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Abstract. Using a method developped in [1] and [2], we prove the existence of weak non trivial solutions to fourth order elliptic equations with singularities and with critical Sobolev growth.

1. Introduction

Fourth order elliptic equations have been widely studied these last years because of their importance in the analysis on manifolds particularly those involving the Paneitz - Branson operators. Many works have been devoted to this subject (see [**1**], [**2**], [**3**], [**4**],[**5**], [**6**], [**7**], [**8**], [**9**] [**10**], [**13**] and [**16**]). Different techniques have been used for the resolution of the fourth order equations as example the variational method which was developed by Yamabe to solve the problem of the prescribed scalar curvature. Let (M, g) a compact smooth Riemannian of dimension $n \geq 5$ with a metric g. We denote by $H_2^2(M)$ the standard Sobolev space which is the completed of the space $C^\infty(M)$ with respect to the norm

$$\|\varphi\|_{2,2} = \sum_{k=0}^{k=2} \left\| \nabla^k \varphi \right\|_2 .$$

$H_2^2(M)$ will be endowed with the suitable equivalent norm

$$\|u\|_{H_2^2(M)} = (\int_M \left((\Delta_g u)^2 + |\nabla_g u|^2 + u^2 \right) dv_g)^{\frac{1}{2}}.$$

In 1979, [**17**], M. Vaugon has proved the existence of real $\lambda > 0$ and a non trivial solution $u \in C^4(M)$ to the equation

$$\Delta_g^2 u - div_g (a(x) \nabla_g u) + b(x) u = \lambda f(t, x)$$

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where a, b are smooth functions on M and $f(t,x)$ is odd and increasing function in t fulfilling the inequality

$$|f(t,x)| < a + b\,|t|^{\frac{n+4}{n-4}}\,.$$

D.E. Edminds, D. Fortunato and E. Jannelli [**14**] have shown that the only solutions in R^n to the equation

$$\Delta^2 u = u^{\frac{n+4}{n-4}}$$

are positive, symmetric, radial and decreasing functions of the form

$$u_\epsilon(x) = \frac{\left((n-4)n(n^2-4)\epsilon^4\right)^{\frac{n-4}{8}}}{(r^2+\epsilon^2)^{\frac{n-4}{2}}}.$$

In 1995, [**15**] Van Der Vorst obtains the same results as D.E. Edminds, D. Fortunato and E. Jannelli to the following problem

$$\begin{cases} \Delta^2 u - \lambda u = u\,|u|^{\frac{8}{n-4}} \text{ in } \Omega, \\ \Delta u = u = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of R^n.
In 1996, [**9**] F. Bernis, J. Garcia-Azorero and I.Peral have obtained the existence at least of two positive solution to the following problem

$$\begin{cases} \Delta^2 u - \lambda u\,|u|^{q-2} = u\,|u|^{\frac{8}{n-4}} \text{ in } \Omega, \\ \Delta u = u = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is bounded domain of R^n,$1 < q < 2$ and $\lambda > 0$ in some interval. In 2001, [**12**], D. Caraffa has obtained the existence of a non trivial solution of class $C^{4,\alpha}$, $\alpha \in (0,1)$ to the following equation

$$\Delta_g^2 u - \nabla^\alpha\left(a(x)\nabla_\alpha u\right) + b(x)u = \lambda f(x)\,|u|^{N-2}\,u$$

with $\lambda > 0$, first for f a constant and next for a positive function f on M.

Recently the first author [**4**], has shown the existence of at least two distinct non trivial solutions in the subcritical case and a non trivial solution in the critical case to the following equation

$$\Delta_g^2 u - \nabla^\alpha\left(a(x)\nabla_\alpha u\right) + b(x)u = f(x)\,|u|^{N-2}\,u$$

where f is a changing sign smooth function and a and b are smooth functions. In [**6**] the same author proved the existence of at least two non trivial solutions to

$$\Delta_g^2 u - \nabla^\alpha\left(a(x)\nabla_\alpha u\right) + b(x)u = f(x)\,|u|^{N-2}\,u + |u|^{q-2}\,u + \varepsilon g(x)$$

where a, b, f, g are smooth functions on M with $f > 0$, $2 < q < N$, $\lambda > 0$ and $\epsilon > 0$ small enough. Let S_g denote the scalar curvature of M. In 2010, [**8**], the authors proved the following result

THEOREM 1. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 6$ and a, b, f smooth functions on M, $\lambda \in (0,\lambda_*)$, $1<q<2$ such that*
1) $f(x)>0$ on M.
2) At the point $x_\circ$ where f attains its maximum, we suppose, for $n=6$

$$S_g(x_\circ)+3a(x_\circ)>0$$

and for $n>6$

$$\left(\frac{\left(n^2+4n-20\right)}{2(n+2)(n-6)}S_g(x_\circ)+\frac{(n-1)}{(n+2)(n-6)}a(x_\circ)-\frac{1}{8}\frac{\Delta f(x_\circ)}{f(x_\circ)}\right)>0\ .$$

Then the equation

$$\Delta_g^2 u+div_g\left(a(x)\nabla_g u\right)+b(x)u=\lambda\left|u\right|^{q-2}u+f(x)\left|u\right|^{N-2}u$$

admits a non trivial solution of class $C^{4,\alpha}(M)$, $\alpha\in(0,1)$.

Recently, F. Madani [**14**], has considered the Yamabe problem with singularities which he solved under some geometric conditions. The first author in [**7**] considered fourth order elliptic equation with singularities of the form

$$\Delta^2 u-\nabla^i\left(a(x)\nabla_i u\right)+b(x)u=f\left|u\right|^{N-2}u \tag{1}$$

where the functions a and b are in $L^s(M)$, $s>\frac{n}{2}$ and in $L^p(M)$, $p>\frac{n}{4}$ respectively, $N=\frac{2n}{n-4}$ is the Sobolev critical exponent in the embedding $H_2^2(M)\hookrightarrow L^N(M)$. He established the following result. Let (M,g) be a compact n-dimensional Riemannian manifold, $n\geq 6$, $a\in L^s(M)$, $b\in L^p(M)$, with $s>\frac{n}{2}$, $p>\frac{n}{4}$, $f\in C^\infty(M)$ a positive function and $x_o\in M$ such that $f(x_o)=\max_{x\in M}f(x)$.

THEOREM 2. *For $n\geq 10$, or $n=8,9$ and $2<p<5$, $\frac{9}{4}<s<11$ or $n=7$, $\frac{7}{2}<s<9$ and $\frac{7}{4}<p<9$ we suppose that*

$$\frac{n^2+4n-20}{6\left(n-6\right)\left(n^2-4\right)}S_g\left(x_o\right)-\frac{n-4}{2n\left(n-2\right)}\frac{\Delta f(x_o)}{f(x_o)}>0.$$

For $n=6$ and $\frac{3}{2}<p<2$, $3<s<4$, we suppose that

$$S_g(x_o)>0.$$

Then the equation (1) has a non trivial weak solution u in $H_2^2(M)$. Moreover if $a\in H_1^s(M)$, then $u\in C^{0,\beta}(M)$, for some $\beta\in\left(0,1-\frac{n}{4p}\right)$.

In this paper, we are concerned with the following problem: let (M,g) be a Riemannian compact manifold of dimension $n\geq 5$. Let $a\in L^r(M)$, $b\in L^s(M)$ where $r>\frac{n}{2}$, $s>\frac{n}{4}$ and f a positive C^∞-function on M; we look for non trivial solution of the equation

$$\Delta_g^2 u+div_g\left(a(x)\nabla_g u\right)+b(x)u=\lambda\left|u\right|^{q-2}u+f(x)\left|u\right|^{N-2}u \tag{2}$$

where $1<q<2$ and $N=\frac{2n}{n-4}$ is the critical Sobolev exponent and $\lambda>0$ a real number. Our main result states as follows

THEOREM 3. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 6$ and f a positive function. Suppose that P_g is coercive and at a point x_o where f attains its maximum the following conditions*

(C)
$$\begin{cases} \frac{\Delta f(x_o)}{f(x_o)} < \left(\frac{n\left(n^2+4n-20\right)}{3(n+2)(n-4)(n-6)} \frac{1}{\left(1+\|a\|_r+\|b\|_s\right)^{\frac{4}{n}}} - \frac{n-2}{3(n-1)} \right) S_g\left(x_o\right) \ \text{ in case } n>6 \\ S_g(x_\circ) > 0 \ \text{ in case } n=6. \end{cases}$$
are true.

Then there is $\lambda^* > 0$ such that for any $\lambda \in (0,\, \lambda^*)$, the equation (2) has a non trivial weak solution.

For fixed $R \in M$, we define the function ρ on M by

$$\rho(Q) = \begin{cases} d(R,Q) & \text{if} \quad d(R,Q) < \delta(M) \\ \delta(M) & \text{if} \quad d(R,Q) \geq \delta(M) \end{cases} \tag{3}$$

where $\delta(M)$ denotes the injectivity radius of M.

For real numbers σ and μ, consider the equation in the distribution sense

$$\Delta^2 u - \nabla^i\left(\frac{a}{\rho^\sigma}\nabla_i u\right) + \frac{bu}{\rho^\mu} = \lambda\left|u\right|^{q-2}u + f(x)\left|u\right|^{N-2}u \tag{4}$$

where the functions a and b are smooth on M,

COROLLARY 1. *Let $0 < \sigma < \frac{n}{r} < 2$ and $0 < \mu < \frac{n}{s} < 4$. Suppose that*

$$\begin{cases} \frac{\Delta f(x_o)}{f(x_o)} < \frac{1}{3}\left(\frac{(n-1)n\left(n^2+4n-20\right)}{(n^2-4)(n-4)(n-6)} \frac{1}{\left(1+\|a\|_r+\|b\|_s\right)^{\frac{4}{n}}} - 1 \right) S_g\left(x_o\right) \ \text{ in case } n>6 \\ S_g(x_\circ) > 0 \ \text{ in case } n=6. \end{cases}$$

Then there is $\lambda_* > 0$ such that if $\lambda \in (0,\, \lambda_*)$, the equation (4) possesses a weak non trivial solution $u_{\sigma,\mu} \in M_\lambda$.

