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F-HARMONIC MAPS AS GLOBAL MAXIMA

MOHAMMED BENALILI AND HAFIDA BENALLAL

ABSTRACT. In this note, we show that some F-harmonic maps into spheres are global maxima of the variations of their energy functional on the conformal group of the sphere. Our result extends partially those obtained in [15] and [17] for harmonic and p-harmonic maps.

1. INTRODUCTION

Harmonic maps have been studied first by J. Eells and J.H.Sampson in the sixties and since then many articles have appeared (see [6], [12], [16], [19], [20], [24]) to cite a few of them. Extensions to the notions of p-harmonic, biharmonic, F-harmonic and f-harmonic maps were introduced and similar research has been carried out (see [1], [2], [3], [7], [15], [18], [21], [23]). Harmonic maps were applied to broad areas in sciences and engineering including the robot mechanics (see [5], [8], [9]).

In this paper for a C^2-function $F : [0, +\infty[\to [0, +\infty[$ such that $F'(t) > 0$ on $t \in]0, +\infty[$, we look for sufficient conditions which present F-harmonic maps into spheres as global maxima of the energy functional. Our result extends similar results obtained in [17] and [18] for harmonic and p-harmonic maps.

Let (M, g) and S^n be, respectively, a compact Riemannian manifold of dimension $m \geq 2$ and the unit n-dimensional Euclidean sphere with $n \geq 2$ endowed with the canonical metric can induced by the inner product of R^{n+1}.

For a C^1- application $\phi : (M, g) \longrightarrow (S^n, can)$, we define the F-energy functional by

$$E_F(\phi) = \int_M F\left(\frac{|d\phi|^2}{2}\right) dv_g,$$

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where $\frac{|d\phi|^2}{2}$ denotes the energy density given by

$$\frac{|d\phi|^2}{2} = \frac{1}{2}\sum_{i=1}^{m} |d\phi(e_i)|^2$$

and where $\{e_i\}$ is an orthonormal basis on the tangent space T_xM and dv_g is the Riemannian measure associated to g on M.

Let $\phi^{-1}TS^n$and $\Gamma\left(\phi^{-1}TS^n\right)$ be, respectively, the pullback vector fiber bundle of TS^n and the space of sections on $\phi^{-1}TS^n$. Denote by ∇^M, $\nabla^{S^n}$and ∇, respectively, the Levi-Civita connections on: TM, $T\ S^n$ and $\phi^{-1}TS^n$. Recall that ∇ is defined by

$$\nabla_X Y = \nabla^{S^n}_{\phi_* X} Y$$

where $X \in TM$ and $Y \in \Gamma\left(\phi^{-1}TS^n\right)$.

Let v be a vector field on S^n and denote by $(\gamma_t^v)_t$ the flow of diffeomorphisms induced by v on S^n i.e.

$$\gamma_0^v = id\ , \quad \frac{d}{dt}\gamma_t^v{}_{t=0} = v\left(\gamma_t^v\right).$$

Denote by $\phi_t = \gamma_t^v o\phi$ the flow generated by v along the map ϕ. The first variation formula of $E_F(\phi)$ is given by

$$\frac{d}{dt}E_F(\phi_t)\mid_{t=0} = \int_M F'\left(\frac{|d\phi_t|^2}{2}\right)\langle\nabla_{\partial t}d\phi_t, d\phi_t\rangle\mid_{t=0} dv_g$$

$$= -\int_M \langle v, \tau_F(\phi)\rangle\, dv_g$$

where $\tau_F(\phi) = trace_g\nabla\left(F'\left(\frac{|d\phi|^2}{2}\right)d\phi\right)$ is the F-tension.

Definition 1.1. ϕ is said F-harmonic if and only if $\tau_F(\phi) = 0$ i.e. ϕ is a critical point of the F-energy functional E_F.

Let $v \in \mathbb{R}^{n+1}$ and set $\bar{v}(y) = v - \langle v, y\rangle\, y$ for any $y \in S^n$. It is known that $\bar{v}$ is a conformal vector field on S^n i.e. $(\gamma_t^v)^*\, can = \alpha_t^2 can$ where $(\gamma_t^v)_t$ denotes the flow induced by the vector field $\bar{v}$. The expression of α_t is given in [17] by

$$\alpha_t = \frac{|v|}{|v|\, cht + \phi_v sht}. \tag{1}$$

where $\phi_v(x) = \langle v, \phi(x)\rangle$ and $\langle .,.\rangle$ the inner product on the Euclidean space $\mathbb{R}^{n+1}$. Denote by $\pounds(\phi)$ the subspace of $\Gamma(\phi^{-1}TS^n)$ given by

$$\pounds(\phi) = \left\{\bar{v}\circ\phi, v \in \mathbb{R}^{n+1}\right\}.$$

Obviously, if ϕ is not constant, $\pounds(\phi)$ is of dimension $n+1$.

