



HAL
open science

Self-adjoint extensions of discrete magnetic Schrödinger operators

Francoise Truc, Ognjen Milatovic

► **To cite this version:**

Francoise Truc, Ognjen Milatovic. Self-adjoint extensions of discrete magnetic Schrödinger operators. 2012. hal-00747698v1

HAL Id: hal-00747698

<https://hal.science/hal-00747698v1>

Preprint submitted on 2 Nov 2012 (v1), last revised 6 Dec 2012 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SELF-ADJOINT EXTENSIONS OF DISCRETE MAGNETIC SCHRÖDINGER OPERATORS

OGNJEN MILATOVIĆ, FRANÇOISE TRUC

ABSTRACT. Using the concept of intrinsic metric on a locally finite weighted graph, we give sufficient conditions for the magnetic Schrödinger operator to be essentially self-adjoint. The present paper is an extension of some recent results proven in the context of graphs of bounded degree.

1. INTRODUCTION AND THE MAIN RESULTS

1.1. **The setting.** Let V be a countably infinite set. We assume that V is equipped with a measure $\mu: V \rightarrow (0, \infty)$. Let $b: V \times V \rightarrow [0, \infty)$ be a function such that

(i) $b(x, y) = b(y, x)$, for all $x, y \in V$;

(ii) $b(x, x) = 0$, for all $x \in V$;

(iii) $\deg(x) := \#\{y \in V: b(x, y) > 0\} < \infty$, for all $x \in V$. Here, $\#S$ denotes the number of elements in the set S .

Vertices $x, y \in V$ with $b(x, y) > 0$ are called *neighbors*, and we denote this relationship by $x \sim y$. We call the triple (V, b, μ) a *locally finite weighted graph*. We assume that (V, b, μ) is connected, that is, for any $x, y \in V$ there exists a path γ joining x and y . Here, γ is a sequence $x_0, x_2, \dots, x_n \in V$ such that $x = x_0$, $y = x_n$, and $x_j \sim x_{j+1}$ for all $0 \leq j \leq n - 1$.

1.2. **Intrinsic metric.** Following [15] we define a *pseudo metric* to be a map $d: V \times V$ such that $d(x, y) = d(y, x)$, for all $x, y \in V$; $d(x, x) = 0$, for all $x \in V$; and $d(x, y)$ satisfies the triangle inequality. A pseudo-metric $d = d_\sigma$ is called a *path pseudo-metric* if there exists a map $\sigma: V \times V \rightarrow [0, \infty)$ such that $\sigma(x, y) = \sigma(y, x)$, for all $x, y \in V$; $\sigma(x, y) > 0$ if and only if $x \sim y$; and

$$d_\sigma = \inf\{l_\sigma(\gamma): \gamma = (x_0, x_1, \dots, x_n), n \geq 1, \text{ is a path connecting } x \text{ and } y\},$$

where the length l_σ of the path $\gamma = (x_0, x_1, \dots, x_n)$ is given by

$$l_\sigma(\gamma) = \sum_{i=0}^{n-1} \sigma(x_i, x_{i+1}). \tag{1.1}$$

As in [15] we make the following definitions.

2000 *Mathematics Subject Classification.* 35J10, 39A12, 47B25.

Definition 1.3. (i) A pseudo metric d on (V, b, μ) is called *intrinsic* if

$$\frac{1}{\mu(x)} \sum_{y \in V} b(x, y) (d(x, y))^2 \leq 1, \quad \text{for all } x \in V.$$

(ii) An intrinsic path pseudo metric $d = d_\sigma$ on (V, b, μ) is called *strongly intrinsic* if

$$\frac{1}{\mu(x)} \sum_{y \in V} b(x, y) (\sigma(x, y))^2 \leq 1, \quad \text{for all } x \in V.$$

Remark 1.4. On a locally finite graph (V, b, μ) , the formula

$$\sigma_1(x, y) = b(x, y)^{-1/2} \min \left\{ \frac{\mu(x)}{\deg(x)}, \frac{\mu(y)}{\deg(y)} \right\}^{1/2}, \quad \text{with } x \sim y, \quad (1.2)$$

where $\deg(x)$ is as in property (iii) of $b(x, y)$, defines a strongly intrinsic path metric; see [15, Example 2.1].

1.5. Cauchy boundary. For a path metric $d = d_\sigma$ on V , we denote the metric completion by $(\widehat{V}, \widehat{d})$. As in [15] we define the *Cauchy boundary* $\partial_C V$ as follows: $\partial_C V := \widehat{V} \setminus V$. Note that (V, d) is metrically complete if and only if $\partial_C V$ is empty. For a path metric $d = d_\sigma$ on V and $x \in V$, we define

$$D(x) := \inf_{z \in \partial_C V} d_\sigma(x, z). \quad (1.3)$$

1.6. Inner product. In what follows, $C(V)$ is the set of complex-valued functions on V , and $C_c(V)$ is the set of finitely supported elements of $C(V)$. By $\ell_\mu^2(V)$ we denote the space of functions $f \in C(V)$ such that

$$\|f\|^2 := \sum_{x \in V} \mu(x) |f(x)|^2 < \infty, \quad (1.4)$$

where $|\cdot|$ denotes the modulus of a complex number.

In particular, the space $\ell_\mu^2(V)$ is a Hilbert space with the inner product

$$(f, g) := \sum_{x \in V} \mu(x) f(x) \overline{g(x)}. \quad (1.5)$$

1.7. Laplacian operator. We define the formal Laplacian $\Delta_b: C(V) \rightarrow C(V)$ on (V, b, μ) by the formula

$$(\Delta_{b, \mu} u)(x) = \frac{1}{\mu(x)} \sum_{y \in V} b(x, y) (u(x) - u(y)). \quad (1.6)$$

1.8. Magnetic Schrödinger operator. We fix a phase function $\theta: V \times V \rightarrow [-\pi, \pi]$ such that $\theta(x, y) = -\theta(y, x)$ for all $x, y \in V$, and denote $\theta_{x,y} := \theta(x, y)$. We define the formal magnetic Laplacian $\Delta_{b,\mu;\theta}: C(V) \rightarrow C(V)$ on (V, b, μ) by the formula

$$(\Delta_{b,\mu;\theta}u)(x) = \frac{1}{\mu(x)} \sum_{y \in V} b(x, y)(u(x) - e^{i\theta_{x,y}}u(y)). \quad (1.7)$$

We define the formal magnetic Schrödinger operator $H: C(V) \rightarrow C(V)$ by the formula

$$Hu := \Delta_{b,\mu;\theta}u + Wu, \quad (1.8)$$

where $W: V \rightarrow \mathbb{R}$.

1.9. Statements of the results. We are ready to state our first result.

Theorem 1. *Assume that (V, b, μ) is a locally finite, weighted, and connected graph. Let $d = d_\sigma$ be an intrinsic path metric on V such that (V, d) is not metrically complete. Assume that there exists a constant C such that*

$$W(x) \geq \frac{1}{2(D(x))^2} - C, \quad \text{for all } x \in V, \quad (1.9)$$

where $D(x)$ is as in (1.3). Then H is essentially self-adjoint on $C_c(V)$.

