

SOMEWHERE DENSE ORBIT OF ABELIAN SUBGROUP OF DIFFEOMORPHISMS MAPS ACTING ON \mathbb{C}^n

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ABSTRACT. In this paper, we give a characterization for any abelian subgroup G of a lie group of diffeomorphisms maps of \mathbb{C}^n , having a somewhere dense orbit $G(x)$, $x \in \mathbb{C}^n$: $G(x)$ is somewhere dense in \mathbb{C}^n if and only if there are $f_1, \dots, f_{2n+1} \in \exp^{-1}(G)$ such that $f_{2n+1} \in \text{vect}(f_1, \dots, f_{2n})$ and $\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)$ is dense in \mathbb{C}^n , where $\text{vect}(f_1, \dots, f_{2n})$ is the vector space over \mathbb{R} generated by f_1, \dots, f_{2n} .

1. Introduction

Denote by $\text{Diff}^r(\mathbb{C}^n)$, $r \geq 1$ the group of all C^r -diffeomorphisms of \mathbb{C}^n . Let Γ be a lie subgroup of $\text{Diff}^r(\mathbb{C}^n)$, $r \geq 1$ and G be an abelian subgroup of Γ , such that $\text{Fix}(G) \neq \emptyset$, where $\text{Fix}(G) = \{x \in \mathbb{C}^n : f(x) = x, \forall f \in G\}$ be the global fixed point set of G . There is a natural action $G \times \mathbb{C}^n \rightarrow \mathbb{C}^n$. $(f, x) \mapsto f(x)$. For a point $x \in \mathbb{C}^n$, denote by $G(x) = \{f(x), f \in G\} \subset \mathbb{C}^n$ the orbit of G through x . A subset $E \subset \mathbb{C}^n$ is called G -invariant if $f(E) \subset E$ for any $f \in G$; that is E is a union of orbits. Denote by \overline{E} (resp. $\overset{\circ}{E}$) the closure (resp. interior) of E .

Recall that $E \subset \mathbb{C}^n$ is somewhere dense in \mathbb{C}^n if the closure \overline{E} has nonempty interior in \mathbb{C}^n . An orbit γ is called somewhere dense (or locally dense) if $\overset{\circ}{\gamma} \neq \emptyset$. The group G is called hypercyclic if it has a dense orbit in \mathbb{C}^n . Hypercyclic is also called topologically transitive.

The purpose of this paper is to give a characterization for any subgroup G of a lie group of diffeomorphisms maps of \mathbb{C}^n , having a dense orbit. In [1], the authors present a global dynamic of every abelian subgroup of $GL(n, \mathbb{C})$ and in [2], they characterize hypercyclic abelian subgroup of $GL(n, \mathbb{C})$. Our main result is viewed as a continuation of [7] and [8].

Denote by:

- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.
- $C^r(\mathbb{C}^n, \mathbb{C}^n)$ the set of all C^r -differentiable maps of \mathbb{C}^n .
- For a subset $E \subset \mathbb{C}^n$ (resp. $E \subset C^r(\mathbb{C}^n, \mathbb{C}^n)$), denote by $\text{vect}(E)$ the vector subspace of \mathbb{C}^n (resp. $C^r(\mathbb{C}^n, \mathbb{C}^n)$) over \mathbb{R} generated by all elements of E .
- $\exp : C^r(\mathbb{C}^n, \mathbb{C}^n) \rightarrow \text{Diff}^r(\mathbb{C}^n)$ the exponential map defined by $\exp(f) = e^f$, $f \in C^r(\mathbb{C}^n, \mathbb{C}^n)$.

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- H the lie algebra associated to Γ .
- $\exp : H \rightarrow \Gamma$ be the exponential map.
- $H_x = \{f(x), B \in H\}$, it is a vector subspace of \mathbb{C}^n over \mathbb{R} .
- $\mathfrak{g} = \exp^{-1}(G)$, it is an additive group because G is abelian.
- $\mathfrak{g}_x = \{f(x), B \in \mathfrak{g}\}$, it is an additive subgroup of \mathbb{C}^n because \mathfrak{g} is an additive group.

Our principal results can be stated as follows:

Theorem 1.1. *Let Γ be an abelian lie subgroup of $\text{Diff}^r(\mathbb{C}^n)$ and $x \in \mathbb{C}^n \setminus \{0\}$. Then the following assertions are equivalent:*

- (i) $H_x = \mathbb{C}^n$.
- (ii) $\overline{\Gamma(x)} \neq \emptyset$.

