial_t(\bar{D}D(cr\bar{D}D\rho))v^2 - \sum_{i=1}^{2}\frac{h^2}{4}\int_{Q_{0i}} D(cr\bar{D}D\rho)(\partial_t\tilde{v})Dv + Y_{23}^{(2)}
\end{aligned}$$

as $v|_{\partial\Omega_0} = 0$.

Lemma A.6. *(Lemma A.2 in [BL12]) Provided $sh \leq \mathfrak{K}$ we have*

$$\begin{aligned}
&h^2\bar{D}D(c_i r_i(\bar{D}D\rho_i)) = s(sh)\mathcal{O}_{\lambda,\mathfrak{K}}(1), \\
&h^2\partial_t(\bar{D}D(c_i r_i\bar{D}D\rho_i)) = Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1), \\
&h\partial_t(D(c_i r_i\bar{D}D\rho_i)) = Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1), \\
&D(c_i r_i\bar{D}D\rho_i) = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}$$

Note that all above terms are done in each domain Ω_{01}, Ω_{02} separately.

We thus obtain

$$Y_{23}^{(2)} = \int_0^T sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2(a).$$

Applying the Young's inequality and using that $|\partial_t\tilde{v}|^2 \leq \widetilde{|\partial_t v|^2}$ in each domain Ω_{01}, Ω_{02}, we have

$$\begin{aligned}\bar{Q}_2^1 &\geq \int_{Q_0'} Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{Q_0'} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\,\widetilde{|\partial_t v|^2} + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2 + Y_{23}^{(2)} \\ &\geq \int_{Q_0'} Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{Q_0} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2|\partial_t v|^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2 + Y_{23}^{(2)}\end{aligned} \tag{A.4}$$

as $\sum_{i=1}^2 \int_{\Omega_{0i}} \mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{|\partial_t v|^2} = \int_\Omega \mathcal{O}_{\lambda,\mathfrak{K}}(1)\,|\partial_t v|^2$.

By using Proposition 3.5, Lemma A.6 in each domain Ω_{01}, Ω_{02} separately yield

$$\begin{aligned}\bar{Q}_2^2 &= -\sum_{i=1}^2 \frac{h^2}{8}\int_{\Omega_{0i}} \bar{D}D(cr\bar{D}D\rho)(v)^2|_{t=0}^{t=T} \\ &\quad + \frac{h^2}{8}v^2(a)\big(D(cr\bar{D}D\rho)\big)_{n+\frac{1}{2}}|_{t=0}^{t=T} - \frac{h^2}{8}v^2(a)\big(D(cr\bar{D}D\rho)\big)_{n+\frac{3}{2}}|_{t=0}^{t=T} \\ &= \int_{\Omega_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(v)^2|_{t=T} + \int_{\Omega_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(v)^2|_{t=0} + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v^2(a)|_{t=0}^{t=T} \\ &= \int_{\Omega_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(v)^2|_{t=T} + \int_{\Omega_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(v)^2|_{t=0} + Y_{23}^{(3)},\end{aligned} \tag{A.5}$$

as $v|_{\partial\Omega_0} = 0$ where

$$Y_{23}^{(3)} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v^2(a)|_{t=0}^{t=T}.$$

Collecting (A.3), (A.4) and (A.5) we obtain

$$I_{23} \geq \int_{\Omega_0} s^2\big(\mathcal{O}_{\lambda,\mathfrak{K}}(1)v_{|_{t=0}}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}(1)v_{|_{t=T}}^2\big) - X_{23} + Y_{23},$$

where X_{23} and Y_{23} are as given in the statement of Lemma 4.9.

A.8 Proof of Lemma 4.10

By means of a discrete integration by parts (Proposition 3.5) in each domain Ω_{01}, Ω_{02} separately, we get

$$\begin{aligned}I_{31} &= -2\tau\int_{Q_{01}} (\partial_t\theta)\varphi cr\overline{D\rho}v\overline{Dv} - 2\tau\int_{Q_{02}} (\partial_t\theta)\varphi cr\overline{D\rho}v\overline{Dv} \\ &= -2\tau\int_{Q_{01}} (\partial_t\theta)\,\widetilde{\varphi cr\overline{D\rho}v}\,Dv - 2\tau\int_{Q_{02}} (\partial_t\theta)\,\widetilde{\varphi cr\overline{D\rho}v}\,Dv + Y_{31}^{(1)},\end{aligned}$$

with

$$Y_{31}^{(1)} = \tau\frac{h}{2}\int_0^T (\partial_t\theta)(cr\overline{D\rho}\varphi v)(a^-)(Dv)_{n+\frac{1}{2}} + \tau\frac{h}{2}\int_0^T (\partial_t\theta)(cr\overline{D\rho}\varphi v)(a^+)(Dv)_{n+\frac{3}{2}}$$

as $v|_{\partial\Omega_0} = 0$.

We have $\widetilde{\varphi cr\overline{D\rho}v} = \widetilde{\varphi cr\overline{D\rho}}\;\tilde{v} + \frac{h^2}{4}D(\varphi cr\overline{D\rho})Dv$ in each domain Ω_{01}, Ω_{02}. It follows that

$$
\begin{aligned}
I_{31} &= -\tau\sum_{i=1}^{2}\int_{Q_{0i}}(\partial_t\theta)\,\widetilde{(cr\overline{D\rho}\varphi)}\,D(v)^2-\sum_{i=1}^{2}\tau\frac{h^2}{2}\int_{Q_{0i}}(\partial_t\theta)D(cr\overline{D\rho}\varphi)(Dv)^2+Y_{31}^{(1)}\\
&= \tau\sum_{i=1}^{2}\int_{Q_{0i}}(\partial_t\theta)\overline{(D(cr\overline{D\rho}\varphi))}v^2-\sum_{i=1}^{2}\tau\frac{h^2}{2}\int_{Q_{0i}}(\partial_t\theta)D(cr\overline{D\rho}\varphi)(Dv)^2+Y_{31}^{(1)}\\
&- \tau\int_0^T(\partial_t\theta)v^2(a)\,\widetilde{(cr\overline{D\rho}\varphi)}_{n+\frac12}+\tau\int_0^T(\partial_t\theta)v^2(a)\,\widetilde{(cr\overline{D\rho}\varphi)}_{n+\frac32}\\
&= \tau\sum_{i=1}^{2}\int_{Q_{0i}}(\partial_t\theta)\overline{(D(cr\overline{D\rho}\varphi))}v^2-\sum_{i=1}^{2}\tau\frac{h^2}{2}\int_{Q_{0i}}(\partial_t\theta)D(cr\overline{D\rho}\varphi)(Dv)^2+Y_{31}^{(1)}+Y_{31}^{(2)}
\end{aligned}
$$

by using a discrete integration by parts in each domain Ω_{01}, Ω_{02} separately and

$$
Y_{31}^{(2)}=-\tau\int_0^T(\partial_t\theta)v^2(a)\,\widetilde{(cr\overline{D\rho}\varphi)}_{n+\frac12}+\tau\int_0^T(\partial_t\theta)v^2(a)\,\widetilde{(cr\overline{D\rho}\varphi)}_{n+\frac32}
$$

as $v|_{\partial\Omega}=0$.