2. F-HARMONIC MAPS AS GLOBAL MAXIMA

For any $\overline{v} \in \pounds(\phi)$, we denote by $(\gamma_t^v)_{t\in R}$ the one parameter group of conformal diffeomorphisms on S^n induced by the vector $\bar{v}$. For a C^2-function $F : [0, +\infty[\to [0, +\infty[$ such that $F'(t) > 0$ in $]0, +\infty[$.

Now we introduce the following tensor field

$$S_g^F(\phi) = F'\left(\frac{|d\phi|^2}{2}\right)\frac{|d\phi|^2}{2}g$$

$$-\left[F'\left(\frac{|d\phi|^2}{2}\right) + \frac{|d\phi|^2}{2}F''\left(\frac{|d\phi|^2}{2}\right)\right]\phi^* can.$$

For $x \in M$, we set

$$S_g^{o,F}(\phi)(x) = \inf\left\{S_{g,v}^F(\phi)(X,X),\ X \in T_xM \text{ such that } g(X,X) = 1\right\}.$$

The tensor $S_g^F(\phi)(x)$ will be said positive (resp. positive defined) at x if $S_g^{o,F}(\phi)(x) \geq 0$ (resp. $S_g^{o,F}(\phi)(x) > 0$).

Example 1. *For $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$, with $p = 2$ or $p \geq 4$, $S_g^p(\phi)$ is the stress-energy tensor introduced, respectively, by Eells and Lemaire for $p = 2$ [12] and modulo a multiplied positive constant by El Soufi for $p \geq 4$ [16], so we may call $S_g^F(\phi)$ the stress-energy tensor of ϕ.*

Indeed if $F(t) = t$ then $F\prime(t) = 1$, $F''(t) = 0$ and

$$S_g^F(\phi) = \frac{|d\phi|^2}{2}g - \phi^* can.$$

In the case $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$, with $p \geq 4$, $F\prime(t) = (2t)^{\frac{p}{2}-1}$, $F''(t) = (p-2)(2t)^{\frac{p}{2}-2}$ and

$$S_g^F(\phi) = \frac{1}{2}|d\phi|^p g - \frac{p}{2}|d\phi|^{p-2}\phi^* can = \frac{p}{2}\left(\frac{1}{p}|d\phi|^p g - |d\phi|^{p-2}\phi^* can\right)$$

The function F is called admissible if F satisfies

$$B = \left(\frac{F''(\alpha_t^2 o\phi.\frac{|d\phi|^2}{2})}{F'(\alpha_t^2 o\phi.\frac{|d\phi|^2}{2})}\alpha_t^2 o\phi - \frac{F''\left(\frac{|d\phi|^2}{2}\right)}{F'\left(\frac{|d\phi|^2}{2}\right)}\right)\phi_v \geq 0$$

and the stress-energy tensor $S_g^F(\phi)$ of ϕ fulfills

$$S_g^F(\gamma_t o\phi) \geq \alpha_t^2 o\phi . S_g^F(\phi)$$

where γ_t is the one parameter group of conformal transformations induced by the vector field $\overline{v}$ (defined above) on the euclidean sphere S^n and α_t is given by (1).

Example 2.

The function $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$ for $p = 2$ and $p \geq 4$ and $t \geq 0$ is admissible.

Indeed, for $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$ we have $B = 0$ and for any conformal diffeomorphism γ on the euclidean sphere, we have

$$S_g^F(\gamma o\phi) = \frac{1}{2}|d(\gamma_t o\phi)|^p g - \frac{p}{2}|d(\gamma_t o\phi)|^{p-2}\phi^* can$$

so if we let $|d(\gamma_t o\phi)|^2 = \alpha_t^2 o\phi . |d\phi|^2$, we get

$$\begin{aligned} S_g^F(\gamma_t o\phi) &= \alpha_t^2 o\phi . \left(\frac{1}{2}|d\phi|^p g - \frac{p}{2}|d\phi|^{p-2}\phi^* can\right) \\ &= \alpha_t^2 o\phi . S_g^F(\phi). \end{aligned}$$

The $F(t) = 1 + at - e^{-t}$, for $t \in [0, +\infty[$ where $a = \max_{x\in M}\frac{|d\phi|^2}{2}$ is admissible provided that the conformal diffeomorphism on the euclidean sphere S^n is contracting that means that the function ϕ_v given in the expression of (1) is nonnegative.

