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On the Müller paradox for thermal-incompressible media

Abstract In his monograph *Thermodynamics*, I. Müller proves that for incompressible media the volume does not change with the temperature. This Müller paradox yields an incompatibility between experimental evidence and the entropy principle. This result has generated much debate within the mathematical and thermodynamical communities as to the basis of Boussinesq approximation in fluid dynamics.
The aim of this paper is to prove that for an appropriate definition of incompressibility, as a limiting case of *quasi thermal-incompressible* body, the entropy principle holds for pressures smaller than a critical pressure value. The main consequence of our result is the physically obvious one, that for very large pressures, no body can be perfectly incompressible. The result is first established in the fluid case. In the case of hyperelastic media subject to large deformations the approach is similar, but with a suitable definition of the pressure associated with convenient stress tensor decomposition.

Keywords Incompressible fluids and solids · Entropy principle for incompressible materials · Boussinesq approximation

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Mathematics Subject Classification (2000) 74F05 · 74A10 · 74A15 · 76Bxx · 76E19

1 Introduction

It is well known that compressible and incompressible bodies have different mathematical treatments: for compressible media the pressure is a constitutive function, while for incompressible media the pressure comes from a Lagrange multiplier associated with the constraint of incompressibility. From an experimental point of view, incompressible medium has no real existence but can be approximated as limit case of the compressible one. Starting from this observation much literature has been devoted by using qualitative analysis and numerical methods to search solutions of the incompressible case; for example, the limit of solutions of the compressible fluids are considered as the Mach number tends to zero under certain assumptions on the initial data (see e.g. the isothermal case [1–4]). When the thermal effects are relevant the limit is more ambiguous and depends on the particular model of

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incompressibility (see e.g. [5]). With the aim to have the same set of equations for compressible and incompressible fluids, Müller [6] and others (see e.g. [7,8]) choose the pressure p as unknown field variable instead of the density ρ. More precisely, in the case of dissipative Navier-Stokes-Fourier fluids, they add to the balance law system of mass, momentum and energy,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0, \tag{1}$$

$$\tag{2}$$

$$\frac{\partial \rho v_j}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i v_j - t_{ij}) = \rho f_j, \tag{3}$$

$$\tag{4}$$

$$\frac{\partial \rho \left(\varepsilon + \frac{1}{2}v^2\right)}{\partial t} + \frac{\partial}{\partial x_i}\left\{\rho\left(\varepsilon + \frac{1}{2}v^2\right)v_i - t_{ij}v_j + q_i\right\} = \rho f_j v_j + \rho r, \tag{5}$$

the following constitutive equations,

$$\rho \equiv \rho(p,T), \quad \varepsilon \equiv \varepsilon(p,T), \tag{6}$$

$$\sigma_{ij} = \nu(p,T)\, d_{kk}\delta_{ij} + 2\mu(p,T)\, d_{ij}, \quad q_i = -\kappa(p,T)\, \partial T/\partial x_i\,, \tag{7}$$

with

$$\mathbf{t} = -p\mathbf{1} + \boldsymbol{\sigma}\,, \tag{8}$$

where $\mathbf{t} \equiv (t_{ij})$ represents the stress tensor, $\boldsymbol{\sigma} \equiv (\sigma_{ij})$ is the viscous shear stress, $\mathbf{1} \equiv (\delta_{ij})$ the identity tensor and δ_{ij} the Kronecker symbol. The other variables have the usual meaning: $\mathbf{v} \equiv (v_j)$, $\mathbf{q} \equiv (q_j)$, ε, T respectively denote the velocity, the heat flux, the internal energy and the temperature while $\mathbf{f} \equiv (f_j)$, r, are the specific body force and the heat supply; κ is the heat conductivity and the scalars ν and μ are the viscosity coefficients. Matrix $\mathbf{d} = \|d_{ij}\|$ denotes the symmetric part of $\nabla\mathbf{v}$ ($d_{kk} = \text{div } \mathbf{v}$). Consequently, system (3)-(7) is a closed system for the unknown variables p, T, $\mathbf{v}$.
Müller defines an incompressible fluid as a medium for which the constitutive equations (6)-(7) are independent of the pressure, and in particular:

$$\rho \equiv \rho(T)\ , \quad \varepsilon \equiv \varepsilon(T)\,. \tag{9}$$

Nonetheless, he proves [6] that the only function $\rho(T)$ compatible with the entropy principle is a constant function $\rho = \rho_o$. Obviously, this result disagrees with experiments showing that the density changes with the temperature (see e.g. [9]) and with all the theoretical results given in particular in the so-called Boussinesq approximation (see e.g. [10,11]). We call this contradiction the *Müller paradox*.
In this paper we prove that for a convenient definition of incompressibility, the entropy principle is compatible with the fact that volume changes with temperature provided that the pressure is smaller than a critical pressure value p_{cr}. These results provide a precise value of the critical pressure under which a fluid can be experimentally similar to an incompressible one and permits to obtain a quantitative measurement for which the Boussinesq approximation can be considered as valid.
The paper is organized as follows:
In Section 2, the Müller paradox is examined in fluid case.
In Section 3, we prove that exists a critical pressure p_{cr} such that for any $p \ll p_{cr}$ the paradox is solved. Then, a numerical evaluation is done allowing to obtain the critical pressure value for which water can be considered as incompressible liquid.
In section 4, the case of hyperelastic media subject to large deformations is considered; thanks to a stress tensor decomposition carrying out a suitable definition of the pressure, an approach similar to fluid case allows to solve the Müller paradox. We perform a numerical evaluation of the critical pressure for pure gum rubber.

2 The Müller paradox for incompressible fluids

The Gibbs relation

$$TdS = d\varepsilon - \frac{p}{\rho^2}d\rho,$$

where S is the entropy density, can be rewritten by using the chemical potential

$$\mu = \varepsilon + pV - TS\,, \tag{10}$$

in the form

$$d\mu = Vdp - SdT\,, \tag{11}$$

where $V = 1/\rho$ is the specific volume. The choice of the chemical potential is natural if we use p and T as variables. In fact from eqs. (10-11) it follows

$$V = \mu_p\,, \quad S = -\mu_T\,, \quad \varepsilon = \mu - p\mu_p - T\mu_T\,, \tag{12}$$

where

$$\mu_p = \left(\frac{\partial\mu}{\partial p}\right)_T, \quad \mu_T = \left(\frac{\partial\mu}{\partial T}\right)_p.$$

From eqs. $(12)_1 - (12)_3$ we get

$$\varepsilon_p = -pV_p - TV_T\,. \tag{13}$$

The Müller definition of incompressibility (9) and Eq. (13) imply V is constant, i.e.

$$\rho = \rho_0 = \text{constant}\,.$$

Obviously, this Müller result [6] (p. 27) disagrees with the experiments proving that the volume of fluids changes with the temperature and also, a little, with the pressure [9,12].

3 Removal of the Müller paradox for fluids

According to the fact that the so-called *compressibility coefficient* $\beta = -V_p/V$ is very small in the case of incompressible body, to remove the Müller paradox we define the notion of quasi thermal-incompressible fluid:
A quasi thermal-incompressible fluid *is a medium for which the only equation independent of* p *among constitutive equations* (6)-(7) *is the density*:

$$\rho \equiv \rho(T)\,. \tag{14}$$

We assume that the condition of independence of p is not necessary for the other constitutive equations; in particular, the internal energy ε remains function of p and T:

$$\varepsilon = \varepsilon(p, T)\,.$$

From Eq. (14), by integration of Eq. $(12)_1$, we obtain

$$\mu(p, T) = V(T)\; p + \mu_0(T)\,, \tag{15}$$

and substituting in Eq. $(12)_3$

$$\varepsilon(p, T) = -TV'(T)\; p + e(T)\,, \tag{16}$$

where

$$e(T) = \mu_0 - T\mu_0'\,,$$

with $' = d/dT$. Following Müller proposal, we name incompressible fluid (or better *perfectly incompressible fluid*) a fluid for which all the constitutive equations are independent of p.
Therefore, from Eq. (16), a *quasi thermal-incompressible fluid* tends to be *perfectly incompressible* if $\varepsilon(p, T)$ can be approximated with $e(T)$, i.e. when:

$$p \ll \frac{e(T)}{|V'|T} = \frac{\rho^2 e(T)}{|\rho'|T}\,. \tag{17}$$

For example, let us consider the classical linear behavior (also typical of Boussinesq approximation):

$$V = V_0[1 + \alpha(T - T_0)], \tag{18}$$

where α is the *thermal expansion*, the constant $V_0 = 1/\rho_0$ is associated with the scale of volume and T_0 is a reference temperature. If we assume that the specific heat at constant pressure $c_p \equiv h_T$ is constant (where $h = \varepsilon + pV$ is the enthalpy), then from Eq. (16) we have

$$e(T) = c_p\, T. \tag{19}$$

Inserting eqs. (18-19) into inequality (17) we obtain:

$$p \ll p_{cr} \quad \text{with} \quad p_{cr} = c_p\, \rho_0/\alpha\,. \tag{20}$$

We call p_{cr} the critical pressure at density ρ_0. As we expect in physical situations, we note that p_{cr} is inversely proportional to α. Critical pressure p_{cr} and inequality (20) characterize the fact that a quasi thermal-incompressible fluid is experimentally similar to a perfectly incompressible fluid. In such a case, the Müller paradox is removed.