In the sharp case $\sigma = 2$ and $\mu = 4$, letting $K(n,2,\gamma)$ the best constant in the Hardy-Sobolev inequality given by Theorem 6 we obtain the following result

THEOREM 4. *Let (M,g) be a Riemannian compact manifold of dimension $n \geq 5$. Let $\left(u_{\sigma_m,\mu_m}\right)_m$ be a sequence in M_λ such that*

$$\begin{cases} J_{\lambda,\sigma,\mu}(u_{\sigma_m,\mu_m}) \leq c_{\sigma,\mu} \\ \nabla J_\lambda(u_{\sigma,\mu}) - \mu_{\sigma,\mu}\nabla\Phi_\lambda(u_{\sigma,\mu}) \to 0 \end{cases}.$$

Suppose that

$$c_{\sigma,\mu} < \frac{2}{n\ K_o^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}$$

and

$$1 + a^- \max\left(K(n,2,\sigma), A\left(\varepsilon,\sigma\right)\right) + b^- \max\left(K(n,2,\mu), A\left(\varepsilon,\mu\right)\right) > 0$$

then the equation

$$\Delta^2 u - \nabla^{\mu}(\frac{a}{\rho^2}\nabla_{\mu}u) + \frac{bu}{\rho^4} = f\left|u\right|^{N-2}u + \lambda\left|u\right|^{q-2}u$$

in the distribution has a weak non trivial solution.

2. Existence of solutions

In this section we focus on the existence of solutions to equation (1); we use a variational method so we consider on $H_2^2(M)$ the functional

$$J_{\lambda}(u) = \frac{1}{2}\int_M \left(\left|\Delta_g u\right|^2 - a(x)\left|\nabla_g u\right|^2 + b(x)u^2\right)dv_g - \frac{\lambda}{q}\int_M \left|u\right|^q dv_g - \frac{1}{N}\int_M f(x)\left|u\right|^N dv_g.$$

First, we put

$$\Phi_{\lambda}(u) = \langle \nabla J_{\lambda}(u),\ u\rangle$$

hence

$$\Phi_{\lambda}(u) = \int_M \left((\Delta_g u)^2 - a(x)\left|\nabla_g u\right|^2 + b(x)u^2\right)dv_g - \lambda\int_M \left|u\right|^q dv_g - \int_M f(x)\left|u\right|^N dv_g.$$

We let

$$M_{\lambda} = \left\{u \in H_2^2(M) :\ \Phi_{\lambda}(u) = 0 \text{ and } \|u\| \geq \tau > 0\right\}.$$

The operator $P_g(u)$ is said coercive if there exits $\Lambda > 0$ such that for any $u \in H_2^2(M)$

$$\int_M uP_g(u)dv_g \geq \Lambda\left\|u\right\|^2_{H_2^2(M)}.$$

Proposition 1. *$\|u\| = (\int_M \left|\Delta_g u\right|^2 - a(x)\left|\nabla_g u\right|^2 + b(x)u^2 dv_g)^{\frac{1}{2}}$ is an equivalent norm to the usual one on $H_2^2(M)$ if and only if P_g is coercive.*

Proof. If P_g is coercive there is $\Lambda > 0$ such that for any $u \in H_2^2(M)$,

$$\int_M P_g(u)udv_g \geq \Lambda\left\|u\right\|^2_{H_2^2(M)}$$

and since $a \in L^r(M)$ and $b \in L^s(M)$ where $r > \frac{n}{2}$ and $s > \frac{n}{4}$, by Hölder's inequality we get

$$\int_M uP_g(u)dv_g \leq \left\|\Delta_g u\right\|_2^2 + \left\|a\right\|_{\frac{n}{2}}\left\|\nabla_g u\right\|^2_{2^*} + \left\|b\right\|_{\frac{n}{4}}\left\|u\right\|_N^2$$

where $2^* = \frac{2n}{n-2}$.

The Sobolev's inequalities lead to : for any $\eta > 0$

$$\left\|\nabla_g u\right\|^2_{2^*} \leq \max((1+\eta)K(n,1)^2, A_{\eta})\int_M \left(\left|\nabla_g^2 u\right|^2 + \left|\nabla_g u\right|^2\right)dv_g$$

where $K(n,1)$ denotes the best Sobolev's constant in the embedding $H_1^2(R^n) \hookrightarrow L^{\frac{2n}{n-2}}(R^n)$, and for any $\epsilon > 0$

$$\left\|u\right\|_N^2 \leq \max((1+\varepsilon)K_o, B_{\varepsilon})\left\|u\right\|^2_{H_2^2(M)}$$

where in this latter inequality K_o is the best Sobolev's constant in the embedding $H_1^2(M) \hookrightarrow L^{\frac{2n}{n-2}}(M)$ and B_ϵ the corresponding see (see [**3**]). Now by the well known formula (see [**3**], page 115)

$$\int_M \left|\nabla_g^2 u\right|^2 dv_g = \int_M \left(|\Delta_g u|^2 - R_{ij}\nabla^i u \nabla^j u\right) dv_g$$

where R_{ij} denote the components of the Ricci curvature, there is a constant $\beta > 0$ such that

$$\int_M \left|\nabla_g^2 u\right|^2 dv_g \leq \int_M |\Delta_g u|^2 + \beta \left|\nabla_g u\right|^2 dv_g$$

so we get

$$\|\nabla_g u\|_{2^*}^2 \leq (\beta+1)\max((1+\eta)K(n,1)^2, A_\eta) \int_M \left(|\Delta_g u|^2 + |\nabla_g u|^2 + u^2\right) dv_g$$

and we infer that

$$\int_M P_g(u) u dv_g \leq \|u\|_{H_2^2(M)}^2 + (\beta+1)\|a\|_{\frac{n}{2}} \max((1+\eta)K(n,1)^2, A_\eta)\|u\|_{H_2^2(M)}^2 +$$

$$\|b\|_{\frac{n}{4}} \max((1+\varepsilon)K_o, B_\varepsilon)\|u\|_{H_2^2(M)}^2.$$

Hence

$$\int_M u P_g(u) dv_g \leq \underbrace{\max\left(1, \|b\|_{\frac{n}{4}} \max((1+\varepsilon)K_o, B_\varepsilon), (\beta+1)\|a\|_{\frac{n}{2}} \max((1+\varepsilon)K(n,1)^2, A_\varepsilon)\right)}_{>0} \|u\|_{H_2^2(M)}^2.$$

. □

LEMMA 1. *The set M_λ is non empty provided that $\lambda \in (0, \lambda_\circ)$ where*

$$\lambda_\circ = \frac{\left(2^{q-2} - 2^{q-N}\right)\Lambda^{\frac{N-q}{N-2}}}{V(M)^{(1-\frac{q}{N})}\left(\max_{x\in M} f(x)\right)^{\frac{2-q}{N-2}}\left(\max((1+\varepsilon)K(n,2), A_\varepsilon)\right)^{\frac{N-q}{N-2}}}.$$

PROOF. The proof of Lemma 1 is the same as in ([**8**]), but we give it here for convenience. Let $t > 0$ and $u \in H_2^2(M) - \{0\}$. Evaluating Φ_λ at tu, we get

$$\Phi_\lambda(tu) = t^2\|u\|^2 - \lambda t^q \|u\|_q^q - t^N \int_M f(x)|u|^N dv_g.$$

Put

$$\alpha(t) = \|u\|^2 - t^{N-2}\int_M f(x)|u|^N dv(g)$$

and

$$\beta(t) = \lambda t^{q-2}\|u\|_q^q;$$

by Sobolev's inequality, we get

$$\alpha(t) \geq \|u\|^2 - \max_{x\in M} f(x)(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}}\|u\|_{H_2^2(M)}^N t^{N-2}.$$

By the coercivity of the operator $P_g = \Delta_g^2 - div_g(a\nabla_g) + b$ there is a constant $\Lambda > 0$ such that

$$\alpha(t) \geq \|u\|^2 - \Lambda^{-\frac{N}{2}} \max_{x\in M} f(x)(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}} \|u\|^N t^{N-2}.$$

Letting

$$\alpha_1(t) = \|u\|^2 - \Lambda^{-\frac{N}{2}} \max_{x\in M} f(x)(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}} \|u\|^N t^{N-2}$$

Hölder and Sobolev inequalities lead to

$$\beta(t) \leq \lambda V(M)^{(1-\frac{q}{N})}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q_{H_2^2(M)} t^{q-2}$$

and the coercivity of P_g assures the existence of a constant $\Lambda > 0$ such that

$$\beta(t) \leq \lambda\Lambda^{-\frac{q}{2}} V(M)^{(1-\frac{q}{N})}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q t^{q-2}.$$

Put

$$\beta_1(t) = \lambda\Lambda^{-\frac{q}{2}} V(M)^{(1-\frac{q}{N})}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q t^{q-2}.$$

Let t_o such $\alpha_1(t_o) = 0$ i.e.

$$t_\circ = \frac{\Lambda^{\frac{N}{2(N-2)}}}{\|u\| (\max_{x\in M} f(x))^{\frac{1}{N-2}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2(N-2)}}}$$