In general, the Lie algebra $\tilde{\mathfrak{g}}$ is not explicitly defined, so we give an explicitly test to the existence of somewhere dense orbit by the following theorem:

Theorem 1.2. *Let G be an abelian subgroup of a lie group $\Gamma \subset \text{Diff}^r(\mathbb{C}^n)$ and $x \in \mathbb{C}^n \setminus \{0\}$. Then $\overline{G(x)} \neq \emptyset$ if and only if there exist $f_1, \dots, f_{2n+1} \in \exp^{-1}(\tilde{G})$ such that $f_{2n+1} \in \text{vect}(f_1, \dots, f_{2n})$ and $\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)$ is a dense additive subgroup of \mathbb{C}^n .*

Let's introduce the arithmetic property: We say that $f_1, \dots, f_{2n+1} \in C^r(\mathbb{C}^n, \mathbb{C}^n)$ satisfy *property $\mathcal{D}(x)$* for some $x \in \mathbb{C}^n$ if f_1, \dots, f_{2n} are linearly independent, $f_{2n+1} \in \text{vect}(f_1, \dots, f_{2n})$ and for every $(s_1, \dots, s_{2n+1}) \in \mathbb{Z}^{2n+1} \setminus \{0\}$:

$$\text{rank} \begin{bmatrix} \text{Re}(f_1(x)) & \dots & \text{Re}(f_{2n+1}(x)) \\ \text{Im}(f_1(x)) & \dots & \text{Im}(f_{2n+1}(x)) \\ s_1 & \dots & s_{2n+1} \end{bmatrix} = 2n + 1.$$

For a vector $v \in \mathbb{C}^n$, we write $v = \text{Re}(v) + i\text{Im}(v)$ where $\text{Re}(v)$ and $\text{Im}(v) \in \mathbb{R}^n$.

As an immediate consequence of Theorem 1.2, we have:

Corollary 1.3. *Let G be an abelian subgroup of a lie group $\Gamma \subset \text{Diff}^r(\mathbb{C}^n)$ and $x \in \mathbb{C}^n \setminus \{0\}$. Then $\overline{G(x)} \neq \emptyset$ if and only if there exist $f_1, \dots, f_{2n+1} \in \exp^{-1}(G)$ and satisfying property $\mathcal{D}(x)$.*

As an important consequence of the Theorem 1.2, we give the following Corollary which simplifies the test given by Theorem 1.3 proved in [2] for the abelian subgroup of $GL(n, \mathbb{C})$:

Corollary 1.4. *Let G be an abelian subgroup of $GL(n, \mathbb{C})$ and $x \in \mathbb{C}^n \setminus \{0\}$. Then $\overline{G(x)} = \mathbb{C}^n$ if and only if there exist $B_1, \dots, B_{2n+1} \in \exp^{-1}(G)$ such that $\mathbb{Z}B_1x + \dots + \mathbb{Z}B_{2n+1}x$ is dense in \mathbb{C}^n .*

This paper is organized as follows: In Section 2 we prove Theorem 1.1. Section 3 is devoted to prove Theorem 1.2 and Corollaries 1.3, 1.4.

2. Proof of Theorem 1.1

We will cite the definition of the exponential map given in [3].

2.1. Exponential map. In this section, we illustrate the theory developed of the group $Diff(\mathbb{C}^n)$ of diffeomorphisms of \mathbb{C}^n . For simplicity, throughout this section we only consider the case of $\mathbb{C} = \mathbb{R}$; however, all results also hold for complex case. The group $Diff(\mathbb{R}^n)$ is not a Lie group (it is infinite-dimensional), but in many way it is similar to Lie groups. For example, it easy to define what a smooth map from some Lie group G to $Diff(\mathbb{R}^n)$ is: it is the same as an action of G on \mathbb{R}^n by diffeomorphisms. Ignoring the technical problem with infinite-dimensionality for now, let us try to see what is the natural analog of the Lie algebra \mathfrak{g} for the group G . It should be the tangent space at the identity; thus, its elements are derivatives of one-parameter families of diffeomorphisms.

Let $\varphi^t : G \rightarrow G$ be one-parameter family of diffeomorphisms. Then, for every point $a \in G$, $\varphi^t(a)$ is a curve in G and thus $\frac{\partial}{\partial t}\varphi^t(a)|_{t=0} = \xi(a) \in T_a G$ is a tangent vector to G at m . In other words, $\frac{\partial}{\partial t}\varphi^t$ is a vector field on G .

The exponential map $exp : \mathfrak{g} \rightarrow G$ is defined by $exp(x) = \gamma_x(1)$ where $\gamma_x(t)$ is the one-parameter subgroup with tangent vector at 1 equal to x .