Remark 1.10. It is possible find μ , b , and a potential W satisfying $W(x) \geq \frac{k}{2(D(x))^2}$ with $0 < k < 1$, such that $H = \Delta_{b,\mu} + W$ is not essentially self-adjoint; see [2, Section 5.3.2].

If the graph (V, b, μ) has a special type of covering, the condition (1.9) on W can be relaxed with the help of “effective potential,” as seen in the next theorem. First, we give a description of this special type of covering. In what follows, for a graph (V, b, μ) , we define the set of unoriented edges as $E := \{\{x, y\}: x, y \in V \text{ and } b(x, y) > 0\}$. Sometimes, when we want to emphasize the set E , instead of $G = (V, b, \mu)$ we will use the notation $G = (V, E)$.

Definition 1.11. Let $m \in \mathbb{N}$. A *good covering of degree m* of $G = (V, E)$ is a family $G_l = (V_l, E_l)_{l \in L}$ of finite connected sub-graphs of G so that

- (i) $V = \cup_{l \in L} V_l$;
- (ii) for any $\{x, y\} \in E$,

$$0 < \#\{l \in L \mid \{x, y\} \in E_l\} \leq m.$$

Remark 1.12. It is known that a graph with bounded vertex degree admits a good covering; see [3, Proposition 2.2]. The graph in Example 5.2 below does not have a bounded vertex degree. Note that this graph has a good covering of degree $m = 1$.

Assume that (V, b, μ) has a good covering $(V_l, E_l)_{l \in L}$. Let θ_l be the restriction of θ to $V_l \times V_l$. Let $\Delta_{1,\mu;\theta}^{(l)}$ be as in (1.7) with $V = V_l$, $\theta = \theta_l$, and $b \equiv 1$. Then $\Delta_{1,\mu;\theta}^{(l)}$ is a bounded and non-negative self-adjoint operator in $\ell_\mu^2(V_l)$. Let p_l denote the lowest eigenvalue of $\Delta_{1,\mu;\theta}^{(l)}$. With

these notations, for a graph (V, b, μ) and the phase function θ , we define the *effective potential* corresponding to a good covering $(V_l, E_l)_{l \in L}$ of degree m as follows:

$$W_e(x) := \frac{1}{m} \sum_{\{l \in L \mid x \in V_l\}} p_l \inf_{\{y, z\} \in E_l} b(y, z). \quad (1.10)$$

We now state our second result.

Theorem 2. *Assume that (V, b, μ) is a locally finite, weighted, and connected graph. Assume that (V, b, μ) has a good covering $(V_l, E_l)_{l \in L}$. Let $d = d_\sigma$ be an intrinsic path metric on V such that (V, d) is not metrically complete. Assume that there exists a constant C such that*

$$W_e(x) + W(x) \geq \frac{1}{2(D(x))^2} - C, \quad \text{for all } x \in V, \quad (1.11)$$

where W_e is as in (1.10) and $D(x)$ is as in (1.3). Then H is essentially self-adjoint on $C_c(V)$.

In the setting of metrically complete graphs, we have the following result:

Theorem 3. *Assume that (V, b, μ) be a locally finite, weighted, and connected graph. Let d_σ be a strongly intrinsic path metric on V . Let $q: V \rightarrow [1, \infty)$ be a function satisfying*

$$|q^{-1/2}(x) - q^{-1/2}(y)| \leq K\sigma(x, y), \quad \text{for all } x, y \in V \text{ such that } x \sim y, \quad (1.12)$$

where K is a constant. Let H be as in (1.8) with $W: V \rightarrow \mathbb{R}$ satisfying

$$W(x) \geq -q(x), \quad \text{for all } x \in V. \quad (1.13)$$

Let

$$\sigma_q(x, y) = \min\{q^{-1/2}(x), q^{-1/2}(y)\} \cdot \sigma(x, y) \quad (1.14)$$

and let d_{σ_q} be the path metric corresponding to σ_q . Assume that (V, d_{σ_q}) is metrically complete. Then H is essentially self-adjoint on $C_c(V)$.

1.13. Some comments on the existing literature. The notion of intrinsic metric allows us to remove the bounded vertex degree assumption present in [2, 3, 20]. More specifically, Theorem 1 extends [2, Theorem 4.2], which was proven in the context of graphs of bounded vertex degree for the operator $\Delta_{b, \mu} + W$, with $\Delta_{b, \mu}$ as in (1.6). Theorem 2 is an extension of [3, Theorem 3.1], which was proven in the context of graphs of bounded vertex degree for the operator $\Delta_{b, \mu; \theta}$. In this regard, the first two results of the present paper answer a question posed in [3, Section 5]. Theorem 3 extends [20, Theorem 1], which was proven in the context of graphs of bounded vertex degree for the operator $\Delta_{b, \mu; \theta} + W$ with W as in (1.13). We should also mention that in the context of locally finite graphs (with an assumption on b and μ originating from [17]), a sufficient condition for the essential self-adjointness of a semi-bounded from below operator $\Delta_{b, \mu; \theta} + W$ is given in [19, Theorem 1.2]. In the setting of locally finite graphs, another sufficient condition for the essential self-adjointness of $\Delta_{b, \mu; \theta} + W$ is given in [9, Proposition 2.2]: Take $\lambda \in \mathbb{R}$ so that

$$\{x \in V : \lambda + \text{Deg}(x) + W(x) = 0\} = \emptyset, \quad (1.15)$$

where $\text{Deg}(x)$ denotes the “weighted degree”

$$\text{Deg}(x) := \frac{1}{\mu(x)} \sum_{y \in V} b(x, y), \quad x \in V. \quad (1.16)$$

Suppose that for every sequence of vertices $\{y_1, y_2, \dots\}$ such that $y_j \sim y_{j+1}$, $j \geq 1$, the following property holds:

$$\sum_{n=1}^{\infty} ((a_n)^2 \mu(y_n)) = \infty, \quad \text{where} \quad a_n := \prod_{j=1}^{n-1} \left(1 + \frac{\lambda + W(y_j)}{\text{Deg}(y_j)} \right), \quad n \geq 1, \quad (1.17)$$

and $a_1 := 1$. Then $\Delta_{b,\mu;\theta} + W$ is essentially self-adjoint on $C_c(V)$.

Note that [9, Proposition 2.2] allows potentials that are unbounded from below. We mention that Example 5.1 below describes a situation where Theorem 2 is applicable, while neither [19, Theorem 1.2] nor [9, Proposition 2.2] is applicable. Additionally, Example 5.2 below describes a situation where Theorem 3 is applicable, while neither [19, Theorem 1.2] nor [9, Proposition 2.2] is applicable.