By using the Lipschitz continuity and Proposition 3.13 we get

$$
\begin{aligned}
&D(cr_i\overline{D\rho_i}\varphi_i)=s\mathcal{O}_{\lambda,\Re}(1),\\
&\overline{D(cr_i\overline{D\rho_i}\varphi_i)}=s\mathcal{O}_{\lambda,\Re}(1),\\
&\widetilde{cr_i\overline{D\rho_i}\varphi_i}=s\mathcal{O}_{\lambda,\Re}(1),\\
&cr_i\overline{D\rho_i}=c(r_i\partial\rho_i+s^2\mathcal{O}_{\lambda,\Re}(sh)^2)=c(-s\lambda\phi_i\psi_i'+s\mathcal{O}_{\lambda,\Re}(sh)^2)=s\mathcal{O}_{\lambda,\Re}(1).
\end{aligned}
$$

The proof is done in each domain Ω_{01}, Ω_{02} separately. Note that $\max_t\partial_t\theta=T\theta^2$.

It thus follows that

$$
I_{31}=\int_{Q_0'}T\theta s^2\mathcal{O}_{\lambda,\Re}(1)v^2+\int_{Q_0'}T\theta\mathcal{O}_{\lambda,\Re}(sh)^2(Dv)^2+Y_{31},
$$

where

$$
\begin{aligned}
&Y_{31}=Y_{31}^{(1)}+Y_{31}^{(2)},\\
&Y_{31}^{(1)}=\int_0^T T\theta s^2\mathcal{O}_{\lambda,\Re}(1)v(a)\frac{h}{2}(Dv)_{n+\frac12}+\int_0^T T\theta s^2\mathcal{O}_{\lambda,\Re}(1)v(a)\frac{h}{2}(Dv)_{n+\frac32},\\
&Y_{31}^{(2)}=\int_0^T T\theta s^2\mathcal{O}_{\lambda,\Re}(1)v^2(a).
\end{aligned}
$$

A.9 Proof of Lemma 4.13

We see that

$$\begin{aligned} & Y_{11}^{(1)} + Y_{21}^{(1,1)} \\ = & \int_0^T \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)(c\bar{c}_d)(1)(r\overline{D\rho})_1(Dv)^2_{n+m+\frac{3}{2}} \\ - & \int_0^T \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)(c\bar{c}_d)(0)(r\overline{D\rho})_0(Dv)^2_{\frac{1}{2}} \\ + & \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(r\overline{D\rho})_0(Dv)^2_{\frac{1}{2}} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(r\overline{D\rho})_1(Dv)^2_{n+m+\frac{3}{2}}. \end{aligned}$$

Moreover, by (4.1) we have $Y_{11}^{(1)} + Y_{21}^{(1,1)} \geq 0$ for sh sufficiently small.

We next focus our attention on the trace term at $'a'$ on $Y_{11}^{(2,1)} + Y_{21}^{(1,21)}$ as follows

A.10 Proof of Lemma 4.14

$$\begin{aligned} (\tilde{v})^2_{n+\frac{3}{2}} &= \big(\frac{v_{n+1}+v_{n+2}}{2}\big)^2 = \big(v_{n+1} + \frac{h}{2}(Dv)_{n+\frac{3}{2}}\big)^2 \\ &= v_{n+1}^2 + \frac{h^2}{4}(Dv)^2_{n+\frac{3}{2}} + hv_{n+1}(Dv)_{n+\frac{3}{2}} \\ &= v_{n+1}^2 + \frac{h^2}{4(c_d)^2_{n+\frac{3}{2}}}(c_dDv)^2_{n+\frac{3}{2}} + v_{n+1}\frac{h}{(c_d)_{n+\frac{3}{2}}}(c_dDv)_{n+\frac{3}{2}}. \end{aligned} \tag{A.6}$$

Similarly, we have

$$(\tilde{v})^2_{n+\frac{1}{2}} = v_{n+1}^2 + \frac{h^2}{4(c_d)^2_{n+\frac{1}{2}}}(c_dDv)^2_{n+\frac{1}{2}} - v_{n+1}\frac{h}{(c_d)_{n+\frac{1}{2}}}(c_dDv)_{n+\frac{1}{2}}. \tag{A.7}$$

We thus write $Y_{21}^{(1,21)}$ as follows:

$$
\begin{aligned}
& Y_{21}^{(1,21)} \\
= & \int_0^T (s\lambda\phi(a))^3 [c^2(\psi')^3 \star |\tilde{v}|^2]_a \\
= & \int_0^T (s\lambda\phi(a))^3 (c^2\psi'^3)(a^+)\Big(v_{n+1}^2 + \frac{h^2}{4(c_d)_{n+\frac{3}{2}}^2}(c_d Dv)_{n+\frac{3}{2}}^2 + v_{n+1}\frac{h}{2(c_d)_{n+\frac{3}{2}}}(c_d Dv)_{n+\frac{3}{2}}\Big) \\
- & \int_0^T (s\lambda\phi(a))^3 (c^2\psi'^3)(a^-)\Big(v_{n+1}^2 + \frac{h^2}{4(c_d)_{n+\frac{1}{2}}^2}(cDv)_{N+\frac{1}{2}}^2 - v_{n+1}\frac{h}{2(c_d)_{n+\frac{1}{2}}}(c_d Dv)_{n+\frac{1}{2}}\Big) \\
= & \int_0^T (s\lambda\phi(a))^3 [c^2\psi'^3 \star]_a v_{n+1}^2 \\
+ & \int_0^T (s\lambda\phi(a))^3 \Big((c^2\psi'^3)(a^+)\frac{h^2}{4(c_d)_{n+\frac{3}{2}}^2}(c_d Dv)_{n+\frac{3}{2}}^2 - (c^2\psi'^3)(a^-)\frac{h^2}{4(c_d)_{n+\frac{1}{2}}^2}(c_d Dv)_{n+\frac{1}{2}}^2\Big) \\
+ & \int_0^T (s\lambda\phi(a))^3 \Big((c^2\psi'^3)(a^+)\frac{h}{2(c_d)_{n+\frac{3}{2}}}(c_d Dv)_{n+\frac{3}{2}} - (c^2\psi'^3)(a^-)\frac{h}{2(c_d)_{n+\frac{1}{2}}}(c_d Dv)_{n+\frac{1}{2}}\Big) v(a).
\end{aligned}
\tag{A.8}
$$