If $\xi \in \mathfrak{g}$ is a vectorfield, then $exp(t\xi)$ should be one-parameter family of diffeomorphisms whose derivative is vector field ξ . So this is the solution of differential equation

$$\frac{\partial}{\partial t}\varphi^t(a)|_{t=0} = \xi(a).$$

In other words, φ^t is the time t flow of the vector field. Thus, it is natural to define the Lie algebra of G to be the space \mathfrak{g} of all smooth vector ξ fields on \mathbb{R}^n such that $exp(t\xi) \in G$ for every $t \in \mathbb{R}$.

We will use the definition of Whitney topology given in [6].

2.2. Whitney Topology on $C^0(\mathbb{C}^n, \mathbb{C}^n)$. For each open subset $U \subset \mathbb{C}^n \times \mathbb{C}^n$ let $\tilde{U} \subset C^0(\mathbb{C}^n, \mathbb{C}^n)$ be the set of continuous functions g , whose graphs $\{(x, g(x)) \in \mathbb{C}^n \times \mathbb{C}^n, x \in \mathbb{C}^n\}$ is contained in U . We want to construct a neighborhood basis of each function $f \in C^0(\mathbb{C}^n, \mathbb{C}^n)$. Let $K_j = \{x \in \mathbb{C}^n, \|x\| \leq j\}$ be a countable family of compact sets (closed balls with center 0) covering \mathbb{C}^n such that K_j is contained

in the interior of K_{j+1} . Consider then the compact subsets $L_j = K_j \setminus \overset{\circ}{K_{j-1}}$, which are compact sets, too. Let $\epsilon = (\epsilon_j)_j$ be a sequence of positive numbers and then define

$$V_{(f;\epsilon)} = \{f \in C^0(\mathbb{C}^n, \mathbb{C}^n) : \|f(x) - g(x)\| < \epsilon_j, \text{ for any } x \in L_j, \forall j\}.$$

We claim this is a neighborhood system of the function f in $C^0(\mathbb{C}^n, \mathbb{C}^n)$. Since L_i is compact, the set $U = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n : \|f(x) - g(x)\| < \epsilon_j, \text{ if } x \in L_j\}$ is open. Thus, $V_{(f;\epsilon)} = \tilde{U}$ is an open neighborhood of f . On the other hand, if O is an open subset of $\mathbb{C}^n \times \mathbb{C}^n$ which contains the graph of f , then since L_j is compact, it follows that there exists $\epsilon_j > 0$ such that if $x \in L_j$ and $\|y - f(x)\| < \epsilon_j$, then $(x; y) \in O$. Thus, taking $\tilde{\epsilon} = (\epsilon_j)_j$ we have $V_{(f;\tilde{\epsilon})} \subset \tilde{O}$, so we have obtained the family $V_{(f;\epsilon)}$ is a neighborhood system of f . Moreover, for each given $\epsilon = (\epsilon_j)_j$,

we can find a C^∞ -function $\epsilon : \mathbb{C}^n \rightarrow \mathbb{R}_+$, such that $\epsilon(x) < \epsilon_j$ for any $x \in L_j$. It follows that the family $V_{(f;\epsilon)} = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n : \|f(x) - g(x)\| < \epsilon(x)\}$ is also a neighborhood system.

Denote by:

- $\tilde{G} = \overline{G} \cap \text{Diff}^r(\mathbb{C}^n)$, where \overline{G} is the closure of G in $C^r(\mathbb{C}^n, \mathbb{C}^n)$ for the Whitney topology defined above. So \tilde{G} is an abelian lie subgroup of Γ .
- $\mathcal{A}(\tilde{G})$ the algebra generated by G . See that $G \subset \mathcal{A}(\tilde{G})$.
- $\Phi_x : \mathcal{A}(\tilde{G}) \rightarrow \mathbb{C}^n$ the linear map given by $\Phi_x(f) = f(x)$, $f \in \mathcal{A}(\tilde{G})$.
- $E(x) = \Phi_x(\mathcal{A}(G))$.

Lemma 2.1. *The linear map $\Phi_x : \mathcal{A}(\tilde{G}) \rightarrow E(x)$ is continuous.*

Proof. Firstly, we take the restriction of the Whitney topology to $\mathcal{A}(\tilde{G})$. Secondly, let $f \in \mathcal{A}(\tilde{G})$ and $\varepsilon > 0$. Then for $\epsilon = (\varepsilon_j)_j$ with $\varepsilon_j = \varepsilon$ and for $V_{(f;\epsilon)}$ be a neighborhood system of f , we obtain: for every $g \in V_{(f;\epsilon)} \cap \mathcal{A}(\tilde{G})$ and for every $y \in L_j$, $\|f(y) - g(y)\| < \varepsilon$, $\forall j$. In particular for $j = j_0$ in which $x \in L_{j_0}$, we have $\|f(x) - g(x)\| < \varepsilon$, so $\|\Phi_x(f) - \Phi_x(g)\| < \varepsilon$. It follows that Φ_x is continuous. \square