The recent study [15] is concerned with the operator $\Delta_{b,\mu}$ as in (1.6), with property (iii) of b (see Section 1.1 above) replaced by the following more general condition:

$$\sum_{y \in V} b(x, y) < \infty, \quad \text{for all } x \in V.$$

Using the notion of intrinsic distance d with finite jump size, the authors of [15] show that if the weighted degree (1.16) is bounded on balls defined with respect to any such distance d , then $\Delta_{b,\mu}$ is essentially self-adjoint. In the context of a locally finite graph, the authors of [15] show that if the graph is metrically complete in any intrinsic path metric with finite jump size, then $\Delta_{b,\mu}$ is essentially self-adjoint. In the metrically incomplete case, one of the results of [15] shows that if the Cauchy boundary has finite capacity, then $\Delta_{b,\mu}$ has a unique Markovian extension if and only if the Cauchy boundary is polar (here, “Cauchy boundary is polar” means that the Cauchy boundary has zero capacity). Another result of [15] shows that if the upper Minkowski codimension of the Cauchy boundary is greater than 2, then the Cauchy boundary is polar. Additionally, we should mention that the authors of [15] prove Hopf–Rinow-type theorem for locally finite weighted graphs with a path pseudo metric.

In recent years, various authors have developed independently the concept of intrinsic metric on a graph. The definition given in the present paper can be traced back to the work [8]. For applications of intrinsic metrics in various contexts, see, for instance, [1, 5, 6, 7, 10, 12, 13, 14, 18].

With regard to the problem of self-adjoint extensions of adjacency, (magnetic) Laplacian and Schrödinger-type operators on infinite graphs, we should mention that there has been a lot of interest this area in the past few years. For pointers to the existing literature on this topic, see, for instance, [2, 3, 9, 11, 15, 17, 20, 24].

2. PROOF OF THEOREM 1

In this section, we modify the proof of [2, Theorem 4.2]. Throughout the section, we assume that the hypotheses of Theorem 1 are satisfied. We begin with the following lemma, whose proof is given in [3, Lemma 3.3].

Lemma 2.1. *Let H be as in (1.8), let $v \in \ell_\mu^2(V)$ be a weak solution of $Hv = 0$, and let $f \in C_c(V)$ be a real-valued function. Then the following equality holds:*

$$(fv, H(fv)) = \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} b(x, y) \operatorname{Re} [e^{-i\theta(x, y)} v(x) \overline{v(y)}] (f(x) - f(y))^2. \quad (2.1)$$

The key ingredient in the proof of Theorem 1 is the Agmon-type estimate given in the next lemma, whose proof, inspired by an idea of [21], is based on the technique developed in [4] for magnetic Laplacians on an open set with compact boundary in \mathbb{R}^n .

Lemma 2.2. *Let $\lambda \in \mathbb{R}$ and let $v \in \ell_\mu^2(V)$ be a weak solution of $(H - \lambda)v = 0$. Assume that there exists a constant $c_1 > 0$ such that, for all $u \in C_c(V)$,*

$$(u, (H - \lambda)u) \geq \frac{1}{2} \sum_{x \in V} \max \left(\frac{1}{D(x)^2}, 1 \right) \mu(x) |u(x)|^2 + c_1 \|u\|^2. \quad (2.2)$$

Then $v \equiv 0$.

Proof. Let ρ and R be numbers satisfying $0 < \rho < 1/2$ and $1 < R < +\infty$. For any $\epsilon > 0$, we define the function $f_\epsilon: V \rightarrow \mathbb{R}$ by $f_\epsilon(x) = F_\epsilon(D(x))$, where $D(x)$ is as in (1.3) and $F_\epsilon: \mathbb{R}^+ \rightarrow \mathbb{R}$ is the continuous piecewise affine function defined by

$$F_\epsilon(s) = \begin{cases} 0 & \text{for } s \leq \epsilon \\ \rho(s - \epsilon)/(\rho - \epsilon) & \text{for } \epsilon \leq s \leq \rho \\ s & \text{for } \rho \leq s \leq 1 \\ 1 & \text{for } 1 \leq s \leq R \\ R + 1 - s & \text{for } R \leq s \leq R + 1 \\ 0 & \text{for } s \geq R + 1 \end{cases}$$

We first note that by the definition of F_ϵ and continuity of $D(x)$, the support of f_ϵ is compact. Now by [15, Lemma A.3(b)] it follows that the support of f_ϵ is finite. Using Lemma 2.1 with $H - \lambda$ in place of H , the inequality

$$\operatorname{Re} [e^{-i\theta(x, y)} v(x) \overline{v(y)}] \leq \frac{1}{2} (|v(x)|^2 + |v(y)|^2),$$

and Definition 1.3(i) we have

$$\begin{aligned} (f_\epsilon v, (H - \lambda)(f_\epsilon v)) &\leq \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} b(x, y) |v(x)|^2 (f_\epsilon(x) - f_\epsilon(y))^2 \\ &\leq \frac{\rho^2}{2(\rho - \epsilon)^2} \sum_{x \in V} \sum_{y \sim x} |v(x)|^2 b(x, y) (d(x, y))^2 \leq \frac{\rho^2}{2(\rho - \epsilon)^2} \sum_{x \in V} \mu(x) |v(x)|^2, \end{aligned} \quad (2.3)$$

where the second inequality uses the fact that f_ϵ is a β -Lipschitz function with $\beta = \rho/(\rho - \epsilon)$.

On the other hand, using the definition of f_ϵ and the assumption (2.2) we have

$$(f_\epsilon v, (H - \lambda)(f_\epsilon v)) \geq \frac{1}{2} \sum_{\rho \leq D(x) \leq R} \mu(x)|v(x)|^2 + c_1 \|f_\epsilon v\|^2. \quad (2.4)$$

We now combine (2.4) and (2.3) to get

$$\frac{1}{2} \sum_{\rho \leq D(x) \leq R} \mu(x)|v(x)|^2 + c_1 \|f_\epsilon v\|^2 \leq \frac{\rho^2}{2(\rho - \epsilon)^2} \sum_{x \in V} \mu(x)|v(x)|^2.$$

We fix ρ and R , and let $\epsilon \rightarrow 0+$. After that, we let $\rho \rightarrow 0+$ and $R \rightarrow +\infty$. As a result, we get $v \equiv 0$. \square

Conclusion of the proof of Theorem 1. Since $\Delta_{b,\mu;\theta}|_{C_c(V)}$ is a non-negative operator, for all $u \in C_c(V)$, we have

$$(u, Hu) \geq \sum_{x \in V} \mu(x)W(x)|u(x)|^2,$$

and, hence, by assumption (1.9) we get:

$$\begin{aligned} (u, (H - \lambda)u) &\geq \frac{1}{2} \sum_{x \in V} \frac{1}{D(x)^2} \mu(x)|u(x)|^2 - (\lambda + C)\|u\|^2 \\ &\geq \frac{1}{2} \sum_{x \in V} \max\left(\frac{1}{D(x)^2}, 1\right) \mu(x)|u(x)|^2 - (\lambda + C + 1/2)\|u\|^2. \end{aligned} \quad (2.5)$$

Choosing, for instance, $\lambda = -C - 3/2$ in (2.5) we get the inequality (2.2) with $c_1 = 1$.

