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F-HARMONIC MAPS AS GLOBAL MAXIMA

MOHAMMED BENALILI AND HAFIDA BENALLAL

Abstract. In this note, we show that some F-harmonic maps into spheres are global maxima of the variations of their energy functional on the conformal group of the sphere. Our result extends partially those obtained in [15] and [17] for harmonic and p-harmonic maps.

1. Introduction

Harmonic maps have been studied first by J. Eells and J.H.Sampson in the sixties and since then many articles have appeared (see [6], [12], [16], [19], [20], [24]) to cite a few of them. Extensions to the notions of p-harmonic, biharmonic, F-harmonic and f-harmonic maps were introduced and similar research has been carried out (see [1], [2], [3], [7], [15], [18], [21], [23]). Harmonic maps were applied to broad areas in sciences and engineering including the robot mechanics (see [5], [8], [9]).

In this paper for a C^2-function $F : [0, +\infty[\to [0, +\infty[$ such that $F'(t) > 0$ on $t \in]0, +\infty[$, we look for sufficient conditions which present F-harmonic maps into spheres as global maxima of the energy functional. Our result extends similar results obtained in [17] and [18] for harmonic and p-harmonic maps.

Let (M, g) and S^n be, respectively, a compact Riemannian manifold of dimension $m \geq 2$ and the unit n-dimensional Euclidean sphere with $n \geq 2$ endowed with the canonical metric can induced by the inner product of R^{n+1}.

For a C^1- application $\phi : (M, g) \longrightarrow (S^n, can)$, we define the F-energy functional by

$$E_F(\phi) = \int_M F\left(\frac{|d\phi|^2}{2}\right) dv_g,$$

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where $\frac{|d\phi|^2}{2}$ denotes the energy density given by

$$\frac{|d\phi|^2}{2} = \frac{1}{2}\sum_{i=1}^{m} |d\phi(e_i)|^2$$

and where $\{e_i\}$ is an orthonormal basis on the tangent space T_xM and dv_g is the Riemannian measure associated to g on M.

Let $\phi^{-1}TS^n$and $\Gamma\left(\phi^{-1}TS^n\right)$ be, respectively, the pullback vector fiber bundle of TS^n and the space of sections on $\phi^{-1}TS^n$. Denote by ∇^M, $\nabla^{S^n}$and ∇, respectively, the Levi-Civita connections on: TM, $T\ S^n$ and $\phi^{-1}TS^n$. Recall that ∇ is defined by

$$\nabla_X Y = \nabla^{S^n}_{\phi_* X} Y$$

where $X \in TM$ and $Y \in \Gamma\left(\phi^{-1}TS^n\right)$.

Let v be a vector field on S^n and denote by $(\gamma_t^v)_t$ the flow of diffeomorphisms induced by v on S^n i.e.

$$\gamma_0^v = id\ , \quad \frac{d}{dt}\gamma_t^v{}_{t=0} = v\left(\gamma_t^v\right).$$

Denote by $\phi_t = \gamma_t^v o\phi$ the flow generated by v along the map ϕ. The first variation formula of $E_F(\phi)$ is given by

$$\begin{aligned}\frac{d}{dt}E_F(\phi_t)\mid_{t=0} &= \int_M F'\left(\frac{|d\phi_t|^2}{2}\right)\langle\nabla_{\partial t}d\phi_t, d\phi_t\rangle\mid_{t=0} dv_g \\ &= -\int_M \langle v, \tau_F(\phi)\rangle\, dv_g\end{aligned}$$

where $\tau_F(\phi) = trace_g\nabla\left(F'\left(\frac{|d\phi|^2}{2}\right)d\phi\right)$ is the F-tension.

Definition 1.1. ϕ is said F-harmonic if and only if $\tau_F(\phi) = 0$ i.e. ϕ is a critical point of the F-energy functional E_F.

Let $v \in \mathbb{R}^{n+1}$ and set $\bar{v}(y) = v - \langle v, y\rangle\, y$ for any $y \in S^n$. It is known that $\bar{v}$ is a conformal vector field on S^n i.e. $(\gamma_t^v)^*\, can = \alpha_t^2 can$ where $(\gamma_t^v)_t$ denotes the flow induced by the vector field $\bar{v}$. The expression of α_t is given in [17] by

$$\alpha_t = \frac{|v|}{|v|\, cht + \phi_v sht}. \tag{1}$$

where $\phi_v(x) = \langle v, \phi(x)\rangle$ and $\langle .,.\rangle$ the inner product on the Euclidean space $\mathbb{R}^{n+1}$. Denote by $\pounds(\phi)$ the subspace of $\Gamma(\phi^{-1}TS^n)$ given by

$$\pounds(\phi) = \left\{\bar{v}\circ\phi, v \in \mathbb{R}^{n+1}\right\}.$$

Obviously, if ϕ is not constant, $\pounds(\phi)$ is of dimension $n+1$.