2.3. Proof of Theorem 1.1.

Proposition 2.2. ([3], Theorem 3.29) *Let G be a Lie group acting on \mathbb{C}^n with lie algebra $\tilde{\mathfrak{g}}$ and let $u \in \mathbb{C}^n$.*

- (i) *The stabilizer $G_x = \{B \in G : Bu = u\}$ is a closed Lie subgroup in G , with Lie algebra $\mathfrak{h}_x = \{B \in \tilde{\mathfrak{g}} : Bu = 0\}$.*
- (ii) *The map $G/G_x \rightarrow \mathbb{C}^n$ given by $B.G_x \mapsto Bu$ is an immersion. Thus, the orbit $G(x)$ is an immersed submanifold in \mathbb{C}^n . In particular $\dim(G(x)) = \dim(\tilde{\mathfrak{g}}) - \dim(\mathfrak{h}_x)$.*

Here $\mathfrak{h}_x = \text{Ker}(\Phi_x)$ since $\text{Ker}(\Phi_x) \subset \tilde{\mathfrak{g}}$. Write:

- \tilde{L} the vector subspace of $\tilde{\mathfrak{g}}$ supplement to $\text{Ker}(\Phi_x)$, (i.e. $\tilde{L} \oplus \text{Ker}(\Phi_x) = \tilde{\mathfrak{g}}$). It is clear that $\dim(\tilde{L}) = \text{cod}(\text{Ker}(\Phi_x)) \leq n$, then \tilde{L} is closed.
- $\exp : \tilde{L} \oplus \text{Ker}(\Phi_x) \rightarrow \tilde{G}$ the exponential map. Since \tilde{G} is abelian, so is $\tilde{\mathfrak{g}}$, then $\exp(f + h) = \exp(f) \circ \exp(h)$ for every $f \in \tilde{L}$ and $h \in \text{Ker}(\Phi_x)$.
- \tilde{G}_x the stabilizer of \tilde{G} on the point u . So it is a lie subgroup of \tilde{G} with lie algebra $\text{Ker}(\Phi_x)$.

As a directly consequence of Proposition 5.13, given in [4], applied to Γ , we have the following Lemma:

Lemma 2.3. ([4], Proposition 5.13) *Let G be an abelian subgroup of a lie group Γ . There exists an open neighborhood U of 0 in H such that $\exp : U \rightarrow \exp(U)$ is a diffeomorphism and $\exp(U \cap \tilde{\mathfrak{g}}) = \exp(U) \cap \tilde{G}$.*

Denote by $V = \exp(U)$, where U is the open set defined in Lemma 2.3.

Lemma 2.4. *We have $\overline{G(x)} = \overline{\tilde{G}(x)}$.*

Proof. It is clear that $\overline{G(x)} \subset \overline{\tilde{G}(x)} \subset \overline{\overline{G(x)}}$. Let $v \in \overline{\overline{G(x)}}$, so $v = \lim_{m \rightarrow +\infty} f_m(x)$ for some sequence $(f_m)_{m \in \mathbb{N}}$ in \overline{G} . Then for every $m \in \mathbb{N}$, there exists a sequence $(f_{m,k})_{k \in \mathbb{N}}$ in G such that $\lim_{k \rightarrow +\infty} f_{m,k} = f_m$, so by continuity of Φ_x (Lemma 2.1), we have $\lim_{k \rightarrow +\infty} f_{m,k}(x) = f_m(x)$, thus for every $\varepsilon > 0$, there exists $M > 0$ and for every $m \geq M$, there exists $k_m > 0$, such that $\|f_m(x) - v\| < \frac{\varepsilon}{2}$ and for every $k \geq k_m$, $\|f_{m,k}(x) - f_m(x)\| < \frac{\varepsilon}{2}$. Then, for every $m > M$,

$$\|f_{m,k_m}(x) - v\| \leq \|f_{m,k_m}(x) - f_m(x)\| + \|f_m(x) - v\| < \varepsilon,$$

therefore $\lim_{m \rightarrow +\infty} f_{m,k_m}(x) = v$. Hence $v \in \overline{G(x)}$. It follows that $\overline{\tilde{G}(x)} \subset \overline{\overline{G(x)}} \subset \overline{G(x)}$. \square

Lemma 2.5. *Let $W = \Phi_x(V)$. Then $\Phi_x^{-1}(\tilde{G}(x) \cap W) \cap V = \tilde{G} \cap V$.*

