

Weighted Improper Colouring.[☆]

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Abstract

In this paper, we study a colouring problem motivated by a practical frequency assignment problem and, up to our best knowledge, new. In wireless networks, a node interferes with other nodes, the level of interference depending on numerous parameters: distance between the nodes, geographical topography, obstacles, etc. We model this with a weighted graph (G, w) where the weight function w on the edges of G represents the noise (interference) between the two end-vertices. The total interference in a node is then the sum of all the noises of the nodes emitting on the same frequency. A weighted t -improper k -colouring of (G, w) is a k -colouring of the nodes of G (assignment of k frequencies) such that the interference at each node does not exceed the threshold t . We consider here the Weighted Improper Colouring problem which consists in determining the weighted t -improper chromatic number defined as the minimum integer k such that (G, w) admits a weighted t -improper k -colouring. We also consider the dual problem, denoted the Threshold Improper Colouring problem, where, given a number k of colours, we want to determine the minimum real t such that (G, w) admits a weighted t -improper k -colouring. We first present general upper bounds for both problems; in particular we show a generalisation of Lovász's Theorem for the weighted t -improper chromatic number. Motivated by the original application, we then study a special interference model on various grids (square, triangular, hexagonal) where a node produces a noise of intensity 1 for its neighbours and a noise of intensity $1/2$ for the nodes at distance two. We derive the weighted t -improper chromatic number for all values of t .

Keywords: graph colouring, improper colouring, interference, radio networks, frequency assignment.

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1. Introduction

Let $G = (V, E)$ be a graph. A k -colouring of G is a function $c : V \rightarrow \{1, \dots, k\}$. The colouring c is *proper* if $uv \in E$ implies $c(u) \neq c(v)$. The *chromatic number* of G , denoted by $\chi(G)$, is the minimum integer k such that G admits a proper k -colouring. The goal of the VERTEX COLOURING problem is to determine $\chi(G)$ for a given graph G . It is a well-known NP-hard problem [14].

A k -colouring c is *l -improper* if $|\{v \in N(u) \mid c(v) = c(u)\}| \leq l$, for all $u \in V$ (as usual in the literature, $N(u)$ stands for the set $\{v \mid uv \in E(G)\}$). Given a non-negative integer l , the *l -improper chromatic number* of a graph G , denoted by $\chi_l(G)$, is the minimum integer k such that G admits an l -improper k -colouring. Given a graph G and an integer l , the IMPROPER COLOURING problem consists in determining $\chi_l(G)$ and is also NP-hard [17, 7]. Indeed, if $l = 0$, observe that $\chi_0(G) = \chi(G)$. Consequently, VERTEX COLOURING is a particular case of IMPROPER COLOURING.

In this work we define and study a new variation of the IMPROPER COLOURING problem for edge-weighted graphs. An edge-weighted graph is a pair (G, w) where $G = (V, E)$ is a graph and $w : E \rightarrow \mathbb{R}_+^*$. Given an edge-weighted graph (G, w) and a colouring c of G , the *interference* of a vertex u in this colouring is defined by

$$I_u(G, w, c) = \sum_{\{v \in N(u) \mid c(v) = c(u)\}} w(u, v).$$

For any non-negative real number t , called *threshold*, we say that c is a *weighted t -improper k -colouring* of (G, w) if c is a k -colouring of G such that $I_u(G, w, c) \leq t$, for all $u \in V$.

Given a threshold $t \in \mathbb{R}_+^*$, the minimum integer k such that the graph G admits a weighted t -improper k -colouring is the *weighted t -improper chromatic number* of (G, w) , denoted by $\chi_t(G, w)$. Given an edge-weighted graph (G, w) and a threshold $t \in \mathbb{R}_+^*$, determining $\chi_t(G, w)$ is the goal of the WEIGHTED IMPROPER COLOURING problem. Note that if $t = 0$ then $\chi_0(G, w) = \chi(G)$, and if $w(e) = 1$ for all $e \in E$, then $\chi_l(G, w) = \chi_l(G)$ for any positive integer l . Therefore, the WEIGHTED IMPROPER COLOURING problem is clearly NP-hard since it generalises VERTEX COLOURING and IMPROPER COLOURING.

On the other hand, given a positive integer k , we define the *minimum k -threshold* of (G, w) , denoted by $T_k(G, w)$ as the minimum real t such that (G, w) admits a weighted t -improper k -colouring. Then, for a given edge-weighted graph (G, w) and a positive integer k , the THRESHOLD IMPROPER COLOURING problem consists in determining $T_k(G, w)$. The THRESHOLD IMPROPER COLOURING problem is also NP-hard. This fact follows from the observation that determining whether $\chi_l(G) \leq k$ is NP-complete, for every $l \geq 2$ and $k \geq 2$ [9, 8, 7]. Consequently, in particular, it is a NP-complete problem to decide whether a graph G admits a weighted t -improper 2-colouring when all the weights of the edges of G are equal to one, for every $t \geq 2$.

1.1. Motivation

Our initial motivation to these problems was the design of satellite antennas for multi-spot MFTDMA satellites [2]. In this technology, satellites transmit signals to areas on the ground called *spots*. These spots form a grid-like structure which is modelled by an hexagonal cell graph. To each spot is assigned a radio channel or colour. Spots are interfering with other spots having the same channel and a spot can use a colour only if the interference level does not exceed a given threshold t . The level of interference between two spots depends on their distance. The authors of [2] introduced a factor of mitigation γ and the interference of remote spots are reduced by a factor $1 - \gamma$. When the interference level is too low, the nodes are considered to not interfere anymore. Considering such types of interference, where nodes at distance at most i interfere, leads to the study of the i -th power of the graph modelling the network and a case of special interest is the power of grid graphs (see Section 3).

1.2. Related Work

Our problems are particular cases of the FREQUENCY ASSIGNMENT problem (FAP). FAP has several variations that were already studied in the literature (see [1] for a survey). In most of these variations, the main constraint to be satisfied is that if two vertices (mobile phones, antennas, spots, etc.) are close, then the difference between the frequencies that are assigned to them must be greater than some function which usually depends on their distance.

There is a strong relationship between most of these variations and the $L(p_1, \dots, p_d)$ -LABELLING problem [18]. In this problem, the goal is to find a colouring of the vertices of a given graph G , in such a way that the difference between the colours assigned to vertices at distance i is at least p_i , for every $i = 1, \dots, d$.

In some other variants, for each non-satisfied interference constraint a penalty must be paid. In particular, the goal of the MINIMUM INTERFERENCE FREQUENCY ASSIGNMENT problem (MI-FAP) is to minimise the total penalties that must be paid, when the number of frequencies to be assigned is given. This problem can also be studied for only *co-channel interference*, in which the penalties are applied only if the two vertices have the same frequency. However, MI-FAP under these constraints does not correspond to WEIGHTED IMPROPER COLOURING, because we consider the co-channel interference, i.e. penalties, just between each vertex and its neighbourhood.

The two closest related works we found in the literature are [16] and [10]. However, they both apply penalties over co-channel interference, but also to the *adjacent channel interference*, i.e. when the colours of adjacent vertices differ by one unit. Moreover, their results are not similar to ours. In [16], they propose an enumerative algorithm for the problem, while in [10] a Branch-and-Cut method is proposed and applied over some instances.

1.3. Results

In this article, we study both parameters $\chi_t(G, w)$ and $T_k(G, w)$. We first present general bounds; in particular we show a generalisation of Lovász's Theorem for $\chi_t(G, w)$.

Motivated by the original application, we then study a special interference model on various grids (square, triangular, hexagonal) where a node produces a noise of intensity 1 for its neighbours and a noise of intensity 1/2 for the nodes that are at distance two. We derive the weighted t -improper chromatic number for all possible values of t . These proofs easily lead to optimal polynomial-time algorithms to find weighted improper colourings of these graphs. We also give an algorithm for trees for this interference model differing from the lower bound by at most 2.

2. General Results

In this section, we present some results for WEIGHTED IMPROPER COLOURING and THRESHOLD IMPROPER COLOURING for general graphs and general interference models.

Let (G, w) be an edge-weighted graph with positive rational weights given by $w : E(G) \rightarrow \mathbb{Q}_+^*$. For any vertex $v \in V(G)$, its *weighted degree* is $d_w(v) = \sum_{u \in N(v)} w(u, v)$. The *maximum weighted degree* of G is $\Delta(G, w) = \max_{v \in V} d_w(v)$.