Indeed, we have

$$B = \left(-.\alpha_t^2 o\phi \frac{e^{-\alpha_t^2 o\phi\frac{|d\phi|^2}{2}}}{a + e^{-\alpha_t^2 o\phi\frac{|d\phi|^2}{2}}} + \frac{e^{-\frac{|d\phi|^2}{2}}}{a + e^{-\frac{|d\phi|^2}{2}}}\right)\phi_v$$

Putting $u = \alpha_t^2 o\phi \in]0, 1]$, we consider the function $\varphi(u) = -.u\frac{e^{-u\frac{|d\phi|^2}{2}}}{a+e^{-u\frac{|d\phi|^2}{2}}} + \frac{e^{-\frac{|d\phi|^2}{2}}}{a+e^{-\frac{|d\phi|^2}{2}}}$,we get

$$\varphi'(u) = \left(\frac{|d\phi|^2}{2}u - a - e^{-|d\phi|^2 u}\right)\frac{e^{-\frac{|d\phi|^2}{2}}}{\left(a + e^{-\frac{|d\phi|^2}{2}u}\right)^2}$$

and it is obvious that $\varphi'(u) \leq 0$, hence φ is a decreasing function on $]0, 1]$ i.e. $\varphi(u) \geq \varphi(1) = 0$. Consequently $B \geq 0$.

Now

$$\begin{aligned} S_g^F(\gamma_t o\phi)(X; X) &= \left(a + e^{-\frac{|d(\gamma_t o\phi)|^2}{2}}\right)\frac{|d(\gamma_t o\phi)|^2}{2} g(X, X) \\ &- \left[\left(a + e^{-\frac{|d(\gamma_t o\phi)|^2}{2}}\right) - \frac{|d(\gamma_t o\phi)|^2}{2} e^{-\frac{|d(\gamma_t o\phi)|^2}{2}}\right](\gamma_t o\phi)^* can(X, X). \\ &= \alpha_t^2 o\phi\left(a + e^{-\alpha_t^2 o\phi\frac{|d\phi|^2}{2}}\right)\frac{|d\phi|^2}{2} g(X, X) \end{aligned}$$

$$-\alpha_t^2 o\phi\left[\left(a+e^{-\alpha_t^2 o\phi\frac{|d\phi|^2}{2}}\right)-\alpha_t^2 o\phi\frac{|d\phi|^2}{2}e^{-\alpha_t^2 o\phi\frac{|d\phi|^2}{2}}\right]\phi^* can\,(X,X)$$
$$=\alpha_t^2 o\phi\left(a+e^{-\frac{|d(\gamma_t o\phi)|^2}{2}}\right)\left[\frac{1}{2}|d\phi|^2 g\,(X,X)-\phi^* can\,(X,X)\right]$$
$$+\alpha_t^2 o\phi\frac{|d\phi|^2}{2}e^{-\alpha_t^2 o\phi\frac{|d\phi|^2}{2}}\phi^* can\,(X,X)\,.$$

An other example is the following function $F(t)=(1+2t)^{\alpha}$ where $0<\alpha<1$, the F-energy is the α-energy of Sacks-Uhlenbeck (see **??**). In fact

$$B=(\alpha-1)\left(\frac{1}{1+\alpha_t^2 o\phi.\,|d\phi|^2}\alpha_t^2 o\phi-\frac{1}{1+|d\phi|^2}\right)\phi_v$$
$$=\frac{(\alpha-1)\,(\alpha_t^2 o\phi-1)}{\left(1+\alpha_t^2 o\phi.\,|d\phi|^2\right)\left(1+|d\phi|^2\right)}\phi_v\geq 0$$

provided that $\phi_v\geq 0$.