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For any $\overline{v} \in \pounds(\phi)$, we denote by $(\gamma_t^v)_{t\in R}$ the one parameter group of conformal diffeomorphisms on S^n induced by the vector $\bar{v}$. For a C^2-function $F : [0, +\infty[\to [0, +\infty[$ such that $F'(t) > 0$ in $]0, +\infty[$.

Now we introduce the following tensor field

$$S_g^F(\phi) = F'\left(\frac{|d\phi|^2}{2}\right)\frac{|d\phi|^2}{2}g$$

$$-\left[F'\left(\frac{|d\phi|^2}{2}\right) + \frac{|d\phi|^2}{2}F''\left(\frac{|d\phi|^2}{2}\right)\right]\phi^* can.$$

For $x \in M$, we set

$$S_g^{o,F}(\phi)(x) = \inf\left\{S_{g,v}^F(\phi)(X,X),\ X \in T_xM \text{ such that } g(X,X) = 1\right\}.$$

The tensor $S_g^F(\phi)(x)$ will be said positive (resp. positive defined) at x if $S_g^{o,F}(\phi)(x) \geq 0$ (resp. $S_g^{o,F}(\phi)(x) > 0$).

Example 1. *For $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$, with $p = 2$ or $p \geq 4$, $S_g^p(\phi)$ is the stress-energy tensor introduced, respectively, by Eells and Lemaire for $p = 2$ [12] and modulo a multiplied positive constant by El Soufi for $p \geq 4$ [16], so we may call $S_g^F(\phi)$ the stress-energy tensor of ϕ.*

Indeed if $F(t) = t$ then $F\prime(t) = 1$, $F''(t) = 0$ and

$$S_g^F(\phi) = \frac{|d\phi|^2}{2}g - \phi^* can.$$

In the case $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$, with $p \geq 4$, $F\prime(t) = (2t)^{\frac{p}{2}-1}$, $F''(t) = (p-2)(2t)^{\frac{p}{2}-2}$ and

$$S_g^F(\phi) = \frac{1}{2}|d\phi|^p g - \frac{p}{2}|d\phi|^{p-2}\phi^* can = \frac{p}{2}\left(\frac{1}{p}|d\phi|^p g - |d\phi|^{p-2}\phi^* can\right)$$

The function F is called admissible if F satisfies

$$B = \left(\frac{F''(\alpha_t^2 o\phi.\frac{|d\phi|^2}{2})}{F'(\alpha_t^2 o\phi.\frac{|d\phi|^2}{2})}\alpha_t^2 o\phi - \frac{F''\left(\frac{|d\phi|^2}{2}\right)}{F'\left(\frac{|d\phi|^2}{2}\right)}\right)\phi_v \geq 0$$

and the stress-energy tensor $S_g^F(\phi)$ of ϕ fulfills

$$S_g^F(\gamma_t o\phi) \geq \alpha_t^2 o\phi.S_g^F(\phi)$$

where γ_t is the one parameter group of conformal transformations induced by the vector field $\overline{v}$ (defined above) on the euclidean sphere S^n and α_t is given by (1).

Example 2.

The function $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$ for $p = 2$ and $p \geq 4$ and $t \geq 0$ is admissible.

Indeed, for $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$ we have $B = 0$ and for any conformal diffeomorphism γ on the euclidean sphere, we have

$$S_g^F(\gamma o\phi) = \frac{1}{2}\left|d(\gamma_t o\phi)\right|^p g - \frac{p}{2}\left|d(\gamma_t o\phi)\right|^{p-2}\phi^* can$$

so if we let $\left|d(\gamma_t o\phi)\right|^2 = \alpha_t^2 o\phi.\left|d\phi\right|^2$, we get

$$\begin{aligned} S_g^F(\gamma_t o\phi) &= \alpha_t^2 o\phi.\left(\frac{1}{2}\left|d\phi\right|^p g - \frac{p}{2}\left|d\phi\right|^{p-2}\phi^* can\right) \\ &= \alpha_t^2 o\phi.S_g^F(\phi). \end{aligned}$$

The $F(t) = 1 + at - e^{-t}$, for $t \in [0, +\infty[$ where $a = \max_{x\in M}\frac{\left|d\phi\right|^2}{2}$ is admissible provided that the conformal diffeomorphism on the euclidean sphere S^n is contracting that means that the function ϕ_v given in the expression of (1) is nonnegative.