Now since $\alpha_1(t)$ is a decreasing and a concave function and $\beta_1(t)$ is a decreasing and convex function, then

$$\min_{t\in(0,\ \frac{t_\circ}{2}]} \alpha_1(t) = \alpha_1(\frac{t_\circ}{2}) = \|u\|^2 (1 - 2^{2-N}) > 0$$

and

$$\min_{t\in(0,\ \frac{t_\circ}{2}]} \beta_1(t) = \beta_1(\frac{t_\circ}{2}) > 0$$

where

$$\beta_1(\frac{t_\circ}{2}) = \frac{2^{2-q}\lambda V(M)^{(1-\frac{q}{N})}\Lambda^{\frac{q-N}{N-2}} \|u\|^2}{(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q-N}{N-2}} (\max_{x\in M} f(x))^{\frac{q-2}{N-2}}}.$$

Consequently $\Phi_\lambda(tu) = 0$ with $t \in (0,\ \frac{t_\circ}{2}]$ has a solution if

$$\min_{t\in(0,\ \frac{t_\circ}{2}]} \alpha_1(t) \geq \max_{t\in(0,\ \frac{t_\circ}{2}]} \beta_1(t)$$

that is to say

$$0 < \lambda < \frac{\left(2^{q-2} - 2^{q-N}\right) (\max_{x\in M} f(x))^{\frac{q-2}{N-2}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q-N}{N-2}}}{\Lambda^{\frac{N-q}{N-2}} V(M)^{(1-\frac{q}{N})}} = \lambda_\circ$$

Let $t_1 \in (0,\ \frac{t_\circ}{2}]$ such that $\Phi_\lambda(t_1 u) = 0$. If we take $u \in H_2^2(M)$ such that $\|u\| \geq \frac{\rho}{t_1}$ and $v = t_1 u$ we get $\Phi_\lambda(v) = 0$ and $\|v\| = t_1 \|u\| \geq \rho$ i.e. $v \in M_\lambda$ provided that $\lambda \in (0,\ \lambda_\circ)$. □

3. Geometric conditions of J_λ

The following lemmas whose proofs are the same as in [**8**] will be useful.

LEMMA 2. *Let (M,g) be a Riemannian compact manifold of dimension $n \geq 5$. For all $u \in M_\lambda$ and all $\lambda \in (0, \min(\lambda_\circ, \lambda_1))$ there is $A > 0$ such that $J_\lambda(u) \geq A > 0$ where*

$$\lambda_1 = \frac{\frac{(N-2)q}{2(N-q)}\Lambda^{\frac{q}{2}}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K(n,2), A_\varepsilon))^{\frac{q}{2}}\tau^{q-2}}.$$

LEMMA 3. *Let (M,g) be a Riemannian compact manifold of dimension $n \geq 5$. The following assertions are true:*
(i) $\langle \nabla\Phi_\lambda(u), u\rangle < 0$ for all $u \in M_\lambda$ and for all $\lambda \in (0, \min(\lambda_\circ, \lambda_1))$.
(ii) The critical points of J_λ are points of M_λ.

4. Existence of non trivial solution in M_λ

In this section, first we show that J_λ satisfies the Palais-Smale condition on M_λ provided that $\lambda > 0$ is sufficiently small.

LEMMA 4. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 5$. Let $(u_m)_m$ be a sequence in M_λ such that*

$$\begin{cases} J_\lambda(u_m) \leq c \\ \nabla J_\lambda(u_m) - \mu_m \nabla\Phi_\lambda(u_m) \to 0 \end{cases}.$$

Suppose that

$$c < \frac{2}{n\ K_o^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}$$

then there is a subsequence $(u_m)_m$ converging strongly in $H_2^2(M)$.

PROOF. Let $(u_m)_m \subset M_\lambda$

$$J_\lambda(u_m) = \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\int_M |u_m|^q\, dv_g$$

As in the proof of Lemma 2, we have

$$J_\lambda(u_m) \geq \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_o, A_\varepsilon))^{\frac{q}{2}}\|u_m\|^q$$

$$J_\lambda(u_m) \geq \|u_m\|^2\,(\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_o, A_\varepsilon))^{\frac{q}{2}}\tau^{q-2}) > 0$$

and since $0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)}\Lambda^{\frac{q}{2}}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K(n,2),A_\varepsilon))^{\frac{q}{2}}\tau^{q-2}}$ and $J_\lambda(u_m) \leq c$, we get

$$c \geq J_\lambda(u_m)$$

$$\geq [\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_o, A_\varepsilon))^{\frac{q}{2}}\tau^{q-2}]\,\|u_m\|^2 > 0$$

so

$$\|u_m\|^2 \leq \frac{c}{\frac{N-2}{2N} - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_o, A_\varepsilon))^{\frac{q}{2}} \tau^{q-2}} < +\infty.$$

$(u_m)_m$ is a bounded in $H_2^2(M)$. By the compactness of the embedding $H_2^2(M) \subset H_p^k(M)$ $(k = 0, 1;\ p < N)$ we get a subsequence still denoted $(u_m)_m$ such that

$u_m \to u$ weakly in $H_2^2(M)$
$u_m \to u$ strongly in $L^p(M)$ where $p < N$
$\nabla u_m \to \nabla u$ strongly in $L^p(M)$ where $p < 2^* = \frac{2n}{n-2}$
$u_m \to u$ a.e. in M.

On the other hand since $\frac{2s}{s-1} < N = \frac{2n}{n-4}$, we obtain

$$\left| \int_M b(x) |u_m - u|^2 \, dv_g \right| \leq \|b\|_s \|u_m - u\|^2_{\frac{2s}{s-1}}$$

$$\leq \|b\|_s \left((K_\circ + \epsilon) \|\Delta (u_m - u)\|_2^2 + A_\epsilon \|u_m - u\|_2^2 \right).$$

Now taking account of

$$K_\circ = \frac{16}{n(n^2-4)(n-4)\omega_n^{\frac{n}{4}}} < 1 \tag{K}$$

we get

$$\int_M b(x) (u_m - u)^2 \, dv_g \leq \|b\|_s \|\Delta (u_m - u)\|_2^2 + o(1).$$

By the same procedure as above we get

$$\int_M a(x) |\nabla (u_m - u)|^2 \, dv_g \leq \|a\|_r \|\Delta (u_m - u)\|_2^2 + o(1).$$

By Brezis-Lieb lemma we write

$$\int_M (\Delta_g u_m)^2 \, dv_g = \int_M (\Delta_g u)^2 \, dv_g + \int_M (\Delta_g (u_m - u))^2 \, dv_g + o(1)$$

and also

$$\int_M f(x) |u_m|^N \, dv_g = \int_M f(x) |u|^N \, dv_g + \int_M f(x) |u_m - u|^N \, dv_g + o(1).$$

Now we claim that $\mu_m \to 0$ as $m \to +\infty$
Testing with u_m we obtain

$$\langle \nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m), u_m \rangle = o(1)$$

$$= \underbrace{\langle \nabla J_\lambda(u_m), u_m \rangle}_{=0} - \mu_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle = o(1)$$

hence

$$\mu_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle = o(1).$$

By Lemma 3, we get $\limsup_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle < 0$ so

$$\mu_m \to 0 \text{ as } m \to +\infty.$$

Our last claim is that $u_m \to u$ strongly in $H_2^2(M)$, indeed

$$J_\lambda(u_m) - J_\lambda(u)$$
$$= \frac{1}{2}\int_M (\Delta_g(u_m-u))^2\, dv_c - \frac{1}{N}\int_M f(x)\,|u_m-u|^N\, dv_g + o(1).$$

Since $u_m - u \to 0$ weakly in $H_2^2(M)$, we test with $\nabla J_\lambda(u_m) - \nabla J_\lambda(u)$

$$\langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u), u_m - u\rangle = o(1)$$

$$= \int_M (\Delta_g(u_m-u))^2\, dv_g - \int_M f(x)\,|u_m-u|^N\, dv_g = o(1) \tag{5}$$

and get

$$\int_M (\Delta_g(u_m-u))^2\, dv_g = \int_M f(x)\,|u_m-u|^N\, dv_g + o(1)$$

and taking account of (5) we obtain

$$J_\lambda(u_m) - J_\lambda(u) = \frac{1}{2}\int_M (\Delta_g(u_m-u))^2\, dv_g - \frac{1}{N}\int_M (\Delta_g(u_m-u))^2\, dv_g + o(1)$$

i.e.

$$J_\lambda(u_m) - J_\lambda(u) = \frac{2}{n}\int_M (\Delta_g(u_m-u))^2\, dv_g + o(1).$$

Independently, by the Sobolev's inequality we have

$$\|u_m-u\|_N^2 \le (1+\varepsilon)K_o \int_M (\Delta_g(u_m-u))^2\, dv_g + o(1). \tag{6}$$

Since

$$\int_M f(x)\,|u_m-u|^N\, dv_g \le \max_{x\in M} f(x)\,\|u_m-u\|_N^N$$

we infer by (6) that

$$\int_M f(x)\,|u_m-u|^N\, dv_g \le (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}}\,\|\Delta_g(u_m-u)\|_2^N + o(1)$$

and appealing equality (5)

$$o(1) \ge \|\Delta_g(u_m-u)\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_o^{\frac{n}{n-4}}\,\|\Delta_g(u_m-u)\|_2^N + o(1)$$

$$\ge \|\Delta_g(u_m-u)\|_2^2 \left(1 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}}\,\|\Delta_g(u_m-u)\|_2^{N-2}\right) + o(1)$$

so if

$$\limsup_{m\to+\infty} \|\Delta_g(u_m-u)\|_2^2 < \frac{1}{K_\circ^{\frac{n}{4}}\left(\max_{x\in M} f(x)\right)^{\frac{n}{4}-1}} \tag{8}$$

then $u_m \to u$ strongly in $H_2^2(M)$. The condition (8) is fulfilled since by Lemma 2 $J_\lambda(u) > 0$ on M_λ with λ is as in Lemma 2 and by hypothesis

$$c \ge J_\lambda(u_m) > (J_\lambda(u_m) - J_\lambda(u)) = \frac{2}{n}\int_M (\Delta_g(u_m-u))^2\, dv_g$$

and

$$c < \frac{2}{n\ K_{\circ}^{\frac{n}{4}} \left(\max_{x\in M} f(x)\right)^{\frac{n}{4}-1}}.$$

It is obvious that

$$\Phi_\lambda(u) = 0 \ \text{ and } \ \|u\| \geq \tau$$

i.e. $u \in M_\lambda$. □

Now we show the existence of a sequence in M_λ satisfying the conditions of Palais-Smale.