And for vector field X on M, we have

$$S_g^F\,(\gamma_t o\phi)\,(X;X)=2\alpha\left(1+\alpha_t^2 o\phi\,|d\phi|^2\right)^{\alpha-1}\alpha_t^2 o\phi\frac{|d\phi|^2}{2}g\,(X,X)-$$
$$\left[2\alpha\left(1+\alpha_t^2 o\phi\,|d\phi|^2\right)^{\alpha-1}+4\alpha\,(\alpha-1)\left(1+\alpha_t^2 o\phi\,|d\phi|^2\right)^{\alpha-2}\alpha_t^2 o\phi\right]\alpha_t^2 o\phi.\phi^* can\,(X,X)$$
$$=\left[2\alpha\left(1+\alpha_t^2 o\phi\,|d\phi|^2\right)^{\alpha-1}\alpha_t^2 o\phi\left(\frac{|d\phi|^2}{2}g\,(X,X)-\phi^* can\,(X,X)\right)+\right.$$
$$\left.4\alpha\,(1-\alpha)\left(1+\alpha_t^2 o\phi\,|d\phi|^2\right)^{\alpha-2}\alpha_t^2 o\phi.\phi^* can\,(X,X)\right]$$

and taking account of the positivity of the stress-energy tensor of ϕ and the fact that $\phi_v\geq 0$, we infer that

$$S_g^F\,(\gamma_t o\phi)\,(X,X)\geq\alpha_t^4 o\phi.S_g^F\phi\,(X,X)\,.$$

Remark 2.1. *$\phi_v\geq 0$ occurs for example if $\phi(M)$ is included in the positive half-sphere $S^{n+}=\{x\in S^n:\langle x,v\rangle\geq 0\}$.*

In this section we state the following result

Theorem 2.2. *Let $F:[0,+\infty[\to[0,+\infty[$ be an admissible function and ϕ be an F-harmonic map from a compact m-Riemannian manifold (M,g) $(m\geq 2)$ into the Euclidean sphere S^n $(n\geq 2)$. Then for any conformal diffeomorphism γ on S^n, $E_F\,(\gamma o\phi)\leq E_F\,(\phi)$ (resp. $E_F\,(\gamma o\phi)<E_F\,(\phi)$) provided that the stress-energy tensor $S_g^F\,(\phi)$ is positive (respectively positive defined).*

Remark 2.3. *In case* $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$, $p = 2$ *or* $p \geq 4$ *the condition* $\phi_v \geq 0$ *is not needed since* $B = 0$, *so our result recover the ones by El-Soufi in* [16] *and* [18].

To prove Theorem 2.2, we need the following lemmas

Lemma 2.4. *Let* $\phi : (M, g) \to (N, h)$ *be a smooth map and* γ *be a conformal diffeomorphism on* N, *then the* F*-tension of the map* $\gamma o\phi$, *is given by*

$$\tau_F(\gamma o\phi) = 2\alpha^{-1} o\phi . F'(\alpha^2 o\phi \frac{|d\phi|^2}{2}) d\gamma \left(d\phi_v - \frac{|d\phi|^2}{2} \nabla \alpha o\phi \right)$$

$$+ f d\gamma(\tau_F(\phi)) + d\gamma \left(F'\left(\frac{|d\phi|^2}{2} \right) d\phi(\nabla f) \right).$$

where $f = \frac{F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2}\right)}{F'\left(\frac{|d\phi|^2}{2}\right)}$ *and* $\gamma^* can = \alpha^2 can$.

Proof. We follow closely the proof in [18]

$$\tau_F(\gamma o\phi) = trace_g \nabla \left(F'\left(\frac{|d(\gamma o\phi)|^2}{2} \right) d(\gamma o\phi) \right) = F'\left(\frac{|d(\gamma o\phi)|^2}{2} \right) trace(\nabla d(\gamma o\phi))$$

$$+ d(\gamma o\phi) \left(\nabla F'\left(\frac{|d(\gamma o\phi)|^2}{2} \right) \right)$$

where $\nabla F'\left(\frac{|d(\gamma o\phi)|^2}{2}\right)$ is the gradient of $F'\left(\frac{|d(\gamma o\phi)|^2}{2}\right)$ in M.

Since γ is a conformal diffeomorphism on S^n, we have

$$\tau_F(\gamma o\phi) = F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2} \right) \tau(\gamma o\phi) + d(\gamma o\phi) \left(\nabla F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2} \right) \right)$$

$$= F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2} \right) (trace_g \nabla^\gamma d\gamma(d\phi, d\phi) + d\gamma.\tau(\phi)) + d(\gamma o\phi) \left(\nabla F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2} \right) \right)$$

$$= F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2} \right) \left(trace_g \nabla^\gamma d\gamma(d\phi, d\phi) + \frac{1}{F'\left(\frac{|d\phi|^2}{2}\right)} d\gamma(\tau_F(\phi)) \right)$$

$$- \frac{F'(\alpha^2 o\phi \frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)} d(\gamma o\phi) \left(\nabla F'\left(\frac{|d\phi|^2}{2} \right) \right) + d(\gamma o\phi) \left(\nabla F'\left(\alpha^2 o\phi \frac{|d\phi|^2}{2} \right) \right).$$