Moreover, the term $Y_{11}^{(2,1)}$ is given by

$$
\begin{aligned}
Y_{11}^{(2,1)} &= \int_0^T s\lambda\phi(a)\Big(- \psi'(a^-)c(a^-)\bar{c}_d(a)(Dv)_{n+\frac{1}{2}}^2 + \psi'(a^+)c(a^+)\bar{c}_d(a)(Dv)_{n+\frac{3}{2}}^2\Big) \\
&= \int_0^T s\lambda\phi(a)\Big(- \psi'(a^-)\frac{c(a^-)\bar{c}_d(a)}{(c_d)_{n+\frac{1}{2}}^2}(c_d Dv)_{n+\frac{1}{2}}^2 + \psi'(a^+)\frac{c(a^+)\bar{c}_d(a)}{(c_d)_{n+\frac{3}{2}}^2}(c_d Dv)_{n+\frac{3}{2}}^2\Big).
\end{aligned}
$$

We estimate as

$$
\begin{aligned}
\frac{c(a^-)\bar{c}_d(a)}{(c_d)_{n+\frac{1}{2}}^2} &= \frac{\big((c_d)_{n+\frac{1}{2}} + \mathcal{O}(h)\big)\big((c_d)_{n+\frac{1}{2}} + (c_d)_{n+\frac{3}{2}}\big)}{2(c_d)_{n+\frac{1}{2}}^2} \\
&= \frac{\big((c_d)_{n+\frac{1}{2}} + \mathcal{O}(h)\big)\big(2(c_d)_{n+\frac{1}{2}} + \mathcal{O}(h)\big)}{2(c_d)_{n+\frac{1}{2}}^2} \\
&= 1 + h\mathcal{O}(1).
\end{aligned}
$$

Similarly,

$$
\frac{c(a^+)\bar{c}_d(a)}{(c_d)_{n+\frac{3}{2}}^2} = 1 + h\mathcal{O}(1).
$$

We thus obtain $Y_{11}^{(2,1)}$

$$
\begin{aligned}
Y_{11}^{(2,1)} &= \int_0^T s\lambda\phi(a)\Big(- \psi'(a^-)\big(1 + h\mathcal{O}(1)\big)(c_d Dv)_{n+\frac{1}{2}}^2 + \psi'(a^+)\big(1 + h\mathcal{O}(1)\big)(c_d Dv)_{n+\frac{3}{2}}^2\Big) \\
&= \int_0^T s\lambda\phi(a)[\psi' \star (c_d Dv)^2]_a \\
&+ \int_0^T s\lambda\phi(a)\psi'(a^+)\mathcal{O}(h)(c_d Dv)_{n+\frac{3}{2}}^2 + \int_0^T s\lambda\phi(a)\psi'(a^-)\mathcal{O}(h)(c_d Dv)_{n+\frac{1}{2}}^2.
\end{aligned}
\tag{A.9}
$$

Combining (A.8) with (A.9) we obtain

$$\begin{aligned}
& Y_{11}^{(2,1)} + Y_{21}^{(1,21)} \\
= & \int_0^T s\lambda\phi(a)[\psi' \star (c_d Dv)^2]_a + \int_0^T s^3\lambda^3\phi^3(a)[(\psi')^3 c^2 \star]_a v_{n+1}^2 \\
+ & \int_0^T s\lambda\phi(a)\psi'(a^+)\mathcal{O}(h)(c_d Dv)_{n+\frac{3}{2}}^2 + \int_0^T s\lambda\phi(a)\psi'(a^-)\mathcal{O}(h)(c_d Dv)_{n+\frac{1}{2}}^2 \\
+ & \int_0^T (s\lambda\phi(a))^3 \Big((c^2\psi'^3)(a^+)\frac{h^2}{4(c_d)_{n+\frac{3}{2}}^2}(c_d Dv)_{n+\frac{3}{2}}^2 - (c^2\psi'^3)(a^-)\frac{h^2}{4(c_d)_{n+\frac{1}{2}}^2}(c_d Dv)_{n+\frac{1}{2}}^2 \Big) \\
+ & \int_0^T (s\lambda\phi(a))^3 \Big((c^2\psi'^3)(a^+)\frac{h}{2(c_d)_{n+\frac{3}{2}}}(c_d Dv)_{n+\frac{3}{2}} - (c^2\psi'^3)(a^-)\frac{h}{2(c_d)_{n+\frac{1}{2}}}(c_d Dv)_{n+\frac{1}{2}} \Big) v(a) \\
= & \mu + \mu_1,
\end{aligned}$$

where

$$\mu = \int_0^T s\lambda\phi(a)[\psi' \star (c_d Dv)^2]_a + \int_0^T s^3\lambda^3\phi^3(a)[c^2(\psi')^3 \star]_a v_{n+1}^2.$$

and μ_1 can be written as

$$\begin{aligned}
\mu_1 = & \int_0^T s\mathcal{O}_{\lambda,\Re}(sh)(c_d Dv)_{n+\frac{3}{2}}^2 + \int_0^T s\mathcal{O}_{\lambda,\Re}(sh)(c_d Dv)_{n+\frac{1}{2}}^2 \\
+ & \int_0^T s^2\mathcal{O}_{\lambda,\Re}(sh)(c_d Dv)_{n+\frac{3}{2}} v_{n+1} + \int_0^T s^2\mathcal{O}_{\lambda,\Re}(sh)(c_d Dv)_{n+\frac{1}{2}} v_{n+1}.
\end{aligned}$$

We can write

$$\begin{aligned}
& [(\psi') \star (c_d Dv)^2]_a \\
= & [(\psi')\star]_a (c_d Dv)_{n+\frac{1}{2}}^2 + [\star(c_d Dv)]_a^2 \psi'(a^+) + 2[\star(c_d Dv)]_a \psi'(a^+)(c_d Dv)_{n+\frac{1}{2}}.
\end{aligned}$$