Proof. Since $W = \Phi_x(V)$, it is obvious that $\tilde{G} \cap V \subset \Phi_x^{-1}(\tilde{G}(x) \cap W) \cap V$. Let $f \in \Phi_x^{-1}(\tilde{G}(x) \cap W)$. Then there exists $g \in \tilde{G} \cap V$ such that $f(x) = g(x)$. So $g^{-1} \circ f(x) = x$. Hence $g^{-1} \circ f \in H_x$, where H_x be the lie group generated by $\{h \in \text{Diff}^r(\mathbb{C}^n) : h(x) = x\} \cap \mathcal{A}(\tilde{G})$. So H_x is contained in the stabilizer of $\text{Diff}^r(\mathbb{C}^n)$ on x . Set L_x be the lie algebra of H_x , so $L_x \subset \{h \in \text{Diff}^r(\mathbb{C}^n) : h(x) = 0\} \cap \mathcal{A}(\tilde{G})$. Therefore $L_x \subset \text{Ker}(\Phi_x) \subset \tilde{\mathfrak{g}}$. Hence $H_x \subset \tilde{G}$. It follows that $g^{-1} \circ f \in \tilde{G}$, so $f \in \tilde{G} \cap V$. This completes the proof. \square

Proof of Theorem 1.1.

Since \tilde{G} is a locally closed sub-manifold of $\text{Diff}^r(\mathbb{C}^n)$. By Proposition 2.2.(ii), $\tilde{G}(x)$ is an immersed submanifold of \mathbb{C}^n with dimension $r = \dim(\tilde{\mathfrak{g}}) - \dim(\text{Ker}(\Phi_x))$. We have $\text{Im}(\Phi_x) = \tilde{\mathfrak{g}}_x$. Then $\dim(\tilde{\mathfrak{g}}_x) = \dim(\tilde{\mathfrak{g}}) - \dim(\text{Ker}(\Phi_x))$. It follows from Proposition 2.2.(ii) that

$$\dim(\tilde{G}(x)) = \dim(\tilde{\mathfrak{g}}_x) \quad (2)$$

Proof of (i) \implies (iii) : The proof results directly from (2), and the fact that $\dim(\tilde{G}(x)) = n$ if and only if $\tilde{G}(x)$ is a non empty open set.

Proof of (iii) \implies (ii) : Since $\tilde{G}(x) \cap W$ is a non empty open set then the proof follows directly from Lemma 2.4.(ii), because $\tilde{G}(x) \cap W \subset \overline{\tilde{G}(x)} \cap W = \overline{G(x)} \cap W$.

Proof of (ii) \implies (i) : Since $\overline{G(x)} \subset \text{Im}(\Phi_x) \subset \mathbb{C}^n$ then the linear map $\Phi_x : \mathcal{A}(\tilde{G}) \rightarrow \mathbb{C}^n$ is surjective, so it is an open map. By Lemma 2.3 there exists two open subsets U and $V = \exp(U)$ respectively of H and Γ such that the exponential map $\exp : U \rightarrow V$ is a diffeomorphism and satisfying $\exp(\tilde{\mathfrak{g}} \cap U) = \tilde{G} \cap V$. So

$$\exp^{-1}(\tilde{G} \cap V) = \tilde{\mathfrak{g}} \cap U. \quad (1)$$

Recall that $W = \Phi_x(V)$. Since Φ_x is an open map and by Lemma 2.4.(i), $\overline{\overline{G(x)}} = \overline{\overline{\tilde{G}(x)}}$, so

$$\begin{aligned} \Phi_x^{-1}(\overline{\overline{G(x)}} \cap W) &= \Phi_x^{-1}(\overline{\overline{\tilde{G}(x)}} \cap W) \\ &\subset \Phi_x^{-1}(\overline{\overline{\tilde{G}(x)}} \cap W) \\ &\subset \overline{\overline{\Phi_x^{-1}(\tilde{G}(x) \cap W)}} \quad (3) \end{aligned}$$

We have

$$\begin{aligned} \Phi_x \circ \exp^{-1}(\Phi_x^{-1}(\overline{\overline{G(x)}} \cap W) \cap V) &\subset \Phi_x \circ \exp^{-1}(\overline{\overline{\Phi_x^{-1}(\tilde{G}(x) \cap W) \cap V}}) \quad (\text{by (3)}) \\ &\subset \Phi_x \circ \exp^{-1}(\overline{\overline{\tilde{G} \cap V}}), \quad (\text{by Lemma 2.5}) \\ &\subset \Phi_x \circ \exp^{-1}(\overline{\overline{\tilde{G} \cap V}}) \\ &\subset \Phi_x(\overline{\overline{\tilde{g} \cap U}}) \quad (\text{by (1)}) \\ &\subset \tilde{g}_x \end{aligned}$$

Since $\overline{\overline{G(x)}} \cap W$ is a non empty open subset of \mathbb{C}^n then $\Phi_x \circ \exp^{-1}(\Phi_x^{-1}(\overline{\overline{G(x)}} \cap W) \cap V)$ is an open subset of \mathbb{C}^n . It follows that $\tilde{g}_x = \mathbb{C}^n$. The proof is completed \square .