Putting $f = \frac{F'(\alpha^2 o\phi \frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)}$ we get

$$\tau_F(\gamma o\phi) = F'(\alpha^2 o\phi \frac{|d\phi|^2}{2}) trace_g \nabla^\gamma d\gamma(d\phi, d\phi) + f d\gamma(\tau_F(\phi)) + F'\left(\frac{|d\phi|^2}{2}\right) d(\gamma o\phi)(\nabla f).$$

Now since $\gamma : (N, \gamma^* can) \to (N, can)$ is an isometry then, if $\widetilde{\nabla}$ denotes the connection corresponding to $\gamma^* can$, we have

$$\nabla^\gamma d\gamma(X, Y) = d\gamma\left(\widetilde{\nabla}_X Y - \nabla_X Y\right)$$

and since (see [18])

$$\widetilde{\nabla}_X Y - \nabla_X Y = \alpha^{-1}(\langle X, \nabla\alpha\rangle Y + \langle Y, \nabla\alpha\rangle X - \langle X, Y\rangle \nabla\alpha)$$

we obtain

$$trace_g \nabla^\gamma d\gamma(d\phi, d\phi) = 2\alpha^{-1} o\phi . d\gamma\left(d\phi(\nabla\alpha o\phi) - \frac{|d\phi|^2}{2}\nabla\alpha o\phi\right).$$

Finally we infer that

$$\tau_F(\gamma o\phi) = 2\alpha^{-1} o\phi . F'(\alpha^2 o\phi \frac{|d\phi|^2}{2}) d\gamma\left(d\phi(\nabla\alpha o\phi) - \frac{|d\phi|^2}{2}\nabla\alpha o\phi\right) + f d\gamma(\tau_F(\phi)) + F'\left(\frac{|d\phi|^2}{2}\right) d\gamma o d\phi(\nabla f).$$

□

Lemma 2.5. *Let ϕ be an F-harmonic map from an m-dimensional Riemannian manifold (M, g) $(m \geq 2)$ into the Euclidean unit sphere (S^n, can) $(n \geq 2)$.*

Then for any $v \in R^{n+1} - \{0\}$ and any $t_o \in R$ we have

$$\frac{d}{dt} E_F(\gamma_t^v o\phi)_{t=t_o} =$$

$$-2\frac{sht_o}{|v|}\int_M \alpha_{t_o}^3 o\phi . F'\left(\alpha_{t_o}^2 o\phi . \frac{|d\phi|^2}{2}\right)\left(|d\phi|^2 |\overline{v} o\phi|^2 - |d\phi_v|^2\right) dv_g$$

$$-\int_M \alpha_{t_o}^2 o\phi . F'\left(\frac{|d\phi|^2}{2}\right)\langle d\phi(\nabla f_{t_o}), \overline{v} o\phi\rangle dv_g$$

where $f_{t_o} = \frac{F'\left(\alpha^2_{t_o} o\phi \frac{|d\phi|^2}{2}\right)}{F'\left(\frac{|d\phi|^2}{2}\right)}$, $\gamma^* can = \alpha^2_{t_o} can$

and

$$\alpha_{t_o} = \frac{|v|}{\phi_v sht_o + |v|\, cht_o}.$$

Proof. Recall that the first variation formula of the F-energy is given by

$$\frac{d}{dt} E_F\left(\gamma^v_t o\phi\right)_{t=t_o} = -\int_M \left\langle \tau_F\left(\gamma^v_{t_o} o\phi\right), \overline{v} o\left(\gamma^v_{to} o\phi\right)\right\rangle dv_g.$$