Given a k -colouring $c : V \rightarrow \{1, \dots, k\}$ of G , we define, for every vertex $v \in V(G)$ and colour $i = 1, \dots, k$, $d_{w,c}^i(v) = \sum_{\{u \in N(v) \mid c(u)=i\}} w(u, v)$. Note that $d_{w,c}^c(v) = I_v(G, w, c)$. We say that a k -colouring c of G is *w-balanced* if c satisfies the following property:

For any vertex $v \in V(G)$, $I_v(G, w, c) \leq d_{w,c}^j(v)$, for every $j = 1, \dots, k$.

We denote by $\gcd(w)$ the greatest common divisor of the weights of w (observe that $\gcd(w) > 0$ because we just consider positive weights). We use here the generalisation of the gcd to non-integer numbers (e.g. in \mathbb{Q}) where a number x is said to divide a number y if the fraction y/x is an integer. The important property of $\gcd(w)$ is that the difference between two interferences is a multiple of $\gcd(w)$; in particular, if for two vertices v and u , $d_{w,c}^i(v) > d_{w,c}^i(u)$, then $d_{w,c}^i(v) \geq d_{w,c}^i(u) + \gcd(w)$.

If t is not a multiple of the $\gcd(w)$, that is, there exists an integer $a \in \mathbb{Z}$ such that $a \gcd(w) < t < (a+1)\gcd(w)$, then $\chi_t^w(G) = \chi_{a \gcd(w)}^w(G)$.

Proposition 1. *Let (G, w) be an edge-weighted graph. For any $k \geq 2$, there exists a w -balanced k -colouring of G .*

Proof. Let us colour $G = (V, E)$ arbitrarily with k colours and then repeat the following procedure: if there exists a vertex v coloured i and a colour j such that $d_{w,c}^i(v) > d_{w,c}^j(v)$, then recolour v with colour j . Observe that this procedure neither increases (we just move a vertex from one colour to another) nor decreases (a vertex without neighbour on its colour is never moved) the

number of colours within this process. Let W be the sum of the weights of the edges having the same colour in their end-vertices. In this transformation, W has increased by $d_{w,c}^j(v)$ (edges incident to v that previously had colour j in its endpoint opposite to v), but decreased by $d_{w,c}^i(v)$ (edges that previously had colour i in both of their end-vertices). So, W has decreased by $d_{w,c}^i(v) - d_{w,c}^j(v) \geq \gcd(w)$. As $W \leq |E| \max_{e \in E} w(e)$ is finite, this procedure finishes and produces a w -balanced k -colouring of G . \square

The existence of a w -balanced colouring gives easily some upper bounds on the weighted t -improper chromatic number and the minimum k -threshold of an edge-weighted graph (G, w) . It is a folklore result that $\chi(G) \leq \Delta(G) + 1$, for any graph G . Lovász [15] extended this result for IMPROPER COLOURING problem using w -balanced colouring. He proved that $\chi_t(G) \leq \lceil \frac{\Delta(G)+1}{t+1} \rceil$. In what follows, we extend this result to weighted improper colouring.

Theorem 2. *Let (G, w) be an edge-weighted graph with $w : E(G) \rightarrow \mathbb{Q}_+$, and t a multiple of $\gcd(w)$. Then*

$$\chi_t(G, w) \leq \left\lceil \frac{\Delta(G, w) + \gcd(w)}{t + \gcd(w)} \right\rceil.$$

Proof. If t, w , and G are such that $\chi_t(G, w) = 1$, then the inequality is trivially satisfied. Thus, consider that $\chi_t(G, w) > 1$.

Observe that, in any w -balanced k -colouring c of a graph G , the following holds:

$$d_w(v) = \sum_{u \in N(v)} w(u, v) \geq k d_{w,c}^{c(v)}(v). \quad (1)$$

Let $k^* = \left\lceil \frac{\Delta(G, w) + \gcd(w)}{t + \gcd(w)} \right\rceil \geq 2$ and c^* be a w -balanced k^* -colouring of G .

We claim that c^* is a weighted t -improper k^* -colouring of (G, w) .

By contradiction, suppose that there is a vertex v in G such that $c^*(v) = i$ and that $d_{w,c^*}^i(v) > t$. Since c^* is w -balanced, $d_{w,c^*}^j(v) > t$, for all $j = 1, \dots, k^*$. By the definition of $\gcd(w)$ and as t is a multiple of $\gcd(w)$, it leads to $d_{w,c^*}^j(v) \geq t + \gcd(w)$ for all $j = 1, \dots, k^*$. Combining this inequality with Inequality (1), we obtain:

$$\Delta(G, w) \geq d_w(v) \geq k^*(t + \gcd(w)),$$

giving

$$\Delta(G, w) \geq \Delta(G, w) + \gcd(w),$$

a contradiction. The result follows. \square

Note that when all weights are unit, we obtain the bound for the improper colouring derived in [15]. Brooks [6] proved that for a connected graph G , $\chi(G) = \Delta(G) + 1$ if, and only if, G is complete or an odd cycle. One could wonder for which edge-weighted graphs the bound we provided in Theorem 2 is tight. However, Correa *et al.* [7] already showed that it is NP-complete to determine

if the improper chromatic number of a graph G attains the upper bound of Lovász, which is a particular case of WEIGHTED IMPROPER COLOURING, i.e. of the bound of Theorem 2.

We now show that w -balanced colourings also yield upper bounds for the minimum k -threshold of an edge-weighted graph (G, w) . When $k = 1$, then all the vertices must have the same colour, and $T_1(G, w) = \Delta(G, w)$. This may be generalised as follows, using w -balanced colourings.

Theorem 3. *Let (G, w) be an edge-weighted graph with $w : E(G) \rightarrow \mathbb{R}_+^*$, and let k be a positive integer. Then*

$$T_k(G, w) \leq \frac{\Delta(G, w)}{k}.$$

Proof. Let c be a w -balanced k -colouring of G . Then, for every vertex $v \in V(G)$:

$$kT_k(G, w) \leq kd_{w,c}^{c(v)}(v) \leq d_w(v) = \sum_{u \in N(v)} w(u, v) \leq \Delta(G, w)$$

□

Because $T_1(G, w) = \Delta(G, w)$, Theorem 3 may be restated as $kT_k(G, w) \leq \dots \leq T_1(G, w)$. This inequality may be generalised as follows.

Theorem 4. *Let (G, w) be an edge-weighted graph with $w : E(G) \rightarrow \mathbb{R}_+$, and let k and p be two positive integers. Then*

$$T_{kp}(G, w) \leq \frac{T_p(G, w)}{k}.$$

Proof. Set $t = T_p(G, w)$. Let c be a t -improper p -colouring of (G, w) . For $i = 1, \dots, p$, let G_i be the subgraph of G induced by the vertices coloured i by c . By definition of improper colouring $\Delta(G_i, w) \leq t$ for all $1 \leq i \leq p$. By Theorem 3, each (G_i, w) admits a t/k -improper k -colouring c_i with colours $\{(i-1)k+1, \dots, ik\}$. The union of the c_i 's is then a t/k -improper kp -colouring of (G, w) . □

Theorem 4 and its proof suggest that to find a kp -colouring with small impropriety, it may be convenient to first find a p -colouring with small impropriety and then to refine it. In addition, such a strategy allows to adapt dynamically the refinement. In the above proof, the vertex set of each part G_i is again partitioned into k parts. However, sometimes, we shall get a better kp -colouring by partitioning each G_i into a number of k_i parts, with $\sum_{i=1}^p k_i = kp$. Doing so, we obtain a T -improper kp -colouring of (G, w) , where $T = \max\{\frac{\Delta(G_i, w)}{k_i}, 1 \leq i \leq p\}$.

One can also find an upper bound on the minimum k -threshold by considering first the $k-1$ edges of largest weight around each vertex. Let (G, w) be an edge-weighted graph, and let v_1, \dots, v_n be an ordering of the vertices of G . The edges of G may be ordered in increasing order of their weight. Furthermore, to

make sure that the edges incident to any particular vertex are totally ordered, we break ties according to the label of the second vertex. Formally, we say that $v_i v_j \leq_w v_i v_{j'}$ if either $w(v_i v_j) < w(v_i v_{j'})$ or $w(v_i v_j) = w(v_i v_{j'})$ and $j < j'$. With such a partial order on the edge set, the set $E_w^k(v)$ of $\min\{|N(v)|, k-1\}$ greatest edges (according to this ordering) around a vertex is uniquely defined. Observe that every edge incident to v and not in $E_w^k(v)$ is smaller than an edge of $E_w^k(v)$ for \leq_w .

