Self-adjoint extensions of discrete magnetic Schrödinger operators

Francoise Truc, Ognjen Milatovic

SELF-ADJOINT EXTENSIONS OF DISCRETE MAGNETIC SCHRÖDINGER OPERATORS

OGNJEN MILATOVIC, FRANÇOISE TRUC

Abstract. Using the concept of intrinsic metric on a locally finite weighted graph, we give sufficient conditions for the magnetic Schrödinger operator to be essentially self-adjoint. The present paper is an extension of some recent results proven in the context of graphs of bounded degree.

1. Introduction and the main results

1.1. **The setting.** Let V be a countably infinite set. We assume that V is equipped with a measure $\mu\colon V \to (0,\infty)$. Let $b\colon V\times V \to [0,\infty)$ be a function such that

(i) $b(x,y)=b(y,x)$, for all x, $y\in V$;

(ii) $b(x,x)=0$, for all $x\in V$;

(iii) $\deg(x) := \sharp\{y\in V\colon b(x,y)>0\}<\infty$, for all $x\in V$. Here, $\sharp S$ denotes the number of elements in the set S.

Vertices x, $y\in V$ with $b(x,y)>0$ are called *neighbors*, and we denote this relationship by $x\sim y$. We call the triple (V,b,μ) a *locally finite weighted graph*. We assume that (V,b,μ) is connected, that is, for any x, $y\in V$ there exists a path γ joining x and y. Here, γ is a sequence x_0, x_2, $\dots, x_n\in V$ such that $x=x_0$, $y=x_n$, and $x_j\sim x_{j+1}$ for all $0\le j\le n-1$.

1.2. **Intrinsic metric.** Following [15] we define a *pseudo metric* to be a map $d\colon V\times V$ such that $d(x,y)=d(y,x)$, for all x, $y\in V$; $d(x,x)=0$, for all $x\in V$; and $d(x,y)$ satisfies the triangle inequality. A pseudo-metric $d=d_\sigma$ is called a path pseudo-metric if there exists a map $\sigma\colon V\times V\to[0,\infty)$ such that $\sigma(x,y)=\sigma(y,x)$, for all x, $y\in V$; $\sigma(x,y)>0$ if and only if $x\sim y$; and

$$d_\sigma=\inf\{l_\sigma(\gamma)\colon \gamma=(x_0,x_1,\dots,x_n), n\ge 1, \text{is a path connecting } x \text{ and } y\},$$

where the length l_σ of the path $\gamma=(x_0,x_1,\dots,x_n)$ is given by

$$l_\sigma(\gamma)=\sum_{i=0}^{n-1}\sigma(x_i,x_{i+1}). \tag{1.1}$$

As in [15] we make the following definitions.

2000 *Mathematics Subject Classification.* 35J10, 39A12, 47B25.

Definition 1.3. (i) A pseudo metric d on (V, b, μ) is called *intrinsic* if

$$\frac{1}{\mu(x)} \sum_{y \in V} b(x,y)(d(x,y))^2 \leq 1, \qquad \text{for all } x \in V.$$

(ii) An intrinsic path pseudo metric $d = d_\sigma$ on (V, b, μ) is called *strongly intrinsic* if

$$\frac{1}{\mu(x)} \sum_{y \in V} b(x,y)(\sigma(x,y))^2 \leq 1, \qquad \text{for all } x \in V.$$

Remark 1.4. On a locally finite graph (V, b, μ), the formula

$$\sigma_1(x,y) = b(x,y)^{-1/2} \min \left\{ \frac{\mu(x)}{\deg(x)}, \frac{\mu(y)}{\deg(y)} \right\}^{1/2}, \qquad \text{with } x \sim y, \tag{1.2}$$

where $\deg(x)$ is as in property (iii) of $b(x,y)$, defines a strongly intrinsic path metric; see [15, Example 2.1].

1.5. **Cauchy boundary.** For a path metric $d = d_\sigma$ on V, we denote the metric completion by $(\widehat{V}, \widehat{d})$. As in [15] we define the *Cauchy boundary* $\partial_C V$ as follows: $\partial_C V := \widehat{V} \backslash V$. Note that (V, d) is metrically complete if and only if $\partial_C V$ is empty. For a path metric $d = d_\sigma$ on V and $x \in V$, we define

$$D(x) := \inf_{z \in \partial_C V} d_\sigma(x,z). \tag{1.3}$$

1.6. **Inner product.** In what follows, $C(V)$ is the set of complex-valued functions on V, and $C_c(V)$ is the set of finitely supported elements of $C(V)$. By $\ell^2_\mu(V)$ we denote the space of functions $f \in C(V)$ such that

$$\|f\|^2 := \sum_{x \in V} \mu(x)|f(x)|^2 < \infty, \tag{1.4}$$

where $|\cdot|$ denotes the modulus of a complex number.