Indeed, we have

$$[(\psi') \star (c_d Dv)^2]_a = (c_d Dv)_{n+\frac{3}{2}}^2 \psi'(a^+) - (c_d Dv)_{n+\frac{1}{2}}^2 \psi'(a^-),$$

and

$$\begin{aligned}
& [(\psi')\star]_a (c_d Dv)_{n+\frac{1}{2}}^2 + [\star(c_d Dv)]_a^2 \psi'(a^+) + 2[\star(c_d Dv)]_a \psi'(a^+)(c_d Dv)_{n+\frac{1}{2}} \\
= & (c_d Dv)_{n+\frac{1}{2}}^2 \psi'(a^+) - (c_d Dv)_{n+\frac{1}{2}}^2 \psi'(a^-) \\
+ & (c_d Dv)_{n+\frac{3}{2}}^2 \psi'(a^+) + (c_d Dv)_{n+\frac{1}{2}}^2 \psi'(a^+) - 2(c_d Dv)_{n+\frac{3}{2}}(c_d Dv)_{n+\frac{1}{2}} \psi'(a^+) \\
+ & 2(c_d Dv)_{n+\frac{3}{2}}(c_d Dv)_{n+\frac{1}{2}} \psi'(a^+) - 2(c_d Dv)_{n+\frac{1}{2}}^2 \psi'(a^+) \\
= & (c_d Dv)_{n+\frac{3}{2}}^2 \psi'(a^+) - (c_d Dv)_{n+\frac{1}{2}}^2 \psi'(a^-).
\end{aligned}$$

Moreover, by using Lemma 3.17, we obtain

$$\begin{aligned}
& [(\psi') \star (c_d Dv)^2]_a \\
= & [(\psi')\star]_a (c_d Dv)_{n+\frac{1}{2}}^2 + [\star(c_d Dv)]_a^2 \psi'(a^+) + 2[\star(c_d Dv)]_a \psi'(a^+)(c_d Dv)_{n+\frac{1}{2}} \\
= & [(\psi')\star]_a (c_d Dv)_{n+\frac{1}{2}}^2 + \Big(\lambda^2 s^2 [\star(c\phi\psi')]_a^2 v_{n+1}^2 + r_0^2 + 2\lambda s r_0 [\star c\phi\psi']_a v_{n+1} \Big) \psi'(a^+) \\
& \qquad + 2\Big(s\lambda [\star c\phi\psi']_a v_{n+1} + r_0 \Big) \psi'(a^+)(c_d Dv)_{n+\frac{1}{2}},
\end{aligned}$$

which gives

$$\begin{aligned}
\mu &= \int_0^T s\lambda\phi(a)[\psi'\star]_a(c_dDv)^2_{n+\frac{1}{2}} \\
&+ \int_0^T 2s^2\lambda^2\phi(a)[\star c\phi\psi']_a\psi'(a^+)v_{N+1}(c_dDv)_{n+\frac{1}{2}} \\
&+ \int_0^T s^3\lambda^3\phi(a)\Big([\star c\phi\psi']^2_a\psi'(a^+) + [c^2(\psi')^3\star]_a\phi^2(a)\Big)v^2_{n+1} \\
&+ \int_0^T s\lambda\phi(a)\psi'(a^+)r_0^2 + 2\int_0^T s^2\lambda^2\phi(a)[\star c\phi\psi']_a\psi'(a^+)r_0v_{n+1} \\
&+ 2\int_0^T s\lambda\phi(a)\psi'(a^+)r_0(c_dDv)_{n+\frac{1}{2}} \quad .
\end{aligned}$$

Moreover, we have:

$$\begin{aligned}
&[\star c\phi\psi']_a = c\phi\psi'|_{n+\frac{3}{2}} - c\phi\psi'|_{n+\frac{1}{2}} = c\phi\psi'|_{a^+} - c\phi\psi'|_{a^-} + h\mathcal{O}_\lambda(1) = \phi(a)[c\psi'\star]_a + h\mathcal{O}_\lambda(1), \\
&[\star c\phi\psi']^2_a = [c\phi\psi'\star]^2_a + 2[c\phi\psi'\star]_a h\mathcal{O}_\lambda(1) + h^2\mathcal{O}_\lambda(1) = \phi^2(a)[c\psi'\star]^2_a + h\mathcal{O}_\lambda(1).
\end{aligned}$$

We thus write μ as

$$\begin{aligned}
\mu &= \int_0^T s\lambda\phi(a)[\psi'\star]_a(c_dDv)^2_{n+\frac{1}{2}} \\
&+ \int_0^T 2s^2\lambda^2\phi^2(a)[c\psi'\star]_a\psi'(a^+)v_{n+1}(c_dDv)_{n+\frac{1}{2}} \\
&+ \int_0^T s^3\lambda^3\phi^3(a)\Big([c\psi'\star]^2_a\psi'(a^+) + [c^2(\psi')^3\star]_a\Big)v^2_{n+1} \\
&+ \int_0^T s\lambda\phi(a)\psi'(a^+)r_0^2 + 2\int_0^T s^2\lambda^2\phi^2(a)[c\psi'\star]_a\psi'(a^+)r_0v_{n+1} \\
&+ 2\int_0^T s\lambda\phi(a)\psi'(a^+)r_0(c_dDv)_{n+\frac{1}{2}} + \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2_{n+1} \\
&+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_dDv)_{n+\frac{1}{2}} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)r_0v_{n+1} \\
&= \int_0^T s\lambda\phi(a)[\psi'\star]_a(c_dDv)^2_{n+\frac{1}{2}} \\
&+ \int_0^T 2s^2\lambda^2\phi^2(a)[c\psi'\star]_a\psi'(a^+)v_{n+1}(c_dDv)_{n+\frac{1}{2}} \\
&+ \int_0^T s^3\lambda^3\phi^3(a)\Big([c\psi'\star]^2_a\psi'(a^+) + [c^2(\psi')^3\star]_a\Big)v^2_{n+1} \quad + \mu_r
\end{aligned}$$

where μ_r can be written as

$$\begin{aligned}
\mu_r &= \int_0^T s\mathcal{O}_\lambda(1)r_0^2 + \int_0^T s^2\mathcal{O}_\lambda(1)r_0v_{n+1} + \int_0^T s\mathcal{O}_\lambda(1)r_0(c_dDv)_{n+\frac{1}{2}} \\
&+ \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2_{n+1} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_dDv)_{n+\frac{1}{2}} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)r_0v_{n+1}.
\end{aligned}$$

We have thus achieved

$$\mu = \int_0^T s\lambda\phi(a)\big(\mathbf{A}u(t,a), u(t,a)\big) + \mu_r,$$

with $u(t,a) = \big((c_d Dv)_{n+\frac{1}{2}}, s\lambda\phi(a)v_{n+1}\big)^t$ and the symmetric matrix $\mathbf{A}$ defined in Lemma 2.1.