3.1 Application to water

To evaluate the magnitude order of the critical pressure (20), we give some numerical results in the case of water.
At temperature $T_0 = 20°$ C, we get [9,13]:

$$\rho_0 \simeq 10^3 \text{kg/m}^3, \quad c_p \simeq 4.2 \cdot 10^3 \text{Joule/kg.K}, \quad \alpha = 207 \cdot 10^{-6}/\text{K},$$

and from Eq. (20) we deduce: $p_{cr} \simeq 2 \cdot 10^{10}$ Pascal $\simeq 2 \cdot 10^5$ atm.
The value of critical pressure is large with respect to the normal pressure conditions. The fact that for usual pressures, a liquid is experimentally incompressible and the volume changes with the temperature (Eq. (18)) does not violate the principles of thermodynamics. We can interpret the result in another way: for very large pressure, perfectly incompressible fluids do not exist. From physical point of view, this observation seems reasonable, for example in astrophysics where very high pressures are present.
We also observe that from experimental data of sound velocity in the water [12,13], the compressibility coefficient β is not zero but very small ($\beta = 4.98 \times 10^{-10}/\texttt{Pascal}$). This is in agreement with the fact that perfect incompressibility is an idealization.

Remark
Our definition of quasi thermal-incompressibility is, for some authors (e.g. [7,8]), the definition of incompressible body in the context of Boussinesq approximation. Nevertheless, we strongly believe that the Müller definition of incompressible fluid is the correct one even if it is a limit case: only in this case, we obtain the same differential equations as for the usual incompressible approach with $\rho, \mathbf{v}, T$ variables. In fact in the case of incompressibility, i.e. when $\rho \equiv \rho(T)$, a generic constitutive quantity $\phi \equiv \phi(\rho, T)$ becomes a function only of T. In our analysis *perfectly incompressible fluid* is considered as limit case of *quasi thermal-incompressible fluid*.
The quasi thermal-incompressibility is obtained as a limit process justifying the compatibility between incompressibility and Gibbs relation when inequality (20) is verified. The quasi thermal-incompressibility does not characterize a real compressible material; for real compressible fluids the chemical potential μ must be a concave function of (p, T). When V depends only on T, the chemical potential is a linear function of p (see Eq. (15)) and consequently cannot be concave. For real compressible materials the volume V necessarily depends on p; quasi thermal-incompressible materials can be considered as an approximation of incompressible materials when the pressure is sufficiently small such that inequality (20) is satisfied.
Moreover, note that we use term of quasi-thermal incompressibility, rather than term of quasi incompressibility, because this last term has a different meaning in the pure isothermal mechanical case.

4 Incompressible hyperelastic media

In the case of elasticity, Müller presented a similar paradox [6] (p. 263). In fact, he proved that

$$J \equiv \det \mathbf{F},$$

where $\mathbf{F}$ denotes the deformation gradient, cannot depend on the temperature T and must be constant as in the pure mechanical case, i.e. $J = 1$. This result also disagrees with experiments proving that the volume of an elastic incompressible solid changes with the temperature.
We use similar arguments as in fluid case, but on the contrary of the Müller procedure, we consider a particular decomposition of the stress tensor allowing to remove the paradox. Moreover, our approach is also valid for non-isotropic materials.