In particular, the space $\ell^2_\mu(V)$ is a Hilbert space with the inner product

$$(f,g) := \sum_{x \in V} \mu(x) f(x) \overline{g(x)}. \tag{1.5}$$

1.7. **Laplacian operator.** We define the formal Laplacian $\Delta_b \colon C(V) \to C(V)$ on (V, b, μ) by the formula

$$(\Delta_{b,\mu} u)(x) = \frac{1}{\mu(x)} \sum_{y \in V} b(x,y)(u(x) - u(y)). \tag{1.6}$$

1.8. **Magnetic Schrödinger operator.** We fix a phase function $\theta\colon V\times V\to[-\pi,\pi]$ such that $\theta(x,y)=-\theta(y,x)$ for all $x\,,y\in V$, and denote $\theta_{x,y}:=\theta(x,y)$. We define the formal magnetic Laplacian $\Delta_{b,\mu;\theta}\colon C(V)\to C(V)$ on (V,b,μ) by the formula

$$(\Delta_{b,\mu;\theta}u)(x)=\frac{1}{\mu(x)}\sum_{y\in V}b(x,y)(u(x)-e^{i\theta_{x,y}}u(y)). \tag{1.7}$$

We define the formal magnetic Schrödinger operator $H\colon C(V)\to C(V)$ by the formula

$$Hu:=\Delta_{b,\mu;\theta}u+Wu, \tag{1.8}$$

where $W\colon V\to\mathbb{R}$.

1.9. **Statements of the results.** We are ready to state our first result.

Theorem 1. *Assume that (V,b,μ) is a locally finite, weighted, and connected graph. Let $d=d_\sigma$ be an intrinsic path metric on V such that (V,d) is not metrically complete. Assume that there exists a constant C such that*

$$W(x)\geq\frac{1}{2(D(x))^2}-C,\qquad \textit{for all } x\in V, \tag{1.9}$$

where $D(x)$ is as in (1.3). Then H is essentially self-adjoint on $C_c(V)$.

Remark 1.10. It is possible find μ, b, and a potential W satisfying $W(x)\geq\frac{k}{2(D(x))^2}$ with $0<k<1$, such that $H=\Delta_{b,\mu}+W$ is not essentially self-adjoint; see [2, Section 5.3.2].

If the graph (V,b,μ) has a special type of covering, the condition (1.9) on W can be relaxed with the help of "effective potential," as seen in the next theorem. First, we give a description of this special type of covering. In what follows, for a graph (V,b,μ), we define the set of unoriented edges as $E:=\{\{x,y\}\colon x,\,y\in V \text{ and } b(x,y)>0\}$. Sometimes, when we want to emphasize the set E, instead of $G=(V,b,\mu)$ we will use the notation $G=(V,E)$.

Definition 1.11. Let $m\in\mathbb{N}$. A *good covering of degree* m of $G=(V,E)$ is a family $G_l=(V_l,E_l)_{l\in L}$ of finite connected sub-graphs of G so that

(i) $V=\cup_{l\in L}V_l$;

(ii) for any $\{x,y\}\in E$,

$$0<\#\{l\in L\mid\{x,y\}\in E_l\}\leq m.$$

Remark 1.12. It is known that a graph with bounded vertex degree admits a good covering; see [3, Proposition 2.2]. The graph in Example 5.2 below does not have a bounded vertex degree. Note that this graph has a good covering of degree $m=1$.

Assume that (V,b,μ) has a good covering $(V_l,E_l)_{l\in L}$. Let θ_l be the restriction of θ to $V_l\times V_l$. Let $\Delta^{(l)}_{1,\mu;\theta}$ be as in (1.7) with $V=V_l$, $\theta=\theta_l$, and $b\equiv1$. Then $\Delta^{(l)}_{1,\mu;\theta}$ is a bounded and non-negative self-adjoint operator in $\ell^2_\mu(V_l)$. Let p_l denote the lowest eigenvalue of $\Delta^{(l)}_{1,\mu;\theta}$. With these notations, for a graph (V,b,μ) and the phase function θ, we define the *effective potential* corresponding to a good covering $(V_l,E_l)_{l\in L}$ of degree m as follows:

$$W_e(x) := \frac{1}{m}\sum_{\{l\in L\,|\,x\in V_l\}} p_l \inf_{\{y,z\}\in E_l} b(y,z). \tag{1.10}$$

We now state our second result.

Theorem 2. *Assume that (V,b,μ) is a locally finite, weighted, and connected graph. Assume that (V,b,μ) has a good covering $(V_l,E_l)_{l\in L}$. Let $d=d_\sigma$ be an intrinsic path metric on V such that (V,d) is not metrically complete. Assume that there exists a constant C such that*

$$W_e(x)+W(x)\geq \frac{1}{2(D(x))^2}-C, \qquad \text{for all } x\in V, \tag{1.11}$$

where W_e is as in (1.10) and $D(x)$ is as in (1.3). Then H is essentially self-adjoint on $C_c(V)$.

In the setting of metrically complete graphs, we have the following result:

Theorem 3. *Assume that (V,b,μ) be a locally finite, weighted, and connected graph. Let d_σ be a strongly intrinsic path metric on V. Let $q\colon V\to[1,\infty)$ be a function satisfying*

$$|q^{-1/2}(x)-q^{-1/2}(y)|\leq K\sigma(x,y), \qquad \text{for all } x,\, y\in V \text{ such that } x\sim y, \tag{1.12}$$

where K is a constant. Let H be as in (1.8) with $W\colon V\to\mathbb{R}$ satisfying

$$W(x)\geq -q(x), \quad \text{for all } x\in V. \tag{1.13}$$

Let

$$\sigma_q(x,y)=\min\{q^{-1/2}(x),q^{-1/2}(y)\}\cdot\sigma(x,y) \tag{1.14}$$

and let d_{σ_q} be the path metric corresponding to σ_q. Assume that (V,d_{σ_q}) is metrically complete. Then H is essentially self-adjoint on $C_c(V)$.

