

# SOMEWHERE DENSE ORBIT OF ABELIAN SUBGROUP OF DIFFEOMORPHISMS MAPS ACTING ON $\mathbb{C}^n$

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ABSTRACT. In this paper, we give a characterization for any abelian subgroup  $G$  of a lie group of diffeomorphisms maps of  $\mathbb{C}^n$ , having a somewhere dense orbit  $G(x)$ ,  $x \in \mathbb{C}^n$ :  $G(x)$  is somewhere dense in  $\mathbb{C}^n$  if and only if there are  $f_1, \dots, f_{2n+1} \in \exp^{-1}(G)$  such that  $f_{2n+1} \in \text{vect}(f_1, \dots, f_{2n})$  and  $\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)$  is dense in  $\mathbb{C}^n$ , where  $\text{vect}(f_1, \dots, f_{2n})$  is the vector space over  $\mathbb{R}$  generated by  $f_1, \dots, f_{2n}$ .

## 1. Introduction

Denote by  $\text{Diff}^r(\mathbb{C}^n)$ ,  $r \geq 1$  the group of all  $C^r$ -diffeomorphisms of  $\mathbb{C}^n$ . Let  $\Gamma$  be a lie subgroup of  $\text{Diff}^r(\mathbb{C}^n)$ ,  $r \geq 1$  and  $G$  be an abelian subgroup of  $\Gamma$ , such that  $\text{Fix}(G) \neq \emptyset$ , where  $\text{Fix}(G) = \{x \in \mathbb{C}^n : f(x) = x, \forall f \in G\}$  be the global fixed point set of  $G$ . There is a natural action  $G \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ .  $(f, x) \mapsto f(x)$ . For a point  $x \in \mathbb{C}^n$ , denote by  $G(x) = \{f(x), f \in G\} \subset \mathbb{C}^n$  the orbit of  $G$  through  $x$ . A subset  $E \subset \mathbb{C}^n$  is called  $G$ -invariant if  $f(E) \subset E$  for any  $f \in G$ ; that is  $E$  is a union of orbits. Denote by  $\overline{E}$  (resp.  $\overset{\circ}{E}$ ) the closure (resp. interior) of  $E$ .

Recall that  $E \subset \mathbb{C}^n$  is somewhere dense in  $\mathbb{C}^n$  if the closure  $\overline{E}$  has nonempty interior in  $\mathbb{C}^n$ . An orbit  $\gamma$  is called somewhere dense (or locally dense) if  $\overset{\circ}{\gamma} \neq \emptyset$ . The group  $G$  is called hypercyclic if it has a dense orbit in  $\mathbb{C}^n$ . Hypercyclic is also called topologically transitive.

The purpose of this paper is to give a characterization for any subgroup  $G$  of a lie group of diffeomorphisms maps of  $\mathbb{C}^n$ , having a dense orbit. In [3], the authors present a global dynamic of every abelian subgroup of  $GL(n, \mathbb{C})$  and in [4], they characterize hypercyclic abelian subgroup of  $GL(n, \mathbb{C})$ . Our main result is viewed as a continuation of [1] and [2].

Denote by:

- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  and  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ .
- $C^r(\mathbb{C}^n, \mathbb{C}^n)$  the set of all  $C^r$ -differentiable maps of  $\mathbb{C}^n$ .
- For a subset  $E \subset \mathbb{C}^n$  (resp.  $E \subset C^r(\mathbb{C}^n, \mathbb{C}^n)$ ), denote by  $\text{vect}(E)$  the vector subspace of  $\mathbb{C}^n$  (resp.  $C^r(\mathbb{C}^n, \mathbb{C}^n)$ ) over  $\mathbb{R}$  generated by all elements of  $E$ .
- $\exp : C^r(\mathbb{C}^n, \mathbb{C}^n) \rightarrow \text{Diff}^r(\mathbb{C}^n)$  the exponential map defined by  $\exp(f) = e^f$ ,  $f \in C^r(\mathbb{C}^n, \mathbb{C}^n)$ .

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- $H$  the lie algebra associated to  $\Gamma$ .
- $\exp : H \rightarrow \Gamma$  be the exponential map.
- $H_x = \{f(x), B \in H\}$ , it is a vector subspace of  $\mathbb{C}^n$  over  $\mathbb{R}$ .
- $\mathfrak{g} = \exp^{-1}(G)$ , it is an additive group because  $G$  is abelian.
- $\mathfrak{g}_x = \{f(x), B \in \mathfrak{g}\}$ , it is an additive subgroup of  $\mathbb{C}^n$  because  $\mathfrak{g}$  is an additive group.

Our principal results can be stated as follows:

**Theorem 1.1.** *Let  $\Gamma$  be an abelian lie subgroup of  $\text{Diff}^r(\mathbb{C}^n)$  and  $x \in \mathbb{C}^n \setminus \{0\}$ . Then the following assertions are equivalent:*

- (i)  $H_x = \mathbb{C}^n$ .
- (ii)  $\overline{\Gamma(x)} \neq \emptyset$ .