Let G_w^k be the graph with vertex set $V(G)$ and edge set $\bigcup_{v \in V(G)} E_w^k(v)$. Observe that every vertex of $E_w^k(v)$ has degree at least $\min\{|N(v)|, k-1\}$, but a vertex may have an arbitrarily large degree. For if any edge incident to v has a greater weight than any edge not incident to v , the degree of v in G_w^k is equal to its degree in G . However we now prove that at least one vertex has degree $k-1$.

Proposition 5. *If (G, w) is an edge-weighted graph, then G_w^k has a vertex of degree at most $k-1$.*

Proof. Suppose for a contradiction, that every vertex has degree at least k , then for every vertex x there is an edge xy in $E(G_w^k) \setminus E_w^k(x)$, and so in $E_w^k(y) \setminus E_w^k(x)$. Therefore, there must be a cycle (x_1, \dots, x_r) such that, for all $1 \leq i \leq r$, $x_i x_{i+1} \in E_w^k(x_{i+1}) \setminus E_w^k(x_i)$ (with $x_{r+1} = x_1$). It follows that $x_1 x_2 \leq_w x_2 x_3 \leq_w \dots \leq_w x_r x_1 \leq_w x_1 x_2$. Hence, by definition, $w(x_1 x_2) = w(x_2 x_3) = \dots = w(x_r x_1) = w(x_1 x_2)$. Let m be the integer such that x_m has maximum index in the ordering v_1, \dots, v_n . Then there exists j and j' such that $x_m = v_j$ and $x_{m+2} = v_{j'}$. By definition of m , we have $j > j'$. But this contradicts the fact that $x_m x_{m+1} \leq_w x_{m+1} x_{m+2}$. \square

Corollary 6. *If (G, w) is an edge-weighted graph, then G_w^k has a proper k -colouring.*

Proof. By induction on the number of vertices. By Proposition 5, G_w^k has a vertex x of degree at most $k-1$. Trivially, $G_w^k - x$ is a subgraph of $(G-x)_w^k$. By the induction hypothesis, $(G-x)_w^k$ has a proper k -colouring, which is also a proper k -colouring of $G_w^k - x$. This colouring can be extended in a proper k -colouring of G_w^k , by assigning to x a colour not assigned to any of its $k-1$ neighbours. \square

Corollary 7. *If (G, w) is an edge-weighted graph, then $T_k(G, w) \leq \Delta(G \setminus E(G_w^k), w)$.*

3. Squares of Particular Graphs

As mentioned in the introduction, WEIGHTED IMPROPER COLOURING is motivated by networks of antennas similar to grids [2]. In these networks, the noise generated by an antenna undergoes an attenuation with the distance it travels. It is often modelled by a decreasing function of d , typically $1/d^\alpha$ or $1/(2^{d-1})$.

Here we consider a simplified model where the noise between two neighbouring antennas is normalised to 1, between antennas at distance two is $1/2$ and 0 when the distance is strictly greater than two. Studying this model of interference corresponds to study the WEIGHTED IMPROPER COLOURING of the square of the graph G , that is the graph G^2 obtained from G by joining every pair of vertices at distance two, and to assign weights $w_2(e) = 1$, if $e \in E(G)$, and $w_2(e) = 1/2$, if $e \in E(G^2) \setminus E(G)$. Observe that in this case the interesting threshold values are the non-negative multiples of $1/2$.

Figure 1 shows some examples of colouring for the square grid. In Figure 1(b), each vertex x has neither a neighbour nor a vertex at distance two coloured with its own colour, so $I_x(G^2, w_2, c) = 0$ and G^2 admits a weighted 0-improper 5-colouring. In Figure 1(c), each vertex x has no neighbour with its colour and at most one vertex of the same colour at distance 2. So $I_x(G^2, w_2, c) = 1/2$ and G^2 admits a weighted 0.5-improper 4-colouring.

For any $t \in \mathbb{R}_+$, we determine the weighted t -improper chromatic number for the square of infinite paths, square grids, hexagonal grids and triangular grids under the interference model w_2 . We also present lower and upper bounds for $\chi_t(T^2, w_2)$, for any tree T and any threshold t .

3.1. Infinite paths and trees

In this section, we characterise the weighted t -improper chromatic number of the square of an infinite path, for all positive real t . Moreover, we present lower and upper bounds for $\chi_t(T^2, w_2)$, for a given tree T .

Theorem 8. *Let $P = (V, E)$ be an infinite path. Then,*

$$\chi_t(P^2, w_2) = \begin{cases} 3, & \text{if } 0 \leq t < 1; \\ 2, & \text{if } 1 \leq t < 3; \\ 1, & \text{if } 3 \leq t. \end{cases}$$

Proof. Let $V = \{v_i \mid i \in \mathbb{Z}\}$ and $E = \{(v_{i-1}, v_i) \mid i \in \mathbb{Z}\}$. Each vertex of P has two neighbours and two vertices at distance two. Consequently, the equivalence $\chi_t(P^2, w_2) = 1$ if, and only if, $t \geq 3$ holds trivially.

There is a 2-colouring c of (P^2, w_2) with maximum interference 1 by just colouring v_i with colour $(i \bmod 2) + 1$. So $\chi_t(P^2, w_2) \leq 2$ if $t \geq 1$. We claim that there is no weighted 0.5-improper 2-colouring of (P^2, w_2) . By contradiction, suppose that c is such a colouring. If $c(v_i) = 1$, for some $i \in \mathbb{Z}$, then $c(v_{i-1}) = c(v_{i+1}) = 2$ and $c(v_{i-2}) = c(v_{i+2}) = 1$. This is a contradiction because v_i would have interference 1.

Finally, the colouring $c(v_i) = (i \bmod 3) + 1$, for every $i \in \mathbb{Z}$, is a feasible weighted 0-improper 3-colouring. \square

Theorem 9. *Let $T = (V, E)$ be a (non-empty) tree. Then, $\left\lceil \frac{\Delta(T) - \lfloor t \rfloor}{2t+1} \right\rceil + 1 \leq \chi_t(T^2, w_2) \leq \left\lceil \frac{\Delta(T) - 1}{2t+1} \right\rceil + 2$.*

Proof. The lower bound is obtained by two simple observations. First, $\chi_t(H, w) \leq \chi_t(G, w)$, for any $H \subseteq G$. Let T be a tree and v be a node of maximum degree in T . Then, observe that the weighted t -improper chromatic number of the subgraph of T^2 induced by v and its neighbourhood is at least $\lceil \frac{\Delta(T) - |t|}{2t+1} \rceil + 1$. Indeed, the colour of v can be assigned to at most $\lfloor t \rfloor$ vertices on its neighbourhood. Any other colour used in the neighbourhood of v cannot appear in more than $2t + 1$ vertices because each pair of vertices in the neighbourhood of v is at distance two.

Let us look now at the upper bound. Choose any node $r \in V$ to be the root of T . Colour r with colour 1. Then, by a breadth-first traversal in the tree, for each visited node v colour all the children of v with the $\lceil \frac{\Delta(T) - 1}{2t+1} \rceil$ colours different from the ones assigned to v and to its parent in such a way that at most $2t + 1$ nodes have the same colour. This is a feasible weighted t -improper k -colouring of T^2 , with $k \leq \lceil \frac{\Delta(T) - 1}{2t+1} \rceil + 2$, since each vertex interferes with at most $2t$ vertices at distance two which are children of its parent. \square

For a tree T and the weighted function w^2 , Theorem 9 provides upper and lower bounds on $\chi_t(T^2, w_2)$, but we do not know the computational complexity of determining $\chi_t(T^2, w_2)$.

3.2. Grids

In this section, we show the optimal values of $\chi_t(G^2, w_2)$, whenever G is an infinite square, hexagonal or triangular grid, for all the possible values of t .

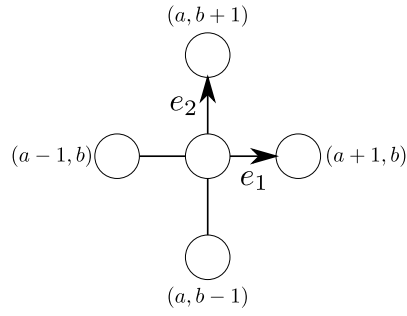
3.2.1. Square Grid

The square grid is the graph \mathfrak{S} in which the vertices are all integer linear combinations $ae_1 + be_2$ of the two vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$, for any $a, b \in \mathbb{Z}$. Each vertex (a, b) has four neighbours: its *down neighbour* $(a, b - 1)$, its *up neighbour* $(a, b + 1)$, its *right neighbour* $(a + 1, b)$ and its *left neighbour* $(a - 1, b)$ (see Figure 1(a)).