From the choice made for the weight function β in Lemma 2.1 we find that:

$$\mu \geq C\alpha_0 \int_0^T s\lambda\phi(a)(c_d Dv)^2_{n+\frac{1}{2}} + C\alpha_0 \int_0^T s^3\lambda^3\phi^3(a)v^2_{n+1} + \mu_r,$$

with $\alpha_0 > 0$.

A.11 Proof of Lemma 4.15

By using Lemma 3.17 we have

$$\begin{aligned}
Y_{13} &= -\int_0^T r\bar{\tilde{\rho}}(a^+)\partial_t v(a)(c_d Dv)_{n+\frac{3}{2}} + \int_0^T r\bar{\tilde{\rho}}(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} \\
&= -\int_0^T r\bar{\tilde{\rho}}(a^+)\partial_t v(a)\Big((c_d Dv)_{n+\frac{1}{2}} + J_1 v_{n+1} + J_2(c_d Dv)_{n+\frac{1}{2}} + J_3 h(rf)_{n+1}\Big) \\
&\quad + \int_0^T r\bar{\tilde{\rho}}(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}},
\end{aligned}$$

where J_1, J_2 and J_3 are given as in Lemma 3.17.

Since $J_2 = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ and $r\bar{\tilde{\rho}} = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ we can write

$$\begin{aligned}
Y_{13} &= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} - \int_0^T r\bar{\tilde{\rho}}(a^+)J_1 v(a)\partial_t v(a) \\
&\quad - \int_0^T r\bar{\tilde{\rho}}(a^+)J_3\partial_t v(a)h(rf)_{n+1}.
\end{aligned}$$

Futhermore, as $f = f_1 - \partial_t(\rho v)$ we thus find

$$\begin{aligned}
Y_{13} &= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} - \int_0^T r\bar{\tilde{\rho}}(a^+)J_1 v(a)\partial_t v(a) \\
&\quad - \int_0^T r\bar{\tilde{\rho}}(a^+)J_3\partial_t v(a)h\big(rf_1 - r\partial_t(\rho v)\big)_{n+1}.
\end{aligned}$$

With an integration by parts w.r.t t for the second term above we obtain

$$
\begin{aligned}
Y_{13} &= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} + \frac{1}{2}\int_0^T \partial_t\big(r\bar{\bar{\rho}}(a^+)J_1\big)v^2(a) \\
&- \frac{1}{2}r\bar{\bar{\rho}}(a^+)J_1 v^2(a)|_{t=0}^{t=T} - \int_0^T r\bar{\bar{\rho}}(a^+)J_3\partial_t v(a)h(rf_1)_{n+1} \\
&+ \int_0^T r\bar{\bar{\rho}}(a^+)J_3\partial_t v(a)hr_{n+1}\big(\rho\partial_t v + \partial_t\rho v\big)_{n+1} \\
&= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} + \frac{1}{2}\int_0^T \partial_t\big(r\bar{\bar{\rho}}(a^+)J_1\big)v^2(a) \\
&+ s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)|_{t=0}^{t=T} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)h(rf_1)_{n+1} \\
&+ \int_0^T \big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)h\big(\partial_t v(a)\big)^2 + \frac{1}{2}\int_0^T r\bar{\bar{\rho}}(a^+)J_3 h(r\partial_t\rho)_{n+1}\partial_t\big(v^2(a)\big),
\end{aligned}
$$

where $r\bar{\bar{\rho}}, J_3$ are of the form $1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ and J_1 of the form $s\mathcal{O}_{\lambda,\mathfrak{K}}(1)$.

We apply an integration by parts in time for the last term

$$
\begin{aligned}
Y_{13} &= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} + \frac{1}{2}\int_0^T \partial_t\big(r\bar{\bar{\rho}}(a^+)J_1\big)v^2(a) \\
&+ s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)|_{t=0}^{t=T} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)h(rf_1)_{n+1} \\
&+ \int_0^T \big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)h\big(\partial_t v(a)\big)^2 - \frac{1}{2}\int_0^T \partial_t\big(r\bar{\bar{\rho}}(a^+)J_3(r\partial_t\rho)_{n+1}\big)hv^2(a) \\
&+ \frac{1}{2}r\bar{\bar{\rho}}(a^+)J_3(r\partial_t\rho)_{n+1}hv^2(a,.)|_{t=0}^{t=T}.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\partial_t s = s(2t-T)\theta = sT\theta\mathcal{O}(1), \\
&\partial_t\rho = -\varphi(x)(\partial_t s)\rho = -\varphi(x)s(2t-T)\theta\rho, \\
&r\partial_t\rho = -\varphi(x)s(2t-T)\theta \\
&\partial_t(r\partial_t\rho) = sT^2\theta^2\mathcal{O}(1),
\end{aligned}
\tag{A.10}
$$

by using (2.2)- (2.3).

Now we estimate the terms $\partial_t\big(r\bar{\bar{\rho}}(a^+)J_1\big)$ and $\partial_t\big(r\bar{\bar{\rho}}(a^+)J_3(r\partial_t\rho)_{n+1}\big)$. By recalling $\partial_t J_1 = sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$, $\partial_t J_3 = T\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ as well as using Proposition 3.14 and (A.10) we obtain

$$
\begin{aligned}
\partial_t\big(r\bar{\bar{\rho}}(a^+)J_1\big) &= \partial_t\big(r\bar{\bar{\rho}}(a^+)\big)J_1 + r\bar{\bar{\rho}}(a^+)\partial_t J_1 \\
&= sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh),
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_t\big(r\bar{\bar{\rho}}(a^+)J_3(r\partial_t\rho)_{n+1}\big) \\
=\ &\partial_t\big(r\bar{\bar{\rho}}(a^+)\big)J_3(r\partial_t\rho)_{n+1} + r\bar{\bar{\rho}}(a^+)\partial_t J_3(r\partial_t\rho)_{n+1} + r\bar{\bar{\rho}}(a^+)J_3\partial_t\big((r\partial_t\rho)_{n+1}\big) \\
=\ &sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}
$$