1.13. **Some comments on the existing literature.** The notion of intrinsic metric allows us to remove the bounded vertex degree assumption present in [2, 3, 20]. More specifically, Theorem 1 extends [2, Theorem 4.2], which was proven in the context of graphs of bounded vertex degree for the operator $\Delta_{b,\mu}+W$, with $\Delta_{b,\mu}$ as in (1.6). Theorem 2 is an extension of [3, Theorem 3.1], which was proven in the context of graphs of bounded vertex degree for the operator $\Delta_{b,\mu;\theta}$. In this regard, the first two results of the present paper answer a question posed in [3, Section 5]. Theorem 3 extends [20, Theorem 1], which was proven in the context of graphs of bounded vertex degree for the operator $\Delta_{b,\mu;\theta}+W$ with W as in (1.13). We should also mention that in the context of locally finite graphs (with an assumption on b and μ originating from [17]), a sufficient condition for the essential self-adjointness of a semi-bounded from below operator $\Delta_{b,\mu;\theta}+W$ is given in [19, Theorem 1.2]. In the setting of locally finite graphs, another sufficient condition for the essential self-adjointness of $\Delta_{b,\mu;\theta}+W$ is given in [9, Proposition 2.2]: *Take $\lambda\in\mathbb{R}$ so that*

$$\{x\in V\colon \lambda+\mathrm{Deg}(x)+W(x)=0\}=\emptyset, \tag{1.15}$$

where $\mathrm{Deg}(x)$ *denotes the "weighted degree"*

$$\mathrm{Deg}(x) := \frac{1}{\mu(x)} \sum_{y \in V} b(x,y), \qquad x \in V. \tag{1.16}$$

Suppose that for every sequence of vertices $\{y_1, y_2, \dots\}$ *such that* $y_j \sim y_{j+1}$, $j \geq 1$, *the following property holds:*

$$\sum_{n=1}^{\infty} ((a_n)^2 \mu(y_n)) = \infty, \qquad \textit{where} \quad a_n := \prod_{j=1}^{n-1} \left(1 + \frac{\lambda + W(y_j)}{\mathrm{Deg}(y_j)}\right), \quad n \geq 1, \tag{1.17}$$

and $a_1 := 1$. *Then* $\Delta_{b,\mu;\theta} + W$ *is essentially self-adjoint on* $C_c(V)$.

Note that [9, Proposition 2.2] allows potentials that are unbounded from below. We mention that Example 5.1 below describes a situation where Theorem 2 is applicable, while neither [19, Theorem 1.2] nor [9, Proposition 2.2] is applicable. Additionally, Example 5.2 below describes a situation where Theorem 3 is applicable, while neither [19, Theorem 1.2] nor [9, Proposition 2.2] is applicable.

The recent study [15] is concerned with the operator $\Delta_{b,\mu}$ as in (1.6), with property (iii) of b (see Section 1.1 above) replaced by the following more general condition:

$$\sum_{y \in V} b(x,y) < \infty, \qquad \text{for all } x \in V.$$

Using the notion of intrinsic distance d with finite jump size, the authors of [15] show that if the weighted degree (1.16) is bounded on balls defined with respect to any such distance d, then $\Delta_{b,\mu}$ is essentially self-adjoint. In the context of a locally finite graph, the authors of [15] show that if the graph is metrically complete in any intrinsic path metric with finite jump size, then $\Delta_{b,\mu}$ is essentially self-adjoint. In the metrically incomplete case, one of the results of [15] shows that if the Cauchy boundary has finite capacity, then $\Delta_{b,\mu}$ has a unique Markovian extension if and only if the Cauchy boundary is polar (here, "Cauchy boundary is polar" means that the Cauchy boundary has zero capacity). Another result of [15] shows that if the upper Minkowski codimension of the Cauchy boundary is greater than 2, then the Cauchy boundary is polar. Additionally, we should mention that the authors of [15] prove Hopf–Rinow-type theorem for locally finite weighted graphs with a path pseudo metric.

In recent years, various authors have developed independently the concept of intrinsic metric on a graph. The definition given in the present paper can be traced back to the work [8]. For applications of intrinsic metrics in various contexts, see, for instance, [1, 5, 6, 7, 10, 12, 13, 14, 18].

With regard to the problem of self-adjoint extensions of adjacency, (magnetic) Laplacian and Schrödinger-type operators on infinite graphs, we should mention that there has been a lot of interest this area in the past few years. For pointers to the existing literature on this topic, see, for instance, [2, 3, 9, 11, 15, 17, 20, 24].

2. Proof of Theorem 1

In this section, we modify the proof of [2, Theorem 4.2]. Throughout the section, we assume that the hypotheses of Theorem 1 are satisfied. We begin with the following lemma, whose proof is given in [3, Lemma 3.3].