Indeed, we have

$$B = \left(-.\alpha_t^2 o\phi\frac{e^{-\alpha_t^2 o\phi\frac{\left|d\phi\right|^2}{2}}}{a + e^{-\alpha_t^2 o\phi\frac{\left|d\phi\right|^2}{2}}} + \frac{e^{-\frac{\left|d\phi\right|^2}{2}}}{a + e^{-\frac{\left|d\phi\right|^2}{2}}}\right)\phi_v$$

Putting $u = \alpha_t^2 o\phi \in]0, 1]$, we consider the function $\varphi(u) = -.u\frac{e^{-u\frac{\left|d\phi\right|^2}{2}}}{a+e^{-u\frac{\left|d\phi\right|^2}{2}}} + \frac{e^{-\frac{\left|d\phi\right|^2}{2}}}{a+e^{-\frac{\left|d\phi\right|^2}{2}}}$,we get

$$\varphi'(u) = \left(\frac{\left|d\phi\right|^2}{2}u - a - e^{-\left|d\phi\right|^2 u}\right)\frac{e^{-\frac{\left|d\phi\right|^2}{2}}}{\left(a + e^{-\frac{\left|d\phi\right|^2}{2}u}\right)^2}$$

and it is obvious that $\varphi'(u) \leq 0$, hence φ is a decreasing function on $]0, 1]$ i.e. $\varphi(u) \geq \varphi(1) = 0$. Consequently $B \geq 0$.

Now

$$\begin{aligned} S_g^F(\gamma_t o\phi)(X;X) &= \left(a + e^{-\frac{\left|d(\gamma_t o\phi)\right|^2}{2}}\right)\frac{\left|d(\gamma_t o\phi)\right|^2}{2}g(X,X) \\ &- \left[\left(a + e^{-\frac{\left|d(\gamma_t o\phi)\right|^2}{2}}\right) - \frac{\left|d(\gamma_t o\phi)\right|^2}{2}e^{-\frac{\left|d(\gamma_t o\phi)\right|^2}{2}}\right](\gamma_t o\phi)^* can(X,X). \\ &= \alpha_t^2 o\phi\left(a + e^{-\alpha_t^2 o\phi\frac{\left|d\phi\right|^2}{2}}\right)\frac{\left|d\phi\right|^2}{2}g(X,X) \end{aligned}$$

$$-\alpha_t^2 o\phi\left[\left(a+e^{-\alpha_t^2 o\phi\frac{|d\phi|^2}{2}}\right)-\alpha_t^2 o\phi\frac{|d\phi|^2}{2}e^{-\alpha_t^2 o\phi\frac{|d\phi|^2}{2}}\right]\phi^* can\left(X,X\right)$$
$$=\alpha_t^2 o\phi\left(a+e^{-\frac{|d(\gamma_t o\phi)|^2}{2}}\right)\left[\frac{1}{2}\left|d\phi\right|^2 g\left(X,X\right)-\phi^* can\left(X,X\right)\right]$$
$$+\alpha_t^2 o\phi\frac{|d\phi|^2}{2}e^{-\alpha_t^2 o\phi\frac{|d\phi|^2}{2}}\phi^* can\left(X,X\right).$$

An other example is the following function $F(t)=(1+2t)^{\alpha}$ where $0<\alpha<1$, the F-energy is the α-energy of Sacks-Uhlenbeck (see **??**). In fact

$$B=(\alpha-1)\left(\frac{1}{1+\alpha_t^2 o\phi.\left|d\phi\right|^2}\alpha_t^2 o\phi-\frac{1}{1+\left|d\phi\right|^2}\right)\phi_v$$
$$=\frac{(\alpha-1)\left(\alpha_t^2 o\phi-1\right)}{\left(1+\alpha_t^2 o\phi.\left|d\phi\right|^2\right)\left(1+\left|d\phi\right|^2\right)}\phi_v\geq 0$$

provided that $\phi_v\geq 0$.

