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The implicative statistical analysis: an interdisciplinary paradigm

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Abstract. In this brief note, which has simply the role of an epistemological survey paper, some of the main basic elements of Implicative Statistical Analysis (ISA) pattern are put into a possible critical comparison with some of the main aspects of Probability Theory, Inductive Inference Theory, Nonparametric and Multivariate Statistics, Optimization Theory and Dynamical System Theory which point out the very interesting multidisciplinary nature of the ISA pattern and related possible hints.

1. Introduction

The *Implicative Statistical Analysis* (ISA) is a new powerful Mathematics Education research tool proposed by Regis Gras in 1990s: in this brief note, after a brief exposition of its main points, from which it emerges its very rich multidisciplinary nature, we simply wish to point out three possible comparative approaches among some of the main aspects of ISA framework and some others of Probability Theory, Inductive Inference Theory, Nonparametric and Multivariate Statistics, Game Theory and Dynamical System Theory.

In what follows, we simply limit ourselves both to identify this specific nature of the ISA pattern and to sketch some possible hints stemming by these possible comparisons.

2. The Implicative Statistical Analysis: brief outlines

According to (Gras, 2000), the fundamental problem of Implicative Statistical Analysis is the following: given two binary variables a and b , what is the measure according to, within a given population E in which they are defined, from every occurrence of a it follows the occurrence of b ? Or, is it true that if a then b ?

The ISA tries to measure the truth degree of the material implication $a \Rightarrow b$ where $a, b : E \rightarrow \{0,1\}$ are two arbitrary binary variables. To this purpose, let $A \doteq \{x; x \in E, a(x) = 1\}$, $B \doteq \{x; x \in E, b(x) = 1\}$ be the truth value sets respectively of the binary variables a and b . Taking into account the fact that an arbitrary material implication $a \Rightarrow b$ is false if and only if a is true and b is false, it follows that the unique counterexamples to $a \Rightarrow b$ are given by $a \wedge \bar{b}$. According to Gras, the probabilistic measure of the implication $a \Rightarrow b$ is given by the following *Gras implication index*

$$(1) \quad \varphi_{X,Y}(a, \bar{b}) \doteq 1 - P[\text{card}(A \cap \bar{B}) \geq \text{card}(X \cap \bar{Y})]$$

where $X, Y \subseteq E$ are randomly chosen according to a given probability distribution, independently the one from the other and in such a way that $\text{card } X = \text{card } A$ and $\text{card } Y = \text{card } B$.

Therefore, we will say that $a \Rightarrow b$ is acceptable at the significance level $\varphi_{X,Y}(a, \bar{b}) = 1 - \alpha$ if and only if $P[\text{card}(A \cap \bar{B}) \geq \text{card}(X \cap \bar{Y})] \leq \alpha$.

3. ISA and Probability Theory

Following (Letta, 1993), in considering an arbitrary *random experiment* in which it is no possible to know a priori its result, the experimenter may however choice a set, say Ω , of possible *eventualities*, disjoint one from another, in which the result of such an experiment might fall, in the sense that it is represented by one and only one element of it. If the information is so poor to obstacle the determination of the related eventuality in Ω , then the prediction of the result may be probabilistically determined considering a possible set of eventualities, say A , in which such a result may fall; in such a case, we say that the result of such a random experiment is linked to the *event* $A \subseteq \Omega$ so that, a priori, every subset of Ω is a possible event for it.

Therefore, in general, there will exist a subclass $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ (= set of all parts of Ω) of events A in which may fall the results of a random experiment and that, for various stability, regularity and symmetry reasons (see (Letta, 1993)), it is required to be a σ -algebra of sets. Moreover, it will be possible to establish a measure of the confidence degree with which the result may fall into a given event $A \in \mathcal{A}$, say $P: \mathcal{A} \rightarrow [0,1]$, which it is also required to be a measure upon the σ -algebra \mathcal{A} . In such a case, we obtain a *probability space* (Ω, \mathcal{A}, P) associated to the given random experiment, which constitutes its formal model upon which work out; for every event $A \in \mathcal{A}$, $P(A)$ is said to be the *probability of the event A* within the given probability space (Ω, \mathcal{A}, P) .