4.1 A decomposition of the stress tensor in the case of hyperelastic medium

Each particle of the continuous medium is labeled by a material variable $\mathbf{X}$, ranging from a reference configuration $\mathcal{D}_0$ into an Euclidian space [14]. The reference density ρ_0 is given as a function on $\mathcal{D}_0$ [15].
The expression $\mathbf{x} = \phi(\mathbf{X}, t)$ of the spatial position describes the motion of the continuous medium. Generally, $\phi(., t)$ is a twice continuously differentiable diffeomorphism from $\mathcal{D}_0$ into a compact oriented manifold $\mathcal{D}_t$ constituting the image of the material at time t. As usual, we denote by $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ the right Cauchy-Green deformation tensor, where superscript ^{T} means the transposition. Let us recall:

$$\rho\ (\det \mathbf{C})^{\frac{1}{2}} = \rho_0. \tag{21}$$

For the sake of simplicity, we consider the case of homogeneous bodies (the case of inhomogeneous bodies can be treated in the same way). The internal energy density is supposed to be function of the tensor $\mathbf{C}$ and temperature T

$$\varepsilon \equiv \varepsilon(\mathbf{C}, T),$$

and the stress tensor can be written [15–17],

$$\mathbf{t} = 2\rho\mathbf{F}\frac{\partial\psi}{\partial\mathbf{C}}\,\mathbf{F}^T, \tag{22}$$

where $\psi = \varepsilon - TS$ is the specific free energy. By writing [18,19]:

$$\widetilde{\mathbf{C}} = \frac{1}{(\det \mathbf{C})^{\frac{1}{3}}}\,\mathbf{C} \quad \text{or} \quad \mathbf{C} = \left(\frac{\rho_0}{\rho}\right)^{\frac{2}{3}}\,\widetilde{\mathbf{C}}\,, \tag{23}$$

the specific free energy can be expressed in the form:

$$\psi \equiv f(\rho, \widetilde{\mathbf{C}}, T).$$

We note that $\mathbf{C}$ is substituted by the independent variables ρ and $\widetilde{\mathbf{C}}$. Since $\det \widetilde{\mathbf{C}} = 1$, the variable ρ corresponds to the change of volume while the tensorial variable $\widetilde{\mathbf{C}}$ represents the distortion of the medium. This point is fundamental for the decomposition of the stress tensor and will be the key of the demonstration. When f is independent of $\widetilde{\mathbf{C}}$ we are back to the fluid case. It is more convenient to introduce the function g such that:

$$g(\rho, \mathbf{C}, T) \equiv f\left(\rho, \frac{1}{(\det \mathbf{C})^{\frac{1}{3}}}\,\mathbf{C}, T\right).$$

Consequently, g is a homogeneous function of degree zero with respect to $\mathbf{C}$.
From Eqs. (21-22) it follows:

$$\mathbf{t} = 2\rho\mathbf{F}\left(\frac{\partial g}{\partial\rho}\frac{\partial\rho}{\partial\mathbf{C}} + \frac{\partial g}{\partial\mathbf{C}}\right)\mathbf{F}^T. \tag{24}$$

Differentiating Eq. (21) and using Jacobi's identity, we obtain

$$d\rho = -\frac{1}{2}\rho\,\mathbf{C}^{-1}\cdot d\mathbf{C}\,,$$

where the dot represents the scalar product between matrices. Hence,

$$\frac{\partial\rho}{\partial\mathbf{C}} = -\frac{1}{2}\rho\,\mathbf{C}^{-1}\,,$$

and consequently, Eq. (24) yields,

$$\mathbf{t} = -\rho^2\frac{\partial g}{\partial\rho}\,\mathbf{1} + 2\rho\mathbf{F}\frac{\partial g}{\partial\mathbf{C}}\mathbf{F}^T.$$

Due to the fact g is homogeneous of degree zero with respect to $\mathbf{C}$, from the Euler's identity we immediately deduce:

$$\frac{\partial g}{\partial\mathbf{C}}\cdot\ \mathbf{C} = 0\,,$$

and

$$\mathbf{t} = -p\mathbf{1} + \boldsymbol{\tau}\,, \tag{25}$$

with

$$p = \rho^2\frac{\partial g}{\partial\rho},\ \ \boldsymbol{\tau} = 2\rho\mathbf{F}\frac{\partial g}{\partial\mathbf{C}}\mathbf{F}^T\ \text{ and }\ \mathrm{tr}\,\boldsymbol{\tau} = 0, \tag{26}$$

where $\mathtt{tr}$ is the trace operator. Let us note that, in relation (25), $\mathbf{t}$ is similar to the fluid decomposition of Eq. (8) and in the solid case, p is analog to a pressure.
The decomposition (25-26) allows to define a pressure also in the case of an elastic body.