Thus Y_{13} can be written

$$\begin{aligned} Y_{13} &= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} + \int_0^T sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2(a) \\ &+ s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)|_{t=0}^{t=T} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)h(rf_1)_{n+1} \\ &+ \int_0^T \big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)h\big(\partial_t v(a)\big)^2 + \int_0^T T^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2(a) \\ &+ \frac{1}{2}(1+\mathcal{O}_{\lambda,\mathfrak{K}}(s(t)h))h\Big(-\varphi(a)s(t)(2t-T)\theta(t)v^2(a,.)\Big)|_{t=0}^{t=T}. \end{aligned}$$

We observe that for $0 < sh < \epsilon_3(\lambda)$ with $\epsilon_3(\lambda)$ sufficiently small we have

$$r\bar{\tilde{\rho}} = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh) > 0.$$

Additionally, $\varphi(x) < 0$ then the last term of Y_{13} are non-negative. From that, we estimate Y_{13} as follows

$$\begin{aligned} Y_{13} &\geq \int_0^T \mathcal{C}_{\lambda,\mathfrak{K}}h(\partial_t v(a))^2 + \int_0^T \big(sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + T^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)v^2(a) \\ &+ s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)|_{t=0}^{t=T} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)h(rf_1)_{n+1}. \end{aligned}$$

A.12 Proof of Lemma 4.16

On the one hands, as $f = f_1 - \partial_t(\rho v)$ we write

$$\begin{aligned} (rf)_{n+1} &= (rf_1)_{n+1} - (r\partial_t(\rho v))_{n+1} \\ &= (rf_1)_{n+1} - \Big((r\rho)\partial_t v + (r\partial_t\rho)v\Big)_{n+1} \\ &= (rf_1)_{n+1} - (\partial_t v)_{n+1} - sT\theta\mathcal{O}_\lambda(1)v_{n+1}. \end{aligned}$$

We thus obtain

$$|(rf)_{n+1}|^2 \leq C\Big((rf_1)_{n+1}^2 + (\partial_t v)_{n+1}^2 + s^2T^2\theta^2\mathcal{O}_\lambda(1)v_{n+1}^2\Big). \tag{A.11}$$

On the other hands,

$$[\rho_1 \star \rho_2] = [\rho_1\star]_a(\rho_2)_{n+\frac{1}{2}} + \rho_1(a^+)[\star\rho_2]_a, \tag{A.12}$$

and we recall

$$(c_d Dv)_{n+\frac{3}{2}} - (c_d Dv)_{n+\frac{1}{2}} = [\star c_d Dv]_a = \lambda s[\star c\phi\psi']_a v_{n+1} + r_0,$$

where r_0 is given in Lemma 3.17 as

$$r_0 = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1} + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{1}{2}} + h\mathcal{O}_{\lambda,\mathfrak{K}}(1)(rf)_{n+1},$$

We then have

$$\begin{aligned}(c_dDv)^2_{n+\frac{3}{2}} &= (c_dDv)^2_{n+\frac{1}{2}} + [\star c_dDv]^2_a + 2[\star c_dDv]_a(c_dDv)_{n+\frac{1}{2}} \\ &= (c_dDv)^2_{n+\frac{1}{2}} + \lambda^2 s^2[\star c\phi\psi']^2_a v^2_{n+1} + r_0^2 + 2\lambda s[\star c\phi\psi']_a r_0 v_{n+1} \\ &\quad +2\lambda s[\star c\phi\psi']_a v_{n+1}(c_dDv)_{n+\frac{1}{2}} + 2r_0(c_dDv)_{n+\frac{1}{2}}. \end{aligned} \tag{A.13}$$

and we compute

$$\begin{aligned} r_0^2 &= s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 v^2_{n+1} + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)^2_{n+\frac{1}{2}} + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(rf)^2_{n+1} \\ &\quad +s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)_{n+\frac{1}{2}} v_{n+1} + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(rf)_{n+1}v_{n+1} \\ &\quad +\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}} h(rf)_{n+1}. \end{aligned}$$

By applying Cauchy-Schwartz inequality we have

$$(c_dDv)^2_{n+\frac{3}{2}} \le \mathcal{O}(1)(c_dDv)^2_{n+\frac{1}{2}} + s^2\mathcal{O}_\lambda(1)v^2_{n+1} + \mathcal{O}(1)r_0^2 \tag{A.14}$$

$$r_0^2 \le s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 v^2_{n+1} + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)^2_{n+\frac{1}{2}} + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(rf)^2_{n+1}, \tag{A.15}$$

$$sr_0v_{n+1} \le \big(s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)+s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\big)v^2_{n+1}+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)^2_{n+\frac{1}{2}}+h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(rf)^2_{n+1}, \tag{A.16}$$

$$s^2r_0v_{n+1} \le \big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)+\epsilon s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)\big)v^2_{n+1}+s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)^2_{n+\frac{1}{2}}+C_\epsilon h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(rf)^2_{n+1}, \tag{A.17}$$

$$\begin{aligned} sT\theta r_0v_{n+1} &\le \big(s^2T\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)v^2_{n+1} \\ &+ s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)^2_{n+\frac{1}{2}} + h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(rf)^2_{n+1}, \end{aligned} \tag{A.18}$$

$$sr_0(cDv)_{n+\frac{1}{2}} \le s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2_{n+1}+\big(s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)+\epsilon s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\big)(c_dDv)^2_{n+\frac{1}{2}}+C_\epsilon\mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(rf)^2_{n+1}, \tag{A.19}$$

$$(\partial_t v(a))r_0 \le \mathcal{O}_{\lambda,\mathfrak{K}}(1)h(\partial_t v(a))^2+s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)^2_{n+\frac{1}{2}}+s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2_{n+1}+\mathcal{O}_{\lambda,\mathfrak{K}}(1)h(rf)^2_{n+1}. \tag{A.20}$$

We estimate following terms

The first term, by using (A.14) we have

$$\begin{aligned}\left|Y_{11}^{(2,2)}\right| &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)^2_{n+\frac{1}{2}} + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)^2_{n+\frac{3}{2}} \\ &\le \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)^2_{n+\frac{1}{2}} + \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v^2_{n+1} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2r_0^2. \end{aligned}$$

Moreover, by using (A.15) we obtain

$$\begin{aligned}&\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 r_0^2\\ \leq\ &\int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4 v_{n+1}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (rf)_{n+1}^2.\end{aligned}$$