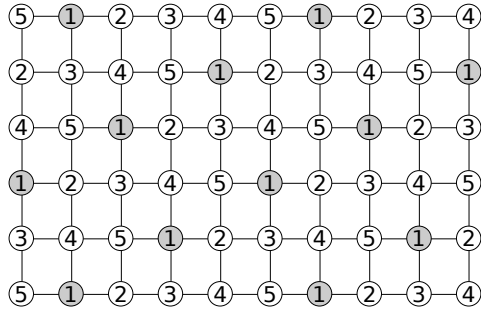
Theorem 10.

$$\chi_t(\mathfrak{S}^2, w_2) = \begin{cases} 5, & \text{if } t = 0; \\ 4, & \text{if } t = 0.5; \\ 3, & \text{if } 1 \leq t < 3; \\ 2, & \text{if } 3 \leq t < 8; \\ 1, & \text{if } 8 \leq t. \end{cases}$$

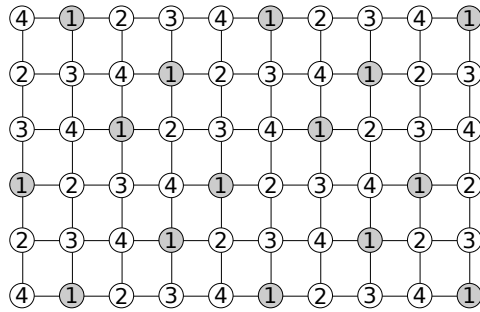
Proof. If $t = 0$, then the colour of vertex (a, b) must be different from the ones used on its four neighbours. Moreover, all the neighbours have different colours, as each pair of neighbours is at distance two. Consequently, at least five colours are needed. The following construction provides a weighted 0-improper 5-colouring of (\mathfrak{S}^2, w_2) : for $0 \leq j \leq 4$, let $A_j = \{(j, 0) + a(5e_1) + b(2e_1 + 1e_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 4$, assign the colour $j + 1$ to all the vertices in A_j (see Figure 1(b)).



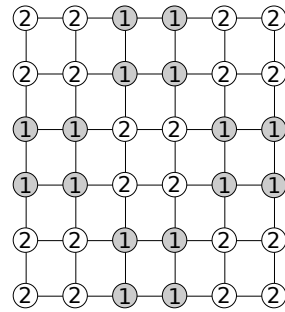
(a)



(b)



(c)



(d)

Figure 1: Optimal colourings of (\mathfrak{S}^2, w_2) : (b) weighted 0-improper 5-colouring of (\mathfrak{S}^2, w_2) , (c) weighted 0.5-improper 4-colouring of (\mathfrak{S}^2, w_2) , and (d) weighted 3-improper 2-colouring of (\mathfrak{S}^2, w_2) .

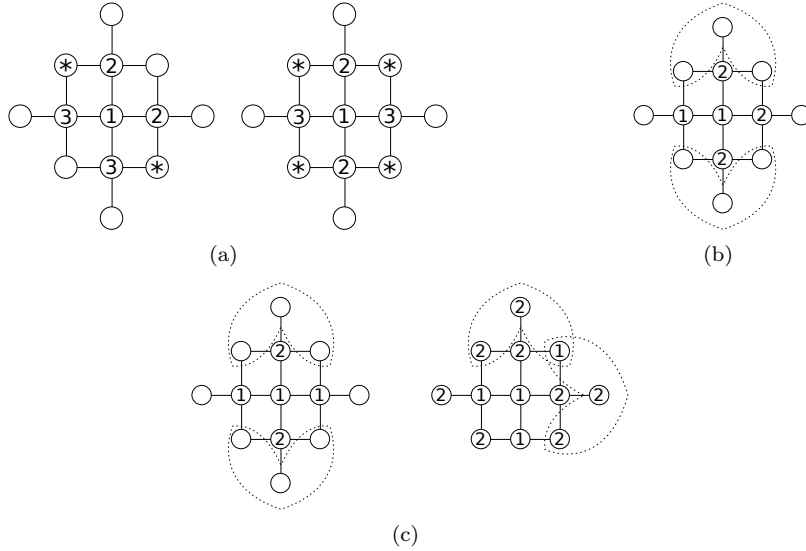


Figure 2: Lower bounds for the square grid: (a) if $t \leq 0.5$ and $k \leq 3$, there is no weighted t -improper k -colouring of (\mathfrak{S}^2, w_2) ; (b) the first case when $t \leq 2.5$ and $k \leq 2$, and (c) the second case.

When $t = 0.5$, we claim that at least four colours are needed to colour (\mathfrak{S}^2, w_2) . The proof is by contradiction. Suppose that there exists a weighted 0.5-improper 3-colouring of it. Let (a, b) be a vertex coloured 1. None of its neighbours is coloured 1, otherwise (a, b) has interference 1. If three neighbours have the same colour, then each of them will have interference 1. So two of its neighbours have to be coloured 2 and the two other ones 3 (see Figure 2(a)). Now consider the four nodes $(a - 1, b - 1)$, $(a - 1, b + 1)$, $(a + 1, b - 1)$ and $(a + 1, b + 1)$. For all configurations, at least two of these four vertices have to be coloured 1 (the ones indicated by a * in Figure 2(a)). But then (a, b) will have interference at least 1, a contradiction. A weighted 0.5-improper 4-colouring of (\mathfrak{S}^2, w_2) can be obtained as follows (see Figure 1(c)): for $0 \leq j \leq 3$, let $B_j = \{(j, 0) + a(4e_1) + b(3e_1 + 2e_2) \mid \forall a, b \in \mathbb{Z}\}$ and $B'_j = \{(j + 1, 2) + a(4e_1) + b(3e_1 + 2e_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 3$, assign the colour $j + 1$ to all the vertices in B_j and in B'_j .

If $t = 1$, there exists a weighted 1-improper 3-colouring of (\mathfrak{S}^2, w_2) given by the following construction: for $0 \leq j \leq 2$, let $C_j = \{(j, 0) + a(3e_1) + b(e_1 + e_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 2$, assign the colour $j + 1$ to all the vertices in C_j .

Now we prove by contradiction that for $t = 2.5$ we still need at least three colours in a weighted 2.5-improper colouring of (\mathfrak{S}^2, w_2) . Consider a weighted 2.5-improper 2-colouring of (\mathfrak{S}^2, w_2) and let (a, b) be a vertex coloured 1. Vertex (a, b) has at most two neighbours of colour 1, otherwise it will have interference 3. We distinguish three cases:

1. Exactly one of its neighbours is coloured 1; let $(a - 1, b)$ be this vertex. Then,

the three other neighbours are coloured 2 (see Figure 2(b)). Consider the two sets of vertices $\{(a-1, b-1), (a+1, b-1), (a, b-2)\}$ and $\{(a-1, b+1), (a+1, b+1), (a, b+2)\}$ (these sets are surrounded by dotted lines in Figure 2(b)); each of them has at least two vertices coloured 1, otherwise the vertex $(a, b-1)$ or $(a, b+1)$ will have interference 3. But then (a, b) having four vertices at distance two coloured 1 has interference 3, a contradiction.

2. Two neighbours of (a, b) are coloured 1.
 - (a) These two neighbours are opposite, say $(a-1, b)$ and $(a+1, b)$ (see Figure 2(c) left). Consider again the two sets $\{(a-1, b-1), (a+1, b-1), (a, b-2)\}$ and $\{(a-1, b+1), (a+1, b+1), (a, b+2)\}$ (these sets are surrounded by dotted lines in Figure 2(c) left); they both contain at least one vertex of colour 1 and therefore (a, b) will have interference 3, a contradiction.
 - (b) The two neighbours of colour 1 are of the form $(a, b-1)$ and $(a-1, b)$ (see Figure 2(c) right). Consider the two sets of vertices $\{(a+1, b-1), (a+1, b+1), (a+2, b)\}$ and $\{(a+1, b+1), (a-1, b+1), (a, b+2)\}$ (these sets are surrounded by dotted lines in Figure 2(c) right); these two sets contain at most one vertex of colour 1, otherwise (a, b) will have interference 3. Moreover, each of these sets cannot be completely coloured 2, otherwise $(a+1, b)$ or $(a, b+1)$ will have interference at least 3. So vertices $(a+1, b-1)$, $(a+2, b)$, $(a, b+2)$ and $(a-1, b+1)$ are of colour 2 and the vertex $(a+1, b+1)$ is of colour 1. But then $(a-2, b)$ and $(a-1, b-1)$ are of colour 2, otherwise (a, b) will have interference 3. Thus, vertex $(a-1, b)$ has exactly one neighbour coloured 1 and we are again in Case 1.
3. All neighbours of (a, b) are coloured 2. If one of these neighbours has itself a neighbour (distinct from (a, b)) of colour 2, we are in Case 1 or 2 for this neighbour. Therefore, all vertices at distance two from (a, b) have colour 1 and the interference in (a, b) is 4, a contradiction.