Given two events, say A and B , we say that the (theoretical) hypothesis A is subordinated to the (experimental) evidence B when it is defined an event $A|B$ (saying that the event A is *subordinated* to the event B) which is true if, being B true, A is true, while it is false if, being B true, A is false; in a certain sense, its truth values correspond to those of the material implication $B \Rightarrow A$ (see (Daboni, 1976, Capitolo 2, § 2.3)). Hence, $P(A|B)$ is the measure of the confidence degree which one sets in the eventuality of A when the hypothesis E be true. Often, $P(\cdot |B)$ is also said to be a *conditional probability* relatively to the event B and it is also written in the form $P_B(\cdot)$: its meaning is what probability measure we believe to be suitable to consider when the event B is realized; as a consequence, the same random experiment will be formalized into the new probability space $(\Omega, \mathcal{A}, P_B)$ in dependence on the new conditioning event B .

One of the crucial points of Probability Theory is the so-called *Bayes theorem* according to which (see (Dall'Aglio, 1987, Capitolo II, § II.10)), given an event B and a finite or denumerable set of independent and exhaustive events A_i $i \in J$ such that $\Omega = \bigcup_{i \in J} A_i$ with $P(B) \neq 0$, then

$$(3) \quad P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i \in J} P(A_i)P(B|A_i)}$$

where the events A_i $i \in J$ may be considered as possible causes of the event B or hypotheses which explain it; the condition $B \subseteq \bigcup_{i \in J} A_i = \Omega$, then this means that the occurrence of B implies the occurrence of, at least, one and only one of the events A_i (given their mutual incompatibility). The various controversial questions which (3) give are due to the possible choices of the prior probabilities $P(A_i)$.

In the case $\text{card } J = 1$, the subordinate event $A|B$ may be written as $A|AB$ (where $AB = A \cap B$) in the form *thesis/hypothesis*, whence

$$(4) \quad P(A|B) = \frac{P(A)P(B|A)}{P(B)} = \frac{P(A)P(B \cap A)}{P(A)P(B)} = \frac{P(B \cap A)}{P(B)}$$

which is the Bayes' theorem for the simple material implication $B \Rightarrow A$ (see (De Finetti, 1970, Volume I, Capitolo IV, § 4.1), providing the probability of the occurrence of A when B occurs.

Therefore, further possible comparisons with (1) might turn out to be of a some interest: one of these will be, for example, traced in the next section, just in connection to the possible choices of the $P(A_i)$. Finally, for further considerations on the Bayesian's aspects involved by ISA, with interesting links with the Boolean algebra, see (Bernard & Poitrenaud, 1999).

4. ISA and Inductive Inference Theory

The two main logical processes to build up the various scientific theories are the *deduction* and the *induction*, the first one being related to the mathematical logic whereas the second one is related to the logic of natural sciences; both are indispensable to the human thought and are irreducible to one another. The induction work out by means of various inference rules, whose main lines are exposed in (Dalla Chiara & Toraldo di Francia, 1981, Capitolo 1) from a physical viewpoint, and in (Dalla Chiara & Toraldo di Francia, 1999, Capitolo V; Toraldo di Francia, 1976, Capitolo IV, § 21; D'Espagnat, 1983, Capitolo 12) from an epistemological viewpoint. For our purposes, among the various currents of the theory of induction, we are mainly interested to the 1944 Rudolf Carnap program on Inductive Logic (see (Niiniluoto, 1983) for a formal survey) because it is strictly correlated with the basic principles of ISA.