By Lemma 2.4 and the fact that ϕ is F-harmonic we get

$$\frac{d}{dt} E_F\left(\gamma^v_t o\phi\right)_{t=t_o} =$$

$$-\int_M 2\alpha_{t_o} o\phi . F'\left(\alpha^2_{t_o} o\phi . \frac{|d\phi|^2}{2}\right)\left\langle \left(\nabla \alpha_{t_o} o\phi\right)^T - \frac{|d\phi|^2}{2} \nabla \alpha_{t_o} o\phi, \overline{v} o\phi \right\rangle dv_g$$

$$-\int_M F'\left(\frac{|d\phi|^2}{2}\right)\left\langle d\phi\left(\nabla f_{t_o}\right), \overline{v} o\phi\right\rangle dv_g$$

and since (see [18])

$$\nabla \alpha^v_{t_o} = -\frac{\left(\alpha^v_{t_o}\right)^2}{|v|} sht_o \overline{v} \tag{2}$$

we have

$$\left\langle \nabla \alpha_{t_o} o\phi, \overline{v} o\phi\right\rangle = -\frac{\left(\alpha^v_{t_o}\right)^2}{|v|} sht_o \left|\overline{v} o\phi\right|^2. \tag{3}$$

Now let $(e_1, ..., e_m)$ be an orthogonal basis on M

$$\left\langle \left(\nabla \alpha_{t_o} o\phi\right)^T, \overline{v} o\phi\right\rangle = \sum_{i=1}^m \left\langle \nabla \alpha_{t_o} o\phi, d\phi(e_i)\right\rangle \left\langle \overline{v} o\phi, d\phi(e_i)\right\rangle$$

$$= -\frac{sht_o}{|v|}\left(\alpha_{t_o} o\phi\right)^2 \sum_{i=1}^m \left\langle \overline{v} o\phi, d\phi(e_i)\right\rangle^2$$

$$= -\frac{sht_o}{|v|}\left(\alpha_{t_o} o\phi\right)^2 \left|d\phi_v\right|^2.$$

Hence

$$\frac{d}{dt} E_F\left(\gamma^v_t o\phi\right)_{t=t_o} =$$

$$-2\frac{sht_o}{|v|}\int_M \alpha^3_{t_o} o\phi . F'\left(\alpha^2_{t_o} o\phi . \frac{|d\phi|^2}{2}\right)\left(\frac{|d\phi|^2}{2}\left|\overline{v} o\phi\right|^2 - \left|d\phi_v\right|^2\right) dv_g$$

$$-\int_M \alpha_{t_o}^2 o\phi . F'\left(\frac{|d\phi|^2}{2}\right)\langle d\phi\left(\nabla f_{t_o}\right), \overline{v} o\phi\rangle\, dv_g.$$

□

We set

$$g(t)=\int_M \alpha_t^2 o\phi . F'\left(\frac{|d\phi|^2}{2}\right)\langle d\phi\left(\nabla f_t\right), \overline{v} o\phi\rangle\, dv_g.$$

Lemma 2.6.

$$g(t)=$$
$$\int_M \alpha_t^3 o\phi . F''(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})\,|d\phi|^2\,\langle d\phi\left(\nabla\left(\alpha_{t_o} o\phi\right)\right), \overline{v} o\phi\rangle\, dv_g$$
$$+\int_M \alpha_t^2 o\phi . \left(F''(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})\alpha_t^2 o\phi - \frac{F'(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)} F''\left(\frac{|d\phi|^2}{2}\right)\right)\frac{\phi_v}{|v|}\,|d\phi|^2\, dv_g.$$

Proof. First, we compute ∇f_t

$$\nabla f_t = \frac{F''(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)}\left(\alpha_t o\phi .\,|d\phi|^2\,\nabla\left(\alpha_t o\phi\right) + \alpha_t^2 o\phi .\,\langle \nabla d\phi, d\phi\rangle\right)$$
$$-\frac{F'(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)^2} F''\left(\frac{|d\phi|^2}{2}\right)\langle \nabla d\phi, d\phi\rangle$$
$$= \frac{F''(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)}\alpha_t o\phi\,|d\phi|^2\,\nabla\left(\alpha_t o\phi\right)$$
$$+\left(\frac{F''(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)}\alpha_t^2 o\phi - \frac{F'(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)^2} F''\left(\frac{|d\phi|^2}{2}\right)\right)\langle \nabla d\phi, d\phi\rangle .$$

Then

$$g(t)=\int_M \alpha_{t_o}^2 o\phi . F'\left(\frac{|d\phi|^2}{2}\right)\langle d\phi\left(\nabla f_{t_o}\right), \overline{v} o\phi\rangle\, dv_g$$
$$=\int_M \alpha_{t_o}^3 o\phi . F''(\alpha_{t_o}^2 o\phi . \frac{|d\phi|^2}{2})\,|d\phi|^2\,\langle d\phi\left(\nabla\left(\alpha_{t_o} o\phi\right)\right), \overline{v} o\phi\rangle\, dv_g$$