A weighted 3-improper 2-colouring of (\mathfrak{S}^2, w_2) can be obtained as follows: a vertex of the grid (a, b) is coloured with colour $(\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor \pmod{2} + 1)$, see Figure 1(d).

Finally, since each vertex has four neighbours and eight vertices at distance two, there is no weighted 7.5-improper 1-colouring of (\mathfrak{S}^2, w_2) and, whenever $t \geq 8$, one colour suffices. \square

3.2.2. Hexagonal Grid

There are many ways to define the system of coordinates of the hexagonal grid. Here, we use grid coordinates as shown in Figure 3. The hexagonal grid graph is then the graph \mathfrak{H} whose vertex set consists of pairs of integers $(a, b) \in \mathbb{Z}^2$ and where each vertex (a, b) has three neighbours: $(a-1, b)$, $(a+1, b)$, and $(a, b+1)$ if $a+b$ is odd, or $(a, b-1)$ otherwise.

Theorem 11.

$$\chi_t(\mathfrak{H}^2, w_2) = \begin{cases} 4, & \text{if } 0 \leq t < 1; \\ 3, & \text{if } 1 \leq t < 2; \\ 2, & \text{if } 2 \leq t < 6; \\ 1, & \text{if } 6 \leq t. \end{cases}$$

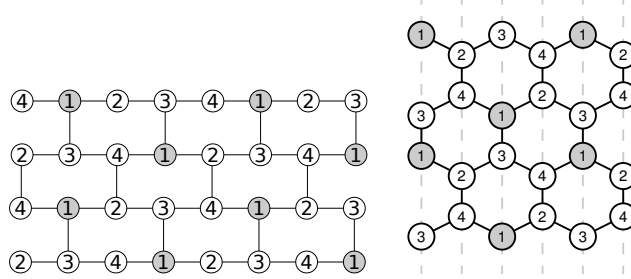


Figure 3: Weighted 0-improper 4-colouring of (\mathfrak{H}^2, w_2) . Left: Graph with coordinates. Right: Corresponding hexagonal grid in the euclidean space.

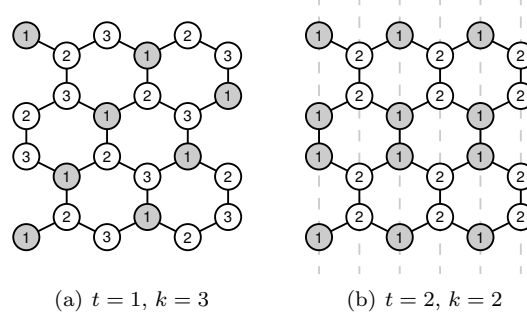


Figure 4: (a) weighted 1-improper 3-colouring of (\mathfrak{H}^2, w_2) and (b) weighted 2-improper 2-colouring of (\mathfrak{H}^2, w_2) .

Proof. Note first, that when $t = 0$, at least four colours are needed to colour the grid, because a vertex and its neighbourhood in \mathfrak{H} form a clique of size four in \mathfrak{H}^2 . The same number of colours are needed if we allow a threshold $t = 0.5$. To prove this fact, let A be a vertex (a, b) of \mathfrak{H} and $B = (a - 1, b)$, $C = (a, b - 1)$ and $D = (a + 1, b)$ be its neighbours in \mathfrak{H} . Denote by $G = (a - 2, b)$, $E = (a - 1, b - 1)$, $F = (a - 2, b - 1)$, $H = (a + 1, b - 1)$, $I = (a + 2, b - 1)$ and $J = (a + 1, b - 2)$ (see Figure 5(a)). By contradiction, suppose there exists a weighted 0.5-improper 3-colouring of \mathfrak{H}^2 . Consider a node A coloured 1. Its neighbours B, C, D cannot be coloured 1 and they cannot all have the same colour. W.l.o.g., suppose that two of them B and C have colour 2 and D has colour 3. Then E, F and G cannot be coloured 2 because of the interference constraint in B and C . If F is coloured 3, then G and E are coloured 1, creating interference 1 in A . So F must be coloured 1 and G and E must be coloured 3. Then, H can be neither

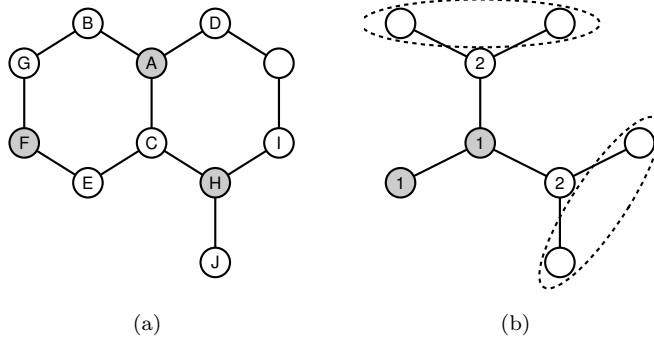


Figure 5: Lower bounds for the hexagonal grid. (a) when $t \leq 0.5$ and $k \leq 3$, there is no weighted t -improper k -colouring of (\mathfrak{H}^2, w_2) ; (b) vertices coloured 2 force a vertex coloured 1 in each ellipse, leading to interference 2 in central node.

coloured 2 (interference in C) nor 3 (interference in E). So H is coloured 1. The vertex I is coloured 3, otherwise the interference constraint in H or in C is not satisfied. Then, J can receive neither colour 1, because of the interference in H , nor colour 2, because of the interference in C , nor colour 3, because of the interference in I .

There exists a construction attaining this bound and the number of colours, i.e. a 0-improper 4-colouring of (\mathfrak{H}^2, w_2) as depicted in Figure 3. We define for $0 \leq j \leq 3$ the sets of vertices $A_j = \{(j, 0) + a(4e_1) + b(2e_1 + e_2) \mid \forall a, b \in \mathbb{Z}\}$. We then assign the colour $j+1$ to the vertices in A_j . This way no vertex experiences any interference as vertices of the same colours are at distance at least three.

For $t = 1.5$ it is not possible to colour the grid with less than three colours. By contradiction, suppose that there exists a weighted 1.5-improper 2-colouring. Consider a vertex A coloured 1. If all of its neighbours are coloured 2, they have already interference 1, so all the vertices at distance two from A need to be coloured 1; this gives interference 3 in A . Therefore one of A 's neighbours, say D , has to be coloured 1 and consider that the other two neighbours B and C are coloured 2. B and C have at most one neighbour of colour 2. It implies that A has at least two vertices at distance two coloured 1. This is a contradiction, because the interference in A would be at least 2 (see Figure 5(b)).

Figure 4(a) presents a weighted 1-improper 3-colouring of (\mathfrak{H}^2, w_2) . To obtain this colouring, let $B_j = \{(j, 0) + a(3e_1) + b(e_1 + e_2) \mid \forall a, b \in \mathbb{Z}\}$, for $0 \leq j \leq 2$. Then, we colour all the vertices in the set B_j with colour $j+1$, for every $0 \leq j \leq 2$.

For $t < 6$, it is not possible to colour the grid with one colour. As a matter of fact, each vertex has three neighbours and six vertices at distance two in \mathfrak{H} . Using one colour leads to an interference equal to 6. There exists a 2-improper 2-colouring of the hexagonal grid as depicted in Figure 4(b). We define for $0 \leq j \leq 1$ the sets of vertices $C_j = \{(j, 0) + a(2e_1) + be_2 \mid \forall a, b \in \mathbb{Z}\}$. We then assign the colour $j+1$ to the vertices in C_j .

□

3.2.3. Triangular Grid

The triangular grid is the graph \mathfrak{T} whose vertices are all the integer linear combinations $af_1 + bf_2$ of the two vectors $f_1 = (1, 0)$ and $f_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Thus we may identify the vertices with the ordered pairs (a, b) of integers. Each vertex $v = (a, b)$ has six neighbours: its *right neighbour* $(a + 1, b)$, its *right-up neighbour* $(a, b + 1)$, its *left-up neighbour* $(a - 1, b + 1)$, its *left neighbour* $(a - 1, b)$, its *left-down neighbour* $(a, b - 1)$ and its *right-down neighbour* $(a + 1, b - 1)$ (see Figure 7(a)).