4.2 Removal of the Müller paradox for hyperelastic media

The Gibbs equation in the case of elastic materials is [15–17]

$$TdS = d\varepsilon - \frac{1}{2\rho_0}\mathbf{S}\cdot d\mathbf{C}\,, \tag{27}$$

where

$$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{t}\,(\mathbf{F}^T)^{-1} \tag{28}$$

is the second Piola-Kirchhoff stress tensor. Inserting Eq. (25) into Eq. (28), we get from Eq. (27) the Gibbs relation:

$$TdS = d\varepsilon - \frac{p}{\rho^2}\,d\rho - \frac{1}{2\,\rho}\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-1^T}\cdot d\mathbf{C}. \tag{29}$$

From Eq. (23) we obtain:

$$d\mathbf{C} = -\frac{2}{3}\frac{\mathbf{C}}{\rho}\,d\rho + \left(\frac{\rho_0}{\rho}\right)^{\frac{2}{3}}d\widetilde{\mathbf{C}},$$

and Eq. (29) leads to:

$$TdS = d\varepsilon - \frac{p}{\rho^2}\,d\rho + \frac{1}{3\,\rho^2}\mathbf{F}^{-1}\boldsymbol{\tau}\,\mathbf{F}^{-1^T}\cdot\mathbf{C}\ d\rho - \frac{1}{2\,\rho}\left(\frac{\rho_0}{\rho}\right)^{\frac{2}{3}}\mathbf{F}^{-1}\boldsymbol{\tau}\,\mathbf{F}^{-1^T}\cdot d\widetilde{\mathbf{C}}.$$

From Eq. (26), we get $\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-1^T}\cdot\mathbf{C} = \mathtt{tr}\left(\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-1^T}\mathbf{C}\right) \equiv \mathtt{tr}\,\boldsymbol{\tau} = 0$. If we define

$$\widetilde{\boldsymbol{\tau}} = -\frac{1}{2\,\rho}\left(\frac{\rho_0}{\rho}\right)^{\frac{2}{3}}\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-1^T}, \tag{30}$$

we obtain the Gibbs relation in the final form:

$$TdS = d\varepsilon - \frac{p}{\rho^2}\, d\rho + \widetilde{\boldsymbol{\tau}} \cdot d\widetilde{\mathbf{C}}. \tag{31}$$

As in the case of fluid, introducing the chemical potential

$$\mu = \varepsilon + pV + \widetilde{\boldsymbol{\tau}} \cdot \widetilde{\mathbf{C}} - TS\,,$$

Eq. (31) implies

$$d\mu = V\,dp - SdT + \widetilde{\mathbf{C}} \cdot d\widetilde{\boldsymbol{\tau}}. \tag{32}$$

Equation (30) together with Eq. $(26)_3$, imply $\mathsf{tr}(\widetilde{\boldsymbol{\tau}}\,\mathbf{C}) = 0$. Therefore $\widetilde{\boldsymbol{\tau}}$ has only five independent components.
From Eq. (32) it is natural to introduce the change of variables $(\rho, T, \widetilde{\mathbf{C}})$ into $(p, T, \widetilde{\boldsymbol{\tau}})$. In fact, Eq. (32) implies:

$$V = \mu_p\,, \quad \widetilde{\mathbf{C}} = \mu_{\widetilde{\boldsymbol{\tau}}}\,, \quad S = -\mu_T\,, \quad \varepsilon = \mu - T\mu_T - p\,\mu_p - \widetilde{\boldsymbol{\tau}} \cdot \mu_{\widetilde{\boldsymbol{\tau}}}\,, \tag{33}$$

where now

$$\mu_p = \left(\frac{\partial \mu}{\partial p}\right)_{T,\,\widetilde{\boldsymbol{\tau}}}, \quad \mu_{\widetilde{\boldsymbol{\tau}}} = \left(\frac{\partial \mu}{\partial \widetilde{\boldsymbol{\tau}}}\right)_{p,\,T}, \quad \mu_T = \left(\frac{\partial \mu}{\partial T}\right)_{p,\,\widetilde{\boldsymbol{\tau}}}.$$

By analogy with the case of fluids, ***a quasi thermal-incompressible elastic medium*** *must verify the condition*:

$$J \equiv J(T) \quad \textit{or} \quad \textit{equivalently} \quad V \equiv V(T)\,,$$

while the other constitutive equations still depend on $(p, T, \widetilde{\boldsymbol{\tau}})$. In particular

$$\varepsilon \equiv \varepsilon(p, T, \widetilde{\boldsymbol{\tau}})\,.$$

By integration of Eq. $(33)_1$, we obtain:

$$\mu(p, T, \widetilde{\boldsymbol{\tau}}) = V(T)\,p + \mu_0(T, \widetilde{\boldsymbol{\tau}})\,,$$

and by substituting in Eq. $(33)_4$, we get

$$\varepsilon = -TV'(T)\,p + e(T, \widetilde{\boldsymbol{\tau}})\,, \tag{34}$$

with

$$e(T, \widetilde{\boldsymbol{\tau}}) = \mu_0 - T\mu_{0T} - \widetilde{\boldsymbol{\tau}} \cdot \mu_{0\widetilde{\boldsymbol{\tau}}}\,.$$