Thus, $(H - \lambda)|_{C_c(V)}$ with $\lambda = -C - 3/2$ is a symmetric operator satisfying $(u, (H - \lambda)u) \geq \|u\|^2$, for all $u \in C_c(V)$. In this case, it is known (see [22, Theorem X.26]) that the essential self-adjointness of $(H - \lambda)|_{C_c(V)}$ is equivalent to the following statement: if $v \in \ell_\mu^2(V)$ satisfies $(H - \lambda)v = 0$, then $v = 0$. Thus, by Lemma 2.2, the operator $(H - \lambda)|_{C_c(V)}$ is essentially self-adjoint. Hence, $H|_{C_c(V)}$ is essentially self-adjoint. \square

3. PROOF OF THEOREM 2

Throughout the section, we assume that the hypotheses of Theorem 2 are satisfied. We begin with the following lemma.

Lemma 3.1. *Let $(V_l, E_l)_{l \in L}$ be a good covering of degree m of (V, b, μ) , let H be as in (1.8), and let W_e be as in (1.10). Then, for all $u \in C_c(V)$ we have*

$$(u, Hu) \geq \sum_{x \in V} \mu(x)(W_e(x) + W(x))|u(x)|^2. \quad (3.1)$$

Proof. It is well known that

$$(u, Hu) = \sum_{\{x,y\} \in E} b(x,y)|u(x) - e^{i\theta(x,y)}u(y)|^2 + \sum_{x \in V} \mu(x)W(x)|u(x)|^2,$$

where E is the set of unoriented edges of (V, b, μ) . Thus, using the definition of the covering $(V_l, E_l)_{l \in L}$ of degree m and the definition of p_l we have

$$\begin{aligned} (u, Hu) &\geq \frac{1}{m} \sum_{l \in L} \sum_{\{x, y\} \in E_l} b(x, y) |u(x) - e^{i\theta(x, y)} u(y)|^2 + \sum_{x \in V} \mu(x) W(x) |u(x)|^2 \\ &\geq \frac{1}{m} \sum_{l \in L} \left(\left(\inf_{\{y, z\} \in E_l} b(y, z) \right) p_l \sum_{x \in V_l} \mu(x) |u(x)|^2 \right) + \sum_{x \in V} \mu(x) W(x) |u(x)|^2, \end{aligned}$$

which together with (1.10) gives (3.1). \square

Conclusion of the proof of Theorem 2. By Lemma 3.1 and assumption (1.11), for all $u \in C_c(V)$ we have

$$\begin{aligned} (u, (H - \lambda)u) &\geq \sum_{x \in V} \mu(x) (W_e(x) + W(x) - \lambda) |u(x)|^2 \\ &\geq \frac{1}{2} \sum_{x \in V} \max \left(\frac{1}{D(x)^2}, 1 \right) \mu(x) |u(x)|^2 - (C + \lambda + 1/2) \|u\|^2. \end{aligned}$$

From hereon we proceed in the same way as in the the proof of Theorem 1. \square

4. PROOF OF THEOREM 3

In this section we modify the proof of [20, Theorem 1], which is based on the technique of [23] in the context of Riemannian manifolds. Throughout the section, we assume that the hypotheses of Theorem 3 are satisfied.

We begin with the definitions of minimal and maximal operators associated with the expression (1.8). We define the operator H_{\min} by the formula $H_{\min}u := Hu$, for all $u \in \text{Dom}(H_{\min}) := C_c(V)$. As W is real-valued, it follows easily that the operator H_{\min} is symmetric in $\ell_\mu^2(V)$. We define $H_{\max} := (H_{\min})^*$, where T^* denotes the adjoint of operator T . Additionally, we define $\mathcal{D} := \{u \in \ell_\mu^2(V) : Hu \in \ell_\mu^2(V)\}$. Then, the following hold: $\text{Dom}(H_{\max}) = \mathcal{D}$ and $H_{\max}u = Hu$ for all $u \in \mathcal{D}$; see, for instance, [20, Section 3] or [24, Section 3] for details. Furthermore, by [16, Problem V.3.10], the operator H_{\min} is essentially self-adjoint if and only if

$$(H_{\max}u, v) = (u, H_{\max}v), \quad \text{for all } u, v \in \text{Dom}(H_{\max}). \quad (4.1)$$

In the setting of graphs of bounded vertex degree, the following proposition was proven in [20, Proposition 12].

Proposition 4.1. *If $u \in \text{Dom}(H_{\max})$, then*

$$\sum_{x, y \in V} b(x, y) \min\{q^{-1}(x), q^{-1}(y)\} |u(x) - e^{i\theta_{x, y}} u(y)|^2 \leq 4(\|Hu\| \|u\| + (K^2 + 1) \|u\|^2), \quad (4.2)$$

where H is as in (1.8) and K is as in (1.12).

Before proving Proposition 4.1, we define a sequence of cut-off functions. Let d_σ and d_{σ_q} be as in the hypothesis of Theorem 3. Fix $x_0 \in V$ and define

$$\chi_n(x) := \left(\left(\frac{2n - d_\sigma(x_0, x)}{n} \right) \vee 0 \right) \wedge 1, \quad x \in V, \quad n \in \mathbb{Z}_+. \quad (4.3)$$

Denote

$$B_n^\sigma(x_0) := \{x \in V : d_\sigma(x_0, x) \leq n\}. \quad (4.4)$$

The sequence $\{\chi_n\}_{n \in \mathbb{Z}_+}$ satisfies the following properties: (i) $0 \leq \chi_n(x) \leq 1$, for all $x \in V$; (ii) $\chi_n(x) = 1$ for $x \in B_n^\sigma(x_0)$ and $\chi_n(x) = 0$ for $x \notin B_{2n}^\sigma(x_0)$; (iii) for all $x \in V$, we have $\lim_{n \rightarrow \infty} \chi_n(x) = 1$; (iv) the functions χ_n have finite support; and (v) the functions χ_n satisfy the inequality

$$|\chi_n(x) - \chi_n(y)| \leq \frac{\sigma(x, y)}{n}, \quad \text{for all } x \sim y.$$

The properties (i)–(iii) and (v) can be checked easily. By hypothesis, we know that (V, d_{σ_q}) is a complete metric space and, thus, balls with respect to d_{σ_q} are finite; see, for instance, [15, Theorem A.1]. Let $B_{2n}^{\sigma_q}(x_0)$ be as in (4.4) with d_σ replaced by d_{σ_q} . Since $q \geq 1$ it follows that $B_{2n}^\sigma(x_0) \subseteq B_{2n}^{\sigma_q}(x_0)$. Thus, property (iv) is a consequence of property (ii) and the finiteness of $B_{2n}^{\sigma_q}(x_0)$.