In general, the Lie algebra  $\tilde{\mathfrak{g}}$  is not explicitly defined, so we give an explicitly test to the existence of somewhere dense orbit by the following theorem:

**Theorem 1.2.** *Let  $G$  be an abelian subgroup of a lie group  $\Gamma \subset \text{Diff}^r(\mathbb{C}^n)$  and  $x \in \mathbb{C}^n \setminus \{0\}$ . Then  $\overline{G(x)} \neq \emptyset$  if and only if there exist  $f_1, \dots, f_{2n+1} \in \exp^{-1}(\tilde{G})$  such that  $f_{2n+1} \in \text{vect}(f_1, \dots, f_{2n})$  and  $\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)$  is a dense additive subgroup of  $\mathbb{C}^n$ .*

Let's introduce the arithmetic property: We say that  $f_1, \dots, f_{2n+1} \in C^r(\mathbb{C}^n, \mathbb{C}^n)$  satisfy *property  $\mathcal{D}(x)$*  for some  $x \in \mathbb{C}^n$  if  $f_1, \dots, f_{2n}$  are linearly independent,  $f_{2n+1} \in \text{vect}(f_1, \dots, f_{2n})$  and for every  $(s_1, \dots, s_{2n+1}) \in \mathbb{Z}^{2n+1} \setminus \{0\}$ :

$$\text{rank} \begin{bmatrix} \text{Re}(f_1(x)) & \dots & \text{Re}(f_{2n+1}(x)) \\ \text{Im}(f_1(x)) & \dots & \text{Im}(f_{2n+1}(x)) \\ s_1 & \dots & s_{2n+1} \end{bmatrix} = 2n + 1.$$

For a vector  $v \in \mathbb{C}^n$ , we write  $v = \text{Re}(v) + i\text{Im}(v)$  where  $\text{Re}(v)$  and  $\text{Im}(v) \in \mathbb{R}^n$ .

As an immediate consequence of Theorem 1.2, we have:

**Corollary 1.3.** *Let  $G$  be an abelian subgroup of a lie group  $\Gamma \subset \text{Diff}^r(\mathbb{C}^n)$  and  $x \in \mathbb{C}^n \setminus \{0\}$ . Then  $\overline{G(x)} \neq \emptyset$  if and only if there exist  $f_1, \dots, f_{2n+1} \in \exp^{-1}(G)$  and satisfying property  $\mathcal{D}(x)$ .*

**Theorem 1.4.** *Let  $G$  be an abelian subgroup of a lie group  $\Gamma \subset \text{Diff}^r(\mathbb{C}^n)$  and  $x \in \mathbb{C}^n \setminus \{0\}$ . If there exist  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ ,  $g \in \exp^{-1}(G)$  and  $f_1, \dots, f_{2n} \in G$  such that  $\alpha g \in \exp^{-1}(G)$  and  $(f_1 \circ g \circ f_1^{-1}(x), \dots, f_{2n} \circ g \circ f_{2n}^{-1}(x))$  is a basis of  $\mathbb{C}^n$  over  $\mathbb{R}$ , then  $\overline{G(x)} \neq \emptyset$ .*

As an important consequence of the Theorem 1.2, we give the following Corollary which simplifies the test given by Theorem 1.3 proved in [4] for the abelian subgroup of  $GL(n, \mathbb{C})$ :

**Corollary 1.5.** *Let  $G$  be an abelian subgroup of  $GL(n, \mathbb{C})$  and  $x \in \mathbb{C}^n \setminus \{0\}$ . Then  $\overline{G(x)} = \mathbb{C}^n$  if and only if there exist  $B_1, \dots, B_{2n+1} \in \exp^{-1}(G)$  such that  $\mathbb{Z}B_1x + \dots + \mathbb{Z}B_{2n+1}x$  is dense in  $\mathbb{C}^n$ .*

This paper is organized as follows: In Section 2 we prove Theorem 1.1. Section 3 is devoted to prove Theorems 1.2, 1.4 and Corollaries 1.3, 1.5.

## 2. Proof of Theorem 1.1

**2.1. Exponential map.** In this section, we illustrate the theory developed of the group  $Diff(\mathbb{C}^n)$  of diffeomorphisms of  $\mathbb{C}^n$ . For simplicity, throughout this section we only consider the case of  $\mathbb{C} = \mathbb{R}$ ; however, all results also hold for complex case. The group  $Diff(\mathbb{R}^n)$  is not a Lie group (it is infinite-dimensional), but in many way it is similar to Lie groups. For example, it easy to define what a smooth map from some Lie group  $G$  to  $Diff(\mathbb{R}^n)$  is: it is the same as an action of  $G$  on  $\mathbb{R}^n$  by diffeomorphisms. Ignoring the technical problem with infinite-dimensionality for now, let us try to see what is the natural analog of the Lie algebra  $\mathfrak{g}$  for the group  $G$ . It should be the tangent space at the identity; thus, its elements are derivatives of one-parameter families of diffeomorphisms.

Let  $\varphi^t : G \rightarrow G$  be one-parameter family of diffeomorphisms. Then, for every point  $a \in G$ ,  $\varphi^t(a)$  is a curve in  $G$  and thus  $\frac{\partial}{\partial t}\varphi^t(a)|_{t=0} = \xi(a) \in T_a G$  is a tangent vector to  $G$  at  $a$ . In other words,  $\frac{\partial}{\partial t}\varphi^t$  is a vector field on  $G$ .

The exponential map  $exp : \mathfrak{g} \rightarrow G$  is defined by  $exp(x) = \gamma_x(1)$  where  $\gamma_x(t)$  is the one-parameter subgroup with tangent vector at 1 equal to  $x$ .