(4)

$$+\int_M \alpha_{t_o}^2 o\phi . \left(F''(\alpha_{t_o}^2 o\phi . \frac{|d\phi|^2}{2})\alpha_{t_o}^2 o\phi - \frac{F'(\alpha_{t_o}^2 o\phi . \frac{|d\phi|^2}{2})}{F'\left(\frac{|d\phi|^2}{2}\right)} F''\left(\frac{|d\phi|^2}{2}\right)\right)$$

$$\times \left\langle d\phi \left(\nabla \left| d\phi \right|^2 \right), \overline{v} o \phi \right\rangle dv_g.$$

Let $\{e_1, ..., e_m\}$ be a basis of T_xM which diagonalizes $\phi^* can$, we have

$$\left\langle d\phi \left(\nabla \frac{|d\phi|^2}{2} \right), \overline{v} o \phi \right\rangle = \langle \nabla_{e_i} d\phi, d\phi \rangle \langle \overline{v} o \phi, d\phi(e_j) \rangle \langle d\phi(e_i), d\phi(e_j) \rangle$$

$$= \langle \nabla_{e_i} d\phi, d\phi \rangle \langle \overline{v} o \phi, d\phi(e_j) \rangle \phi^* can (e_i, e_j)$$

$$= \langle \nabla_{\overline{v} o \phi} d\phi (e_j), d\phi (e_j) \rangle$$

$$= \left\langle \nabla_{d\phi(e_j)} \overline{v} o \phi, d\phi (e_j) \right\rangle + \langle [\overline{v} o \phi, d\phi (e_j)], d\phi (e_j) \rangle .$$

Likewise we get

$$\langle [\overline{v} o \phi, d\phi (e_j)], d\phi (e_j) \rangle = -\frac{1}{2} \frac{d}{dt} \mid_{t=0} \gamma_t^* \left| d\phi (e_j) \right|^2$$

$$= -\frac{1}{2} \frac{d}{dt} \mid_{t=0} \alpha_t^2 \left| d\phi (e_j) \right|^2$$

and taking account of (1) we obtain that

$$\langle [\overline{v} o \phi, d\phi (e_j)], d\phi (e_j) \rangle = \frac{\phi_v}{|v|} \left| d\phi (e_j) \right|^2$$

so we infer that

$$\left\langle d\phi \left(\nabla \frac{|d\phi|^2}{2} \right), \overline{v} o \phi \right\rangle = \frac{\phi_v}{|v|} |d\phi|^2 + \left\langle \nabla_{e_j} \overline{v} o \phi, d\phi (e_j) \right\rangle$$

and

$$\left\langle \nabla_{e_j} \overline{v} o \phi, d\phi (e_j) \right\rangle = \nabla_{e_j} \langle \overline{v} o \phi, d\phi (e_j) \rangle - \left\langle \overline{v} o \phi, \nabla_{e_j} d\phi (e_j) \right\rangle$$

$$= \nabla_{e_j} \langle v, d\phi (e_j) \rangle - \left\langle \overline{v} o \phi, \nabla_{e_j} d\phi (e_j) \right\rangle$$

$$= \left\langle v, \nabla_{e_j} d\phi (e_j) \right\rangle - \left\langle \overline{v} o \phi, \nabla_{e_j} d\phi (e_j) \right\rangle$$

$$= \left\langle v - \overline{v} o \phi, \nabla_{e_j} d\phi (e_j) \right\rangle = 0.$$

Hence

$$\left\langle d\phi \left(\nabla \frac{|d\phi|^2}{2} \right), \overline{v} o \phi \right\rangle = \frac{\phi_v}{|v|} |d\phi|^2 . \tag{5}$$

□

Now set

$$\varphi(t_o) = 2 \frac{sht_o}{|v|} \int_M \alpha_{t_o}^3 o\phi . F' \left(\alpha_{t_o}^2 o\phi . \frac{|d\phi|^2}{2} \right) \left(-\frac{|d\phi|^2}{2} |\overline{v} o \phi|^2 + |d\phi_v|^2 \right) dv_g$$

$$- \int_M \alpha_{t_o}^3 o\phi . F''(\alpha_{t_o}^2 o\phi . \frac{|d\phi|^2}{2}) \frac{|d\phi|^2}{2} \langle d\phi (\nabla (\alpha_{t_o} o\phi)), \overline{v} o \phi \rangle dv_g$$