According to Carnap (see (Niiniluoto, 1983, Section III)), all inductive reasoning, in the wide sense of nondeductive reasoning, is reasoning in probabilistic terms; inductive logic, the theory of the principles of inductive reasoning, is the same as probability logic; and, the concept of probability on which inductive logic

is to be based is a logical relation between two statements or propositions, given by the degree of confirmation of a hypothesis (or conclusion) on the basis of some given evidence (or premises). From what has been said in the previous section, Carnap starts from the fact that $P(A|B)$ could be considered as the degree of confirmation of the hypothesis A on the basis of the evidence B , or it measures the degree of validity of the assertion « B confirms or supports A » or the degree of confirmation of A on B , deducing a his own axiomatic theory whose main formal points are well exposed in (Niiniluoto, 1983) of which we want above all to stress one of its founding axioms (see (Niiniluoto, 1983, Section V)), namely the *axiom of finite regularity* according to which, for singular sentences B and A , we have $P(A|B) = 1$ if and only if $\vdash B \Rightarrow A$, this last being the logical implication, and not the simple material one, of the deductive logic. Moreover, Carnap argues upon the so-called concepts of *firmness* and of *increase in firmness* according to which the qualitative notion « A is confirmed by B » should be defined via the positivity condition $P(A|B) \geq P(A)$ through to which to define such a degree of confirmation by $P(A|B) - P(A)$.

In conclusion, the inductive logic is inclined to give the probability of an assertion or statement through the so-called *confirmation function* $P(A|B)$ which gives the probability that the hypothesis¹ A be true when it is known the occurrence of B for instance from an experimental evidence.

5. ISA and Nonparametric Statistics

5.1 Introduction

One of the central problems in the theory of data elaboration of Experimental Physics is that of determining the probability distributions of the various measures obtained from an experimental measurement which, in general, belongs to the class formed by the Gauss, Bernoulli and Poisson distributions. The tests for the deviation of the observed distribution of experimental data from a presumed theoretical distributions provide criteria for deciding with what approximation the former follows the latter.

There exist specific tests for each distribution (as, for instance, the β -skewness and the β_2 -flatness for the Gauss distribution), as well as some related to a general theoretical distribution, like the well-known (among the non-parametric² statistical tests for goodness fit) χ^2 -test and the Kolmogorov one, which result to be independent from the presumed distributions (distribution free).

In what follows about subsection 5.2, we recall the main outlines on the χ^2 -test, introduced by K. Pearson at the very beginning of the 1900, mainly following the exposition given by (Taylor, 1986).

5.2 The quantitative χ^2 -test: brief outlines

Let X be a random variable defined in $[a,b] \subseteq \mathbb{R}$ and \mathcal{F} a partition of it into n subintervals, and let O_k and E_k be respectively the observed and the expected values of X which fall into the k -th subinterval. The hypothesis (often denoted with H_0) according to which the observed values follow a given preassigned theoretical probability distribution which predict the given expected values, clearly depends by the various related deviations $O_k - E_k$, so that it results to be natural to consider the following number

$$(5) \quad \chi^2 = \sum_{k=1}^n \frac{(O_k - E_k)^2}{E_k}$$

said to be *chi-squared*, which is a good estimate for the accordance between the observed distribution and the presumed one. Following (Montanari & Poppi, 1982), this test is significative at least when $n \geq 40$ and allows to decide whether the deviations between the experimental and hypothesized theoretical distributions (*hypothesis* H_0) are due to the casualness or not.

Since it is expected that each term of this sum be about 1, then we would have approximately $\chi^2 \leq n$ if there is a good accordance between the observed distribution and the hypothesized one, otherwise we would have $\chi^2 \gg n$. A better accordance estimate is carried out if one compares the chi-squared with the statistical

¹ Often, when one speaks of hypothesis in the inductive logic context, not necessarily it has the same meaning of the deductive logic. Instead, it should be intended as a theoretical hypothesis A which requires confirmation by some experimental evidence E , according to the general program of the theory of induction.