If  $\xi \in \mathfrak{g}$  is a vectorfield, then  $exp(t\xi)$  should be one-parameter family of diffeomorphisms whose derivative is vector field  $\xi$ . So this is the solution of differential equation

$$\frac{\partial}{\partial t}\varphi^t(a)|_{t=0} = \xi(a).$$

In other words,  $\varphi^t$  is the time  $t$  flow of the vector field. Thus, it is natural to define the Lie algebra of  $G$  to be the space  $\mathfrak{g}$  of all smooth vector  $\xi$  fields on  $\mathbb{R}^n$  such that  $exp(t\xi) \in G$  for every  $t \in \mathbb{R}$ .

We will use the definition of Whitney topology given in [16].

**2.2. Whitney Topology on  $C^0(\mathbb{C}^n, \mathbb{C}^n)$ .** For each open subset  $U \subset \mathbb{C}^n \times \mathbb{C}^n$  let  $\tilde{U} \subset C^0(\mathbb{C}^n, \mathbb{C}^n)$  be the set of continuous functions  $g$ , whose graphs  $\{(x, g(x)) \in \mathbb{C}^n \times \mathbb{C}^n, x \in \mathbb{C}^n\}$  is contained in  $U$ . We want to construct a neighborhood basis of each function  $f \in C^0(\mathbb{C}^n, \mathbb{C}^n)$ . Let  $K_j = \{x \in \mathbb{C}^n, \|x\| \leq j\}$  be a countable family of compact sets (closed balls with center 0) covering  $\mathbb{C}^n$  such that  $K_j$  is contained

in the interior of  $K_{j+1}$ . Consider then the compact subsets  $L_j = K_j \setminus \overset{\circ}{K_{j-1}}$ , which are compact sets, too. Let  $\epsilon = (\epsilon_j)_j$  be a sequence of positive numbers and then define

$$V_{(f;\epsilon)} = \{f \in C^0(\mathbb{C}^n, \mathbb{C}^n) : \|f(x) - g(x)\| < \epsilon_j, \text{ for any } x \in L_j, \forall j\}.$$

We claim this is a neighborhood system of the function  $f$  in  $C^0(\mathbb{C}^n, \mathbb{C}^n)$ . Since  $L_j$  is compact, the set  $U = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n : \|f(x) - g(x)\| < \epsilon_j, \text{ if } x \in L_j\}$  is open. Thus,  $V_{(f;\epsilon)} = \tilde{U}$  is an open neighborhood of  $f$ . On the other hand, if  $O$  is an open subset of  $\mathbb{C}^n \times \mathbb{C}^n$  which contains the graph of  $f$ , then since  $L_j$  is compact, it follows that there exists  $\epsilon_j > 0$  such that if  $x \in L_j$  and  $\|y - f(x)\| < \epsilon_j$ , then

$(x; y) \in O$ . Thus, taking  $\tilde{\epsilon} = (\epsilon_j)_j$  we have  $V_{(f; \tilde{\epsilon})} \subset \tilde{O}$ , so we have obtained the family  $V_{(f; \epsilon)}$  is a neighborhood system of  $f$ . Moreover, for each given  $\epsilon = (\epsilon_j)_j$ , we can find a  $C^\infty$ -function  $\epsilon : \mathbb{C}^n \rightarrow \mathbb{R}_+$ , such that  $\epsilon(x) < \epsilon_j$  for any  $x \in L_j$ . It follows that the family  $V_{(f; \epsilon)} = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n : \|f(x) - g(x)\| < \epsilon(x)\}$  is also a neighborhood system.

Denote by:

- $\tilde{G} = \overline{G} \cap Diffr(\mathbb{C}^n)$ , where  $\overline{G}$  is the closure of  $G$  in  $C^r(\mathbb{C}^n, \mathbb{C}^n)$  for the Whitney topology defined above. So  $\tilde{G}$  is an abelian lie subgroup of  $\Gamma$ .
- $\mathcal{A}(\tilde{G})$  the algebra generated by  $G$ . See that  $G \subset \mathcal{A}(\tilde{G})$ .
- $\Phi_x : \mathcal{A}(\tilde{G}) \rightarrow \mathbb{C}^n$  the linear map given by  $\Phi_x(f) = f(x)$ ,  $f \in \mathcal{A}(\tilde{G})$ .
- $E(x) = \Phi_x(\mathcal{A}(G))$ .

**Lemma 2.1.** *The linear map  $\Phi_x : \mathcal{A}(\tilde{G}) \rightarrow E(x)$  is continuous.*

*Proof.* Firstly, we take the restriction of the Whitney topology to  $\mathcal{A}(\tilde{G})$ . Secondly, let  $f \in \mathcal{A}(\tilde{G})$  and  $\epsilon > 0$ . Then for  $\epsilon = (\epsilon_j)_j$  with  $\epsilon_j = \epsilon$  and for  $V_{(f; \epsilon)}$  be a neighborhood system of  $f$ , we obtain: for every  $g \in V_{(f; \epsilon)} \cap \mathcal{A}(\tilde{G})$  and for every  $y \in L_j$ ,  $\|f(y) - g(y)\| < \epsilon$ ,  $\forall j$ . In particular for  $j = j_0$  in which  $x \in L_{j_0}$ , we have  $\|f(x) - g(x)\| < \epsilon$ , so  $\|\Phi_x(f) - \Phi_x(g)\| < \epsilon$ . It follows that  $\Phi_x$  is continuous.  $\square$