And for vector field X on M, we have

$$S_g^F\left(\gamma_t o\phi\right)\left(X;X\right)=2\alpha\left(1+\alpha_t^2 o\phi\left|d\phi\right|^2\right)^{\alpha-1}\alpha_t^2 o\phi\frac{|d\phi|^2}{2}g\left(X,X\right)-$$
$$\left[2\alpha\left(1+\alpha_t^2 o\phi\left|d\phi\right|^2\right)^{\alpha-1}+4\alpha\left(\alpha-1\right)\left(1+\alpha_t^2 o\phi\left|d\phi\right|^2\right)^{\alpha-2}\alpha_t^2 o\phi\right]\alpha_t^2 o\phi.\phi^* can\left(X,X\right)$$
$$=\left[2\alpha\left(1+\alpha_t^2 o\phi\left|d\phi\right|^2\right)^{\alpha-1}\alpha_t^2 o\phi\left(\frac{|d\phi|^2}{2}g\left(X,X\right)-\phi^* can\left(X,X\right)\right)+\right.$$
$$\left.4\alpha\left(1-\alpha\right)\left(1+\alpha_t^2 o\phi\left|d\phi\right|^2\right)^{\alpha-2}\alpha_t^2 o\phi.\phi^* can\left(X,X\right)\right]$$

and taking account of the positivity of the stress-energy tensor of ϕ and the fact that $\phi_v\geq 0$, we infer that

$$S_g^F\left(\gamma_t o\phi\right)\left(X,X\right)\geq\alpha_t^4 o\phi.S_g^F\phi\left(X,X\right).$$

Remark 2.1. *$\phi_v\geq 0$ occurs for example if $\phi(M)$ is included in the positive half-sphere $S^{n+}=\{x\in S^n:\langle x,v\rangle\geq 0\}$.*

In this section we state the following result

Theorem 2.2. *Let $F:[0,+\infty[\to[0,+\infty[$ be an admissible function and ϕ be an F-harmonic map from a compact m-Riemannian manifold (M,g) $(m\geq 2)$ into the Euclidean sphere S^n $(n\geq 2)$. Then for any conformal diffeomorphism γ on S^n, $E_F\left(\gamma o\phi\right)\leq E_F\left(\phi\right)$ (resp. $E_F\left(\gamma o\phi\right)<E_F\left(\phi\right)$) provided that the stress-energy tensor $S_g^F\left(\phi\right)$ is positive (respectively positive defined).*

Remark 2.3. *In case* $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$, $p = 2$ *or* $p \geq 4$ *the condition* $\phi_v \geq 0$ *is not needed since* $B = 0$, *so our result recover the ones by El-Soufi in* [16] *and* [18].

To prove Theorem 2.2, we need the following lemmas

Lemma 2.4. *Let* $\phi : (M, g) \to (N, h)$ *be a smooth map and* γ *be a conformal diffeomorphism on* N, *then the* F*-tension of the map* $\gamma o \phi$, *is given by*

$$\tau_F(\gamma o\phi) = 2\alpha^{-1} o\phi . F'(\alpha^2 o\phi \frac{|d\phi|^2}{2}) d\gamma \left(d\phi_v - \frac{|d\phi|^2}{2} \nabla \alpha o\phi \right)$$

$$+ f d\gamma(\tau_F(\phi)) + d\gamma \left(F'\left(\frac{|d\phi|^2}{2}\right) d\phi(\nabla f) \right).$$

where $f = \frac{F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2}\right)}{F'\left(\frac{|d\phi|^2}{2}\right)}$ *and* $\gamma^* can = \alpha^2 can$.

Proof. We follow closely the proof in [18]

$$\tau_F(\gamma o\phi) = trace_g \nabla \left(F'\left(\frac{|d(\gamma o\phi)|^2}{2}\right) d(\gamma o\phi) \right) = F'\left(\frac{|d(\gamma o\phi)|^2}{2}\right) trace(\nabla d(\gamma o\phi))$$

$$+ d(\gamma o\phi) \left(\nabla F'\left(\frac{|d(\gamma o\phi)|^2}{2}\right) \right)$$

where $\nabla F'\left(\frac{|d(\gamma o\phi)|^2}{2}\right)$ is the gradient of $F'\left(\frac{|d(\gamma o\phi)|^2}{2}\right)$ in M.

Since γ is a conformal diffeomorphism on S^n, we have

$$\tau_F(\gamma o\phi) = F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2}\right) \tau(\gamma o\phi) + d(\gamma o\phi) \left(\nabla F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2}\right) \right)$$

$$= F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2}\right) (trace_g \nabla^\gamma d\gamma(d\phi, d\phi) + d\gamma.\tau(\phi)) + d(\gamma o\phi) \left(\nabla F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2}\right) \right)$$

$$= F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2}\right) \left(trace_g \nabla^\gamma d\gamma(d\phi, d\phi) + \frac{1}{F'\left(\frac{|d\phi|^2}{2}\right)} d\gamma(\tau_F(\phi)) \right)$$

$$- \frac{F'(\alpha^2 o\phi \frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)} d(\gamma o\phi) \left(\nabla F'\left(\frac{|d\phi|^2}{2}\right) \right) + d(\gamma o\phi) \left(\nabla F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2}\right) \right).$$