LEMMA 5. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$, then there is a couple $(u_m, \mu_m) \in M_\lambda \times R$ such that $\nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m) \to 0$ strongly in $(H_2^2(M))^*$ and $J_\lambda(u_m)$ is bounded provide that $\lambda \in (0, \lambda_*)$ with $\lambda_* = \{\min(\lambda_\circ, \lambda_1), 0\}$.*

PROOF. Since J_λ is Gateau differentiable and by Lemma 1 bounded below on M_λ it follows from Ekeland's principle that there is a couple $(u_m,\ \mu_m) \in M_\lambda \times R$ such that $\nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m) \to 0$ strongly in $(H_2^2(M))'$ and $J_\lambda(u_m)$ is bounded i.e. $(u_m,\ \mu_m)_m$ is a Palais-Smale sequence on M_λ. □

Now we are in position to establish the following generic existence result.

THEOREM 5. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$ and f a positive function. Suppose that P_g is coercive and*

$$c < \frac{2}{n\ K_{o}^{\frac{n}{4}} \left(f(x_\circ)\right)^{\frac{n-4}{4}}}. \tag{C1}$$

Then there is $\lambda^* > 0$ such that for any $\lambda \in (0,\ \lambda^*)$, the equation (2) has a non trivial weak solution.

PROOF. By Lemma 4 and 5 there is $u \in H_2^2(M)$ such that

$$J_\lambda(u) = \min_{\varphi \in M_\lambda} J_\lambda(\varphi).$$

By Lagrange multiplicative theorem there is a real number μ such that for any $\varphi \in H_2^2(M)$

$$\langle \nabla J_\lambda(u), \varphi\rangle = \mu \langle \nabla \Phi_\lambda(u), \varphi\rangle \tag{9}$$

and letting $\varphi = u$ in the equation (9), we get

$$\Phi_\lambda(u) = \langle \nabla J_\lambda(u), u\rangle = \mu \langle \nabla \Phi_\lambda(u), u\rangle .$$

By Lemma 3 we get that $\mu = 0$ and by equation (9), we infer that for any $\varphi \in H_2^2(M)$

$$\langle \nabla J_\lambda(u), \varphi\rangle = 0$$

hence u is weak non trivial solution to equation (2) and since by Lemma 2 critical points of J_λ, we conclude that $u \in M_\lambda$. □

5. Application

Let $P \in M$, we define a function on M by

$$\rho_{_P}(Q) = \begin{cases} d(P,Q) \text{ if } d(P,Q) < \delta(M) \\ \delta(M) \text{ if } d(P,Q) \geq \delta(M) \end{cases} \tag{10}$$

where $\delta(M)$ is the injectivity radius of M. For brevity we denote this function by ρ. The weighted $L^p(M,\rho^\gamma)$ space will be the set of measurable functions u on M such that $\rho^\gamma |u|^p$ are integrable where $p \geq 1$. We endow $L^p(M,\rho^\gamma)$ with the norm

$$\|u\|_{p,\rho} = \left(\int_M \rho^\gamma |u|^p \, dv_g \right)^{\frac{1}{p}}.$$

In this section we will be in need of the following Hardy-Sobolev inequality and Releich-Kondrakov embedding respectively whose proofs are given in ([**7**]).

THEOREM 6. *Let (M,g) be a Riemannian compact manifold of dimension $n \geq 5$ and p, q , γ are real numbers such that $\frac{\gamma}{p} = \frac{n}{q} - \frac{n}{p} - 2$ and $2 \leq p \leq \frac{2n}{n-4}$.*
For any $\epsilon > 0$, there is $A(\epsilon,q,\gamma)$ such that for any $u \in H_2^2(M)$

$$\|u\|^2_{p,\rho^\gamma} \leq (1+\epsilon) K(n,2,\gamma)^2 \|\Delta_g u\|_2^2 + A(\epsilon,q,\gamma) \|u\|_2^2 \tag{11}$$

where $K(n,2,\gamma)$ is the optimal constant.

In case $\gamma = 0$, $K(n,2,0) = K(n,2) = K_o^{\frac{1}{2}}$ is the best constant in the Sobolev's embedding of $H_2^2(M)$ in $L^N(M)$ where $N = \frac{2n}{n-4}$.

THEOREM 7. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 5$ and p, q , γ are real numbers satisfying $1 \leq q \leq p \leq \frac{nq}{n-2q}$, $\gamma < 0$ and $l = 1,2$.*
If $\frac{\gamma}{p} = n\,(\frac{1}{q} - \frac{1}{p}) - l$ then the inclusion $H_l^q(M) \subset L^p(M,\rho^\gamma)$ is continuous.
If $\frac{\gamma}{p} > n\,(\frac{1}{q} - \frac{1}{p}) - l$ then inclusion $H_l^q(M) \subset L^p(M,\rho^\gamma)$ is compact.

We consider the following equation

$$\Delta_g^2 u + div_g \left(\frac{a(x)}{\rho^\sigma} \nabla_g u \right) + \frac{b(x)}{\rho^\mu} u = \lambda |u|^{q-2} u + f(x) |u|^{N-2} u \tag{12}$$

where a and b are smooth function and ρ denotes the distance function defined by (10), $\lambda > 0$ in some interval $(0,\lambda_*)$, $1 < q < 2$, σ, μ will be precise later and we associate to (12) on $H_2^2(M)$ the functional

$$J_\lambda(u) = \frac{1}{2} \int_M \left((\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma} |\nabla_g u|^2 + \frac{b(x)}{\rho^\mu} u^2 \right) dv_g - \frac{\lambda}{q} \int_M |u|^q \, dv_g - \frac{1}{N} \int_M f(x) |u|^N \, dv_g.$$

If we put

$$\Phi_\lambda(u) = \langle \nabla J_\lambda(u),\ u \rangle$$

we get

$$\Phi_\lambda(u)=\int_M (\Delta_g u)^2-\frac{a(x)}{\rho^\sigma}\left|\nabla_g u\right|^2+\frac{b(x)}{\rho^\mu}u^2 dv_g-\lambda\int_M |u|^q\, dv_g-\int_M f(x)\,|u|^N\, dv_g.$$

THEOREM 8. *Let $0<\sigma<\frac{n}{s}<2$ and $0<\mu<\frac{n}{p}<4$. Suppose that*

$$\sup_{u\in H_2^2(M)} J_{\lambda,\sigma,\mu}(u)<\frac{2}{n\ K_o^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}$$

then there is $\lambda_*>0$ such that if $\lambda\in(0,\ \lambda_*)$, the equation (12) possesses a weak non trivial solution $u_{\sigma,\mu}\in M_\lambda$.

PROOF. Let $\tilde{a}=\frac{a(x)}{\rho^\sigma}$ and $\tilde{b}=\frac{b(x)}{\rho^\mu}$, so if $\sigma\in(0,\min\left(2,\frac{n}{s}\right))$ and $\mu\in(0,\min(4,\frac{n}{p}))$, obliviously $\tilde{a}\in L^s(M)$, $\tilde{b}\in L^p(M)$, where $s>\frac{n}{2}$ and $p>\frac{n}{4}$ .Theorem 8 is a consequence of Theorem 5. □

6. The critical cases $\sigma=2$ and $\mu=4$

By section four, for any $\sigma\in(0,\min\left(2,\frac{n}{s}\right))$ and $\mu\in(0,\min(4,\frac{n}{p}))$, there is a solution $u_{\sigma,\mu}\in M_\lambda$ of equation (2). Now we are going to show that the sequence $\left(u_{\sigma,\mu}\right)_{\sigma,\mu}$ is bounded in $H_2^2(M)$. Evaluating $J_{\lambda,\sigma,\mu}$ at $u_{\sigma,\mu}$

$$J_{\lambda,\sigma,\mu}(u_{\sigma,\mu})=\frac{1}{2}\left\|u_{\sigma,\mu}\right\|^2-\frac{1}{N}\int_M f(x)\left|u_{\sigma,\mu}\right|^N dv_g-\frac{1}{q}\lambda\int_M |u_{\sigma,\mu}|^q\, dv_g$$

and taking account of $u_{\sigma,\mu}\in M_\lambda$, we infer that

$$J_{\lambda,\sigma,\mu}(u_{\sigma,\mu})=\frac{N-2}{2N}\left\|u_{\sigma,\mu}\right\|^2-\lambda\frac{N-q}{Nq}\int_M |u_{\sigma,\mu}|^q\, dv_g.$$

For a smooth function a on M, denotes by $a^-=\min\left(0,\min_{x\in M}(a(x)\right)$. Let $K(n,2,\sigma)$ the best constant and $A\left(\varepsilon,\sigma\right)$ the corresponding constant in the Hardy- Sobolev inequality given in Theorem 6.