### 2.3. Proof of Theorem 1.1.

**Proposition 2.2.** ([9], Theorem 3.29) *Let  $G$  be a Lie group acting on  $\mathbb{C}^n$  with lie algebra  $\tilde{\mathfrak{g}}$  and let  $u \in \mathbb{C}^n$ .*

- (i) *The stabilizer  $G_x = \{B \in G : Bu = u\}$  is a closed Lie subgroup in  $G$ , with Lie algebra  $\mathfrak{h}_x = \{B \in \tilde{\mathfrak{g}} : Bu = 0\}$ .*
- (ii) *The map  $G/G_x \rightarrow \mathbb{C}^n$  given by  $B.G_x \mapsto Bu$  is an immersion. Thus, the orbit  $G(x)$  is an immersed submanifold in  $\mathbb{C}^n$ . In particular  $\dim(G(x)) = \dim(\tilde{\mathfrak{g}}) - \dim(\mathfrak{h}_x)$ .*

Here  $\mathfrak{h}_x = Ker(\Phi_x)$  since  $Ker(\Phi_x) \subset \tilde{\mathfrak{g}}$ . Write:

- $\tilde{L}$  the vector subspace of  $\tilde{\mathfrak{g}}$  supplement to  $Ker(\Phi_x)$ , (i.e.  $\tilde{L} \oplus Ker(\Phi_x) = \tilde{\mathfrak{g}}$ ). It is clear that  $\dim(\tilde{L}) = cod(Ker(\Phi_x)) \leq n$ , then  $\tilde{L}$  is closed.
- $exp : \tilde{L} \oplus Ker(\Phi_x) \rightarrow \tilde{G}$  the exponential map. Since  $\tilde{G}$  is abelian, so is  $\tilde{\mathfrak{g}}$ , then  $exp(f + h) = exp(f) \circ exp(h)$  for every  $f \in \tilde{L}$  and  $h \in Ker(\Phi_x)$ .
- $\tilde{G}_x$  the stabilizer of  $\tilde{G}$  on the point  $u$ . So it is a lie subgroup of  $\tilde{G}$  with lie algebra  $Ker(\Phi_x)$ .

As a directly consequence of Proposition 5.13, given in [17], applied to  $\Gamma$ , we have the following Lemma:

**Lemma 2.3.** ([17], Proposition 5.13) *Let  $G$  be an abelian subgroup of a lie group  $\Gamma$ . There exists an open neighborhood  $U$  of 0 in  $H$  such that  $exp : U \rightarrow exp(U)$  is a diffeomorphism and  $exp(U \cap \tilde{\mathfrak{g}}) = exp(U) \cap \tilde{G}$ .*

Denote by  $V = exp(U)$ , where  $U$  is the open set defined in Lemma 2.3.

**Lemma 2.4.** *We have  $\overline{G(x)} = \overline{\tilde{G}(x)}$ .*

*Proof.* It is clear that  $\overline{G(x)} \subset \overline{\tilde{G}(x)} \subset \overline{\overline{G(x)}}$ . Let  $v \in \overline{\overline{G(x)}}$ , so  $v = \lim_{m \rightarrow +\infty} f_m(x)$  for some sequence  $(f_m)_{m \in \mathbb{N}}$  in  $\overline{G}$ . Then for every  $m \in \mathbb{N}$ , there exists a sequence  $(f_{m,k})_{k \in \mathbb{N}}$  in  $G$  such that  $\lim_{k \rightarrow +\infty} f_{m,k} = f_m$ , so by continuity of  $\Phi_x$  (Lemma 2.1), we have  $\lim_{k \rightarrow +\infty} f_{m,k}(x) = f_m(x)$ , thus for every  $\varepsilon > 0$ , there exists  $M > 0$  and for every  $m \geq M$ , there exists  $k_m > 0$ , such that  $\|f_m(x) - v\| < \frac{\varepsilon}{2}$  and for every  $k \geq k_m$ ,  $\|f_{m,k}(x) - f_m(x)\| < \frac{\varepsilon}{2}$ . Then, for every  $m > M$ ,

$$\|f_{m,k_m}(x) - v\| \leq \|f_{m,k_m}(x) - f_m(x)\| + \|f_m(x) - v\| < \varepsilon,$$

therefore  $\lim_{m \rightarrow +\infty} f_{m,k_m}(x) = v$ . Hence  $v \in \overline{G(x)}$ . It follows that  $\overline{\tilde{G}(x)} \subset \overline{\overline{G(x)}} \subset \overline{G(x)}$ .  $\square$

**Lemma 2.5.** *Let  $W = \Phi_x(V)$ . Then  $\Phi_x^{-1}(\tilde{G}(x) \cap W) = \tilde{G} \cap V$ .*

*Proof.* Since  $W = \Phi_x(V)$ , it is obvious that  $\tilde{G} \cap V \subset \Phi_x^{-1}(\tilde{G}(x) \cap W)$ . Let  $f \in \Phi_x^{-1}(\tilde{G}(x) \cap W)$ . Then there exists  $g \in \tilde{G} \cap V$  such that  $f(x) = g(x)$ . So  $g^{-1} \circ f(x) = x$ . Hence  $g^{-1} \circ f \in H_x$ , where  $H_x$  be the lie group generated by  $\{h \in \text{Diff}^r(\mathbb{C}^n) : h(x) = x\} \cap \tilde{A}(\tilde{G})$ . So  $H_x$  is contained in the stabilizer of  $\text{Diff}^r(\mathbb{C}^n)$  on  $x$ . Set  $L_x$  be the lie algebra of  $H_x$ , so  $L_x \subset \{h \in \text{Diff}^r(\mathbb{C}^n) : h(x) = 0\} \cap \mathcal{A}(\tilde{G})$ . Therefore  $L_x \subset \text{Ker}(\Phi_x) \subset \tilde{\mathfrak{g}}$ . Hence  $H_x \subset \tilde{G}$ . It follows that  $g^{-1} \circ f \in \tilde{G}$ , so  $f \in \tilde{G} \cap V$ . This completes the proof.  $\square$