Proof of Proposition 4.1. Let $u \in \text{Dom}(H_{\max})$ and let $\phi \in C_c(V)$ be a real-valued function. Define

$$I := \left(\sum_{x, y \in V} b(x, y) |u(x) - e^{i\theta_{x, y}} u(y)|^2 ((\phi(x))^2 + (\phi(y))^2) \right)^{1/2}. \quad (4.5)$$

We will first show that

$$\begin{aligned} I^2 &\leq 4|(\phi^2 H u, u)| + 4(\phi^2 q u, u) \\ &\quad + \sqrt{2} I \left(\sum_{x, y \in V} b(x, y) (\phi(x) - \phi(y))^2 |u(x) + e^{i\theta_{x, y}} u(y)|^2 \right)^{1/2}. \end{aligned} \quad (4.6)$$

To do this, we first note that

$$\begin{aligned} I^2 &= 4(\phi^2 H u, u) - 4(\phi^2 W u, u) \\ &\quad + \sum_{x, y \in V} b(x, y) (e^{i\theta_{x, y}} u(y) - u(x)) (e^{-i\theta_{x, y}} \overline{u(y)} + \overline{u(x)}) ((\phi(x))^2 - (\phi(y))^2), \end{aligned} \quad (4.7)$$

which can be checked by expanding the terms under summations on both sides of the equality and using the properties $b(x, y) = b(y, x)$ and $\theta(x, y) = -\theta(y, x)$. The details of this computation can be found in the proof of [20, Proposition 12].

The inequality (4.6) is obtained from (4.7) by using (1.13), the factorization

$$(\phi(x))^2 - (\phi(y))^2 = (\phi(x) - \phi(y))(\phi(x) + \phi(y)),$$

Cauchy–Schwarz inequality, and

$$(\phi(x) + \phi(y))^2 \leq 2(\phi^2(x) + \phi^2(y)).$$

Let χ_n be as in (4.3) and let q be as in (1.13). Define

$$\phi_n(x) := \chi_n(x)q^{-1/2}(x). \quad (4.8)$$

By property (iv) of χ_n it follows that ϕ_n has finite support. By property (i) of χ_n and since $q \geq 1$, we have

$$0 \leq \phi_n(x) \leq q^{-1/2}(x) \leq 1, \quad \text{for all } x \in V. \quad (4.9)$$

By property (iii) of χ_n we have

$$\lim_{n \rightarrow \infty} \phi_n(x) = q^{-1/2}(x), \quad \text{for all } x \in V. \quad (4.10)$$

By (1.12), properties (i) and (v) of χ_n , and the inequality $q \geq 1$, we have

$$|\phi_n(x) - \phi_n(y)| \leq \left(\frac{1}{n} + K\right) \sigma(x, y), \quad \text{for all } x \sim y, \quad (4.11)$$

where K is as in (1.12). We will also use the inequality

$$|e^{i\theta_{x,y}}u(y) + u(x)|^2 \leq 2(|u(x)|^2 + |u(y)|^2). \quad (4.12)$$

By (4.11), (4.12), and Definition 1.3(ii), we get

$$\begin{aligned} & \left(\sum_{x,y \in V} b(x,y) (\phi_n(x) - \phi_n(y))^2 |u(x) + e^{i\theta_{x,y}}u(y)|^2 \right)^{1/2} \\ & \leq \sqrt{2} \left(\frac{1}{n} + K \right) \left(\sum_{x,y \in V} b(x,y) (\sigma(x,y))^2 (|u(x)|^2 + |u(y)|^2) \right)^{1/2} \\ & = 2 \left(\frac{1}{n} + K \right) \left(\sum_{x,y \in V} b(x,y) (\sigma(x,y))^2 |u(x)|^2 \right)^{1/2} \\ & \leq 2 \left(\frac{1}{n} + K \right) \left(\sum_{x \in V} \mu(x) |u(x)|^2 \right)^{1/2} \end{aligned} \quad (4.13)$$

By (4.6) with $\phi = \phi_n$, (4.13), and (4.9), we obtain

$$I_n^2 \leq 4\|Hu\| \|u\| + 4\|u\|^2 + 2\sqrt{2}I_n \left(\frac{1}{n} + K\right) \|u\|, \quad (4.14)$$

for all $u \in \text{Dom}(H_{\max})$, where I_n is as in (4.5) with $\phi = \phi_n$.

Using the inequality $ab \leq \frac{a^2}{4} + b^2$ with $a = \sqrt{2}I_n$ in the third term on the right-hand side of (4.14) and rearranging, we obtain

$$I_n^2 \leq 8 \left(\|Hu\| \|u\| + \left(\left(\frac{1}{n} + K \right)^2 + 1 \right) \|u\|^2 \right). \quad (4.15)$$

Letting $n \rightarrow \infty$ in (4.15) and using (4.10) together with Fatou's lemma, we get

$$\begin{aligned} & \sum_{x,y \in V} b(x,y) |u(x) - e^{i\theta_{x,y}} u(y)|^2 (q^{-1}(x) + q^{-1}(y)) \\ & \leq 8 (\|Hu\| \|u\| + (K^2 + 1) \|u\|^2). \end{aligned} \quad (4.16)$$

Since

$$2 \min\{q^{-1}(x), q^{-1}(y)\} \leq q^{-1}(x) + q^{-1}(y), \quad \text{for all } x, y \in V,$$

the inequality (4.2) follows directly from (4.16). \square

Continuation of the proof of Theorem 3. Our final goal is to prove (4.1). Let d_{σ_q} be as in the hypothesis of Theorem 3. Fix $x_0 \in V$ and define

$$P(x) := d_{\sigma_q}(x_0, x), \quad x \in V. \quad (4.17)$$

In what follows, for a function $f: V \rightarrow \mathbb{R}$ we define $f^+(x) := \max\{f(x), 0\}$. Let $u, v \in \text{Dom}(H_{\max})$, let $s > 0$, and define

$$J_s := \sum_{x \in V} \left(1 - \frac{P(x)}{s} \right)^+ \left((Hu)(x) \overline{v(x)} - u(x) \overline{(Hv)(x)} \right) \mu(x), \quad (4.18)$$

where P is as in (4.17) and H is as in (1.8).

Since (V, d_{σ_q}) is a complete metric space, by [15, Theorem A.1] it follows that the set

$$U_s := \{x \in V : P(x) \leq s\}$$

is finite. Thus, for all $s > 0$, the summation in (4.18) is performed over finitely many vertices.

The following lemma follows easily from the definition of J_s and the dominated convergence theorem; see the proof of [20, Lemma 13] for details.