3. Proof of Theorem 1.2 and Corollaries 1.3, 1.4

Under the notation of Lemma 2.3, recall that there exists an open subset U of $\mathcal{A}(\tilde{G})$ such that $\exp : U \rightarrow \exp(U)$ is a diffeomorphism. Now, by using the restriction of the Whitney topology to $\mathcal{A}(\tilde{G})$, denote by:

- $B_{(0,r)} = \{f \in \mathcal{A}(\tilde{G}) : \|f\| < r\}$, the open ball with center 0 and radius $r > 0$.
- $r_G = \sup\{r \in]0, 1[: B_{(0,r)} \subset U\}$, it is dependent of G since is U .

Theorem 3.1. *Let G be a subgroup of $\text{Diff}^r(\mathbb{C}^n)$ and $x \in \mathbb{C}^n$. If there exist $f_1, \dots, f_{2n} \in \exp^{-1}(\tilde{G})$ with $\|f_k\| < r_{\tilde{G}}$, for every $k = 1, \dots, 2n$ such that $(f_1(x), \dots, f_{2n}(x))$ is a basis of \mathbb{C}^n over \mathbb{R} , then $\overline{\overline{G(x)}} \neq \emptyset$.*

Proof. We have $f_k \in \exp^{-1}(\tilde{G})$ with $\|f_k\| < r_{\tilde{G}}$ for every $k = 1, \dots, 2n$, then $f_1, \dots, f_{2n} \in U$ and so $e^{f_k} \in \tilde{G} \cap V$. By Lemma 2.3, $\tilde{G} \cap V = \exp(U \cap \tilde{g})$, hence $f_k \in \tilde{g}$, for every $k = 1, \dots, 2n$. As $(f_1(x), \dots, f_{2n}(x))$ is a basis of \mathbb{C}^n over \mathbb{R} then $\tilde{g}_x = \mathbb{C}^n$. It follows by Theorem 1.1 that $\overline{\overline{G(x)}} \neq \emptyset$. \square

Lemma 3.2. *Let H be a vector space with dimension $2n$ over \mathbb{R} and $x_1, \dots, x_{2n+1} \in H$, such that $\mathbb{Z}x_1 + \dots + \mathbb{Z}x_{2n+1}$ is dense in H . Then for every $1 \leq k \leq 2n+1$, $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{2n+1})$ is a basis of H over \mathbb{R} .*

Proof. We have H is isomorphic to \mathbb{C}^n . Let $1 \leq k \leq 2n + 1$ be a fixed integer and take

$$K = \text{vect}(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{2n+1}).$$

Suppose that $\dim(K) = p < 2n$. Let $(x_{k_1}, \dots, x_{k_p})$ be a basis of K . Then $\mathbb{Z}x_1 + \dots + \mathbb{Z}x_{2n+1} \subset K + \mathbb{Z}x_k$ which cannot be dense in H , a contradiction. \square

Recall that \tilde{L} is the vector subspace of $\tilde{\mathfrak{g}}$ supplement to $\text{Ker}(\Phi_x)$, (i.e. $\tilde{L} \oplus \text{Ker}(\Phi_x) = \tilde{\mathfrak{g}}$). Denote by:

- $p_x : \tilde{L} \oplus \text{Ker}(\Phi_x) \longrightarrow \tilde{L}$ given by $p_x(f + h) = f$, $f \in \tilde{L}$ and $h \in \text{Ker}(\Phi_x)$.

Lemma 3.3. *Under above notations, we have:*

- (i) *The linear map $\Phi_x : \tilde{L} \longrightarrow E(x)$ defined by $\Phi_x(f) = f(x)$, is an isomorphism.*
- (ii) *for every $f \in \tilde{\mathfrak{g}}$ one has $\Phi_x^{-1}(f(x)) = p_x(f)$.*

Proof. (i) By construction Φ_x is surjective and restreint to \tilde{L} it became injective. By Lemma 2.1 Φ_x is continuous and bijective. Hence it is an isomorphism because it is linear.