*Proof of Theorem 1.1.*

Since  $\tilde{G}$  is a locally closed sub-manifold of  $\text{Diff}^r(\mathbb{C}^n)$ . By Proposition 2.2.(ii),  $\tilde{G}(x)$  is an immersed submanifold of  $\mathbb{C}^n$  with dimension  $r = \dim(\tilde{\mathfrak{g}}) - \dim(\text{Ker}(\Phi_x))$ . Now, we let  $\Phi'_x : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}_x$  the restriction of  $\Phi_x$  to  $\tilde{\mathfrak{g}}$  (i.e.  $\Phi'_x(f) = f(x)$ ,  $f \in \tilde{\mathfrak{g}}$ ). We have  $\text{Ker}(\Phi'_x) = \text{Ker}(\Phi_x) = \{f \in \tilde{\mathfrak{g}} : f(x) = 0\}$ . Then

$$\begin{aligned} \dim(\tilde{\mathfrak{g}}) &= \dim(\text{Ker}(\Phi'_x)) + \dim(\text{Im}(\Phi'_x)) \\ &= \dim(\text{Ker}(\Phi_x)) + \dim(\tilde{\mathfrak{g}}_x). \end{aligned}$$

So  $\dim(\tilde{\mathfrak{g}}_x) = \dim(\tilde{\mathfrak{g}}) - \dim(\text{Ker}(\Phi_x))$ . It follows from Proposition 2.2.(ii) that

$$\dim(\tilde{G}(x)) = \dim(\tilde{\mathfrak{g}}_x) \quad (2)$$

*Proof of (i)  $\implies$  (iii) :* The proof results directly from (2), and the fact that  $\dim(\tilde{G}(x)) = n$  if and only if  $\tilde{G}(x)$  is a non empty open set.

*Proof of (iii)  $\implies$  (ii) :* Since  $\tilde{G}(x) \cap W$  is a non empty open set then the proof follows directly from Lemma 2.4.(ii), because  $\tilde{G}(x) \cap W \subset \overline{(x)} \cap W = \overline{G(x)} \cap W$ .

*Proof of (ii)  $\implies$  (i) :* Since  $\overline{G(x)} \subset \text{Im}(\Phi_x) \subset \mathbb{C}^n$  then the linear map  $\Phi_x : \mathcal{A}(\tilde{G}) \rightarrow \mathbb{C}^n$  is surjective, so it is an open map. By Lemma 2.3 there exists two

open subsets  $U$  and  $V = \exp(U)$  respectively of  $H$  and  $\Gamma$  such that the exponential map  $\exp : U \rightarrow V$  is a diffeomorphism and satisfying  $\exp(\tilde{g} \cap U) = \tilde{G} \cap V$ . So

$$\exp^{-1}(\tilde{G} \cap V) = \tilde{g} \cap U. \quad (1)$$

Let  $W = \Phi_x(V)$ . Now, since  $\Phi_x$  is an open map and by Lemma 2.4.(i),  $\overline{\overset{\circ}{G(x)}} = \overline{\overset{\circ}{\tilde{G}(x)}}$ , so

$$\begin{aligned} \Phi_x^{-1}(\overline{\overset{\circ}{G(x)}} \cap W) &= \Phi_x^{-1}(\overline{\overset{\circ}{\tilde{G}(x)}} \cap W) \cap \tilde{G} \cap V \\ &\subset \Phi_x^{-1}(\overline{\tilde{G}(x) \cap W}) \cap \tilde{G} \cap V \\ &\subset \overline{\Phi_x^{-1}(\tilde{G}(x) \cap W)} \cap \tilde{G} \cap V \\ &\subset \overline{\Phi_x^{-1}(\tilde{G}(x) \cap W) \cap \tilde{G} \cap V} \\ &\subset \overline{\Phi_x^{-1}(\tilde{G}(x) \cap W)} \quad (3) \end{aligned}$$

we have

$$\begin{aligned} \Phi_x \circ \exp^{-1}(\Phi_x^{-1}(\overline{\overset{\circ}{G(x)}} \cap W)) &\subset \Phi_x \circ \exp^{-1}(\overline{\Phi_x^{-1}(\tilde{G}(x) \cap W)}) && \text{(by (3))} \\ &\subset \Phi_x \circ \exp^{-1}(\overline{\Phi_x^{-1}(\tilde{G}(x) \cap W)}), && \text{by Lemma 2.5, we obtain} \\ &\subset \Phi_x \circ \exp^{-1}(\overline{\tilde{G} \cap V}), && \text{by (1), we have} \\ &\subset \Phi_x(\overline{\tilde{g} \cap U}) \\ &\subset \tilde{g}_x \end{aligned}$$