Following Müller proposal again, we define an *incompressible elastic medium* (or better *perfectly incompressible elastic medium*) as a solid for which all the constitutive equations are independent of p. Therefore, taking account of Eq. (34), a thermal-incompressible elastic medium tends to be *perfectly incompressible* if $\varepsilon(p, T, \widetilde{\boldsymbol{\tau}})$ can be approximated by $e(T, \widetilde{\boldsymbol{\tau}})$. This assumption is verified when

$$p \ll \frac{e(T, \widetilde{\boldsymbol{\tau}})}{|\,V'|\,T}\,.$$

We consider the case:

$$e(T, \widetilde{\boldsymbol{\tau}}) = c_p\,T\,, \quad V(T) = V_0[1 + \alpha(T - T_0)]\,,$$

where now, α and V_0 are positive constants and c_p can be a function of $\widetilde{\boldsymbol{\tau}}$. As well as for fluids, we obtain:

$$p \ll c_p\rho_0/\alpha. \tag{35}$$

We assume that $c_p\,\rho_0/\alpha$, varying with $\widetilde{\boldsymbol{\tau}}$, has a minimal value p_{cr}; we again denote by p_{cr} the critical pressure. In this case the critical pressure p_{cr} and inequality $p \ll p_{cr}$ are characteristic of quasi thermal-incompressible elastic medium to be experimentally similar to perfectly incompressible one and the Müller paradox is removed.

4.3 Application to pure gum rubber

The most famous incompressible hyperelastic medium is the pure gum rubber. This material was studied by many authors and in particular by P.J. Flory, Nobel prize of Chemistry in 1974 [20].
In the range of temperatures $[50^\circ\,\mathrm{C}, 85^\circ\,\mathrm{C}]$ physical constants are [13]:

$$\rho_0 \simeq 930\,\mathrm{kg/m^3}, \quad c_p \simeq 1.9 \cdot 10^3 \mathrm{Joule/kg.K}, \quad \alpha = 6.7 \cdot 10^{-3}/\mathrm{K}.$$

Equation (35) allows to obtain: $p_{cr} \simeq 2.7 \cdot 10^8$ Pascal $\simeq 2.7 \cdot 10^3$ atm. The critical pressure is very large and therefore pure gum rubber is a good incompressible body in normal conditions with usual pressures.

5 Conclusions

In the case of fluids and elastic media, we showed that, for pressures smaller than a critical value, the volume can depend on the temperature if incompressibility is defined as limit case of quasi thermal-incompressibility. Quasi thermal-incompressible materials can be considered as an approximation of incompressible materials - in the sense of [7,8] where $\beta = 0$ - if the pressure is small enough such that inequality (20) is satisfied.
To obtain these results, we used temperature as natural thermodynamical variable. Nevertheless, Manacorda [21] first noted (see also [22,23]) that in the case $V \equiv V(T)$, instabilities occur in wave propagations. The instabilities are due to the chemical potential non-concavity (see Remark in previous section) and the sound velocity c becomes complex. For this reason, some authors consider the volume as function of entropy $V \equiv V(S)$ instead of function of temperature [24,25]. This assumption does not seem realistic: $V = V(S)$ cannot be measured because entropy is not an observable and moreover, in this case $1/c = 0$ (i.e., the sound velocity is infinite); as a consequence the mathematical structure of Euler fluids becomes parabolic.
Our goal was to present the simplest model for removing the thermodynamical paradox when the volume depends only on T. Nevertheless, a more realistic definition of quasi thermal-incompressibility needs to suppose that the compressibility coefficient β is small but not zero; the concavity of the chemical potential can be restored and the sound velocity can be real with consequence that incompressible body can be seen as a limit case of a compressible one but the present result cannot quantitatively change. This will be the subject of a forthcoming study.
Finally thanks to decomposition (25), the technique used to remove the Müller paradox is available both for fluids and elastic media. This similarity allows to forecast a possible Boussinesq approximation in case of elastic media.

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