Lemma 2.1. *Let H be as in (1.8), let $v \in \ell^2_\mu(V)$ be a weak solution of $Hv = 0$, and let $f \in C_c(V)$ be a real-valued function. Then the following equality holds:*

$$(fv,\, H(fv)) = \frac{1}{2}\sum_{x\in V}\sum_{y\sim x} b(x,y)\,\mathrm{Re}\,[e^{-i\theta(x,y)}v(x)\overline{v(y)}](f(x)-f(y))^2. \tag{2.1}$$

The key ingredient in the proof of Theorem 1 is the Agmon-type estimate given in the next lemma, whose proof, inspired by an idea of [21], is based on the technique developed in [4] for magnetic Laplacians on an open set with compact boundary in $\mathbb{R}^n$.

Lemma 2.2. *Let $\lambda \in \mathbb{R}$ and let $v \in \ell^2_\mu(V)$ be a weak solution of $(H-\lambda)v = 0$. Assume that that there exists a constant $c_1 > 0$ such that, for all $u \in C_c(V)$,*

$$(u,\, (H-\lambda)u) \geq \frac{1}{2}\sum_{x\in V}\max\left(\frac{1}{D(x)^2}, 1\right)\mu(x)|u(x)|^2 + c_1\|u\|^2. \tag{2.2}$$

Then $v \equiv 0$.

Proof. Let ρ and R be numbers satisfying $0 < \rho < 1/2$ and $1 < R < +\infty$. For any $\epsilon > 0$, we define the function $f_\epsilon\colon V \to \mathbb{R}$ by $f_\epsilon(x) = F_\epsilon(D(x))$, where $D(x)$ is as in (1.3) and $F_\epsilon\colon \mathbb{R}^+ \to \mathbb{R}$ is the continuous piecewise affine function defined by

$$F_\epsilon(s) = \begin{cases} 0 \text{ for } s \leq \epsilon \\ \rho(s-\epsilon)/(\rho-\epsilon) \text{ for } \epsilon \leq s \leq \rho \\ s \text{ for } \rho \leq s \leq 1 \\ 1 \text{ for } 1 \leq s \leq R \\ R+1-s \text{ for } R \leq s \leq R+1 \\ 0 \text{ for } s \geq R+1 \end{cases}$$

We first note that by the definition of F_ϵ and continuity of $D(x)$, the support of f_ϵ is compact. Now by [15, Lemma A.3(b)] it follows that the support of f_ϵ finite. Using Lemma 2.1 with $H-\lambda$ in place of H, the inequality

$$\mathrm{Re}\,[e^{-i\theta(x,y)}v(x)\overline{v(y)}] \leq \frac{1}{2}(|v(x)|^2 + |v(y)|^2),$$

and Definition 1.3(i) we have

$$(f_\epsilon v,\, (H-\lambda)(f_\epsilon v)) \leq \frac{1}{2}\sum_{x\in V}\sum_{y\sim x} b(x,y)|v(x)|^2(f_\epsilon(x)-f_\epsilon(y))^2$$

$$\leq \frac{\rho^2}{2(\rho-\epsilon)^2}\sum_{x\in V}\sum_{y\sim x}|v(x)|^2 b(x,y)(d(x,y))^2 \leq \frac{\rho^2}{2(\rho-\epsilon)^2}\sum_{x\in V}\mu(x)|v(x)|^2, \tag{2.3}$$

where the second inequality uses the fact that f_ϵ is a β-Lipschitz function with $\beta = \rho/(\rho-\epsilon)$.

On the other hand, using the definition of f_ϵ and the assumption (2.2) we have

$$(f_\epsilon v,\, (H-\lambda)(f_\epsilon v)) \geq \frac{1}{2}\sum_{\rho\leq D(x)\leq R}\mu(x)|v(x)|^2 + c_1\|f_\epsilon v\|^2. \tag{2.4}$$

We now combine (2.4) and (2.3) to get

$$\frac{1}{2}\sum_{\rho\leq D(x)\leq R}\mu(x)|v(x)|^2 + c_1\|f_\epsilon v\|^2 \leq \frac{\rho^2}{2(\rho-\epsilon)^2}\sum_{x\in V}\mu(x)|v(x)|^2.$$

We fix ρ and R, and let $\epsilon \to 0+$. After that, we let $\rho \to 0+$ and $R \to +\infty$. As a result, we get $v \equiv 0$. $\square$

Conclusion of the proof of Theorem 1. Since $\Delta_{b,\mu;\theta}|_{C_c(V)}$ is a non-negative operator, for all $u \in C_c(V)$, we have

$$(u,\, Hu) \geq \sum_{x\in V}\mu(x)W(x)|u(x)|^2,$$

and, hence, by assumption (1.9) we get:

$$\begin{aligned}(u,\, (H-\lambda)u) &\geq \frac{1}{2}\sum_{x\in V}\frac{1}{D(x)^2}\mu(x)|u(x)|^2 - (\lambda + C)\|u\|^2\\ &\geq \frac{1}{2}\sum_{x\in V}\max\left(\frac{1}{D(x)^2}, 1\right)\mu(x)|u(x)|^2 - (\lambda + C + 1/2)\|u\|^2.\end{aligned} \tag{2.5}$$

Choosing, for instance, $\lambda = -C - 3/2$ in (2.5) we get the inequality (2.2) with $c_1 = 1$.