Since  $\overline{\overset{\circ}{G(x)}} \cap W$  is an open subset of  $\mathbb{C}^n$  then  $\Phi_x \circ \exp^{-1}(\Phi_x^{-1}(\overline{\overset{\circ}{G(x)}} \cap W))$  is an open subset of  $\mathbb{C}^n$ . It follows that  $\tilde{g}_x = \mathbb{C}^n$ . The proof is completed  $\square$

### 3. Proof of Theorems 1.2, 1.4 and Corollaries 1.3, 1.5

Under the notation of Lemma 2.3, recall that there exists an open subset  $U$  of  $\mathcal{A}(\tilde{G})$  such that  $\exp : U \rightarrow \exp(U)$  is a diffeomorphism. Now, by using the restriction of the Whitney topology to  $\mathcal{A}(\tilde{G})$ , denote by:

- $B_{(0,r)} = \{f \in \mathcal{A}(\tilde{G}) : \|f\| < r\}$ , the open ball with center 0 and radius  $r > 0$ .
- $r_G = \sup\{r \in ]0, 1[ : B_{(0,r)} \subset U\}$ , it is dependent of  $G$  since is  $U$ .

**Theorem 3.1.** *Let  $G$  be a subgroup of  $\text{Diff}^r(\mathbb{C}^n)$  and  $x \in \mathbb{C}^n$ . If there exist  $f_1, \dots, f_{2n} \in \exp^{-1}(\tilde{G})$  with  $\|f_k\| < r_{\tilde{G}}$ , for every  $k = 1, \dots, 2n$  such that  $(f_1(x), \dots, f_{2n}(x))$  is a basis of  $\mathbb{C}^n$  over  $\mathbb{R}$ , then  $\overline{\overset{\circ}{G(x)}} \neq \emptyset$ .*

*Proof.* We have  $f_k \in \exp^{-1}(\tilde{G})$  with  $\|f_k\| < r_{\tilde{G}}$  for every  $k = 1, \dots, 2n$ , then  $f_1, \dots, f_{2n} \in U$  and so  $e^{f_k} \in \tilde{G} \cap V$ . By Lemma 2.3,  $\tilde{G} \cap V = \exp(U \cap \tilde{g})$ , hence

$f_k \in \tilde{\mathfrak{g}}$ , for every  $k = 1, \dots, 2n$ . As  $(f_1(x), \dots, f_{2n}(x))$  is a basis of  $\mathbb{C}^n$  over  $\mathbb{R}$  then  $\tilde{\mathfrak{g}}_x = \mathbb{C}^n$ . It follows by Theorem 1.1 that  $\overline{\tilde{\mathfrak{G}}(x)} \neq \emptyset$ .  $\square$

**Lemma 3.2.** *Let  $H$  be a vector space with dimension  $2n$  over  $\mathbb{R}$  and  $x_1, \dots, x_{2n+1} \in H$ , such that  $\mathbb{Z}(x)_1 + \dots + \mathbb{Z}(x)_{2n+1}$  is dense in  $H$ . Then for every  $1 \leq k \leq 2n+1$ ,  $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{2n+1})$  is a basis of  $H$  over  $\mathbb{R}$ .*

*Proof.* We have  $H$  is isomorphic to  $\mathbb{C}^n$ . Let  $1 \leq k \leq 2n+1$  be a fixed integer and take

$$K = \text{vect}(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{2n+1}).$$

Suppose that  $\dim(K) = p < 2n$ . Let  $(x_{k_1}, \dots, x_{k_p})$  be a basis of  $K$ . Then  $\mathbb{Z}x_1 + \dots + \mathbb{Z}x_{2n+1} \subset K + \mathbb{Z}x_k$  which cannot be dense in  $H$ , a contradiction.  $\square$

Recall that  $\tilde{L}$  is the vector subspace of  $\tilde{\mathfrak{g}}$  supplement to  $\text{Ker}(\Phi_x)$ , (i.e.  $\tilde{L} \oplus \text{Ker}(\Phi_x) = \tilde{\mathfrak{g}}$ ). Denote by:  
-  $p_x : \tilde{L} \oplus \text{Ker}(\Phi_x) \longrightarrow \tilde{L}$  given by  $p_x(f+h) = f$ ,  $f \in \tilde{L}$  and  $h \in \text{Ker}(\Phi_x)$ .

**Lemma 3.3.** *Under above notations, we have:*

- (i) *The linear map  $\Phi_x : \tilde{L} \longrightarrow E(x)$  defined by  $\Phi_x(f) = f(x)$ , is an isomorphism.*
- (ii) *for every  $f \in \tilde{\mathfrak{g}}$  one has  $\Phi_x^{-1}(f(x)) = p_x(f)$ .*

*Proof.* (i) By construction  $\Phi_x$  is surjective and restreint to  $\tilde{L}$  it became injective. By Lemma 2.1  $\Phi_x$  is continuous and bijective. Hence it is an isomorphism because it is linear.