² Even if such a term is quite incorrect: see (Girone & Salvemini, 2000, § 25.2).

freedom degrees $d = n - c$ instead of the simple interval number n , where c is the number of parameters and/or relations related to the experimental data, which is also called *constrained* number³. Therefore, it is possible to prove as the expected value of χ^2 is just d , so that if $\chi^2 \gg d$ then it is very likely that the hypothesized distribution does not agree with the experimental one. Often it is considered the so-called *reduced chi-squared* defined as $\bar{\chi}^2 = \chi^2/d$, so that if one obtains an estimate of the $\bar{\chi}^2$ less or equal to 1, then the initial hypothesis is valid, whereas if $\bar{\chi}^2$ is much greater than 1 it is not.

Now, the quantitative χ^2 -test tries to determine how a reduced chi-squared greater than one may be considered invalidating of our hypothesis H_0 . To answer to such a question, one considers the probability⁴ $P(\bar{\chi}^2 \geq \bar{\chi}_0^2)$ of obtaining a reduced chi-squared greater or equal to that experimentally computed: if this is quite high, then our value⁵ $\bar{\chi}_0^2$ is acceptable because is very likely to obtain an higher value of it, so that the latter is one of the possible lower values. Hence, it is important at first to determine at what lower value of the probability⁶ $P(\bar{\chi}^2 \geq \bar{\chi}_0^2)$ the given hypothesis H_0 is refusable. In general, it is chosen two (but not the only ones possible) rejection confidence levels P_{sign} under to which there is disagreement, that is to say when $P(\bar{\chi}^2 \geq \bar{\chi}_0^2) < P_{sign}$: the $P_{sign} = 5\%$ level and the $P_{sign} = 1\%$ level.

In practice, chosen the confidence level $P_{sign} = \int_{\chi_0^2}^{+\infty} f(\chi^2)d(\chi^2)$ once known the value χ_0^2 computed through (5), then it is possible to determine a value of χ^2 such that⁷ $P(\chi^2 \geq \chi_0^2) = P_{sign}$, the left hand side of this equation being computed through proper probability tables in dependence on d and on the χ^2 probability distribution $f(\chi^2) d(\chi^2) = 2^{-\frac{d}{2}} \left(\Gamma\left(\frac{d}{2}\right) \right)^{-1} (\chi^2)^{\frac{d-2}{2}} e^{-\frac{\chi^2}{2}} d(\chi^2)$. Thus, if it results to be $\chi^2 < \chi_0^2$ then H_0 is rejected at the given confidence level P_{sign} , whereas if $\chi^2 > \chi_0^2$ then H_0 is considered to be valid at the confidence level P_{sign} , at least till to more information⁸ (see (Porto, 1987, Capitolo VIII, § 3)); furthermore, in this last case, if the related $P(\chi^2 \geq \chi_0^2)$ is also low, then this suggests the presence of systematic errors.

5.3 ISA and χ^2 -test

In Section 2, we have put that $a \Rightarrow b$ is acceptable at the significance level $\varphi_{X,Y}(a, \bar{b}) = 1 - \alpha$ if and only if $P[\text{card}(A \cap \bar{B}) \geq \text{card}(X \cap \bar{Y})] \leq \alpha$. Therefore, if one considers the counterexample random variable defined on $\mathcal{P}(E) \times \mathcal{P}(E)$ and whose nonnegative values are given by $\text{card}(X \cap \bar{Y})$, that is to say, by the number of counterexamples to the material implication $X \Rightarrow Y$, then $P[\text{card}(A \cap \bar{B}) \geq \text{card}(X \cap \bar{Y})] \leq \alpha$ might be interpreted in the context of the statistical hypothesis tests to ascertain what probability distribution such counterexamples follow since the relation⁹

$$1 - P[\text{card}(A \cap \bar{B}) \geq \text{card}(X \cap \bar{Y})] = P[\text{card}(X \cap \bar{Y}) \geq \text{card}(A \cap \bar{B})]$$

might be interpreted as a measure of the probability to obtain a number of random counterexamples $\text{card}(X \cap \bar{Y})$ greater than the observed one $\text{card}(A \cap \bar{B})$ according to a given probability distribution, following therefore a philosophy analogous to that of statistical hypothesis tests of nonparametric statistics, like the χ^2 -test that, among other things, may be also used to ascertain which degree of correlation might subsist between two variables, one chosen independent the other dependent.