Thus, $(H-\lambda)|_{C_c(V)}$ with $\lambda = -C-3/2$ is a symmetric operator satisfying $(u,\, (H-\lambda)u) \geq \|u\|^2$, for all $u \in C_c(V)$. In this case, it is known (see [22, Theorem X.26]) that the essential self-adjointness of $(H-\lambda)|_{C_c(V)}$ is equivalent to the following statement: if $v \in \ell^2_\mu(V)$ satisfies $(H-\lambda)v = 0$, then $v = 0$. Thus, by Lemma 2.2, the operator $(H-\lambda)|_{C_c(V)}$ is essentially self-adjoint. Hence, $H|_{C_c(V)}$ is essentially self-adjoint. $\square$

3. Proof of Theorem 2

Throughout the section, we assume that the hypotheses of Theorem 2 are satisfied. We begin with the following lemma.

Lemma 3.1. *Let $(V_l, E_l)_{l\in L}$ be a good covering of degree m of (V, b, μ), let H be as in (1.8), and let W_e be as in (1.10). Then, for all $u \in C_c(V)$ we have*

$$(u, Hu) \geq \sum_{x\in V}\mu(x)(W_e(x) + W(x))|u(x)|^2. \tag{3.1}$$

Proof. It is well known that

$$(u, Hu) = \sum_{\{x,y\}\in E} b(x,y)|u(x) - e^{i\theta(x,y)}u(y)|^2 + \sum_{x\in V}\mu(x)W(x)|u(x)|^2,$$

where E is the set of unoriented edges of (V,b,μ). Thus, using the definition of the covering $(V_l,E_l)_{l\in L}$ of degree m and the definition of p_l we have

$$(u,Hu)\geq \frac{1}{m}\sum_{l\in L}\sum_{\{x,y\}\in E_l} b(x,y)|u(x)-e^{i\theta(x,y)}u(y)|^2+\sum_{x\in V}\mu(x)W(x)|u(x)|^2$$
$$\geq \frac{1}{m}\sum_{l\in L}\left(\left(\inf_{\{y,z\}\in E_l} b(y,z)\right)p_l\sum_{x\in V_l}\mu(x)|u(x)|^2\right)+\sum_{x\in V}\mu(x)W(x)|u(x)|^2,$$

which together with (1.10) gives (3.1). □

Conclusion of the proof of Theorem 2. By Lemma 3.1 and assumption (1.11), for all $u\in C_c(V)$ we have

$$(u,\,(H-\lambda)u)\geq \sum_{x\in V}\mu(x)(W_e(x)+W(x)-\lambda)|u(x)|^2$$
$$\geq \frac{1}{2}\sum_{x\in V}\max\left(\frac{1}{D(x)^2},1\right)\mu(x)|u(x)|^2-(C+\lambda+1/2)\|u\|^2.$$

From hereon we proceed in the same way as in the the proof of Theorem 1. □

4. Proof of Theorem 3

In this section we modify the proof of [20, Theorem 1], which is based on the technique of [23] in the context of Riemannian manifolds. Throughout the section, we assume that the hypotheses of Theorem 3 are satisfied.

We begin with the definitions of minimal and maximal operators associated with the expression (1.8). We define the operator $H_{\min}$ by the formula $H_{\min}u:=Hu$, for all $u\in \operatorname{Dom}(H_{\min}):=C_c(V)$. As W is real-valued, it follows easily that the operator $H_{\min}$ is symmetric in $\ell^2_\mu(V)$. We define $H_{\max}:=(H_{\min})^*$, where T^* denotes the adjoint of operator T. Additionally, we define $\mathcal{D}:=\{u\in\ell^2_\mu(V)\colon Hu\in\ell^2_\mu(V)\}$. Then, the following hold: $\operatorname{Dom}(H_{\max})=\mathcal{D}$ and $H_{\max}u=Hu$ for all $u\in\mathcal{D}$; see, for instance, [20, Section 3] or [24, Section 3] for details. Furthermore, by [16, Problem V.3.10], the operator $H_{\min}$ is essentially self-adjoint if and only if

$$(H_{\max}u,v)=(u,H_{\max}v),\qquad \text{for all } u\,,v\in\operatorname{Dom}(H_{\max}). \tag{4.1}$$

In the setting of graphs of bounded vertex degree, the following proposition was proven in [20, Proposition 12].

Proposition 4.1. *If $u\in\operatorname{Dom}(H_{\max})$, then*

$$\sum_{x,y\in V} b(x,y)\min\{q^{-1}(x),q^{-1}(y)\}|u(x)-e^{i\theta_{x,y}}u(y)|^2\leq 4(\|Hu\|\|u\|+(K^2+1)\|u\|^2), \tag{4.2}$$

where H is as in (1.8) and K is as in (1.12).