(ii) Let  $f \in \tilde{\mathfrak{g}}$ . Write  $f = f_1 + f_0$  with  $f_1 = p_x(f) \in \tilde{L}$  and  $f_0 \in \text{Ker}(\Phi_x)$ . Since  $f_0(x) = 0$ , so  $f(x) = f_1(x)$ . By (i),  $\Phi_x$  is an isomorphism, then  $\Phi_x^{-1}(f(x)) = \Phi_x^{-1}(f_1(x)) = f_1 = p_x(f)$ . This completes the proof.  $\square$

Let  $f_1, \dots, f_{2n+1} \in \tilde{\mathfrak{g}}$  and suppose that  $(p_x(f_1), \dots, p_x(f_{2n}))$  is a basis of  $\tilde{L}$  over  $\mathbb{R}$  and  $f_{n+1} \in \text{vect}(f_1, \dots, f_{2n})$ . Denote by  $\Psi : \tilde{L} \longrightarrow \tilde{\mathfrak{g}}$  the linear map given by

$$\Psi \left( \sum_{k=1}^{2n} \alpha_k p_x(f_k) \right) = \sum_{k=1}^{2n} \alpha_k f_k.$$

**Lemma 3.4.** *Under above notations, we have:*

- (i) *If  $\overline{\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)} = \mathbb{C}^n$  then  $\overline{\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})} = \tilde{L}$ .*
- (ii)  *$\Psi(\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})) = \mathbb{Z}f_1 + \dots + \mathbb{Z}f_{2n+1}$ .*

*Proof.* (i) Here  $E(x) = \mathbb{C}^n$ . By Lemma 3.3,(i),  $\Phi_x : \tilde{L} \rightarrow \mathbb{C}^n$  is an isomorphism and by Lemma 3.3,(ii), we have  $\Phi_x^{-1}(f_k(x)) = p_x(f_k)(x)$  for every  $k = 1, \dots, 2n+1$ , so

$$\Phi_x^{-1}(\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)) = \mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1}).$$

Then

$$\begin{aligned} \tilde{L} &= \Phi_x^{-1}(\mathbb{C}^n) \\ &= \Phi_x^{-1}(\overline{\mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)}) \\ &= \overline{\mathbb{Z}\Phi_x^{-1}(f_1(x)) + \dots + \mathbb{Z}\Phi_x^{-1}(f_{2n+1}(x))} \\ &= \overline{\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})} \end{aligned}$$

(ii) Let  $k_1, \dots, k_{2n+1} \in \mathbb{Z}$  and  $f = k_1 p_x(f_1) + \dots + k_{2n+1} p_x(f_{2n+1})$ . Write  $f_{2n+1} = \sum_{k=1}^{2n} \alpha_k f_k$ ,  $\alpha_1, \dots, \alpha_{2n} \in \mathbb{R}$ , then

$$f = (k_1 + \alpha_1 k_{2n+1}) p_x(f_1) + \dots + (k_{2n} + \alpha_{2n} k_{2n+1}) p_x(f_{2n}),$$

so

$$\begin{aligned} \Psi(A) &= \Psi((k_1 + \alpha_1 k_{2n+1}) p_x(f_1) + \dots + (k_{2n} + \alpha_{2n} k_{2n+1}) p_x(f_{2n})) \\ &= (k_1 + \alpha_1 k_{2n+1}) f_1 + \dots + (k_{2n} + \alpha_{2n} k_{2n+1}) f_{2n} \\ &= k_1 f_1 + \dots + k_{2n+1} f_{2n+1} \end{aligned}$$

Then  $\Psi(\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})) \subset \mathbb{Z}f_1 + \dots + \mathbb{Z}f_{2n+1}$ . The same proof is used for the converse, by replacing  $\Psi$  by  $\Psi^{-1}$ .  $\square$

**Proposition 3.5.** ([8], Proposition 4.3). Let  $H = \mathbb{Z}x_1 + \dots + \mathbb{Z}x_p$  with  $x_k \in \mathbb{R}^n$ . Then  $H$  is dense in  $\mathbb{R}^n$  if and only if for every  $(s_1, \dots, s_p) \in \mathbb{Z}^p \setminus \{0\}$  :

$$\text{rank} \begin{bmatrix} x_1 & \dots & x_p \\ s_1 & \dots & s_p \end{bmatrix} = n + 1.$$

*Proof of Theorem 1.2.* Write  $H_x = \mathbb{Z}f_1(x) + \dots + \mathbb{Z}f_{2n+1}(x)$ . Since  $\overline{H_x} = \mathbb{C}^n$ , by Lemma 3.2,  $(f_1(x), \dots, f_{2n}(x))$  is a basis of  $\mathbb{C}^n$ , so  $f_1, \dots, f_{2n}$  are linearly independent over  $\mathbb{R}$ . Denote by  $E = \text{vect}(f_1, \dots, f_{2n})$ , then  $E = \Psi(\tilde{L})$  and it has a dimension equal to  $2n$  over  $\mathbb{R}$ , so  $\Psi : \tilde{L} \rightarrow E$  is an isomorphism. Since  $f_{2n+1} \in \text{vect}(f_1, \dots, f_{2n})$  then by Lemma 3.4,(i),  $\overline{\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})} = \tilde{L}$ . Therefore:

$$\begin{aligned} E &= \Psi(\tilde{L}) \\ &= \Psi(\overline{\mathbb{Z}p_x(f_1) + \dots + \mathbb{Z}p_x(f_{2n+1})}) \\ &= \overline{\mathbb{Z}\Psi(p_x(f_1)) + \dots + \mathbb{Z}\Psi(p_x(f_{2n+1}))} \\ &= \overline{\mathbb{Z}f_1 + \dots + \mathbb{Z}f_{2n+1}} \quad (1) \end{aligned}$$

Let  $1 \leq k \leq 2n$  and  $t_k \in \mathbb{R}^*$  such that  $|t_k| < \frac{r_{\tilde{G}}}{\|f_k\|}$ .