³ Roughly speaking, d provides the number of independent random variables.

⁴ Which is computed respect to the same theoretical probability distribution already considered.

⁵ With χ_0^2 (or χ_{exp}^2) we denote the observed (or experimental) value of the chi-squared computed by means of (5), whereas with χ_{th}^2 , or simply with χ^2 , we denote the theoretical one; likewise for $\bar{\chi}^2$.

⁶ The value $P(\bar{\chi}^2 \geq \bar{\chi}_0^2)$ is given by the so-called *Helmert-Pearson χ^2 probability distribution*; see, for instance, (Stoka, 1991, Capitolo VIII, § 8.1).

⁷ This simply means that P_{sign} is the probability with which χ^2 may overcomes χ_0^2 .

⁸ Indeed, the χ^2 -test is definitive only when it leads to a rejection of the assumed hypotheses, as the muon AMM in Standard Model story shows (see (Jegerlehner, 2008), (Melnikov & Vainshtein, 2006)). In this last sense, such a test might also be considered as a sort of partial confirmation of the main Karl R. Popper *falsificationism* theses applied to Physics.

⁹ In the probability space (Ω, \mathcal{A}, P) , if $\bar{A} = \Omega \setminus A$ is the opposite event to the event $A \in \mathcal{A}$, then $P(\bar{A}) = 1 - P(A)$.

Following (Gras and Kuntz, 2008) and (Gras, 2005), put $n = \text{card } E$, $n_a = \text{card } A$, $n_b = \text{card } B$, $n_{\bar{b}} = \text{card } \bar{B}$ and $n_{a\bar{b}} = \text{card } (A \cap \bar{B})$, when $n_{\bar{b}} \neq 0$ to the counterexample random variable $\text{card } (X \cap \bar{Y})$ it is possible to associate the following standardized random variable ¹⁰

$$Q_{X,Y}(a, \bar{b}) \doteq \frac{\text{card } (X \cap \bar{Y}) - \frac{n_a n_{\bar{b}}}{n}}{\sqrt{\frac{n_a n_{\bar{b}}}{n}}};$$

further, if we also set

$$q_{X,Y}(a, \bar{b}) \doteq \frac{\text{card } (A \cap \bar{B}) - \frac{n_a n_{\bar{b}}}{n}}{\sqrt{\frac{n_a n_{\bar{b}}}{n}}},$$

then this last may be considered as the experimental value observed for $Q_{X,Y}(a, \bar{b})$ and it measures a deviation between the contingency and the expected value when a and b are independent (Gras & Kuntz, 2008). Thus, the material implication $a \Rightarrow b$ will be acceptable at the confidence level $1 - \alpha$ or at the threshold α , when

$$\varphi_{X,Y}(a, \bar{b}) = 1 - P[Q_{X,Y}(a, \bar{b}) \leq q_{X,Y}(a, \bar{b})] = P[q_{X,Y}(a, \bar{b}) \leq Q_{X,Y}(a, \bar{b})] \geq 1 - \alpha$$

which seems to resemble the χ^2 -test if we consider $Q_{X,Y}(a, \bar{b})$ as a kind of χ^2 parameter, as regards a given, prefixed probability distribution (Gaussian, Poissonian or Bernoullian) which we suppose to be followed by the counterexamples to $a \Rightarrow b$. Thereafter, $\varphi_{X,Y}(a, \bar{b})$ is an inductive and informative quality measure. Finally, for useful modern application of ISA to educational context, see (Di Fazio et al. 2012).