and since by (2) we have

$$\langle d\phi\left(\nabla\left(\alpha_{t_o}o\phi\right)\right),\overline{v}o\phi\rangle=-\frac{sht_o}{|v|}\alpha_{t_o}^2o\phi\left|d\phi_v\right|^2$$

we get

(6)

$$\varphi(t_o)=2\frac{sht_o}{|v|}\int_M\alpha_{t_o}^3o\phi.\left[\left(F'\left(\alpha_{t_o}^2o\phi.\frac{\left|d\phi\right|^2}{2}\right)+\alpha_{t_o}^2o\phi.\frac{\left|d\phi\right|^2}{2}F''\left(\alpha_{t_o}^2o\phi.\frac{\left|d\phi\right|^2}{2}\right)\right)\left|d\phi_v\right|^2\right.$$

$$\left.-F'\left(\alpha_{t_o}^2o\phi.\frac{\left|d\phi\right|^2}{2}\right)\frac{\left|d\phi\right|^2}{2}\left|\overline{v}o\phi\right|^2\right]dv_g.$$

or

$$\varphi(t_o)=-2\frac{sht_o}{|v|}\int_M\alpha_{t_o}^3o\phi.S_g^F\left(\gamma_t^vo\phi\right)dv_g.$$

Proof. (of Theorem 2.2) Recall (see [13]) that for any conformal diffeomorphism γ of the unit sphere S^n there exist an isometry $r\in O\left(n+1\right)$, a real number $t\geq0$ and a vector $v\in R^{n+1}-\{0\}$ such that $\gamma=ro\gamma_t^v$, so it suffices to consider γ_t^v with $t\geq0$ and $v\in R^{n+1}-\{0\}$.

On the other hand

$$\frac{d}{dt}E_F\left(\gamma_t^vo\phi\right)=\varphi(t)+\chi(t)$$

where

$$\chi(t)=-\int_M\alpha_t^2o\phi.\left(F''(\alpha_t^2o\phi.\frac{\left|d\phi\right|^2}{2})\alpha_t^2o\phi-\frac{F'(\alpha_t^2o\phi.\frac{\left|d\phi\right|^2}{2})}{F'\left(\frac{\left|d\phi\right|^2}{2}\right)}F''\left(\frac{\left|d\phi\right|^2}{2}\right)\right)\frac{\phi_v}{|v|}\left|d\phi\right|^2dv_g$$

and $\varphi\left(t\right)$ is given by (6). Now, since the function F is admissible we infer that $\chi(t)\leq0$. Since the energy stress-tensor $S_g^F\left(\gamma o\phi\right)=S_g^F\left(\gamma_to\phi\right)$ of $\gamma o\phi$ is positive (resp. positive defined) by assumption and

$$S_g^F\left(\gamma_to\phi\right)=F'\left(\frac{\left|d\left(\gamma_to\phi\right)\right|^2}{2}\right)\frac{\left|d\left(\gamma_to\phi\right)\right|^2}{2}g$$

$$-\left[F'\left(\frac{\left|d\left(\gamma_to\phi\right)\right|^2}{2}\right)+\frac{\left|d\left(\gamma_to\phi\right)\right|^2}{2}F''\left(\frac{\left|d\left(\gamma_to\phi\right)\right|^2}{2}\right)\right]\left(\gamma_to\phi\right)^*can$$

$$=\alpha_t^2o\phi\left[F'\left(\alpha_t^2o\phi\frac{\left|d\phi\right|^2}{2}\right)\frac{\left|d\phi\right|^2}{2}g-F'\left(\alpha_t^2o\phi\frac{\left|d\phi\right|^2}{2}\right)\right.$$

$$-\alpha_t^2 o\phi \frac{|d\phi|^2}{2} F''\left(\alpha_t^2 o\phi \frac{|d\phi|^2}{2}\right)\Bigg] \phi^* can$$

so the tensor

$$F'\left(\alpha_t^2 o\phi \frac{|d\phi|^2}{2}\right)\frac{|d\phi|^2}{2} g - \left(F'\left(\alpha_t^2 o\phi \frac{|d\phi|^2}{2}\right) + \alpha_t^2 o\phi \frac{|d\phi|^2}{2} F''\left(\alpha_t^2 o\phi \frac{|d\phi|^2}{2}\right)\right)\phi^* can$$

is positive (resp. positive defined). Consequently $\varphi(t) \leq 0$ (resp. $\varphi(t) < 0$) for any $t \geq 0$ and the proof of Theorem 2.2 is complete. □

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