Putting $f=\frac{F'(\alpha^2 o\phi\frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)}$ we get

$$\tau_F(\gamma o\phi)=F'(\alpha^2 o\phi\frac{|d\phi|^2}{2})trace_g\nabla^\gamma d\gamma(d\phi,d\phi)+fd\gamma(\tau_F(\phi))$$
$$+F'\left(\frac{|d\phi|^2}{2}\right)d(\gamma o\phi)(\nabla f).$$

Now since $\gamma:(N,\gamma^*can)\to(N,can)$ is an isometry then, if $\widetilde{\nabla}$ denotes the connection corresponding to γ^*can, we have

$$\nabla^\gamma d\gamma(X,Y)=d\gamma\left(\widetilde{\nabla}_XY-\nabla_XY\right)$$

and since (see [18])

$$\widetilde{\nabla}_XY-\nabla_XY=\alpha^{-1}\left(\langle X,\nabla\alpha\rangle Y+\langle Y,\nabla\alpha\rangle X-\langle X,Y\rangle\nabla\alpha\right)$$

we obtain

$$trace_g\nabla^\gamma d\gamma(d\phi,d\phi)=2\alpha^{-1}o\phi.d\gamma\left(d\phi(\nabla\alpha o\phi)-\frac{|d\phi|^2}{2}\nabla\alpha o\phi\right).$$

Finally we infer that

$$\tau_F(\gamma o\phi)=2\alpha^{-1}o\phi.F'(\alpha^2 o\phi\frac{|d\phi|^2}{2})d\gamma\left(d\phi(\nabla\alpha o\phi)-\frac{|d\phi|^2}{2}\nabla\alpha o\phi\right)$$
$$+fd\gamma(\tau_F(\phi))+F'\left(\frac{|d\phi|^2}{2}\right)d\gamma o d\phi(\nabla f).$$

□

Lemma 2.5. *Let ϕ be an F-harmonic map from an m-dimensional Riemannian manifold (M,g) $(m\geq 2)$ into the Euclidean unit sphere (S^n,can) $(n\geq 2)$.*

Then for any $v\in R^{n+1}-\{0\}$ and any $t_o\in R$ we have

$$\frac{d}{dt}E_F(\gamma_t^v o\phi)_{t=t_o}=$$
$$-2\frac{sht_o}{|v|}\int_M\alpha_{t_o}^3 o\phi.F'\left(\alpha_{t_o}^2 o\phi.\frac{|d\phi|^2}{2}\right)\left(|d\phi|^2|\overline{v}o\phi|^2-|d\phi_v|^2\right)dv_g$$
$$-\int_M\alpha_{t_o}^2 o\phi.F'\left(\frac{|d\phi|^2}{2}\right)\langle d\phi(\nabla f_{t_o}),\overline{v}o\phi\rangle dv_g$$

where $f_{t_o} = \frac{F'\left(\alpha_{t_o}^2 o\phi \frac{|d\phi|^2}{2}\right)}{F'\left(\frac{|d\phi|^2}{2}\right)}$, $\gamma^* can = \alpha_{t_o}^2 can$

and

$$\alpha_{t_o} = \frac{|v|}{\phi_v sht_o + |v|\, cht_o}.$$

Proof. Recall that the first variation formula of the F-energy is given by

$$\frac{d}{dt} E_F\left(\gamma_t^v o\phi\right)_{t=t_o} = -\int_M \left\langle \tau_F\left(\gamma_{t_o}^v o\phi\right), \overline{v} o\left(\gamma_{to}^v o\phi\right)\right\rangle dv_g.$$

By Lemma 2.4 and the fact that ϕ is F-harmonic we get

$$\frac{d}{dt} E_F\left(\gamma_t^v o\phi\right)_{t=t_o} =$$

$$-\int_M 2\alpha_{t_o} o\phi . F'\left(\alpha_{t_o}^2 o\phi . \frac{|d\phi|^2}{2}\right)\left\langle \left(\nabla \alpha_{t_o} o\phi\right)^T - \frac{|d\phi|^2}{2}\nabla \alpha_{t_o} o\phi, \overline{v} o\phi\right\rangle dv_g$$

$$-\int_M F'\left(\frac{|d\phi|^2}{2}\right)\left\langle d\phi\left(\nabla f_{t_o}\right), \overline{v} o\phi\right\rangle dv_g$$

and since (see [18])

$$\nabla \alpha_{t_o}^v = -\frac{\left(\alpha_{t_o}^v\right)^2}{|v|} sht_o \overline{v} \tag{2}$$

we have

$$\left\langle \nabla \alpha_{t_o} o\phi, \overline{v} o\phi\right\rangle = -\frac{\left(\alpha_{t_o}^v\right)^2}{|v|} sht_o \left|\overline{v} o\phi\right|^2. \tag{3}$$