Before proving Proposition 4.1, we define a sequence of cut-off functions. Let d_σ and d_{σ_q} be as in the hypothesis of Theorem 3. Fix $x_0 \in V$ and define

$$\chi_n(x) := \left(\left(\frac{2n - d_\sigma(x_0, x)}{n}\right) \vee 0\right) \wedge 1, \qquad x \in V, \quad n \in \mathbb{Z}_+. \tag{4.3}$$

Denote

$$B_n^\sigma(x_0) := \{x \in V \colon d_\sigma(x_0, x) \leq n\}. \tag{4.4}$$

The sequence $\{\chi_n\}_{n\in\mathbb{Z}_+}$ satisfies the following properties: (i) $0 \leq \chi_n(x) \leq 1$, for all $x \in V$; (ii) $\chi_n(x) = 1$ for $x \in B_n^\sigma(x_0)$ and $\chi_n(x) = 0$ for $x \notin B_{2n}^\sigma(x_0)$; (iii) for all $x \in V$, we have $\lim\limits_{n\to\infty} \chi_n(x) = 1$; (iv) the functions χ_n have finite support; and (v) the functions χ_n satisfy the inequality

$$|\chi_n(x) - \chi_n(y)| \leq \frac{\sigma(x, y)}{n}, \qquad \text{for all } x \sim y.$$

The properties (i)–(iii) and (v) can be checked easily. By hypothesis, we know that (V, d_{σ_q}) is a complete metric space and, thus, balls with respect to d_{σ_q} are finite; see, for instance, [15, Theorem A.1]. Let $B_{2n}^{\sigma_q}(x_0)$ be as in (4.4) with d_σ replaced by d_{σ_q}. Since $q \geq 1$ it follows that $B_{2n}^\sigma(x_0) \subseteq B_{2n}^{\sigma_q}(x_0)$. Thus, property (iv) is a consequence of property (ii) and the finiteness of $B_{2n}^{\sigma_q}(x_0)$.

Proof of Proposition 4.1. Let $u \in \mathrm{Dom}(H_{\max})$ and let $\phi \in C_c(V)$ be a real-valued function. Define

$$I := \left(\sum_{x,y\in V} b(x, y)|u(x) - e^{i\theta_{x,y}} u(y)|^2((\phi(x))^2 + (\phi(y))^2)\right)^{1/2}. \tag{4.5}$$

We will first show that

$$\begin{aligned} I^2 &\leq 4|(\phi^2 Hu, u)| + 4(\phi^2 qu, u) \\ &\quad + \sqrt{2} I \left(\sum_{x,y\in V} b(x, y)(\phi(x) - \phi(y))^2|(u(x) + e^{i\theta_{x,y}} u(y)|^2\right)^{1/2}. \end{aligned} \tag{4.6}$$

To do this, we first note that

$$\begin{aligned} I^2 &= 4(\phi^2 Hu, u) - 4(\phi^2 Wu, u) \\ &\quad + \sum_{x,y\in V} b(x, y)(e^{i\theta_{x,y}} u(y) - u(x))(e^{-i\theta_{x,y}} \overline{u(y)} + \overline{u(x)})((\phi(x))^2 - (\phi(y))^2), \end{aligned} \tag{4.7}$$

which can be checked by expanding the terms under summations on both sides of the equality and using the properties $b(x, y) = b(y, x)$ and $\theta(x, y) = -\theta(y, x)$. The details of this computation can be found in the proof of [20, Proposition 12].

The inequality (4.6) is obtained from (4.7) by using (1.13), the factorization

$$(\phi(x))^2 - (\phi(y))^2 = (\phi(x) - \phi(y))(\phi(x) + \phi(y)),$$

Cauchy–Schwarz inequality, and

$$(\phi(x)+\phi(y))^2 \le 2(\phi^2(x)+\phi^2(y)).$$

Let χ_n be as in (4.3) and let q be as in (1.13). Define

$$\phi_n(x) := \chi_n(x)q^{-1/2}(x). \tag{4.8}$$

By property (iv) of χ_n it follows that ϕ_n has finite support. By property (i) of χ_n and since $q \ge 1$, we have

$$0 \le \phi_n(x) \le q^{-1/2}(x) \le 1, \qquad \text{for all } x \in V. \tag{4.9}$$

By property (iii) of χ_n we have

$$\lim_{n\to\infty} \phi_n(x) = q^{-1/2}(x), \qquad \text{for all } x \in V. \tag{4.10}$$

By (1.12), properties (i) and (v) of χ_n, and the inequality $q \ge 1$, we have

$$|\phi_n(x)-\phi_n(y)| \le \left(\frac{1}{n}+K\right)\sigma(x,y), \qquad \text{for all } x \sim y, \tag{4.11}$$

where K is as in (1.12). We will also use the inequality

$$|e^{i\theta_{x,y}}u(y)+u(x)|^2 \le 2(|u(x)|^2+|u(y)|^2). \tag{4.12}$$

By (4.11), (4.12), and Definition 1.3(ii), we get

$$\begin{aligned}
&\left(\sum_{x,y\in V} b(x,y)(\phi_n(x)-\phi_n(y))^2|(u(x)+e^{i\theta_{x,y}}u(y)|^2\right)^{1/2}\\
&\le \sqrt{2}\left(\frac{1}{n}+K\right)\left(\sum_{x,y\in V} b(x,y)(\sigma(x,y))^2(|u(x)|^2+|u(y)|^2)\right)^{1/2}\\
&= 2\left(\frac{1}{n}+K\right)\left(\sum_{x,y\in V} b(x,y)(\sigma(x,y))^2|u(x)|^2\right)^{1/2}\\
&\le 2\left(\frac{1}{n}+K\right)\left(\sum_{x\in V} \mu(x)|u(x)|^2\right)^{1/2}
\end{aligned} \tag{4.13}$$