THEOREM 9. *Let (M,g) be a Riemannian compact manifold of dimension $n\geq 5$. Let $(u_m)_m=\left(u_{\sigma_m,\mu_m}\right)_m$ be a sequence in M_λ such that*

$$\begin{cases} J_{\lambda,\sigma,\mu}(u_m)\leq c_{\sigma,\mu} \\ \nabla J_\lambda(u_m)-\mu_{_{\sigma,\mu}}\nabla\Phi_\lambda(u_m)\to 0 \end{cases}.$$

Suppose that

$$c_{\sigma,\mu}<\frac{2}{n\ K\left(n,2\right)^{\frac{n}{4}}\left(\max_{x\in M} f(x)\right)^{\frac{n-4}{4}}}$$

and

$$1+a^-\max\left(K(n,2,\sigma),A\left(\varepsilon,\sigma\right)\right)+b^-\max\left(K(n,2,\mu),A\left(\varepsilon,\mu\right)\right)>0$$

then the equation

$$\Delta^2 u-\nabla^\mu\left(\frac{a}{\rho^2}\nabla_\mu u\right)+\frac{bu}{\rho^4}=f\left|u\right|^{N-2}u+\lambda\left|u\right|^{q-2}u$$

has a non trivial solution in the distribution.

PROOF. Let $(u_m)_m \subset M_{\lambda,\sigma,\mu}$,

$$J_{\lambda,\sigma,\mu}(u_m) = \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\int_M |u_m|^q \, dv_g$$

As in proof of Theorem 5, we get

$$J_{\lambda,\sigma,\mu}(u_m) \geq \|u_m\|^2 \left(\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda_{\sigma,\mu}^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} \left(\max((1+\varepsilon)K(n,2), A_\varepsilon)\right)^{\frac{q}{2}} \tau^{q-2}\right) > 0$$

where $0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)}\Lambda_{\sigma,\mu}^{\frac{q}{2}}}{V(M)^{1-\frac{q}{N}}\left(\max((1+\varepsilon)K(n,2),A_\varepsilon)\right)^{\frac{q}{2}}\tau^{q-2}}$.

First we claim that

$$\lim_{(\sigma,\mu)\to(2^-,4^-)} \inf \Lambda_{\sigma,\mu} > 0.$$

Indeed, if $\nu_{1,\sigma,\mu}$ denotes the first nonzero eigenvalue of the operator $P_g = \Delta_g^2 - div\left(\frac{a}{\rho^\sigma}\nabla_g\right) + \frac{b}{\rho^\mu}$, then clearly $\Lambda_{\sigma,\mu} \geq \nu_{1,\sigma,\mu}$. Suppose by absurd that $\lim_{(\sigma,\mu)\to(2^-,4^-)} \inf \Lambda_{\sigma,\mu} = 0$, then $\liminf_{(\sigma,\mu)\to(2^-,4^-)} \nu_{1,\sigma,\mu} = 0$. Independently, if $u_{\sigma,\mu}$ is the corresponding eigenfunction to $\nu_{1,\sigma,\mu}$ we have

$$\nu_{1,\sigma,\mu} = \|\Delta u\|_2^2 + \int_M \frac{a\,|\nabla u|^2}{\rho^\sigma} dv_g + \int_M \frac{bu^2}{\rho^\mu} dv_g$$

$$\geq \|\Delta u\|_2^2 + a^- \int \frac{|\nabla u|^2}{\rho^\sigma} dv_g + b^- \int_M \frac{u^2}{\rho^\mu} dv_g \tag{13}$$

where $a^- = \min(0, \min_{x\in M} a(x))$ and $b^- = \min(0, \min_{x\in M} b(x))$. The Hardy- Sobolev's inequality given by Theorem 6 leads to

$$\int_M \frac{|\nabla u|^2}{\rho^\sigma} dv_g \leq C(\|\nabla|\nabla u|\|^2 + \|\nabla u\|^2)$$

and since

$$\|\nabla|\nabla u|\|^2 \leq \left\|\nabla^2 u\right\|^2 \leq \|\Delta u\|^2 + \beta\|\nabla u\|^2$$

where $\beta > 0$ is a constant and it is well known that for any $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that

$$\|\nabla u\|^2 \leq \varepsilon\|\Delta u\|^2 + c\|u\|^2.$$

Hence

$$\int_M \frac{|\nabla u|^2}{\rho^\sigma} dv_g \leq C(1+\varepsilon)\|\Delta u\|^2 + A(\varepsilon)\|u\|^2 \tag{14}$$

Now if $K(n,2,\sigma)$ denotes the best constant in inequality (14) we get for any $\varepsilon > 0$

$$\int_M \frac{|\nabla u|^2}{\rho^\sigma} dv_g \leq \left(K(n,2,\sigma)^2 + \varepsilon\right)\|\Delta u\|^2 + A(\varepsilon,\sigma)\|u\|^2. \tag{15}$$

By the inequalities (11), (13) and (15), we have

$$\nu_{1,\sigma,\mu} \geq \left(1 + a^- \max(K(n,2,\sigma), A(\varepsilon,\sigma)) + b^- \max(K(n,2,\mu), A(\varepsilon,\mu))\right)\left(\|\Delta u_{\sigma,\mu}\|^2 + \|u_{\sigma,\mu}\|^2\right)$$

So if

$$1+a^{-}\max\left(K(n,2,\sigma),A\left(\varepsilon,\sigma\right)\right)+b^{-}\max\left(K(n,2,\mu),A\left(\varepsilon,\mu\right)\right)>0$$

then we get $\lim_{\sigma,\mu}\left(u_{\sigma,\mu}\right)=0$ and $\left\|u_{\sigma,\mu}\right\|=1$ a contradiction.
The reflexivity of $H_2^2(M)$ and the compactness of the embedding $H_2^2(M)\subset H_p^k(M)$ ($k\ =0{,}1$; $p<N$), imply that up to a subsequence we have

$u_m\rightarrow u$ weakly in $H_2^2(M)$
$u_m\rightarrow u$ strongly in $L^p(M)$, $p<N$
$\nabla u_m\rightarrow\nabla u$ strongly in $L^p(M)$, $p<2^*=\frac{2n}{n-2}$
$u_m\rightarrow u$ a.e. in M.

The Brézis-Lieb lemma allows us to write

$$\int_M\left(\Delta_g u_m\right)^2dv_g=\int_M\left(\Delta_g u\right)^2dv_g+\int_M\left(\Delta_g(u_m-u)\right)^2dv_g+o(1)$$

and also

$$\int_M f(x)\left|u_m\right|^N dv_g=\int_M f(x)\left|u\right|^N dv_g+\int_M f(x)\left|u_m-u\right|^N dv_g+o(1).$$

Now by the boundedness of the sequence $(u_m)_m$, we have that $u_m\rightarrow u$ weakly in $H_2^2(M)$,

$\nabla u_m\rightarrow\nabla u$ weakly in $L^2(M,\rho^{-2})$ and $u_m\rightarrow u$ weakly in $L^2(M,\rho^{-4})$ i.e. for any $\varphi\in L^2(M)$

$$\int_M\frac{a(x)}{\rho^2}\nabla u_m\nabla\varphi dv_g=\int_M\frac{a(x)}{\rho^2}\nabla u\nabla\varphi dv_g+o(1)$$

and

$$\int_M\frac{b(x)}{\rho^4}u_m\varphi dv_g=\int_M\frac{b(x)}{\rho^4}u\varphi dv_g+o(1).$$

For every $\phi\in H_2^2(M)$ we have

(16)

$$\int_M\left(\Delta_g^2u_m+div_g\left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m\right)+\frac{b(x)}{\rho^{\delta_m}}u_m\right)\phi dv_g=\int_M\left(\lambda\left|u_m\right|^{q-2}u_m+f(x)\left|u_m\right|^{N-2}u_m\right)\phi dv_g.$$

By the weak convergence in $H_2^2(M)$ we have immediately

$$\int_M\phi\Delta_g^2u_m dv_g=\int_M\phi\Delta_g^2u dv_g+o(1)$$

and

$$\int_M\left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m-\frac{a(x)}{\rho^2}\nabla_g u\right)\phi dv_g=\int_M\left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m+\frac{a(x)}{\rho^2}\left(\nabla_g u_m-\nabla_g u_m\right)-\frac{a(x)}{\rho^2}\nabla_g u\right)\phi dv_g$$

Then

$$\left|\int_M\left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m-\frac{a(x)}{\rho^2}\nabla_g u\right)\phi dv_g\right|\leq$$

$$\left|\int_M\left(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m-\frac{a(x)}{\rho^2}\nabla_g u_m\right)\phi dv_g\right|+\left|\int_M\left(\frac{a(x)}{\rho^2}\nabla_g u_m-\frac{a(x)}{\rho^2}\nabla_g u\right)\phi dv_g\right|$$

$$\leq \int_M |a(x)\phi\nabla_g u_m| \left|\frac{1}{\rho^{\sigma_m}} - \frac{1}{\rho^2}\right| dv_g + \left|\int_M \frac{a(x)}{\rho^2}\nabla_g\left(u_m - u\right)\phi dv_g\right|. \tag{17}$$

The weak convergence in $L^2(M,\rho^{-2})$ and the Lebesgue's dominated convergence theorem imply that the second right hand side of (17) goes to 0. For the third term of the left hand side of (15), we write

$$\int_M \left(\frac{b(x)}{\rho^{\delta_m}}u_m - \frac{b(x)}{\rho^4}u\right)\phi dv_g = \int_M \left(\frac{b(x)}{\rho^{\delta_m}}u_m - \frac{b(x)}{\rho^4}u_m + \frac{b(x)}{\rho^4}u_m - \frac{b(x)}{\rho^4}u\right)\phi dv_g$$

and

$$\left|\int_M \left(\frac{b(x)}{\rho^{\delta_m}}u_m - \frac{b(x)}{\rho^4}u\right)\phi dv_g\right|$$

$$\leq \int_M |b(x)\phi u_m| \left|\frac{1}{\rho^{\delta_m}} - \frac{1}{\rho^4}\right| dv_g + \left|\int_M \frac{b(x)}{\rho^4}\left(u_m - u\right)\phi dv_g\right|. \tag{18}$$

Here also the weak convergence in $L^2(M,\rho^{-4})$ and the Lebesgue's dominated convergence allows us to affirm that the left hand side of (18) converges to 0.