Lemma 4.2. *Let J_s be as in (4.18). Then*

$$\lim_{s \rightarrow +\infty} J_s = (Hu, v) - (u, Hv). \quad (4.19)$$

In what follows, for $u \in \text{Dom}(H_{\max})$, define

$$T_u := \left(\sum_{x,y \in V} b(x,y) \min\{q^{-1}(x), q^{-1}(y)\} |u(x) - e^{i\theta_{x,y}} u(y)|^2 \right)^{1/2}. \quad (4.20)$$

Note that T_u is finite by Proposition 4.1.

Lemma 4.3. *Let $u, v \in \text{Dom}(H_{\max})$, let T_u and T_v be as in (4.20), and let J_s be as in (4.18). Then*

$$|J_s| \leq \frac{1}{2s} (\|v\|T_u + \|u\|T_v). \quad (4.21)$$

Proof. A computation shows that

$$2J_s = \sum_{x,y \in V} \left((1 - P(x)/s)^+ - (1 - P(y)/s)^+ \right) b(x,y) \left((e^{-i\theta_{x,y}} \overline{v(y)} - \overline{v(x)})u(x) - (e^{i\theta_{x,y}} u(y) - u(x))\overline{v(x)} \right),$$

which, together with the triangle inequality and property

$$|f^+(x) - g^+(x)| \leq |f(x) - g(x)|,$$

leads to the following estimate:

$$2|J_s| \leq \frac{1}{s} \sum_{x,y \in V} b(x,y) |P(x) - P(y)| \left(|e^{i\theta_{x,y}} v(y) - v(x)| |u(x)| + |e^{i\theta_{x,y}} u(y) - u(x)| |v(x)| \right). \quad (4.22)$$

By (4.17) and (1.14), for all $x \sim y$ we have

$$|P(x) - P(y)| \leq d_{\sigma_q}(x,y) \leq \sigma_q(x,y) = \min\{q^{-1/2}(x), q^{-1/2}(y)\} \cdot \sigma(x,y). \quad (4.23)$$

To obtain (4.21), we combine (4.22) and (4.23) and use Cauchy–Schwarz inequality together with Definition 1.3(ii). \square

The end of the proof of Theorem 3. Let $u \in \text{Dom}(H_{\max})$ and $v \in \text{Dom}(H_{\max})$. By the definition of H_{\max} , it follows that $Hu \in \ell_\mu^2(V)$ and $Hv \in \ell_\mu^2(V)$. Letting $s \rightarrow +\infty$ in (4.21) and using the finiteness of T_u and T_v , it follows that $J_s \rightarrow 0$ as $s \rightarrow +\infty$. This, together with (4.19), shows (4.1). \square

5. EXAMPLES

In this section we give some examples that illustrate the main results of the paper. In what follows, for $x \in \mathbb{R}$, the notation $\lceil x \rceil$ denotes the smallest integer N such that $N \geq x$. Additionally, $\lfloor x \rfloor$ denotes the greatest integer N such that $N \leq x$.

Example 5.1. In this example we consider the graph $G = (V, E)$ whose vertices $x_{j,k}$ are arranged in a “triangular” pattern so that the first row contains $x_{1,1}$; for $2 \leq j \leq 4$, the j -th row contains $x_{j,1}$ and $x_{j,2}$; for $5 \leq j \leq 9$, the j -th row contains $x_{j,1}$, $x_{j,2}$, and $x_{j,3}$; for $10 \leq j \leq 16$, the j -th row contains $x_{j,1}$, $x_{j,2}$, $x_{j,3}$, and $x_{j,4}$; and so on. There are two types of edges in the graph: (i) for every $j \geq 1$, we have $x_{j,1} \sim x_{j+1,k}$ for all $1 \leq k \leq \lceil (j+1)^{1/2} \rceil$; (ii) for every $j \geq 2$, we have the “horizontal” edges $x_{j,k} \sim x_{j,k+1}$, for all $1 \leq k \leq \lfloor j^{1/2} \rfloor - 1$. Clearly, G does not have a bounded vertex degree.

Let $T = (V_T, E_T)$ be the subgraph of G whose set of edges E_T consists of type-(i) edges of G described above. Note that T is a spanning tree of G . Additionally, note that for every type-(ii) edge e of G the following are true: (i) $e \notin E_T$ and (ii) there is a unique 3-cycle (a cycle with 3 vertices) that contains e . Thus, by [3, Lemma 2.1], the corresponding 3-cycles, which we enumerate by $\{C_l\}_{l \in \mathbb{Z}_+}$, form a basis for the space of cycles of G . Furthermore, by Definition 1.11, the family $\{C_l = (V_l, E_l)\}_{l \in \mathbb{Z}_+}$ is a good covering of degree $m = 2$ of G . Following [3, Proposition 2.1(a)] and [3, Lemma 2.3], we define the phase function $\theta: V_l \times V_l \rightarrow [-\pi, \pi]$ satisfying the following properties: (i) if an edge $\{x, y\}$ belongs to $E_l \setminus E_T$, we have $\theta(x, y) = -\theta(y, x)$; (ii) if $\{x, y\} \in E_T$, we have $\theta(x, y) = 0$; and (iii) $p_l = |1 - e^{i\pi/3}|^2 = 1$, where p_l is as in (1.10) with G_l replaced by C_l .

With this choice of p_l and using the good covering $\{C_l\}_{l \in \mathbb{Z}_+}$ of degree $m = 2$, the definition of the effective potential (1.10) simplifies to

$$W_e(x) := \frac{1}{2} \sum_{\{l \in L \mid x \in V_l\}} \inf_{\{y, z\} \in E_l} b(y, z). \quad (5.1)$$

Let $\{b_j\}_{j \in \mathbb{Z}_+}$ be an increasing sequence of positive numbers. We define (i) $b(x, y) = b_j$ if $x \sim y$ and x is in the j -th row and y is in the $(j+1)$ -st row; (ii) $b(x, y) = b_j$ if $x \sim y$ and x and y are both in the $(j+1)$ -st row; (iii) $b(x, y) = 0$, otherwise. With this choice of $b(x, y)$, we have $W_e(x_{1,1}) = b_1/2$. Additionally, since b_j is an increasing sequence of positive numbers, using (5.1) it is easy to see that if a vertex x is in the j -th row, then

$$W_e(x) \geq \frac{1}{2} b_{j-1}, \quad \text{for all } j \geq 2. \quad (5.2)$$

Let $0 < \beta < 3/4$, and set $\mu(x) := j^{-2\beta}$ if the vertex x is in the j -th row. Let $\alpha > 0$ satisfy $\alpha + 2\beta > 3/2$, and set $b_j := j^\alpha$, for all $j \in \mathbb{Z}_+$. With this choice of $b(x, y)$ and $\mu(x)$, let $\sigma_1(x, y)$ be as in (1.2) and let d_{σ_1} be the intrinsic path metric associated with σ_1 as in Section 1.2. As there are $\lfloor \sqrt{j} \rfloor + 3$ edges departing from the vertex $x_{j,1}$, we have

$$\sigma_1(x_{j,1}; x_{j+1,1}) = j^{-\alpha/2} (j+1)^{-\beta} (\lfloor \sqrt{j+1} \rfloor + 3)^{-1/2}, \quad \text{for all } j \in \mathbb{Z}_+. \quad (5.3)$$