By (4.6) with $\phi=\phi_n$, (4.13), and (4.9), we obtain

$$I_n^2 \le 4\|Hu\|\|u\| + 4\|u\|^2 + 2\sqrt{2}I_n\left(\frac{1}{n}+K\right)\|u\|, \tag{4.14}$$

for all $u \in \mathrm{Dom}(H_{\max})$, where I_n is as in (4.5) with $\phi=\phi_n$.

Using the inequality $ab \le \frac{a^2}{4} + b^2$ with $a = \sqrt{2} I_n$ in the third term on the right-hand side of (4.14) and rearranging, we obtain

$$I_n^2 \le 8 \left(\|Hu\| \|u\| + \left(\left(\frac{1}{n} + K \right)^2 + 1 \right) \|u\|^2 \right). \tag{4.15}$$

Letting $n \to \infty$ in (4.15) and using (4.10) together with Fatou's lemma, we get

$$\begin{aligned} &\sum_{x,y \in V} b(x,y) |u(x) - e^{i\theta_{x,y}} u(y)|^2 (q^{-1}(x) + q^{-1}(y)) \\ &\le 8 \left(\|Hu\| \|u\| + (K^2 + 1) \|u\|^2 \right). \end{aligned} \tag{4.16}$$

Since

$$2 \min\{q^{-1}(x), q^{-1}(y)\} \le q^{-1}(x) + q^{-1}(y), \qquad \text{for all } x,\, y \in V,$$

the inequality (4.2) follows directly from (4.16). □

Continuation of the proof of Theorem 3. Our final goal is to prove (4.1). Let d_{σ_q} be as in the hypothesis of Theorem 3. Fix $x_0 \in V$ and define

$$P(x) := d_{\sigma_q}(x_0, x), \qquad x \in V. \tag{4.17}$$

In what follows, for a function $f \colon V \to \mathbb{R}$ we define $f^+(x) := \max\{f(x), 0\}$. Let $u, v \in \operatorname{Dom}(H_{\max})$, let $s > 0$, and define

$$J_s := \sum_{x \in V} \left(1 - \frac{P(x)}{s} \right)^+ \left((Hu)(x) \overline{v(x)} - u(x) \overline{(Hv)(x)} \right) \mu(x), \tag{4.18}$$

where P is as in (4.17) and H is as in (1.8).

Since (V, d_{σ_q}) is a complete metric space, by [15, Theorem A.1] it follows that the set

$$U_s := \{x \in V \colon P(x) \le s\}$$

is finite. Thus, for all $s > 0$, the summation in (4.18) is performed over finitely many vertices.

The following lemma follows easily from the definition of J_s and the dominated convergence theorem; see the proof of [20, Lemma 13] for details.

Lemma 4.2. *Let J_s be as in (4.18). Then*

$$\lim_{s \to +\infty} J_s = (Hu, v) - (u, Hv). \tag{4.19}$$

In what follows, for $u \in \operatorname{Dom}(H_{\max})$, define

$$T_u := \left(\sum_{x,y \in V} b(x,y) \min\{q^{-1}(x), q^{-1}(y)\} |u(x) - e^{i\theta_{x,y}} u(y)|^2 \right)^{1/2}. \tag{4.20}$$

Note that T_u is finite by Proposition 4.1.

Lemma 4.3. *Let u, $v \in \mathrm{Dom}(H_{\max})$, let T_u and T_v be as in (4.20), and let J_s be as in (4.18). Then*

$$|J_s| \leq \frac{1}{2s}(\|v\|T_u + \|u\|T_v). \tag{4.21}$$

Proof. A computation shows that

$$2J_s = \sum_{x,y\in V} \left((1-P(x)/s)^+ - (1-P(y)/s)^+\right) b(x,y) \Big((e^{-i\theta_{x,y}}\overline{v(y)} - \overline{v(x)})u(x)$$
$$-(e^{i\theta_{x,y}}u(y) - u(x))\overline{v(x)}\Big),$$

which, together with the triangle inequality and property

$$|f^+(x) - g^+(x)| \leq |f(x) - g(x)|,$$

leads to the following estimate:

$$2|J_s| \leq \frac{1}{s}\sum_{x,y\in V} b(x,y)|P(x)-P(y)| \Big(|e^{i\theta_{x,y}}v(y) - v(x)||u(x)|$$
$$+|e^{i\theta_{x,y}}u(y) - u(x)||v(x)|\Big). \tag{4.22}$$

By (4.17) and (1.14), for all $x \sim y$ we have

$$|P(x)-P(y)| \leq d_{\sigma_q}(x,y) \leq \sigma_q(x,y) = \min\{q^{-1/2}(x), q^{-1/2}(y)\} \cdot \sigma(x,y). \tag{4.23}$$