Now let $(e_1, ..., e_m)$ be an orthogonal basis on M

$$\left\langle \left(\nabla \alpha_{t_o} o\phi\right)^T, \overline{v} o\phi\right\rangle = \sum_{i=1}^m \left\langle \nabla \alpha_{t_o} o\phi, d\phi(e_i)\right\rangle \left\langle \overline{v} o\phi, d\phi(e_i)\right\rangle$$

$$= -\frac{sht_o}{|v|}\left(\alpha_{t_o} o\phi\right)^2 \sum_{i=1}^m \left\langle \overline{v} o\phi, d\phi(e_i)\right\rangle^2$$

$$= -\frac{sht_o}{|v|}\left(\alpha_{t_o} o\phi\right)^2 \left|d\phi_v\right|^2.$$

Hence

$$\frac{d}{dt} E_F\left(\gamma_t^v o\phi\right)_{t=t_o} =$$

$$-2\frac{sht_o}{|v|}\int_M \alpha_{t_o}^3 o\phi . F'\left(\alpha_{t_o}^2 o\phi . \frac{|d\phi|^2}{2}\right)\left(\frac{|d\phi|^2}{2}\left|\overline{v} o\phi\right|^2 - \left|d\phi_v\right|^2\right) dv_g$$

$$-\int_M \alpha_{t_o}^2 o\phi.F'\left(\frac{|d\phi|^2}{2}\right)\langle d\phi\left(\nabla f_{t_o}\right),\overline{v}o\phi\rangle\, dv_g.$$

□

We set

$$g(t)=\int_M \alpha_t^2 o\phi.F'\left(\frac{|d\phi|^2}{2}\right)\langle d\phi\left(\nabla f_t\right),\overline{v}o\phi\rangle\, dv_g.$$

Lemma 2.6.

$$g(t)=$$
$$\int_M \alpha_t^3 o\phi.F''(\alpha_t^2 o\phi.\frac{|d\phi|^2}{2})\,|d\phi|^2\,\langle d\phi\left(\nabla\left(\alpha_{t_o}o\phi\right)\right),\overline{v}o\phi\rangle\, dv_g$$
$$+\int_M \alpha_t^2 o\phi.\left(F''(\alpha_t^2 o\phi.\frac{|d\phi|^2}{2})\alpha_t^2 o\phi-\frac{F'(\alpha_t^2 o\phi.\frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)}F''\left(\frac{|d\phi|^2}{2}\right)\right)\frac{\phi_v}{|v|}\,|d\phi|^2\, dv_g.$$

Proof. First, we compute ∇f_t

$$\nabla f_t=\frac{F''(\alpha_t^2 o\phi.\frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)}\left(\alpha_t o\phi.\,|d\phi|^2\,\nabla\left(\alpha_t o\phi\right)+\alpha_t^2 o\phi.\,\langle\nabla d\phi,d\phi\rangle\right)$$
$$-\frac{F'(\alpha_t^2 o\phi.\frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)^2}F''\left(\frac{|d\phi|^2}{2}\right)\langle\nabla d\phi,d\phi\rangle$$
$$=\frac{F''(\alpha_t^2 o\phi.\frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)}\alpha_t o\phi\,|d\phi|^2\,\nabla\left(\alpha_t o\phi\right)$$
$$+\left(\frac{F''(\alpha_t^2 o\phi.\frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)}\alpha_t^2 o\phi-\frac{F'(\alpha_t^2 o\phi.\frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)^2}F''\left(\frac{|d\phi|^2}{2}\right)\right)\langle\nabla d\phi,d\phi\rangle.$$