Additionally, note that the path $\gamma = (x_{1,1}; x_{2,1}; x_{3,1}; \dots)$ is a geodesic with respect to the path metric d_{σ_1} , that is, $d_{\sigma_1}(x_{1,1}; x_{j,1}) = l_{\sigma_1}(x_{1,1}; x_{2,1}; \dots; x_{j,1})$ for all $j \in \mathbb{Z}_+$, where l_{σ_1} is as in (1.1). Since $\alpha + 2\beta > 3/2$, it follows that

$$\sum_{j=1}^{\infty} j^{-\alpha/2} (j+1)^{-\beta} (\lfloor \sqrt{j+1} \rfloor + 3)^{-1/2} < \infty;$$

hence, by [15, Theorem A.1] the space (V, d_{σ_1}) is not metrically complete. Let $D(x)$ be as in (1.3) corresponding to d_{σ_1} . If a vertex x is in the n -th row, using (5.3) and

$$\lfloor \sqrt{j+1} \rfloor + 3 \leq 3\sqrt{j+1}, \quad \text{for all } j \in \mathbb{Z}_+,$$

we have

$$D(x) \geq \frac{1}{\sqrt{3}} \sum_{k=n}^{\infty} (k+1)^{-\beta-\alpha/2-1/4} \geq \frac{(n+1)^{-\beta-\alpha/2+3/4}}{\sqrt{3}(\beta+\alpha/2-3/4)},$$

which leads to

$$\frac{1}{2D(x)^2} \leq \frac{3(4\beta + 2\alpha - 3)^2(n+1)^{2\beta+\alpha-3/2}}{32}, \quad (5.4)$$

for all vertices x in the n -th row, where $n \geq 1$. Define $W(x) = -n^{2\beta+\alpha-3/2}$ for all vertices x in the n -th row, where $n \geq 1$. Using (5.2) and $W_e(x_{1,1}) = b_1/2$, together with (5.4) and the assumption $0 < \beta < 3/4$, it follows that there exists a constant $C > 0$ (depending on α and β) such that (1.11) is satisfied. Thus, by Theorem 2 the operator $\Delta_{b,\mu;\theta} + W$ is essentially self-adjoint on $C_c(V)$. Clearly, Theorem 2 is also applicable in the case $W(x) = 0$ for all $x \in V$, that is, the operator $\Delta_{b,\mu;\theta}$ is essentially self-adjoint on $C_c(V)$. A calculation shows that μ and b in this example do not satisfy [19, Assumption A]; hence, we cannot use [19, Theorem 1.2].

We will now show that under more restrictive assumption $1/2 < \beta < 3/4$, we cannot apply [9, Proposition 2.2] to this example with $W(x) \equiv 0$. To see this, using (1.16) and the fact that among the $\lfloor \sqrt{j} \rfloor + 3$ edges departing from the vertex $x_{j,1}$, there are $\lfloor \sqrt{j} \rfloor + 1$ edges with weight b_j and 2 edges with weight b_{j-1} , we first note that

$$\text{Deg}(x_{1,1}) = 2, \quad \text{Deg}(x_{j,1}) = j^{2\beta}((\lfloor \sqrt{j} \rfloor + 1)j^\alpha + 2(j-1)^\alpha), \quad \text{for all } j \geq 2.$$

Let $\lambda \in \mathbb{R}$ be such that (1.15) is satisfied, with $W(x) \equiv 0$. Let a_n be as in (1.17) corresponding to the path $\gamma = (x_{1,1}; x_{2,1}; x_{3,1}; \dots)$, the potential $W \equiv 0$, and λ . Then $a_1 = 1$,

$$(a_2)^2 = (1 + \lambda/2)^2 = (\lambda + 2)^2/4,$$

and

$$\begin{aligned} (a_n)^2 &= \frac{(\lambda + 2)^2}{4} \prod_{j=2}^{n-1} \left(1 + \frac{\lambda}{j^{\alpha+2\beta}(\lfloor \sqrt{j} \rfloor + 1) + 2j^{2\beta}(j-1)^\alpha} \right)^2 \\ &= \frac{(\lambda + 2)^2}{4} \prod_{j=2}^{n-1} \left(\frac{j^{\alpha+2\beta}(\lfloor \sqrt{j} \rfloor + 1) + 2j^{2\beta}(j-1)^\alpha + \lambda}{j^{\alpha+2\beta}(\lfloor \sqrt{j} \rfloor + 1) + 2j^{2\beta}(j-1)^\alpha} \right)^2, \quad n \geq 3. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} (a_n)^2 \mu(x_{n,1}) = 1 + \frac{(\lambda + 2)^2}{4(2)^{2\beta}} + \sum_{n=3}^{\infty} \frac{(a_n)^2}{n^{2\beta}}.$$

Using Raabe's test, it can be checked that the series on the right hand side of this equality converges. (Here, we used the more restrictive assumption $1/2 < \beta < 3/4$.) Hence, looking at (1.17), we see that [9, Proposition 2.2] cannot be used in this example.

Example 5.2. Consider the graph whose vertices are arranged in a “triangular” pattern so that $x_{1,1}$ is in the first row, $x_{2,1}$ and $x_{2,2}$ are in the second row, $x_{3,1}$, $x_{3,2}$, and $x_{3,3}$ are in the third row, and so on. The vertex $x_{1,1}$ is connected to $x_{2,1}$ and $x_{2,2}$. The vertex $x_{2,i}$, where $i = 1, 2$, is connected to every vertex $x_{3,j}$, where $j = 1, 2, 3$. The pattern continues so that each of k vertices in the k -th row is connected to each of $k + 1$ vertices in the $(k + 1)$ -st row. Note that for all $k \geq 1$ and $j \geq 1$ we have $\text{deg}(x_{k,j}) = 2k$, where $\text{deg}(x)$ is as in (1.2). Let $\mu(x) = n^{-1/2}$

for every vertex x in the n -th row, and let $b(x, y) \equiv 1$ for all vertices $x \sim y$. Following (1.2), for every vertex x in the n -th row and every vertex y in the $(n + 1)$ -st row, define

$$\sigma(x, y) := \min \left\{ \frac{n^{-1/2}}{2n}, \frac{(n+1)^{-1/2}}{2(n+1)} \right\}^{1/2} = 2^{-1/2}(n+1)^{-3/4}.$$

For all vertices x in the n -th row, define $W(x) = -n^{1/2}$ and $q(x) = n^{1/2}$. Clearly, the inequality (1.13) is satisfied. With this choice of q , following (1.14), for every vertex x in the n -th row and every vertex y in the $(n + 1)$ -st row, define

$$\sigma_q(x, y) := \min\{n^{-1/4}, (n+1)^{-1/4}\} \cdot \sigma(x, y) = 2^{-1/2}(n+1)^{-1}.$$

Since

$$\sum_{j=1}^{\infty} 2^{-1/2}(j+1)^{-1} = \infty,$$

by [15, Theorem A.1] it follows that the space (V, d_{σ_q}) is metrically complete. Additionally, it is easily checked that (1.12) is satisfied with $K = \sqrt{2}$. Therefore, by Theorem 3 the operator $\Delta_{b,\mu} + W$ is essentially self-adjoint on $C_c(V)$. Furthermore, it is easy to see that for every $k \in \mathbb{R}$, there exists a function $u \in C_c(V)$ such that the inequality

$$((\Delta_{b,\mu} + W)u, u) \geq k\|u\|^2$$

is not satisfied. Thus, the operator $\Delta_{b,\mu} + W$ is not semi-bounded from below, and we cannot use [19, Theorem 1.2].