- First, let's prove that  $e^{t_k f_k} \in G$



Since  $t_k f_k \in E$ , then by (1),  $t_k f_k \in \overline{\mathbb{Z}f_1 + \cdots + \mathbb{Z}f_{2n+1}}$ . Thus there exists a sequence  $(g_j)_{j \in \mathbb{N}} \subset \mathbb{Z}f_1 + \cdots + \mathbb{Z}f_{2n+1}$  such that  $\lim_{j \rightarrow +\infty} g_j = t_k f_k$ . By continuity of the exponential map we have  $\lim_{j \rightarrow +\infty} e^{g_j} = e^{t_k f_k}$ . Since  $\mathbb{Z}f_1 + \cdots + \mathbb{Z}f_{2n+1} \subset \exp^{-1}(\tilde{G})$  then  $g_j \in \exp^{-1}(\tilde{G})$ , so  $e^{t_k f_k} \in \tilde{G}$ , since  $\tilde{G}$  is closed in  $\text{Diff}^r(\mathbb{C}^n)$ .

• Second, as  $|t_k| < \frac{r_G}{\|f_k\|}$ , then  $\|t_k f_k\| < r_{\tilde{G}}$ . Since  $|t_k| \neq 0$  and  $(f_1(x), \dots, f_{2n}(x))$  is a basis of  $\mathbb{C}^n$ , so is  $(t_1 f_1(x), \dots, t_{2n} f_{2n}(x))$ . By the first step we conclude that  $e^{t_k B_k} \in G$  for every  $k = 1, \dots, 2n$ . The proof follows then from Theorem 3.1.  $\square$

The complex form of Proposition 3.5 is given in the following:

**Proposition 3.6.** ([8], page 35). *Let  $H = \mathbb{Z}z_1 + \cdots + \mathbb{Z}z_p$  with  $z_k \in \mathbb{C}^n$  and  $z_k = \text{Re}(z_k) + i\text{Im}(z_k)$ ,  $k = 1, \dots, p$ . Then  $H$  is dense in  $\mathbb{C}^n$  if and only if for every  $(s_1, \dots, s_p) \in \mathbb{Z}^p \setminus \{0\}$ :*

$$\text{rank} \begin{bmatrix} \text{Re}(z_1) & \cdots & \cdots & \text{Re}(z_p) \\ \text{Im}(z_1) & \cdots & \cdots & \text{Im}(z_p) \\ s_1 & \cdots & \cdots & s_p \end{bmatrix} = 2n + 1.$$

*Proof of Corollary 1.3.* The proof results directly, from Theorem 1.2 and Proposition 3.6.  $\square$

**Lemma 3.7.** ([3], Corollary 1.3). *Let  $G$  be an abelian subgroup of  $\text{Diff}^r(\mathbb{C}^n)$ . If  $G$  has a locally dense orbit  $\gamma$  in  $\mathbb{C}^n$  then  $\gamma$  is dense in  $\mathbb{C}^n$ .*

*Proof of Corollary 1.5.* Since the matrices  $B_j$ ,  $1 \leq j \leq 2n+1$  commute then  $\mathbb{Z}B_1 + \cdots + \mathbb{Z}B_{2n+1} \subset \exp^{-1}(G)$ . Hence the proof of Corollary 1.5 results directly from Corollary 1.3 and Lemma 3.7.  $\square$

*Proof of Theorem 1.4.* Firstly, see that  $(\mathbb{Z} + \alpha\mathbb{Z})A \subset \exp^{-1}(G)$ , because  $A$  and  $\alpha A \in \exp^{-1}(G)$  and they commute. Since  $\alpha$  is irrational number then  $\overline{\mathbb{Z} + \alpha\mathbb{Z}} = \mathbb{R}$ . As the exponential map is continuous and  $G$  is closed in  $\text{Diff}^r(\mathbb{C}^n)$  then  $\exp^{-1}(G)$  is closed in  $M_n(\mathbb{C})$ , so  $\mathbb{R}A \subset \exp^{-1}(G)$ .

Secondly, let  $1 \leq k \leq 2n$  be a fixed integer and  $t_k \in \mathbb{R}^*$  such that  $|t_k| < \frac{r_G}{\|B_k A B_k^{-1}\|}$ . By above,  $e^{t_k A} \in G$ . Then  $B_k e^{t_k A} B_k^{-1} \in G$ . Therefore  $t_k B_k A B_k^{-1} \in \exp^{-1}(G)$ . Moreover,  $\|t_k B_k A B_k^{-1}\| < r_G$ .

Finally, since  $t_k \neq 0$  for every  $k = 1, \dots, 2n$  and  $(B_1 A B_1^{-1}(x), \dots, B_{2n} A B_{2n}^{-1}(x))$  is a basis of  $\mathbb{C}^n$  over  $\mathbb{R}$ , so is  $(t_1 B_1 A B_1^{-1}(x), \dots, t_{2n} B_{2n} A B_{2n}^{-1}(x))$ . Then by above and Theorem 3.1, the proof of Theorem 1.4 follows.  $\square$

**Question1:** *How can we characterize explicitly  $\mathfrak{g} = \exp^{-1}(G)$  for any finitely generated abelian subgroup  $G$  of a lie group  $\Gamma \subset \text{Diff}^r(\mathbb{C}^n)$ ?*

**Question2:** *A somewhere dense orbit of a non abelian subgroup of  $\text{Diff}^r(\mathbb{C}^n)$  can always be dense in  $\mathbb{C}^n$ ?*

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