(ii) Let $f \in \tilde{\mathfrak{g}}$. Write $f = f_1 + f_0$ with $f_1 = p_x(f) \in \tilde{L}$ and $f_0 \in \text{Ker}(\Phi_x)$. Since $f_0(x) = 0$, so $f(x) = f_1(x)$. By (i), Φ_x is an isomorphism, then $\Phi_x^{-1}(f(x)) = \Phi_x^{-1}(f_1(x)) = f_1 = p_x(f)$. This completes the proof. \square

Let $f_1, \dots, f_{2n+1} \in \tilde{\mathfrak{g}}$ and suppose that $(p_x(f_1), \dots, p_x(f_{2n}))$ is a basis of \tilde{L} over \mathbb{R} and $f_{n+1} \in \text{vect}(f_1, \dots, f_{2n})$. Denote by $\Psi : \tilde{L} \longrightarrow \tilde{\mathfrak{g}}$ the linear map given by

$$\Psi \left(\sum_{k=1}^{2n} \alpha_k p_x(f_k) \right) = \sum_{k=1}^{2n} \alpha_k f_k.$$

Lemma 3.4. *Under above notations, we have:*

- (i) *If $\overline{\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)} = \mathbb{C}^n$ then $\overline{\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})} = \tilde{L}$.*
- (ii) *$\Psi(\overline{\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})}) = \overline{\mathbb{Z}f_1 + \dots + \mathbb{Z}f_{2n+1}}$.*

Proof. (i) Here $E(x) = \mathbb{C}^n$. By Lemma 3.3,(i), $\Phi_x : \tilde{L} \longrightarrow \mathbb{C}^n$ is an isomorphism and by Lemma 3.3,(ii), we have $\Phi_x^{-1}(f_k(x)) = p_x(f_k)(x)$ for every $k = 1, \dots, 2n + 1$, so

$$\Phi_x^{-1}(\overline{\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)}) = \overline{\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})}.$$

Then

$$\begin{aligned} \tilde{L} &= \Phi_x^{-1}(\mathbb{C}^n) \\ &= \Phi_x^{-1}(\overline{\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)}) \\ &= \overline{\mathbb{Z}\Phi_x^{-1}(f_1(x)) + \dots + \mathbb{Z}\Phi_x^{-1}(f_{2n+1}(x))} \\ &= \overline{\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})} \end{aligned}$$

(ii) Let $k_1, \dots, k_{2n+1} \in \mathbb{Z}$ and $f = k_1 p_x(f_1) + \dots + k_{2n+1} p_x(f_{2n+1})$. Write $f_{2n+1} = \sum_{k=1}^{2n} \alpha_k f_k$, $\alpha_1, \dots, \alpha_{2n} \in \mathbb{R}$, then

$$f = (k_1 + \alpha_1 k_{2n+1}) p_x(f_1) + \dots + (k_{2n} + \alpha_{2n} k_{2n+1}) p_x(f_{2n}),$$

so

$$\begin{aligned} \Psi(f) &= \Psi((k_1 + \alpha_1 k_{2n+1}) p_x(f_1) + \dots + (k_{2n} + \alpha_{2n} k_{2n+1}) p_x(f_{2n})) \\ &= (k_1 + \alpha_1 k_{2n+1}) f_1 + \dots + (k_{2n} + \alpha_{2n} k_{2n+1}) f_{2n} \\ &= k_1 f_1 + \dots + k_{2n+1} f_{2n+1} \end{aligned}$$

Then $\Psi(\mathbb{Z} p_x(f_1) + \dots + \mathbb{Z} p_x(f_{2n+1})) \subset \mathbb{Z} f_1 + \dots + \mathbb{Z} f_{2n+1}$. The same proof is used for the converse, by replacing Ψ by Ψ^{-1} . \square

Proposition 3.5. ([5], Proposition 4.3). *Let $H = \mathbb{Z} x_1 + \dots + \mathbb{Z} x_p$ with $x_k \in \mathbb{R}^n$. Then H is dense in \mathbb{R}^n if and only if for every $(s_1, \dots, s_p) \in \mathbb{Z}^p \setminus \{0\}$:*

$$\text{rank} \begin{bmatrix} x_1 & \dots & \dots & x_p \\ s_1 & \dots & \dots & s_p \end{bmatrix} = n + 1.$$