Theorem 12.

$$\chi_t(\mathfrak{T}^2, w_2) = \begin{cases} 7, & \text{if } t = 0; \\ 6, & \text{if } t = 0.5; \\ 5, & \text{if } t = 1; \\ 4, & \text{if } 1.5 \leq t < 3; \\ 3, & \text{if } 3 \leq t < 5; \\ 2, & \text{if } 5 \leq t < 12; \\ 1, & \text{if } 12 \leq t. \end{cases}$$

Proof. If $t = 0$, there is no weighted 0-improper 6-colouring of (\mathfrak{T}^2, w_2) , since in \mathfrak{T}^2 there is a clique of size seven induced by each vertex and its neighbourhood. There is a weighted 0-improper 7-colouring of (\mathfrak{T}^2, w_2) as depicted in Figure 6(a). This colouring can be obtained by the following construction: for $0 \leq j \leq 6$, let $A_j = \{(j, 0) + a(7f_1) + b(2f_1 + f_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 6$, assign the colour $j + 1$ to all the vertices in A_j .

In what follows, we denote by V_0 a vertex coloured 1 and by $N_0, N_1, N_2, N_3, N_4, N_5$ the six neighbours of V_0 in \mathfrak{T} in a cyclic order. Let Γ^2 be the set of twelve vertices at distance two of V_0 in which $N_{i(i+1)}$ denotes the vertex adjacent to both N_i and N_{i+1} and N_{ii} the vertex joined only to N_i . For every $0 \leq i \leq 5$, $i + 1$ is taken modulo 6 (see Figure 7(b)).

We claim that there is no weighted 0.5-improper 5-colouring of (\mathfrak{T}^2, w_2) . We prove it by contradiction, thus let c be such a colouring. No neighbour of V_0 can be coloured 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 1$. As two consecutive neighbours are adjacent, they cannot have the same colour. Furthermore, there cannot be three neighbours with the same colour (each of them will have an interference at least 1). As there are four colours different from 1, exactly two of them, say 2 and 3, are repeated twice among the six neighbours. So, there exists a sequence of three consecutive neighbours the first one with a colour different from 2 and 3 and the two others coloured 2 and 3. W.l.o.g., let $c(N_5) = 4$, $c(N_0) = 2$, $c(N_1) = 3$.

Note that the vertices coloured 2 and 3 have already an interference of 0.5, and so none of their vertices at distance two can be coloured 2 or 3. In particular, let $A = \{N_{50}, N_{00}, N_{01}, N_{11}, N_{12}\}$; the vertices of A cannot be coloured 2 or 3. At most one vertex in Γ^2 can be coloured 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 1$. If

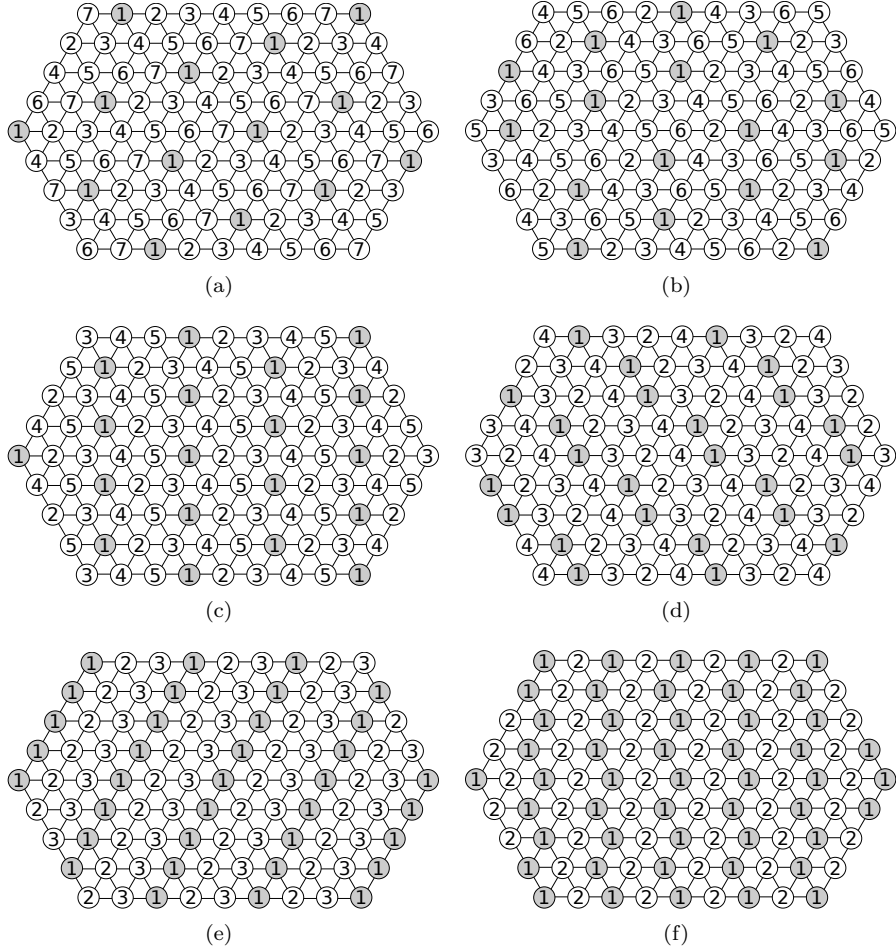


Figure 6: Optimal colourings of (\mathbb{T}^2, w_2) : (a) weighted 0-improper 7-colouring, (b) weighted 0.5-improper 6-colouring, (c) weighted 1-improper 5-colouring, (d) weighted 1.5-improper 4-colouring, (e) weighted 3-improper 3-colouring, and (f) weighted 5-improper 2-colouring.

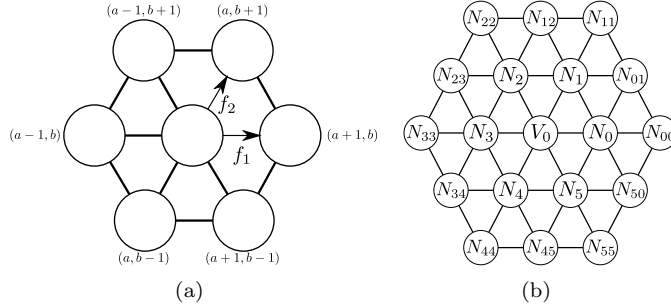


Figure 7: Notations used in proofs: (a) of existence, and (b) of non-existence of weighted improper colourings of (\mathfrak{T}^2, w_2) .

there is no vertex coloured 1 in A , we have a contradiction as we cannot have a sequence of five vertices uniquely coloured 4 and 5 (indeed colours should alternate and the vertex in the middle N_{01} will have interference at least 1). Suppose N_4 is coloured 3, then N_{45} and N_{55} can only be coloured 1 and 5; but, as they have different colours, one is coloured 1 and so there is no vertex coloured 1 in A . So the second vertex coloured 3 in the neighbourhood of V_0 is necessarily N_3 (it cannot be N_2 neighbour of N_1 coloured 3). Then, N_4 cannot be also coloured 5, otherwise N_{45} is coloured 1 and again there is no vertex coloured 1 in A . In summary $c(N_4) = 2$, $c(N_3) = 3$ and the vertex of Γ^2 coloured 1 is in A . But then the five consecutive vertices $A' = \{N_{23}, N_{33}, N_{34}, N_{44}, N_{45}\}$ can only be coloured 4 and 5. A contradiction as $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 1$.

A weighted 0.5-improper 6-colouring of (\mathfrak{T}^2, w_2) can be obtained by the following construction (see Figure 6(b)): for $0 \leq j \leq 11$, let $B_j = \{(j, 0) + a(12f_1) + b(2f_1 + f_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 5$, assign the colour $j + 1$ to all the vertices in B_j , B_6 with colour 2, B_7 with colour 1, B_8 with colour 4, B_9 with colour 3, B_{10} with colour 6 and B_{11} with colour 5.

Now we prove that (\mathfrak{T}^2, w_2) does not admit a weighted 1-improper 4-colouring. Again, by contradiction, suppose that there exists a weighted 1-improper 4-colouring c of (\mathfrak{T}^2, w_2) . We analyse some cases:

1. There exist two adjacent vertices in \mathfrak{T} with the same colour.

Let V_0 and one of its neighbours be both coloured 1. Note that no other neighbour of V_0 , nor the vertices at distance two from V_0 are coloured 1 (otherwise, $I_{V_0}(\mathfrak{T}^2, w_2, c) > 1$). We use intensively the following facts:

Fact 1. *There do not exist three consecutive vertices with the same colour (otherwise the vertex in the middle would have interference at least 2).*

Fact 2. *In a path of five vertices there cannot be four of the same colour (otherwise the second or the fourth vertex in this path would have interference at least 1.5).*

One colour other than 1 should appear at least twice in the neighbourhood of V_0 . Let this colour be denoted 2 (the other colours being denoted 3 and 4).