It remains to show that $\mu_m \to 0$ as $m \to +\infty$ and $u_m \to u$ strongly in $H_2^2(M)$ but this is the same as in the proof of Theorem 5 which implies also $u \in M_\lambda$. □

7. Test Functions

In this section, we give the proof of the main theorem to do so, we consider a normal geodesic coordinate system centred at x_o. Denote by $S_{x_o}(\rho)$ the geodesic sphere centred at x_o and of radius ρ ($\rho < d$ =the injectivity radius). Let $d\Omega$ be the volume element of the $n-1$-dimensional Euclidean unit sphere S^{n-1} and put

$$G(\rho) = \frac{1}{\omega_{n-1}} \int\limits_{S(\rho)} \sqrt{|g(x)|}d\Omega$$

where ω_{n-1} is the volume of S^{n-1} and $|g(x)|$ the determinant of the Riemannian metric g. The Taylor's expansion of $G(\rho)$ in a neighborhood of x_o is given by

$$G(\rho) = 1 - \frac{S_g(x_\circ)}{6n}\rho^2 + o(\rho^2)$$

where $S_g(x_\circ)$ denotes the scalar curvature of M at $x_\circ$.
Let $B(x_\circ,\delta)$ be the geodesic ball centred at $x_\circ$ and of radius δ such that $0 < 2\delta < d$ and denote by η a smooth function on M such that

$$\eta(x) = \begin{cases} 1 & \text{on } B(x_o,\delta) \\ 0 & \text{on } M - B(x_o,2\delta) \end{cases}.$$

Consider the following radial function

$$u_\epsilon(x) = \left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{8}} \frac{\eta(\rho)}{\left((\rho\theta)^2+\epsilon^2\right)^{\frac{n-4}{2}}}$$

with

$$\theta = (1 + \|a\|_r + \|b\|_s)^{\frac{1}{n}}$$

where $\rho = d(x_o, x)$ is the distance from x_o to x and $f(x_\circ) = \max_{x\in M} f(x)$. For further computations we need the following integrals: for any real positive numbers p, g such that $p - q > 1$ we put

$$I_p^q = \int_0^{+\infty} \frac{t^q}{(1+t)^p} dt$$

and the following relations are immediate

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q \quad \text{and} \quad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q.$$

8. Application to compact Riemannian manifolds of dimension $n > 6$

THEOREM 10. *Let (M, g) be a compact Riemannian manifold of dimension $n > 6$. Suppose that at a point x_o where f attains its maximum the following condition*

$$\frac{\Delta f(x_o)}{f(x_o)} < \frac{1}{3}\left(\frac{(n-1)n\left(n^2+4n-20\right)}{\left(n^2-4\right)(n-4)(n-6)} \frac{1}{(1+\|a\|_r + \|b\|_s)^{\frac{4}{n}}} - 1\right) S_g(x_o)$$

holds . Then the equation (1) has a non trivial solution with energy

$$J_\lambda(u) < \frac{1}{K_\circ^{\frac{n}{4}} \left(\max_{x\in M} f(x)\right)^{\frac{n}{4}-1}}.$$

PROOF. The proof of Theorem 10 reduces to show that the condition (C1) of Theorem 5 is satisfied and since by Lemma 1 there is a $t_o > 0$ such that $t_o u_\epsilon \in M_\lambda$ for sufficiently small λ, so it suffices to show that

$$\sup_{t>0} J_\lambda(t u_\epsilon) < \frac{1}{K_\circ^{\frac{n}{4}} \left(\max_{x\in M} f(x)\right)^{\frac{n}{4}-1}}.$$

To compute the term $\int_M f(x) |u_\epsilon(x)|^N dv_g$, we need the following Taylor's expansion of f at the point x_o

$$f(x) = f(x_\circ) + \frac{\partial^2 f(x_\circ)}{2\partial y^i \partial y^j} y^i y^j + o(\rho^2)$$

and also that of the Riemannian measure

$$dv_g = 1 - \frac{1}{6} R_{ij}(x_\circ) y^i y^j + o(\rho^2)$$

where $R_{ij}(x_\circ)$ denotes the Ricci tensor at $x_\circ$. The expression of $\int_M f(x) |u_\epsilon(x)|^N dv_g$ is well known (see for example [**11**]) and is given in case $n > 6$ by

$$\int_M f(x) |u_\epsilon(x)|^N dv_g = \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \left(1 - \left(\frac{\Delta f(x_\circ)}{2(n-2) f(x_\circ)} + \frac{S_g(x_\circ)}{6(n-2)}\right)\epsilon^2 + o(\epsilon^2)\right)$$

where K_o is given by (K) and $\omega_n = 2^{n-1} I_n^{\frac{n}{2}-1} \omega_{n-1}$ and ω_n is the volume of S^n, the standard unit sphere of R^{n+1} endowed with its round metric.

Now the restriction of $\left|\frac{\partial u_\epsilon}{\partial \rho}\right|$ to the geodesic ball $B(x_\circ, \delta)$ is computed as follows

$$\left|\frac{\partial u_\epsilon}{\partial \rho}\right|_{B(x_\circ,\delta)} = |\nabla u_\epsilon| = \theta^{-2}(n-4)\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{8}} \frac{\rho}{\left(\left(\frac{\rho}{\theta}\right)^2+\epsilon^2\right)^{\frac{n-2}{2}}}$$

and Since $a \in L^r(M)$ with $r > \frac{n}{2}$ we have

$$\int_{B(x_\circ,\delta)} a(x)\,|\nabla u_\epsilon|^2\, dv_g \leq \theta^{-4}(n-4)^2\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}} \|a\|_r\, \omega_{n-1}^{1-\frac{1}{r}}$$

$$\times \left(\int_0^\delta \frac{\rho^{\frac{2r}{r-1}+n-1}}{\left(\left(\frac{\rho}{\theta}\right)^2+\epsilon^2\right)^{\frac{(n-2)r}{r-1}}}\left(\int_{S(\rho)} \sqrt{|g(x)|}d\Omega\right) d\rho\right)^{\frac{r-1}{r}}$$

Since

$$\int\limits_{S(\rho)} \sqrt{|g(x)|}d\Omega = \omega_{n-1}\left(1 - \frac{S_g(x_\circ)}{6n}\rho^2 + o(\rho^2)\right)$$

we get

$$\int_{B(x_\circ,\delta)} a(x)\,|\nabla u_\epsilon|^2\, dv_g \leq \theta^{-4}(n-4)^2\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}} \|a\|_r\, \omega_{n-1}^{1-\frac{1}{r}}$$

$$\times \left(\int_0^\delta \frac{\rho^{\frac{2r}{r-1}+n-1}}{\left((\rho\theta)^2+\epsilon^2\right)^{\frac{(n-2)r}{r-1}}} d\rho\left(1 - \frac{S_g(x_\circ)}{6n}\rho^2 + o(\rho^2)\right)\right)^{\frac{r-1}{r}}$$

and by the following change of variable

$$t = (\frac{\rho\theta}{\epsilon})^2 \text{ i.e. } \rho = \frac{\epsilon}{\theta}\sqrt{t}$$

we obtain

$$\int_{B(x_\circ,\delta)} a(x)\,|\nabla u_\epsilon|^2\, dv_g \leq \theta^{-n\frac{r}{r-1}}(n-4)^2\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}} \|a\|_r\, \omega_{n-1}^{1-\frac{1}{r}}\epsilon^{-(n-4)+2-\frac{n}{r}}$$

$$\times \left(\int_0^{(\frac{\delta\theta}{\epsilon})^2} \frac{t^{\frac{n-2}{2}+\frac{r}{r-1}}}{(t+1)^{\frac{(n-2)r}{r-1}}} dt - \frac{S_g(x_\circ)}{6n}\theta^{-2}\epsilon^2 \int_0^{(\frac{\delta\theta}{\epsilon})^2} \frac{t^{\frac{n}{2}+\frac{r}{r-1}}}{(t+1)^{\frac{(n-2)r}{r-1}}} dt + o(\epsilon^2)\right)^{\frac{r-1}{r}}$$

letting $\epsilon \to 0$ we get

$$\int_{B(x_\circ,\delta)} a(x)\,|\nabla u_\epsilon|^2\, dv_g \leq 2^{-1+\frac{1}{r}}\theta^{-n\left(1-\frac{1}{r}\right)}(n-4)^2\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}} \|a\|_r\, \omega_{n-1}^{1-\frac{1}{r}}\epsilon^{-(n-4)+2-\frac{n}{r}}$$

$$\times \left(I_{\frac{(n-2)r}{r-1}}^{\frac{n-2}{2}+\frac{r}{r-1}} - \theta^{-2}\frac{S_g(x_\circ)}{6n} I_{\frac{(n-2)r}{r-1}}^{\frac{n}{2}+\frac{r}{r-1}}\epsilon^2 + o(\epsilon^2)\right)^{\frac{r-1}{r}}$$