It turns out that [9, Proposition 2.2] is not applicable in this example. To see this, using (1.16) we first note that $\text{Deg}(x_{k,j}) = 2k^{3/2}$, for all $k \geq 1$ and all $j \geq 1$. Let $\lambda \in \mathbb{R}$ be such that (1.15) is satisfied, with W as in this example. Let a_n be as in (1.17) corresponding to the path $\gamma = (x_{1,1}; x_{2,1}; x_{3,1}; \dots)$, the potential $W(x_{k,1}) = -k^{1/2}$, and λ . Then $a_1 = 1$,

$$(a_2)^2 = (1 + (\lambda - 1)/2)^2 = (\lambda + 1)^2/4,$$

and for $n \geq 3$ we have

$$(a_n)^2 = \frac{(\lambda + 1)^2}{4} \prod_{k=2}^{n-1} \left(1 + \frac{\lambda - k^{1/2}}{2k^{3/2}} \right)^2 = \frac{(\lambda + 1)^2}{4} \prod_{k=2}^{n-1} \left(\frac{2k^{3/2} - k^{1/2} + \lambda}{2k^{3/2}} \right)^2.$$

Therefore,

$$\sum_{n=1}^{\infty} (a_n)^2 \mu(x_{n,1}) = 1 + \frac{(\lambda + 1)^2}{4\sqrt{2}} + \sum_{n=3}^{\infty} \frac{(a_n)^2}{\sqrt{n}}.$$

Using Raabe's test, it can be checked that the series on the right hand side of this equality converges. Hence, looking at (1.17), we see that [9, Proposition 2.2] cannot be used in this example.

Acknowledgment. The second author is grateful to Daniel Lenz for fruitful discussions.

REFERENCES

- [1] Bauer, F., Keller, M., Wojciechowski, R. K.: Cheeger inequalities for unbounded graph Laplacians. arXiv:1209.4911
- [2] Colin de Verdière, Y., Torki-Hamza, N., Truc, F.: Essential self-adjointness for combinatorial Schrödinger operators II. *Math. Phys. Anal. and Geom.* **14** (2011) 21–38
- [3] Colin de Verdière, Y., Torki-Hamza, N., Truc, F.: Essential self-adjointness for combinatorial Schrödinger operators III- Magnetic fields. *Ann. Fac. Sci. Toulouse Math.* (6) **20** (2011) 599–611
- [4] Colin de Verdière, Y., Truc, F.: Confining quantum particles with a purely magnetic field. *Ann. Inst. Fourier (Grenoble)* **60** (7) (2010) 2333–2356
- [5] Folz, M.: Gaussian upper bounds for heat kernels of continuous time simple random walks. *Electron. J. Probab.* **16** (2011) 1693–1722
- [6] Folz, M.: Volume growth and stochastic completeness of graphs. arXiv:1201.5908. To appear in: *Trans. Amer. Math. Soc.*
- [7] Folz, M.: Volume growth and spectrum for general graph Laplacians. arXiv:1204.4770
- [8] Frank, R. L., Lenz, D., Wingert, D.: Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory. arXiv:1012.5050
- [9] Golénia, S.: Hardy inequality and Weyl asymptotic for discrete Laplacians. arXiv:1106.0658
- [10] Grigor'yan, A., Huang, X., Masamune, J.: On stochastic completeness of jump processes. *Math. Z.* **271** (2012) 1211–1239
- [11] Haeseler, S., Keller, M., Lenz, D., Wojciechowski, R. K.: Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions, *J. Spectr. Theory* **2** (2012) 397–432
- [12] Haeseler, S., Keller, M., Wojciechowski, R. K.: Volume growth and bounds for the essential spectrum for Dirichlet forms. arXiv:1205.4985
- [13] Huang, X.: On uniqueness class for a heat equation on graphs. *J. Math. Anal. Appl.* **393** (2012) 377–388
- [14] Huang, X.: A note on the volume growth criterion for stochastic completeness of weighted graphs. arXiv:1209.2069
- [15] Huang, X., Keller, M., Masamune, J., Wojciechowski, R. K.: A note on self-adjoint extensions of the Laplacian on weighted graphs. arXiv:1208.6358
- [16] Kato, T.: *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin (1980)
- [17] Masamune, J.: A Liouville property and its application to the Laplacian of an infinite graph. In: *Contemp. Math.* vol. 484, Amer. Math. Soc., Providence (2009) pp. 103–115
- [18] Masamune, J., Uemura, T.: Conservation property of symmetric jump processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** (2011) 650–662
- [19] Milatovic, O.: Essential self-adjointness of magnetic Schrödinger operators on locally finite graphs. *Integr. Equ. Oper. Theory* **71** (2011) 13–27
- [20] Milatovic, O.: A Sears-type self-adjointness result for discrete magnetic Schrödinger operators. *J. Math. Anal. Appl.* **396** (2012) 801–809
- [21] Nenciu, G., Nenciu, I.: On confining potentials and essential self-adjointness for Schrödinger operators on bounded domains in \mathbb{R}^n . *Ann. Henri Poincaré* **10** (2009) 377–394
- [22] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics II: Fourier analysis, self-adjointness*. Academic Press, New York (1975)
- [23] Shubin, M. A.: Essential self-adjointness for magnetic Schrödinger operators on non-compact manifolds. In: *Séminaire Équations aux Dérivées Partielles (Polytechnique)* (1998–1999), Exp. No. XV, Palaiseau (1999) pp. XV-1–XV-22
- [24] Torki-Hamza, N.: Laplacians de graphes infinis I Graphes métriquement complets. *Confluentes Math.* **2** (2010) 333–350

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NORTH FLORIDA, JACKSONVILLE, FL 32224, USA.

E-mail address: `omilatov@unf.edu`

GRENOBLE UNIVERSITY, INSTITUT FOURIER, UNITÉ MIXTE DE RECHERCHE CNRS-UJF 5582, BP 74, 38402-SAINTE MARTIN D'HÈRES CEDEX, FRANCE.

E-mail address: `francoise.truc@ujf-grenoble.fr`