Then, by using (A.11) we estimate $Y_{11}^{(2,2)}$

$$\begin{aligned}\left|Y_{11}^{(2,2)}\right| &\leq \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 v_{n+1}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (rf)_{n+1}^2\\ &\leq \int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4\Big) v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 h(\partial_t v)_{n+1}^2\\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (rf_1)_{n+1}^2.\end{aligned}$$

For the next term, using (A.12) and Lemma 3.17 we obtain

$$\begin{aligned}Y_{12} &= \int_0^T s\mathcal{O}_\lambda(1) v(a) [c\psi'^2 \star (c_d Dv)]_a\\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) (cDv)_{n+\frac{1}{2}} + s\mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) (cDv)_{n+\frac{3}{2}}\\ &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) (c_d Dv)_{n+\frac{1}{2}} + s\mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) \Big((c_d Dv)_{n+\frac{1}{2}} + s\mathcal{O}_\lambda(1) v(a) + r_0\Big)\\ &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) (c_d Dv)_{n+\frac{1}{2}} + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2(a) + s\mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) r_0.\end{aligned}$$

Using (A.16) yields

$$\begin{aligned}&\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) r_0\\ \leq\ &\int_0^T \Big(s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\Big) v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh) (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh) (rf)_{n+1}^2.\end{aligned}$$

By using (A.11) we obtain

$$\begin{aligned}|Y_{12}| &\leq \int_0^T \big(s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\big) v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh) h(\partial_t v)_{n+1}^2\\ &+ \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1) (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh) h (rf_1)_{n+1}^2.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\tilde{v}_{n+\frac{1}{2}} = v_{n+1} - \frac{h}{2(c_d)_{n+\frac{1}{2}}} (c_d Dv)_{n+\frac{1}{2}} = v_{n+1} + \mathcal{O}(h)(c_d Dv)_{n+\frac{1}{2}},\\ &\tilde{v}_{n+\frac{3}{2}} = v_{n+1} + \mathcal{O}(h)(c_d Dv)_{n+\frac{3}{2}}.\end{aligned} \tag{A.21}$$

By using (A.21), (A.14) we obtain

$$\begin{aligned}(\tilde{v})^2_{n+\frac{1}{2}} + (\tilde{v})^2_{n+\frac{3}{2}} &\leq \mathcal{O}(1)v^2_{n+1} + \mathcal{O}(h^2)(cDv)^2_{n+\frac{3}{2}} + \mathcal{O}(h^2)(cDv)^2_{n+\frac{1}{2}} \\ &\leq \big(\mathcal{O}(1) + \mathcal{O}(sh)^2\big)v^2_{n+1} + \mathcal{O}(h^2)(cDv)^2_{n+\frac{1}{2}} + \mathcal{O}(h^2)r_0^2\end{aligned}$$

Thus, we have the following estimate

$$\begin{aligned}\Big|Y_{21}^{(1,22)}\Big| &= \int_0^T \Big(s^2\mathcal{O}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\Big)\Big((\tilde{v})^2_{n+\frac{1}{2}} + (\tilde{v})^2_{n+\frac{3}{2}}\Big) \\ &= \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)\Big((\tilde{v})^2_{n+\frac{1}{2}} + (\tilde{v})^2_{n+\frac{3}{2}}\Big) \\ &\leq \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2_{n+1} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)^2_{n+\frac{1}{2}} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2r_0^2.\end{aligned}$$

Futhermore, using (A.15) we have

$$\begin{aligned}&\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2r_0^2 \\ \leq\ &\int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4v^2_{n+1} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4(c_dDv)^2_{n+\frac{1}{2}} + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(rf)^2_{n+1}.\end{aligned}$$

By using (A.11) we get

$$\begin{aligned}\Big|Y_{21}^{(1,22)}\Big| &\leq \int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4\Big)v^2_{n+1} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3h(\partial_tv)^2_{n+1} \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(c_dDv)^2_{n+\frac{1}{2}} + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(rf_1)^2_{n+1}.\end{aligned}$$

For the term $Y_{22}^{(1)}$ we have

$$\begin{aligned}Y_{22}^{(1)} &= \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h^2}{2}(Dv)_{n+\frac{1}{2}} + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h^2}{2}(Dv)_{n+\frac{3}{2}} \\ &= \int_0^T \Big(s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)_{n+\frac{1}{2}} + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)_{n+\frac{3}{2}}\Big)v(a) \\ &= \int_0^T \Big(s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)_{n+\frac{1}{2}} + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v(a) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2r_0\Big)v(a) \\ &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v(a)(c_dDv)_{n+\frac{1}{2}} + \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v^2(a) + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v(a)r_0.\end{aligned}$$

Using (A.16) we achieve

$$\begin{aligned}&\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v(a)r_0 \\ \leq\ &\int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v^2_{n+1} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(c_dDv)^2_{n+\frac{1}{2}} + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(rf)^2_{n+1}.\end{aligned}$$

Using (A.11), we estimate $Y_{22}^{(1)}$ as:

$$\begin{aligned}\left|Y_{22}^{(1)}\right| &\leq \int_0^T \Big(s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4\Big)v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 h(\partial_t v)_{n+1}^2 \\ &+ \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (rf_1)_{n+1}^2.\end{aligned}$$

And, using (A.21) and Lemma 3.17 we obtain

$$\begin{aligned}Y_{23}^{(1)} &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(\partial_t v(a))\frac{h}{2}\tilde{v}_{n+\frac{1}{2}} + \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(\partial_t v(a))\frac{h}{2}\tilde{v}_{n+\frac{3}{2}} \\ &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(\partial_t v(a))\frac{h}{2}\Big(v_{n+1} + \mathcal{O}(h)(c_d Dv)_{n+\frac{1}{2}}\Big) \\ &+ \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(\partial_t v(a))\frac{h}{2}\Big(v_{n+1} + \mathcal{O}(h)(c_d Dv)_{n+\frac{3}{2}}\Big) \\ &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v(a))v(a) + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v(a))(c_d Dv)_{n+\frac{1}{2}} \\ &+ \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v(a))(c_d Dv)_{n+\frac{3}{2}} \\ &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v(a))v(a) + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v(a))(c_d Dv)_{n+\frac{1}{2}} \\ &+ \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v(a))\Big((c_d Dv)_{n+\frac{1}{2}} + \lambda s[\star c\phi\psi']_a v_{n+1} + r_0\Big).\end{aligned}$$