Then

$$\int_{B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv_g \le 2^{-1+\frac{1}{r}}\theta^{-n\frac{r}{r-1}}(n-4)^2\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}}\left\|a\right\|_r\omega_{n-1}^{1-\frac{1}{r}}\epsilon^{\epsilon^{-(n-4)+2-\frac{n}{r}}}$$
$$\times I_{\frac{(n-2)r}{r-1}}^{1+\frac{n-2}{2}\cdot\frac{r-1}{r}}\left[1-\frac{r-1}{r}\theta^2\frac{S_g(x_\circ)}{6n}I_{\frac{(n-2)r}{r-1}}^{\frac{n}{2}+\frac{r}{r-1}}I_{\frac{(n-2)r}{r-1}}^{-\frac{n-2}{2}-\frac{r}{r-1}}\epsilon^2+o(\epsilon^2)\right].$$

It remains to compute the integral $\int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv_g$.
First we remark that

$$\left|\int_{(\frac{\delta\theta}{\epsilon})^2}^{(\frac{2\delta\theta}{\epsilon})^2} h(t)\frac{t^q}{(t+1)^p}dt\right| \le C\left(\frac{1}{\epsilon}\right)^{2(q-p+1)} = C\epsilon^{2(p-q-1)}$$

and since $p-q=n-4\ge 3$, we obtain

$$\int_{(\frac{\delta\theta}{\epsilon})^2}^{(\frac{2\delta\theta}{\epsilon})^2} h(t)\frac{i^q}{(t+1)^p}dt = o(\epsilon^2)$$

and then

$$\int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} a(x)\left|\nabla u_\epsilon\right|^2 dv_g = o(\epsilon^2). \tag{19}$$

Finally we get

$$\int_M a(x)\left|\nabla u_\epsilon\right|^2 dv_g \le 2^{-1+\frac{1}{r}}\theta^{-n\frac{r}{r-1}}(n-4)^2\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}}\left\|a\right\|_r\omega_{n-1}^{1-\frac{1}{r}}\epsilon^{-(n-4)+2-\frac{n}{r}}$$
$$\times\left(I_{\frac{(n-2)r}{r-1}}^{1+\frac{n-2}{2}\cdot\frac{r-1}{r}}+o(\epsilon^2)\right).$$

Letting

$$A = K_\circ^{\frac{n}{4}}\frac{(n-4)^{\frac{n}{4}+1}\times(\omega_{n-1})^{\frac{r-1}{r}}}{2^{\frac{r-1}{r}}}(n(n^2-4))^{\frac{n-4}{4}}\left(I_{\frac{(n-2)r}{r-1}}^{\frac{n-2}{2}+\frac{r}{r-1}}\right)^{\frac{r-1}{r}} \tag{20}$$

we obtain

$$\int_M a(x)\left|\nabla u_\epsilon\right|^2 dv_g \le \epsilon^{2-\frac{n}{r}}\theta^{-n\frac{r}{r-1}}\frac{A}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left\|a\right\|_r\left(1+o(\epsilon^2)\right).$$

Now we compute

$$\int_M b(x)u_\epsilon^2 dv_g = \int_{B(x_\circ,\delta)} b(x)u_\epsilon^2 dv_g + \int_{B(x_\circ,2\delta)-B(x_\circ,\delta)} b(x)u_\epsilon^2 dv_g$$

and since $b\in L^s(M)$ with $s>\frac{n}{4}$, we have

$$\int_M b(x)u_\epsilon^2 dv_g \le \left\|b\right\|_s\left\|u_\epsilon\right\|_{\frac{2s}{s-1}}^2.$$

Independently

$$= \left(\frac{(n-4)\,n\left(n^2-4\right)\epsilon^4}{f(x_o)}\right)^{\frac{n-4}{4}}$$

$$\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_o,\delta)} = \left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}} \left(\int_0^\delta \frac{\rho^{n-1}}{\left((\rho\theta)^2+\epsilon^2\right)^{\frac{(n-4)s}{(s-1)}}} \left(\int_{S(r)} \sqrt{|g(x)|}d\Omega\right) dr\right)^{\frac{s-1}{s}}$$

and

$$\int\limits_{S(r)} \sqrt{|g(x)|}d\Omega = \omega_{n-1}\left(1 - \frac{S_g(x_\circ)}{6n}\rho^2 + o(\rho^2)\right).$$

consequently

$$\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_o,\delta)} = \left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}} \omega_{n-1}^{\frac{s-1}{s}} \times$$

$$\left(\int_0^\delta \frac{\rho^{n-1}}{\left((\rho\theta)^2+\epsilon^2\right)^{\frac{(n-4)s}{(s-1)}}} \left(1 - \frac{S_g(x_\circ)}{6n}\rho^2 + o(\rho^2)\right) d\rho\right)^{\frac{s-1}{s}}.$$

And putting $t = (\frac{\rho\theta}{\epsilon})^2$, we get

$$\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_o,\delta)} = \left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}} (\omega_{n-1})^{\frac{s-1}{s}} \epsilon^{-n+4+4-\frac{n}{s}} \times$$

$$\left(\frac{\epsilon^n\theta^{-n}}{2}\int_0^{(\frac{\delta\theta}{\epsilon})^2} \frac{t^{\frac{n}{2}-1}}{(t+1)^{\frac{(n-4)s}{(s-1)}}}dt - \frac{\theta^{-n-2}S_g(x_\circ)}{12n}\epsilon^{n+2}\int_0^{(\frac{\delta\theta}{\epsilon})^2} \frac{t^{\frac{n}{2}}}{(t+1)^{\frac{(n-4)s}{(s-1)}}}dt + o(\epsilon^{n+2})\right)^{\frac{s-1}{s}}.$$

Letting $\epsilon \to 0$, we get

$$\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_o,\delta)} = \left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}} (\omega_{n-1})^{\frac{s-1}{s}} \epsilon^{-n+4+4-\frac{n}{s}}$$

$$\times\theta^{-n\frac{s}{s-1}}(\frac{\epsilon^n}{2})^{\frac{s-1}{s}} \left(\int_0^{+\infty} \frac{t^{\frac{n}{2}}}{(t+1)^{\frac{(n-4)s}{(s-1)}}}dt - \frac{S_g(x_\circ)}{12n}\epsilon^2\theta^{-2}\int_0^{+\infty} \frac{t^{\frac{n}{2}+1}}{(t+1)^{\frac{(n-4)s}{(s-1)}}}dt + o(\epsilon^2)\right)^{\frac{s-1}{s}}.$$

Hence

$$\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_o,\delta)} = \left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}} (\omega_{n-1})^{\frac{s-1}{s}} \epsilon^{-n+4+4-\frac{n}{s}}$$

$$\times\theta^{-n\frac{s}{s-1}}(\frac{\epsilon^n}{2})^{\frac{s-1}{s}} \left(\int_0^{+\infty} \frac{t^{\frac{n}{2}}}{(t+1)^{\frac{(n-4)s}{(s-1)}}}dt - \theta^{-2}\frac{S_g(x_\circ)}{12n}\epsilon^2\int_0^{+\infty} \frac{t^{\frac{n}{2}+1}}{(t+1)^{\frac{(n-4)s}{(s-1)}}}dt + o(\epsilon^2)\right)^{\frac{s-1}{s}}.$$

Or

$$\|u_\epsilon\|^2_{\frac{2s}{s-1}} = \left(\frac{(n-4)n(n^2-4)}{f(x_\circ)}\right)^{\frac{n-4}{4}} \left(\frac{\omega_{n-1}}{2}\right)^{\frac{s-1}{s}} \epsilon^{4-\frac{n}{s}}\theta^{-n\frac{s}{s-1}}$$

$$\times\left[\left(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}}\right)^{\frac{s-1}{s}} - \frac{\theta^{-2}(s-1)S_g(x_\circ)}{12n\ s}\left(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}}\right)^{-\frac{1}{s}} I^{\frac{n}{2}+1}_{\frac{(n-4)s}{(s-1)}}\epsilon^2 + o(\epsilon^2)\right]$$

Finally, by the same manner as in equality (19) we get

$$\int_M b(x)u_\epsilon^2 dv_g \leq \|b\|_s \left(\frac{(n-4)n(n^2-4)}{f(x_\circ)}\right)^{\frac{n-4}{4}} (\frac{\omega_{n-1}}{2})^{\frac{s-1}{s}} \epsilon^{4-\frac{n}{s}} \theta^{-n\frac{s}{s-1}} \times \left(\left(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}}\right)^{\frac{s-1}{s}} + o(\epsilon^2)\right)$$

Putting

$$B = K_\circ^{\frac{n}{4}}((n-4)n(n^2-4))^{\frac{n-4}{4}}(\frac{\omega_{n-1}}{2})^{\frac{s-1}{s}} \left(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}}\right)^{\frac{s-1}{s}} \tag{21}$$

we get

$$\int_M b(x)u_\epsilon^2 dv_g \leq \epsilon^{4-\frac{n}{s}}\theta^{-n\frac{s}{s-1}} \frac{\|b\|_s B}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}} \left(1+o(\epsilon^2)\right).$$

The computation of $\int_M (\Delta u_\epsilon)^2 dv_g$ is well known see for example ([**11**]) and is given by

$$\int_M (\Delta u_\epsilon)^2 dv_g = \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}} \left(1 - \frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_\circ)\epsilon^2 + o(\epsilon^2)\right).$$