- (a) Two neighbours of V_0 coloured 2 are consecutive, say N_0 and N_1 . By Fact 1, N_2 is coloured 3 w.l.o.g. None of $N_{05}, N_{00}, N_{01}, N_{11}, N_{12}, N_{22}$ and N_{23} can be coloured 2, otherwise $I_{N_1}(\mathfrak{X}^2, w_2, c) > 1$. One of N_{12}, N_{22} and N_{23} is coloured 3, otherwise we contradict Fact 1 with colour 4 and at most one of $N_{01}, N_{11}, N_{12}, N_{22}$ and N_{23} is coloured 3, otherwise $I_{N_2}(\mathfrak{X}^2, w_2, c) > 1$; but we have a contradiction with Fact 2.
- (b) Two neighbours of V_0 coloured 2 are at distance two, say N_0 and N_2 . Then N_{50}, N_{00} and N_{01} (respectively N_{12}, N_{22} and N_{23}) are not coloured 2, otherwise $I_{N_0}(\mathfrak{X}^2, w_2, c) > 1$ (respectively $I_{N_2}(\mathfrak{X}^2, w_2, c) > 1$). One of N_3 and N_5 is not coloured 1, say N_3 . It is not coloured 2, otherwise $I_{N_3}(\mathfrak{X}^2, w_2, c) > 1$. Let $c(N_3) = 3$. If N_4 or N_{11} is coloured 2, then N_{33} and N_{34} are not coloured 2, otherwise $I_{N_2}(\mathfrak{X}^2, w_2, c) > 1$ and we have a sequence of five vertices $N_{12}, N_{22}, N_{23}, N_{33}$ and N_{34} contradicting Fact 2 as four are of colour 4 (indeed, at most one is coloured 3 due to interference in colour 3 with N_3 or N_{22}). So N_{11} is coloured 3 or 4. If N_1 also is coloured 3 or 4, we have a contradiction with Fact 2 applied to the five vertices $N_{00}, N_{01}, N_{11}, N_{12}$ and N_{22} , by the same previous argument. So $c(N_1) = 1$; furthermore N_4 is not coloured 1 (at most one neighbour coloured 1), nor 2 as we have seen above, nor 3, otherwise we are in the case (a). Therefore $c(N_4) = 4$ and $c(N_5) = 3$, by the same reason. But then $c(N_{23}) = 4$, otherwise the interference in V_0 or N_2 or N_3 is greater than 1. N_{33} and N_{34} can be only coloured 2, otherwise V_0, N_3, N_4 or N_{23} will have interference strictly greater than 1, but N_{33} has interference greater than 1, a contradiction.
- (c) Two neighbours of V_0 coloured 2 are at distance three say N_0 and N_3 . Then N_{50}, N_{00} and N_{01} (respectively N_{23}, N_{33} and N_{34}) are not coloured 2, otherwise $I_{N_0}(\mathfrak{X}^2, w_2, c) > 1$ (respectively $I_{N_3}(\mathfrak{X}^2, w_2, c) > 1$). W.l.o.g., let N_1 be the vertex coloured 1. Among the four vertices N_{12}, N_{22}, N_{44} and N_{45} at most one is coloured 2, otherwise $I_{N_3}(\mathfrak{X}^2, w_2, c) > 1$. So, w.l.o.g, we can suppose N_{44} and N_{45} are coloured 3 or 4; but we have a set of five consecutive vertices $N_{23}, N_{33}, N_{34}, N_{44}, N_{45}$, contradicting Fact 2 (indeed at most one can be of the colour of N_4).

2. No colour appears in two adjacent vertices of \mathfrak{X} .

Let V_0 be coloured 1. No colour can appear four or more times among the neighbours of V_0 , otherwise there are two adjacent neighbours with the same colour.

- (a) One colour appears three times among the neighbours of V_0 , say $c(N_0) = c(N_2) = c(N_4) = 2$. W.l.o.g., let $c(N_1) = 3$. No vertex at distance two can be coloured 2. N_{01}, N_{11} and N_{12} being neighbours of N_1 cannot be coloured 3 and they cannot be all coloured 4. So one of N_{01}, N_{11}, N_{12} is coloured 1. Similarly one of N_{23}, N_{33}, N_{34} is coloured 1 (same reasoning with N_3 instead of N_1) and one of N_{45}, N_{55}, N_{50} is coloured 1, so $I_{V_0}(\mathfrak{X}^2, w_2, c) > 1$.
- (b) The three colours appear each exactly twice in the neighbourhood of V_0 .

- i. The same colour appears in some N_i and N_{i+2} , $0 \leq i \leq 3$. W.l.o.g., let $c(N_0) = c(N_2) = 2$ and $c(N_1) = 3$. Then, $c(N_3) = c(N_5) = 4$ and $c(N_4) = 3$. Then, $c(N_{50}) = 1$ or 3 , $c(N_{01}) = 1$ or 4 . If $c(N_{50}) = 3$ and $c(N_{01}) = 4$, then $c(N_{00}) = 1$. Among N_{50}, N_{00}, N_{01} , at least one has colour 1. Similarly one of N_{12}, N_{22}, N_{23} has colour 1. So $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 1$ and $c(N_{34}) = c(N_{45}) = 2$. Consequently, no matter the colour of N_{44} some vertex will have interference greater than 1.
- ii. We have $c(N_0) = c(N_3) = 2$, $c(N_1) = c(N_4) = 3$ and $c(N_2) = c(N_5) = 4$. Here we find in each of the sets $\{N_{50}, N_{00}, N_{01}\}$, $\{N_{12}, N_{22}, N_{23}\}$ and $\{N_{34}, N_{44}, N_{45}\}$ a vertex coloured 1. Therefore $I_{V_0}(\mathfrak{T}^2, w_2, c) > 1$, a contradiction.

To obtain a weighted 1-improper 5-colouring of (\mathfrak{T}^2, w_2) , for $0 \leq j \leq 4$, let $C_j = \{(j, 0) + a(5f_1) + b(2f_1 + f_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 4$, assign the colour $j + 1$ to all the vertices in C_j . See Figure 6(c).

(\mathfrak{T}^2, w_2) has a weighted 1.5-improper 4-colouring as depicted in Figure 6(d). Formally, this colouring can be obtained by the following construction: for $0 \leq j \leq 3$, let $D_j = \{(j, 0) + a(4f_1) + b(f_1 + 2f_2) \mid \forall a, b \in \mathbb{Z}\}$; then assign colour 4 to all the vertices in D_0 , 1 to all the vertices in D_1 , 3 to all the vertices in D_2 and 2 to all the vertices in D_3 . Now, for $0 \leq j \leq 3$, let $D'_j = \{(j, 1) + a(4f_1) + b(f_1 + 2f_2) \mid \forall a, b \in \mathbb{Z}\}$. Then, for $0 \leq j \leq 3$, assign colour $j + 1$ to all the vertices in D'_j .

The proof that (T^2, w_2) does not admit a weighted 2.5-improper 3-colouring can be found in [3]. The proof uses the same tools as for the other cases, but we have to distinguish a great number of cases and subcases and being around 30 pages long we omitted it. It is worth noting that for the proof we only need to consider vertices at distance at most three from the vertex V_0 .

Now we present the colouring providing the corresponding upper bound.

For a weighted 3-improper 3-colouring of (\mathfrak{T}^2, w_2) set, for $0 \leq j \leq 2$, $E_j = \{(j, 0) + a(3f_1) + b(f_2) \mid \forall a, b \in \mathbb{Z}\}$. Then, for $0 \leq j \leq 2$, assign the colour $j + 1$ to all the vertices in E_j . See Figure 6(e).