6. ISA and Optimization Theory

Following (Gras, 2005, § 1.3), in the case of continuous variables, problems arise both of maximization of the degree of the given material implication and of minimization of the number of counterexamples which may occur: these aims are attainable for instance by extremizing the various implication indices, so appealing to the various optimization tools like Operation Research, Graph Theory and Game Theory. In this case, it is also used the Edwin Diday *dynamical cloud method* which lead to interesting relationships with factorial statistical analysis (see (Spagnolo, 1997; Spagnolo, 2005)).

It is clear as the variability of the possible formal nature of the environment space and framework setting in which it is possible to formalize the problem of the ISA study of the material implication may be supplied by the various suitable methods and tools of Linear and Nonlinear Optimization Theory, including Game Theory, Graph Theory, Operation Research, and so on. For instance, the ISA methods include the analyses of the so-called *implicative* and *inclusive graph*, in which it is necessary to choice a certain purposive winning strategy, hence involving the Game Theory context.

7. ISA and Multivariate Statistics

From what said in the previous section, it clearly emerges what important role might play the relationships between the ISA and the factorial statistical analysis in the multidimensional setting¹¹ and from the geometrical perspective: just from this last point of view, the related geometrical tools are widely used in the ISA context (see (Spagnolo, 2005)), for instance in formal treating of the *dynamical clouds* of Diday, also using rational mechanics tools and methods from a statistical standpoint, in which, besides, it arise too

¹⁰ Which can be also written as $[\text{card } (X \cap \bar{Y})/n - p_a p_{\bar{b}}]/\sqrt{p_a p_{\bar{b}}/n}$, where $p_a p_{\bar{b}}$ is the composite probability of the two events a and \bar{b} assumed to be independent each from the other, whereas $\text{card } (X \cap \bar{Y})/n$ is the probability of counterexample occurrence.

¹¹ Even if the ISA pattern works out into the inferential statistics context whereas the Factorial Analysis is into the realm of descriptive statistics.

further optimization problems, going back again to what said in the previous section. For instance, the analogical use of the main concepts and methods of rigid mechanics (like moments of inertia, principal axes of inertia, etc) have led both to the so-called *principal component analysis* (PCA) and to the *factorial analysis of correspondences* (FAC) (see (Spagnolo, 2005)): in both it is used well-precise data matrix distances of the related multidimensional data space (see (Fabbris, 1997)), as, for example, the χ^2 -distance.

Nevertheless, what does matter of consequences may lead the choice of another distance amongst the possible ones?

8. ISA and Dynamical System Theory

Another interesting point of the ISA pattern concerns some Information Theory questions just regarding the information content related to the material implication under examination and its counterexamples. To this purpose, Regis Gras uses a classical Claude Shannon entropy function (see (Gras, 2005)) for trying to measure such a counterexample information and the degree of incompatibility (or reciprocal disorder) between the two variables involved in such a material implication, hence, indirectly, their degree of cohesion.

Therefore, into the ISA pattern also enter basic questions concerning Dynamical System Theory and Ergodic Theory.

9. Conclusions

In short, into the ISA pattern are involved many and different powerful tools coherently used in a synergic manner to experimentally study a material implication of the type $a \Rightarrow b$ when does not subsist the inclusion $B \subseteq A$ but a partial form of it, speaking therefore of a *partial implication* $a \rightarrow b$ statistically linked to the previous (exact) material one.

Thus, it is evident the great interest which may played the ISA pattern, with its wide application range.