Then

$$g(t)=\int_M \alpha_{t_o}^2 o\phi.F'\left(\frac{|d\phi|^2}{2}\right)\langle d\phi\left(\nabla f_{t_o}\right),\overline{v}o\phi\rangle\, dv_g$$
$$=\int_M \alpha_{t_o}^3 o\phi.F''(\alpha_{t_o}^2 o\phi.\frac{|d\phi|^2}{2})\,|d\phi|^2\,\langle d\phi\left(\nabla\left(\alpha_{t_o}o\phi\right)\right),\overline{v}o\phi\rangle\, dv_g$$
(4)
$$+\int_M \alpha_{t_o}^2 o\phi.\left(F''(\alpha_{t_o}^2 o\phi.\frac{|d\phi|^2}{2})\alpha_{t_o}^2 o\phi-\frac{F'(\alpha_{t_o}^2 o\phi.\frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)}F''\left(\frac{|d\phi|^2}{2}\right)\right)$$

$$\times \left\langle d\phi\left(\nabla\left|d\phi\right|^{2}\right),\overline{v}o\phi\right\rangle dv_{g}.$$

Let $\{e_1,...,e_m\}$ be a basis of T_xM which diagonalizes ϕ^*can, we have

$$\left\langle d\phi\left(\nabla\frac{\left|d\phi\right|^{2}}{2}\right),\overline{v}o\phi\right\rangle=\left\langle\nabla_{e_i}d\phi,d\phi\right\rangle\left\langle\overline{v}o\phi,d\phi(e_j)\right\rangle\left\langle d\phi(e_i),d\phi(e_j)\right\rangle$$

$$=\left\langle\nabla_{e_i}d\phi,d\phi\right\rangle\left\langle\overline{v}o\phi,d\phi(e_j)\right\rangle\phi^*can\left(e_i,e_j\right)$$

$$=\left\langle\nabla_{\overline{v}o\phi}d\phi\left(e_j\right),d\phi\left(e_j\right)\right\rangle$$

$$=\left\langle\nabla_{d\phi(e_j)}\overline{v}o\phi,d\phi\left(e_j\right)\right\rangle+\left\langle\left[\overline{v}o\phi,d\phi\left(e_j\right)\right],d\phi\left(e_j\right)\right\rangle.$$

Likewise we get

$$\left\langle\left[\overline{v}o\phi,d\phi\left(e_j\right)\right],d\phi\left(e_j\right)\right\rangle=-\frac{1}{2}\frac{d}{dt}\mid_{t=0}\gamma_t^*\left|d\phi\left(e_j\right)\right|^2$$

$$=-\frac{1}{2}\frac{d}{dt}\mid_{t=0}\alpha_t^2\left|d\phi\left(e_j\right)\right|^2$$

and taking account of (1) we obtain that

$$\left\langle\left[\overline{v}o\phi,d\phi\left(e_j\right)\right],d\phi\left(e_j\right)\right\rangle=\frac{\phi_v}{|v|}\left|d\phi\left(e_j\right)\right|^2$$

so we infer that

$$\left\langle d\phi\left(\nabla\frac{\left|d\phi\right|^{2}}{2}\right),\overline{v}o\phi\right\rangle=\frac{\phi_v}{|v|}\left|d\phi\right|^2+\left\langle\nabla_{e_j}\overline{v}o\phi,d\phi\left(e_j\right)\right\rangle$$

and

$$\left\langle\nabla_{e_j}\overline{v}o\phi,d\phi\left(e_j\right)\right\rangle=\nabla_{e_j}\left\langle\overline{v}o\phi,d\phi\left(e_j\right)\right\rangle-\left\langle\overline{v}o\phi,\nabla_{e_j}d\phi\left(e_j\right)\right\rangle$$

$$=\nabla_{e_j}\left\langle v,d\phi\left(e_j\right)\right\rangle-\left\langle\overline{v}o\phi,\nabla_{e_j}d\phi\left(e_j\right)\right\rangle$$

$$=\left\langle v,\nabla_{e_j}d\phi\left(e_j\right)\right\rangle-\left\langle\overline{v}o\phi,\nabla_{e_j}d\phi\left(e_j\right)\right\rangle$$

$$=\left\langle v-\overline{v}o\phi,\nabla_{e_j}d\phi\left(e_j\right)\right\rangle=0.$$

Hence

$$\left\langle d\phi\left(\nabla\frac{\left|d\phi\right|^{2}}{2}\right),\overline{v}o\phi\right\rangle=\frac{\phi_v}{|v|}\left|d\phi\right|^2. \tag{5}$$