Proof of Theorem 1.2. Write $H_x = \mathbb{Z} f_1(x) + \dots + \mathbb{Z} f_{2n+1}(x)$. Since $\overline{H_x} = \mathbb{C}^n$, by Lemma 3.2, $(f_1(x), \dots, f_{2n}(x))$ is a basis of \mathbb{C}^n , so f_1, \dots, f_{2n} are linearly independent over \mathbb{R} . Denote by $E = \text{vect}(f_1, \dots, f_{2n})$, then $E = \Psi(\tilde{L})$ and it has a dimension equal to $2n$ over \mathbb{R} , so $\Psi : \tilde{L} \rightarrow E$ is an isomorphism. Since $f_{2n+1} \in \text{vect}(f_1, \dots, f_{2n})$ then by Lemma 3.4, (i), $\overline{\mathbb{Z} p_x(f_1) + \dots + \mathbb{Z} p_x(f_{2n+1})} = \tilde{L}$. Therefore:

$$\begin{aligned} E &= \Psi(\tilde{L}) \\ &= \overline{\Psi(\mathbb{Z} p_x(f_1) + \dots + \mathbb{Z} p_x(f_{2n+1}))} \\ &= \overline{\mathbb{Z} f(p_x(f_1)) + \dots + \mathbb{Z} f(p_x(f_{2n+1}))} \\ &= \overline{\mathbb{Z} f_1 + \dots + \mathbb{Z} f_{2n+1}} \quad (1) \end{aligned}$$

Let $1 \leq k \leq 2n$ and $t_k \in \mathbb{R}^*$ such that $|t_k| < \frac{r_{\tilde{G}}}{\|f_k\|}$.

• First, let's prove that $e^{t_k f_k} \in G$. Since $t_k f_k \in E$, then by (1), $t_k f_k \in \overline{\mathbb{Z} f_1 + \dots + \mathbb{Z} f_{2n+1}}$. Thus there exists a sequence $(g_j)_{j \in \mathbb{N}} \subset \mathbb{Z} f_1 + \dots + \mathbb{Z} f_{2n+1}$ such that $\lim_{j \rightarrow +\infty} g_j = t_k f_k$. By continuity of the exponential map we have $\lim_{j \rightarrow +\infty} e^{g_j} = e^{t_k f_k}$. Since $\mathbb{Z} f_1 + \dots + \mathbb{Z} f_{2n+1} \subset \exp^{-1}(\tilde{G})$ then $g_j \in \exp^{-1}(\tilde{G})$, so $e^{t_k f_k} \in \tilde{G}$, since \tilde{G} is closed in $\text{Diff}^r(\mathbb{C}^n)$.

• Second, as $|t_k| < \frac{r_{\tilde{G}}}{\|f_k\|}$, then $\|t_k f_k\| < r_{\tilde{G}}$. Since $|t_k| \neq 0$ and $(f_1(x), \dots, f_{2n}(x))$ is a basis of \mathbb{C}^n , so is $(t_1 f_1(x), \dots, t_{2n} f_{2n}(x))$. By the first step we conclude that $e^{t_k f_k} \in G$ for every $k = 1, \dots, 2n$. The proof follows then from Theorem 3.1. \square

The complex form of Proposition 3.5 is given in the following:

Proposition 3.6. ([5], page 35). Let $H = \mathbb{Z}z_1 + \cdots + \mathbb{Z}z_p$ with $z_k \in \mathbb{C}^n$ and $z_k = \operatorname{Re}(z_k) + i\operatorname{Im}(z_k)$, $k = 1, \dots, p$. Then H is dense in \mathbb{C}^n if and only if for every $(s_1, \dots, s_p) \in \mathbb{Z}^p \setminus \{0\}$:

$$\operatorname{rank} \begin{bmatrix} \operatorname{Re}(z_1) & \dots & \dots & \operatorname{Re}(z_p) \\ \operatorname{Im}(z_1) & \dots & \dots & \operatorname{Im}(z_p) \\ s_1 & \dots & \dots & s_p \end{bmatrix} = 2n + 1.$$

Proof of Corollary 1.3. The proof results directly, from Theorem 1.2 and Proposition 3.6. \square

Lemma 3.7. ([1], Corollary 1.3). Let G be an abelian subgroup of $GL(n, \mathbb{C})$. If G has a locally dense orbit γ in \mathbb{C}^n then γ is dense in \mathbb{C}^n .

Proof of Corollary 1.4. Since the matrices B_j , $1 \leq j \leq 2n+1$ commute then $\mathbb{Z}B_1 + \cdots + \mathbb{Z}B_{2n+1} \subset \exp^{-1}(G)$. Hence the proof of Corollary 1.4 results directly from Corollary 1.3 and Lemma 3.7. \square

Question1: How can we characterize explicitly $\mathfrak{g} = \exp^{-1}(G)$ for any finitely generated abelian subgroup G of a lie group $\Gamma \subset \operatorname{Diff}^r(\mathbb{C}^n)$?

Question2: A somewhere dense orbit of a non abelian subgroup of $\operatorname{Diff}^r(\mathbb{C}^n)$ can always be dense in \mathbb{C}^n ?

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