Resuming we get

$$\int_M (\Delta u_\epsilon)^2 - a(x)|\nabla u_\epsilon|^2 + b(x)u_\epsilon^2 dv_g \leq \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}} f(x_\circ)^{\frac{n-4}{4}}} \times$$

$$\left(1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}} A\|a\|_r + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}} B\|b\|_s - \frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_\circ)\epsilon^2 + o(\epsilon^2)\right).$$

Now, we have

$$J_\lambda(tu_\epsilon) \leq J_o(tu_\epsilon) = \frac{t^2}{2}\|u_\epsilon\|^2 - \frac{t^N}{N}\int_M f(x)|u_\epsilon(x)|^N dv_g$$

$$\leq \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}} f(x_\circ)^{\frac{n-4}{4}}} \left\{\frac{1}{2}t^2\left(1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}} A\|a\|_r + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}} B\|b\|_s\right) - \frac{t^N}{N}\right.$$

$$\left. + \left[\left(\frac{\Delta f(x_o)}{2(n-2)f(x_o)} + \frac{S_g(x_o)}{6(n-1)}\right)\frac{t^N}{N} - \frac{1}{2}t^2\frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_o)\right]\epsilon^2\right\}$$

$$+ o\left(\epsilon^2\right)$$

and letting ϵ small enough so that

$$1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}} A\|a\|_r + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}} B\|b\|_s \leq (1 + \|a\|_r + \|b\|_s)^{\frac{4}{n}}$$

and since the function $\varphi(t) = \alpha\frac{t^2}{2} - \frac{t^N}{N}$, with $\alpha > 0$ and $t > 0$, attains its maximum at $t_o = \alpha^{\frac{1}{N-2}}$ and

$$\varphi(t_o) = \frac{2}{n}\alpha^{\frac{n}{4}}.$$

Consequently, we get

$$J_\lambda(tu_\epsilon) \leq \frac{2\theta^{-n}}{nK_\circ^{\frac{n}{4}} f(x_\circ)^{\frac{n-4}{4}}} \{1 + \|a\|_r + \|b\|_s$$
$$+ \left[\left(\frac{\Delta f(x_o)}{2(n-2)f(x_o)} + \frac{S_g(x_o)}{6(n-1)}\right)\frac{t_o^N}{N} - \frac{1}{2}t_o^2\frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_o)\right]\epsilon^2\Big\}$$
$$+o\left(\epsilon^2\right).$$

Taking account of the value of θ and putting

$$R(t) = \left(\frac{\Delta f(x_o)}{2(n-2)f(x_o)} + \frac{S_g(x_o)}{6(n-1)}\right)\frac{t^N}{N} - \frac{1}{2}\frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_o)\,t^2$$

we obtain

$$\sup_{t\geq 0} J_\lambda(tu_\epsilon) < \frac{2}{nK_\circ^{\frac{n}{4}}\left(\max_{x\in M} f(x)\right)^{\frac{n}{4}-1}}$$

provided that $R(t_o) < 0$ i.e.

$$\frac{\Delta f(x_o)}{f(x_o)} < \left(\frac{n\left(n^2+4n-20\right)}{3(n+2)(n-4)(n-6)}\frac{1}{\left(1+\|a\|_r+\|b\|_s\right)^{\frac{4}{n}}} - \frac{n-2}{3(n-1)}\right)S_g(x_o).$$

Which completes the proof. □

9. Application to compact Riemannian manifolds of dimension $n = 6$

THEOREM 11. *In case $n = 6$, we suppose that at a point x_o where f attains its maximum $S_g(x_o) > 0$. Then the equation (1) has a non trivial solution.*

PROOF. The same calculations as in case $n > 6$ gives us

$$\int_M f(x)\,|u_\epsilon(x)|^N\,dv_g = \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\left(1 - \left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)} + \frac{S_g(x_\circ)}{6(n-2)}\right)\epsilon^2 + o(\epsilon^2)\right).$$

Also, we have

$$\int_M a(x)\,|\nabla u_\epsilon|^2\,dv_g \leq \frac{\|a\|_r A}{K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\epsilon^{2-\frac{n}{r}\theta^{-\frac{r}{r-1}}}\left(1+o\left(\epsilon^2\right)\right)$$

and

$$\int_M b(x)u_\epsilon^2 dv_g \leq \frac{\|b\|_s B}{K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\epsilon^{4-\frac{n}{s}}\theta^{-\frac{s}{s-1}} + \left(1+o\left(\epsilon^2\right)\right).$$

where A and B are given by (20) and (21) respectively for $n = 6$. The computations of the term $\int_M (\Delta u_\epsilon)^2\,dv_g$ are well known (see for example [**11**])

$$\int_M (\Delta u_\epsilon)^2\,dv(g) = \theta^{-n}(n-4)^2\left(\frac{(n-4)n(n^2-4)}{f(x_\circ)}\right)^{\frac{n-4}{4}}\frac{\omega_{n-1}}{2}$$

$$\times \left(\frac{n(n+2)(n-2)}{(n-4)} I_n^{\frac{n}{2}-1} - \frac{2}{n}\theta^{-2} S_g(x_\circ)\epsilon^2 \log(\frac{1}{\epsilon^2}) + O(\epsilon^2) \right).$$

$$\int_M (\Delta u_\epsilon)^2 \, dv_g = \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \left(1 - \frac{2\,(n-4)}{n^2(n^2-4) I_n^{\frac{n}{2}-1}} S_g(x_\circ)\epsilon^2 \log\left(\frac{1}{\epsilon^2}\right) + O(\epsilon^2) \right).$$

Now resuming and letting ϵ so that

$$1 + \epsilon^{2-\frac{n}{r}} \theta^{-\frac{n}{r-1}} A \left\|b\right\|_s + \epsilon^{4-\frac{n}{s}} \theta^{-\frac{n}{s-1}} B \left\|a\right\|_r \leq \left(1 + \left\|a\right\|_r + \left\|b\right\|_s\right)^{\frac{4}{n}}$$

we get

$$J_\lambda(u_\epsilon) \leq \frac{1}{2}\left\|u_\epsilon\right\|^2 - \frac{1}{N}\int_M f(x)\left|u_\epsilon(x)\right|^N dv_g$$

$$\leq \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \left[\frac{t^2}{2}\left(1 + \left\|a\right\|_r + \left\|b\right\|_s\right)^{1-\frac{4}{n}} - \frac{t^N}{N} \right.$$

$$\left. - \frac{n-4}{n^2\,(n^2-4)\,I_n^{\frac{n}{2}-1}} \theta^{-2} S_g(x_\circ) t^2 \epsilon^2 \log\left(\frac{1}{\epsilon^2}\right) \right] + O(\epsilon^2).$$

The same arguments as in the case $n > 6$ allow us to infer that

$$\max_{t\geq 0} J_\lambda(tu_\epsilon) < \frac{2}{n\ K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}}$$

if

$$S_g(x_\circ) > 0.$$

Which achieves the proof. □

References

[1] A. Ambrosetti, Critical points and nonlinear variational problems. Soc.Mathem.de France, mémoire, 49,vol. 20, fascicule 2, (1992).

[2] A. Ambrosetti, J. G. Azorero, I. Peral, Multiplicity results for nonlinear elliptic equations. J. Funct. Anal.137 (1996), 219-242.

[3] T. Aubin, Some nonlinear problems in Riemannian geometry, Springer (1998).

[4] M. Benalili, Existence and multiplicity of solutions to elliptic equations of fourth order on compact manifolds. Dynamics of PDE, vol.6, 3 (2009), 203-225.

[5] M. Benalili, H. Boughazi, On the second Paneitz-Branson invariant. Houston Journal of Mathematics, vol., 36, 2, 2010, 393-420.

[6] M. Benalili, Existence and multiplicity of solutions to fourth order elliptic equations with critical exponent on compact manifolds, Bull. Belg.Math. Soc. Simon Stevin 17 (2010).

[7] M. Benalili, On singular Q-curvature type equations (Submitted).

[8] M. Benalili, T. Kamel, Nonlinear elliptic fourth order equations existence and multiplicity results. NoDEA, Nonlinear diff. Equ. Appl. 18, No. 5 (2011), 539-556.

[9] T.P. Branson, Group representation arising from Lorentz conformal geometry. J. Funct.Anal. 74 (1987), 199-291.

[10] H. Brézis, E.A. Lieb, A relation between pointwise convergence of functions and convergence of functionals., Proc.A.m.s.88 (1983), 486-490.

[11] D. Caraffa, Equations elliptiques du quatrième ordre avec un exposent critiques sur les variétés Riemanniennes compactes, J. Math. Pures appl.80(9)(2001) 941-960.

[12] Z. Djadli, E. Hebey and M. Ledoux, Paneitz-Type operators and applications, Duke.Mat.Journal.104-1(2000), 129-169.
[13] D.E. Edmunds, F. Furtunato, E. Janelli, Critical exponents, critical dimensions and biharmonic operators, Arch. Rational Mech.Anal.1990, 269-289.
[14] F. Madani, Le problème de Yamabé avec singularités, ArXix: 1717v1 [mathAP] 10 Apr. 2008.
[15] R.C.A.M. Van der Vorst, Fourth order elliptic equations, with critical growth, C.R.Acad. Sci. Paris t.320, série I, (1995), 295-299.
[16] R.C.A.M. Van der Vorst, Best constant for the embedding of the space $H^2 \cap H_0^1$ into $L^{\frac{2N}{N-4}}$, Diff.& Int. Eq. bf 6(2) (1993), 259-276.
[17] M. Vaugon, Equations différentielles non linéaires sur les variétés riemanniennes compates, Bull. Sci. Math. 103,3, (1979) 263-272.

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