References

- Bernard, J-M. and Pointreud, S. (1999), L'analyse implicative bayésienne multivariée d'un questionnaire binaire: quasi-implications et treillis de galois simplifié, *Mathématiques et Sciences Humaines*, 147, 25-46.
- Daboni, L. (1976), *Calcolo delle probabilità*, Torino: Boringhieri Editore.
- Dall'Aglio, L. (1987), *Calcolo delle probabilità*, Bologna: Zanichelli Editore.
- Dalla Chiara, M.L. and Toraldo di Francia, G. (1981), *Le teorie fisiche: un'analisi formale*, Torino: Editore Boringhieri.
- Dalla Chiara, M.L. and Toraldo di Francia, G. (1999), *Introduzione alla filosofia della scienza*, Bari: Laterza Editori.
- De Finetti, B. (1970), *Teoria delle probabilità*, 2 Voll., Torino: Giulio Einaudi editore.
- D'Espagnat, B. (1983), *Alla ricerca del reale. Fisica e oggettività*, Torino: Editore Boringhieri.
- Di Fazio, C., Di Paola, B. and Guastella, I. (2012), Prospective elementary teachers' perceptions of the processes of modeling: A case study, *Physical Review Special Topics – Physics Education Research*, 8, 010110-1/010110-18.
- Fabbris, L. (1997), *Statistica multivariata. Analisi esplorativa dei dati*, Milano: McGraw-Hill Libri Italia.
- Girone, G. and Salvemini, T. (2000), *Lezioni di Statistica*, Bari: Cacucci Editore.
- Gras, R. (2005), Panorama du développement de l'Analyse Statistique Implicative à partir de situations fondatrices, *Quaderni di Ricerca in Didattica della Matematica (GRIM Palermo)*, 15 (2), 9-33.
- Gras, R. and Kuntz, P. (2008), An overview of the Statistical Implicative Analysis (SIA) development, in: Gras, R., Suzuki, E., Guillet, F. and Spagnolo, F. (2008), *Statistical Implicative Analysis: Theory and Applications*, Studies in Computational Intelligence Volume No. 127, Berlin & Heidelberg: Springer-Verlag.
- Jegerlehner, F. (2008), *The Anomalous Magnetic Moment of the Muon*, Springer Tracts in Modern Physics No. 226, Berlin & Heidelberg: Springer-Verlag.
- Letta, G. (1993), *Probabilità elementare*, Bologna: Zanichelli Editore.
- Melnikov, K. and Vainshtein, A. (2006), *Theory of the Muon Anomalous Magnetic Moment*, Springer Tracts in Modern Physics No. 216, Berlin & Heidelberg: Springer-Verlag.
- Montanari, T. and Poppi, M. (1982), *Guida alle esercitazioni di Laboratorio di Fisica*, Padova: Piccin Editore.

- Niiniluoto, I. (1983), Inductive Logic as a Methodological Research Programme, in: Dalla Chiara, M.L. et al., (1983), *Logic in the 20th-Century*, A Series of Papers on the Present State and Tendencies of Studies, Milano: Scientia.
- Porto, F. (1987), *Lezioni di Esperimentazioni di Fisica I*, Catania: CULC Editrice.
- Severi, M. (1985), *Introduzione alla sperimentazione fisica*, Bologna: Nicola Zanichelli Editore.
- Spagnolo, F. (1997), L'analisi a-priori e l'indice di implicazione statistica di Gras, *Quaderni di Ricerca in Didattica della Matematica (GRIM Palermo)*, 7, 110-125.
- Spagnolo, F. (2005), L'Analisi Statistica Implicativa: uno dei metodi di analisi dei dati nella ricerca in didattica delle Matematiche, *Quaderni di Ricerca in Didattica della Matematica (GRIM Palermo)*, 15 (2), 35-51.
- Stoka, M. (1991), *Calcolo delle Probabilità e Statistica Matematica*, Torino: Libreria Editrice Universitaria Levrotto & Bella.
- Taylor, J.R. (1982), *An Introduction to Error Analysis. The Study of Uncertainties in Physical Measurements*, Mill Valley, CA: University Science Books (Italian Translation: (1986), *Introduzione all'analisi degli errori. Lo studio delle incertezze nelle misure fisiche*, Bologna: Nicola Zanichelli Editore).
- Toraldo di Francia, G. (1976), *L'indagine del mondo fisico*, Torino: Giulio Einaudi editore.