In addition, with s, λ enough large, sh enough small and with applying Young's inequality and (A.20) yield

$$\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v(a))v(a) \leq \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(\partial_t v(a))^2 + \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a).$$

$$\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v(a))(c_d Dv)_{n+\frac{1}{2}} \leq \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(\partial_t v(a))^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_d Dv)_{n+\frac{1}{2}}^2.$$

$$\begin{aligned}\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v(a))r_0 &\leq \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 h(\partial_t v(a))^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(c_d Dv)_{n+\frac{1}{2}}^2 \\ &+ \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(v)_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 h(rf)_{n+1}^2.\end{aligned}$$

Using (A.11), we estimate $Y_{23}^{(1)}$

$$\begin{aligned}Y_{23}^{(1)} &\leq \int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3\Big)v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(\partial_t v(a))^2 \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 h(rf_1)_{n+1}^2.\end{aligned}$$

Applying Lemma 3.17 we have

$$\begin{aligned} Y_{31}^{(1)} &= \int_0^T T\theta s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) \frac{h}{2} (Dv)_{n+\frac{1}{2}} + \int_0^T T\theta s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) \frac{h}{2} (Dv)_{n+\frac{3}{2}} \\ &= \int_0^T sT\theta \mathcal{O}_{\lambda,\mathfrak{K}}(sh) v(a) (c_d Dv)_{n+\frac{1}{2}} \\ &+ \int_0^T sT\theta \mathcal{O}_{\lambda,\mathfrak{K}}(sh) v(a) \Big(s\mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) + r_0 \Big) \end{aligned}$$

By using (A.18) we obtain

$$\begin{aligned} \int_0^T sT\theta \mathcal{O}_{\lambda,\mathfrak{K}}(sh) v_{n+1} r_0 &\leq \int_0^T \Big(s^2 T\theta \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 + sT^2\theta^2 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 \Big) v_{n+1}^2 \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 h (rf)_{n+1}^2. \end{aligned}$$

We have

$$\int_0^T sT\theta \mathcal{O}_{\lambda,\mathfrak{K}}(sh) (c_d Dv)_{n+\frac{1}{2}} v_{n+1} \leq \int_0^T sT^2\theta^2 \mathcal{O}_{\lambda,\mathfrak{K}}(sh) v_{n+1}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) (c_d Dv)_{n+\frac{1}{2}}^2.$$

With (A.11) we thus estimate $Y_{31}^{(1)}$ as

$$\begin{aligned} \Big| Y_{31}^{(1)} \Big| &\leq \int_0^T \Big(s^2 T\theta \mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2 \mathcal{O}_{\lambda,\mathfrak{K}}(sh) \Big) v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh) h (\partial_t v)_{n+1}^2 \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh) h (rf_1)_{n+1}^2. \end{aligned}$$

Next, by using (A.14) we estimate $\mu_1^{(1)}$ as

$$\begin{aligned} \mu^{(1)} &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) (c_d Dv)_{n+\frac{3}{2}}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) (c_d Dv)_{n+\frac{1}{2}}^2 \\ &\leq \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh) v_{n+1}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) r_0^2 \end{aligned}$$

By making use of (A.15) we have

$$\begin{aligned} &\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) r_0^2 \\ \leq &\int_0^T s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 v_{n+1}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 h (rf)_{n+1}^2. \end{aligned}$$

Using (A.11) we obtain

$$\begin{aligned} \mu_1^{(1)} &\leq \int_0^T \Big(s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 \Big) v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 h (\partial_t v)_{n+1}^2 \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 h (rf_1)_{n+1}^2. \end{aligned}$$

By making use Lemma 3.17 we have

$$\begin{aligned}\mu_1^{(2)} &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_dDv)_{n+\frac{3}{2}} + \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_dDv)_{n+\frac{1}{2}} \\ &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_dDv)_{n+\frac{1}{2}} + \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}^2 + \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)r_0v_{n+1}.\end{aligned}$$

Applying Young's inequality and using (A.17) yield

$$\int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_dDv)_{n+\frac{1}{2}} \leq \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}^2.$$

$$\begin{aligned}&\int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)r_0v_{n+1} \\ \leq\ &\int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v_{n+1}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)_{n+\frac{1}{2}}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(rf)_{n+1}^2.\end{aligned}$$

Using (A.11) we have

$$\begin{aligned}\mu_1^{(2)} &\leq \int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3\Big)v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(\partial_tv)_{n+1}^2 \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(rf_1)_{n+1}^2.\end{aligned}$$

We thus obtain

$$\begin{aligned}\mu_1 &\leq \int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3\Big)v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(\partial_tv)_{n+1}^2 \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(rf_1)_{n+1}^2.\end{aligned}$$

Now, we estimate some terms of μ_r. By using (A.15)- (A.19) we have

$$\begin{aligned}&\int_0^T s\mathcal{O}_{\lambda}(1)r_0^2 \\ \leq\ &\int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v_{n+1}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)_{n+\frac{1}{2}}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(rf)_{n+1}^2.\end{aligned}$$

$$\begin{aligned}\int_0^T s^2\mathcal{O}_{\lambda}(1)r_0v_{n+1} &\leq \int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + \epsilon s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)\Big)v_{n+1}^2 \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}^2 + C_\epsilon\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(rf)_{n+1}^2.\end{aligned}$$

$$\begin{aligned}\int_0^T s\mathcal{O}_\lambda(1) r_0 (c_d Dv)_{n+\frac{1}{2}} &\leq \int_0^T s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh) v_{n+1}^2 + C_\epsilon \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh) h (rf)_{n+1}^2 \\ &+ \int_0^T \Big(s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + \epsilon s \mathcal{O}_{\lambda,\mathfrak{K}}(1) \Big) (c_d Dv)_{n+\frac{1}{2}}^2.\end{aligned}$$

$$\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) v_{n+1} (c_d Dv)_{n+\frac{1}{2}} \leq \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) v_{n+1}^2.$$

Using (A.11) we have:

$$\begin{aligned}\mu_r &\leq \int_0^T \Big(s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2 \mathcal{O}_{\epsilon,\lambda,\mathfrak{K}}(sh)^2 + \epsilon s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(1) \Big) v_{n+1}^2 \\ &+ \int_0^T \mathcal{O}_{\epsilon,\lambda,\mathfrak{K}}(sh) h (\partial_t v)_{n+1}^2 + \int_0^T \mathcal{O}_{\epsilon,\lambda,\mathfrak{K}}(sh) h (rf_1)_{n+1}^2 \\ &+ \int_0^T \Big(s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + \epsilon s \mathcal{O}_{\lambda,\mathfrak{K}}(1) \Big) (c_d Dv)_{n+\frac{1}{2}}^2.\end{aligned}$$

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