Now we prove that (\mathfrak{T}^2, w_2) does not admit a weighted 4.5-improper 2-colouring. Again, by contradiction, suppose that there exists a weighted 4.5-improper 2-colouring c of (\mathfrak{T}^2, w_2) with the interference function w_2 . A vertex can have at most four neighbours of the same colour as it. We analyse some cases:

1. There exists a vertex V_0 with four of its neighbours coloured with its own colour, say 1. Therefore among the vertices of Γ^2 at most one is coloured 1. Consider the two neighbours of V_0 coloured 2. First, consider the case in which they are adjacent and let them be N_0 and N_1 . In Γ^2 , N_0 has three neighbours and four vertices at distance two; since at most one being of colour 1, these vertices produce in N_0 an interference equal to 4 and as N_1 is also of colour 2, then $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction. In case the two neighbours of V_0 coloured 2 are non adjacent, let them be N_i and N_j . At least one of them, say N_i has its three neighbours in Γ^2 coloured 2 and it has also at least three vertices at

distance two in Γ^2 coloured 2; taking into account that N_j is coloured 2 and at distance two from N_i , we get $I_{N_i}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

2. No vertex has four neighbours with its colour and there exists at least one vertex V_0 coloured 1 that has three neighbours of the same colour 1.
 - (a) The three other neighbours of V_0 coloured 2 are consecutive and let them be N_0, N_1 and N_2 . N_{34}, N_{44} and N_{45} are all coloured 2, otherwise N_4 would have four neighbours coloured 1 and we would be in Case 1. At most one of N_{01}, N_{11} and N_{12} has colour 2, otherwise N_1 would have four neighbours coloured 2 and we would be again in Case 1.
 - i. N_{11} is coloured 2. Then $c(N_{01}) = c(N_{12}) = 1$. As already $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 4$, there is at most another vertex in Γ^2 coloured 1. So either the three vertices N_{22}, N_{23} and N_{33} or the three vertices N_{00}, N_{50} and N_{55} are all coloured 2 and then $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 5$ or $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.
 - ii. N_{01} is coloured 2 (the case N_{12} is symmetric). Then, $c(N_{11}) = c(N_{12}) = 1$. One of N_{00} and N_{50} is of colour 1 otherwise, N_0 has four neighbours of colour 2. But then $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 4.5$ so all the other vertices of Γ^2 are coloured 2. Therefore, $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.
 - iii. N_{01}, N_{11} and N_{12} all have colour 1. In that case $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 4.5$. Therefore all the other vertices of Γ^2 are coloured 2 and $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 4.5$. So the other vertices at distance two of N_0 are coloured 1 and then $I_{N_{01}}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.
 - (b) Among the three vertices of colour 2, only two are consecutive. W.l.o.g., let the three vertices of colour 2 be N_0, N_1 and N_3 . At least one vertex of N_{50}, N_{00}, N_{01} is coloured 1, otherwise N_0 has four neighbours of the same colour as it and we would be in the previous case. Similarly at least one vertex of N_{01}, N_{11}, N_{12} is coloured 1, otherwise N_1 has four neighbours with its colour and we would be in the previous case. At least one vertex of N_{23}, N_{33}, N_{34} is coloured 1, otherwise N_3 has three consecutive neighbours of the same colour as it and we are in the previous case. Suppose N_{01} is coloured 2, then $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 4.5$ and exactly one of N_{50}, N_{00} and one of N_{11}, N_{12} is coloured 1 and N_{45}, N_{55} are coloured 2, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$. Then $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction. So, $c(N_{01}) = 1$. If both N_{50}, N_{00} are coloured 2, then $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 5$ with three neighbours coloured 2 and at least four vertices at distance two coloured 2, namely N_3 and three vertices among $N_{45}, N_{55}, N_{11}, N_{12}$ (at most one vertex of these could be of colour 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$). So, one of N_{50}, N_{00} is coloured 1 and all the other vertices in $\{N_{11}, N_{12}, N_{22}, N_{44}, N_{45}, N_{55}\}$ are coloured 2 implying that $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.
 - (c) No two vertices of colour 2 are consecutive. W.l.o.g, let these vertices be N_0, N_2, N_4 . The three neighbours of N_0 (resp. N_1, N_2) in Γ^2 that are not neighbours of V_0 cannot be all coloured 2, otherwise we are in Case (a). So exactly one neighbour of N_0, N_1, N_2 in Γ^2 is coloured 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$. Furthermore all the other vertices of Γ^2 are coloured 2. Then, if $c(N_{12}) =$

$c(N_{45}) = 2$, we conclude that $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction. Consequently, w.l.o.g., suppose that $c(N_{12}) = 1$. In this case, N_{23} has at least three neighbours coloured 2 and we are in some previous case.

3. No vertex has three neighbours coloured with its own colour, but there exists at least one vertex, say V_0 , of colour 1 that has two neighbours coloured 1.
 - (a) These two neighbours are consecutive say N_0 and N_1 . The neighbours of N_3 and N_4 in Γ^2 are all coloured 1, otherwise they would have at least three neighbours with the same colour. Similarly, at least one of N_{12} and N_{22} is coloured 1, otherwise N_2 would have at least three neighbours also coloured 2. Then, $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.
 - (b) These two neighbours are of the form N_i and N_{i+2} , for some $0 \leq i \leq 3$. W.l.o.g., let these neighbours be N_0 and N_2 . Thus, the three neighbours of N_4 in Γ^2 , N_{34} , N_{44} and N_{45} are coloured 1 and at least one vertex of N_{23} and N_{33} (resp. N_{55} and N_{50}) is coloured 1. Moreover, at least one vertex of N_{01} , N_{11} and N_{12} must be coloured 1, otherwise N_1 would have three neighbours with its colour. Consequently, $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.
 - (c) These two neighbours are of the form N_i and N_{i+3} , for some $0 \leq i \leq 2$. W.l.o.g., let these neighbours be N_0 and N_3 . Again, at least three vertices among N_{01} , N_{11} , N_{12} , N_{22} and N_{23} and at least three other vertices among N_{34} , N_{44} , N_{45} , N_{55} and N_{50} are coloured 1. Consequently, $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.
4. No vertex has two neighbours of the same colour. Suppose V_0 is coloured 1 and has only one neighbour N_0 coloured 1. Then, its other five neighbours are coloured 2 and N_2 has two neighbours of the colour 2, a contradiction.

A weighted 5-improper 2-colouring of (\mathfrak{T}^2, w_2) is obtained as follows: for $0 \leq j \leq 1$, let $F_j = \{(j, 0) + a(2f_1) + b(f_1 + 2f_2) \mid \forall a, b \in \mathbb{Z}\}$ and $F'_j = \{(j - 1, 1) + a(2f_1) + b(f_1 + 2f_2) \mid \forall a, b \in \mathbb{Z}\}$. Then, for $0 \leq j \leq 1$, assign the colour $j + 1$ to all the vertices in F_j and in F'_j . See Figure 6(f).

Since each vertex has six neighbours and twelve vertices at distance two in \mathfrak{T} , there is no weighted t -improper 1-colouring of (\mathfrak{T}^2, w_2) , for any $t < 12$. Obviously, there is a weighted 12-improper 1-colouring of \mathfrak{T}^2 . \square

4. Conclusion, Open Problems and Future Directions

In this paper, we introduced and studied a new colouring problem, WEIGHTED IMPROPER COLOURING. This problem is motivated by the design of telecommunication antenna networks in which the interference between two vertices depends on different factors and can take various values. For each vertex, the sum of the interferences it receives should be less than a given threshold value.

We first give general bounds on the weighted-improper chromatic number. We then study the particular case of infinite paths, trees and grids: square,

hexagonal and triangular. For these graphs, we provide their weighted-improper chromatic number for all possible values of t .

Many problems remain to be solved:

- In the study of the grid graphs, we considered a specific function where vertices at distance one interfere by 1 and vertices at distance two by $1/2$. Other weight functions should be considered. e.g. $1/d^2$ or $1/(2^{d-1})$, where d is the distance between vertices.
- Other families of graphs could be considered, for example hypercubes.
- We showed in [3] that the THRESHOLD IMPROPER COLOURING problem can be transformed into a problem with only two possible weights on the edges 1 and ∞ , that is a mix of proper and improper colouring. This simplifies the nature of the graph interferences but at the cost of an important increase of instance sizes. We want to further study this. In particular, let $G = (V, E, w)$ be an edge-weighted graph where the weights are all equal to 1 or M . Let G_M be the subgraph of G induced by the edges of weight M ; is it true that if $\Delta(G_M) \ll \Delta(G)$, then $\chi_t(G, w) \leq \chi_t(G) \leq \left\lceil \frac{\Delta(G, w) + 1}{t + 1} \right\rceil$? A similar result for $L(p, 1)$ -labelling [13] suggests it could be true.

Note that the problem can also be solved *algorithmically* for other classes of graphs and for other functions of interference. We started looking in this direction in [3]. The problem can be expressed as a linear program and then be solved exactly using solvers such as CPLEX¹ or Glpk² for small instances of graphs. For larger instances, we propose a heuristic algorithm inspired by DSATUR [5] but adapted to the specifics of our colouring problem. We used it to derive colourings with few colours for Poisson-Voronoi tessellations as they are good models of antenna networks [4, 11, 12]. We plan to further investigate the algorithmic side of our colouring problem.

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