□

Now set

$$\varphi(t_o)=2\frac{sht_o}{|v|}\int_M\alpha_{t_o}^3o\phi.F'\left(\alpha_{t_o}^2o\phi.\frac{\left|d\phi\right|^2}{2}\right)\left(-\frac{\left|d\phi\right|^2}{2}\left|\overline{v}o\phi\right|^2+\left|d\phi_v\right|^2\right)dv_g$$

$$-\int_M\alpha_{t_o}^3o\phi.F''(\alpha_{t_o}^2o\phi.\frac{\left|d\phi\right|^2}{2})\frac{\left|d\phi\right|^2}{2}\left\langle d\phi\left(\nabla\left(\alpha_{t_o}o\phi\right)\right),\overline{v}o\phi\right\rangle dv_g$$

and since by (2) we have

$$\langle d\phi\left(\nabla\left(\alpha_{t_o}o\phi\right)\right),\overline{v}o\phi\rangle=-\frac{sht_o}{|v|}\alpha_{t_o}^2o\phi\left|d\phi_v\right|^2$$

we get
(6)

$$\varphi(t_o)=2\frac{sht_o}{|v|}\int_M\alpha_{t_o}^3o\phi.\left[\left(F'\left(\alpha_{t_o}^2o\phi.\frac{|d\phi|^2}{2}\right)+\alpha_{t_o}^2o\phi.\frac{|d\phi|^2}{2}F''\left(\alpha_{t_o}^2o\phi.\frac{|d\phi|^2}{2}\right)\right)|d\phi_v|^2\right.$$
$$\left.-F'\left(\alpha_{t_o}^2o\phi.\frac{|d\phi|^2}{2}\right)\frac{|d\phi|^2}{2}\left|\overline{v}o\phi\right|^2\right]dv_g.$$

or

$$\varphi(t_o)=-2\frac{sht_o}{|v|}\int_M\alpha_{t_o}^3o\phi.S_g^F\left(\gamma_t^vo\phi\right)dv_g.$$

Proof. (of Theorem 2.2) Recall (see [13]) that for any conformal diffeomorphism γ of the unit sphere S^n there exist an isometry $r\in O\left(n+1\right)$, a real number $t\geq 0$ and a vector $v\in R^{n+1}-\{0\}$ such that $\gamma=ro\gamma_t^v$, so it suffices to consider γ_t^v with $t\geq 0$ and $v\in R^{n+1}-\{0\}$.

On the other hand

$$\frac{d}{dt}E_F\left(\gamma_t^vo\phi\right)=\varphi(t)+\chi(t)$$

where

$$\chi(t)=-\int_M\alpha_t^2o\phi.\left(F''(\alpha_t^2o\phi.\frac{|d\phi|^2}{2})\alpha_t^2o\phi-\frac{F'(\alpha_t^2o\phi.\frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)}F''\left(\frac{|d\phi|^2}{2}\right)\right)\frac{\phi_v}{|v|}\left|d\phi\right|^2dv_g$$

and $\varphi\left(t\right)$ is given by (6). Now, since the function F is admissible we infer that $\chi(t)\leq 0$. Since the energy stress-tensor $S_g^F\left(\gamma o\phi\right)=S_g^F\left(\gamma_to\phi\right)$ of $\gamma o\phi$ is positive (resp. positive defined) by assumption and

$$S_g^F\left(\gamma_to\phi\right)=F'\left(\frac{\left|d\left(\gamma_to\phi\right)\right|^2}{2}\right)\frac{\left|d\left(\gamma_to\phi\right)\right|^2}{2}g$$
$$-\left[F'\left(\frac{\left|d\left(\gamma_to\phi\right)\right|^2}{2}\right)+\frac{\left|d\left(\gamma_to\phi\right)\right|^2}{2}F''\left(\frac{\left|d\left(\gamma_to\phi\right)\right|^2}{2}\right)\right]\left(\gamma_to\phi\right)^*can$$
$$=\alpha_t^2o\phi\left[F'\left(\alpha_t^2o\phi\frac{|d\phi|^2}{2}\right)\frac{|d\phi|^2}{2}g-F'\left(\alpha_t^2o\phi\frac{|d\phi|^2}{2}\right)\right.$$

$$-\alpha_t^2 o\phi \frac{|d\phi|^2}{2} F''\left(\alpha_t^2 o\phi \frac{|d\phi|^2}{2}\right)\Bigg] \phi^* can$$

so the tensor

$$F'\left(\alpha_t^2 o\phi \frac{|d\phi|^2}{2}\right)\frac{|d\phi|^2}{2} g - \left(F'\left(\alpha_t^2 o\phi \frac{|d\phi|^2}{2}\right) + \alpha_t^2 o\phi \frac{|d\phi|^2}{2} F''\left(\alpha_t^2 o\phi \frac{|d\phi|^2}{2}\right)\right)\phi^* can$$

is positive (resp. positive defined). Consequently $\varphi(t) \leq 0$ (resp. $\varphi(t) < 0$) for any $t \geq 0$ and the proof of Theorem 2.2 is complete. □

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