To obtain (4.21), we combine (4.22) and (4.23) and use Cauchy–Schwarz inequality together with Definition 1.3(ii). □

The end of the proof of Theorem 3. Let $u \in \mathrm{Dom}(H_{\max})$ and $v \in \mathrm{Dom}(H_{\max})$. By the definition of $H_{\max}$, it follows that $Hu \in \ell^2_\mu(V)$ and $Hv \in \ell^2_\mu(V)$. Letting $s \to +\infty$ in (4.21) and using the finiteness of T_u and T_v, it follows that $J_s \to 0$ as $s \to +\infty$. This, together with (4.19), shows (4.1). □

5. Examples

In this section we give some examples that illustrate the main results of the paper. In what follows, for $x \in \mathbb{R}$, the notation $\lceil x \rceil$ denotes the smallest integer N such that $N \geq x$. Additionally, $\lfloor x \rfloor$ denotes the greatest integer N such that $N \leq x$.

Example 5.1. In this example we consider the graph $G = (V, E)$ whose vertices $x_{j,k}$ are arranged in a "triangular" pattern so that the first row contains $x_{1,1}$; for $2 \leq j \leq 4$, the j-th row contains $x_{j,1}$ and $x_{j,2}$; for $5 \leq j \leq 9$, the j-th row contains $x_{j,1}$, $x_{j,2}$, and $x_{j,3}$; for $10 \leq j \leq 16$, the j-th row contains $x_{j,1}$, $x_{j,2}$, $x_{j,3}$, and $x_{j,4}$; and so on. There are two types of edges in the graph: (i) for every $j \geq 1$, we have $x_{j,1} \sim x_{j+1,k}$ for all $1 \leq k \leq \lceil (j+1)^{1/2} \rceil$; (ii) for every $j \geq 2$, we have the "horizontal" edges $x_{j,k} \sim x_{j,k+1}$, for all $1 \leq k \leq \lceil j^{1/2} \rceil - 1$. Clearly, G does not have a bounded vertex degree.

Let $T = (V_T, E_T)$ be the subgraph of G whose set of edges E_T consists of type-(i) edges of G described above. Note that T is a spanning tree of G. Additionally, note that for every type-(ii) edge e of G the following are true: (i) $e \notin E_T$ and (ii) there is a unique 3-cycle (a cycle with 3 vertices) that contains e. Thus, by [3, Lemma 2.1], the corresponding 3-cycles, which we enumerate by $\{C_l\}_{l\in\mathbb{Z}_+}$, form a basis for the space of cycles of G. Furthermore, by Definition 1.11, the family $\{C_l = (V_l, E_l)\}_{l\in\mathbb{Z}_+}$ is a good covering of degree $m = 2$ of G. Following [3, Proposition 2.1(a)] and [3, Lemma 2.3], we define the phase function $\theta\colon V_l \times V_l \to [-\pi, \pi]$ satisfying the following properties: (i) if an edge $\{x, y\}$ belongs to $E_l \backslash E_T$, we have $\theta(x, y) = -\theta(y, x)$; (ii) if $\{x, y\} \in E_T$, we have $\theta(x, y) = 0$; and (iii) $p_l = |1 - e^{i\pi/3}|^2 = 1$, where p_l is as in (1.10) with G_l replaced by C_l.

With this choice of p_l and using the good covering $\{C_l\}_{l\in\mathbb{Z}_+}$ of degree $m = 2$, the definition of the effective potential (1.10) simplifies to

$$W_e(x) := \frac{1}{2} \sum_{\{l\in L\,|\,x\in V_l\}} \inf_{\{y,z\}\in E_l} b(y, z). \tag{5.1}$$

Let $\{b_j\}_{j\in\mathbb{Z}_+}$ be an increasing sequence of positive numbers. We define (i) $b(x, y) = b_j$ if $x \sim y$ and x is in the j-th row and y is in the $(j + 1)$-st row; (ii) $b(x, y) = b_j$ if $x \sim y$ and x and y are both in the $(j + 1)$-st row; (iii) $b(x, y) = 0$, otherwise. With this choice of $b(x, y)$, we have $W_e(x_{1,1}) = b_1/2$. Additionally, since b_j is an increasing sequence of positive numbers, using (5.1) it is easy to see that if a vertex x is in the j-th row, then

$$W_e(x) \geq \frac{1}{2} b_{j-1}, \quad \text{for all } j \geq 2. \tag{5.2}$$

Let $0 < \beta < 3/4$, and set $\mu(x) := j^{-2\beta}$ if the vertex x is in the j-th row. Let $\alpha > 0$ satisfy $\alpha + 2\beta > 3/2$, and set $b_j := j^\alpha$, for all $j \in \mathbb{Z}_+$. With this choice of $b(x, y)$ and $\mu(x)$, let $\sigma_1(x, y)$ be as in (1.2) and let d_{σ_1} be the intrinsic path metric associated with σ_1 as in Section 1.2. As there are $\lfloor\sqrt{j}\rfloor + 3$ edges departing from the vertex $x_{j,1}$, we have

$$\sigma_1(x_{j,1};\, x_{j+1,1}) = j^{-\alpha/2}(j + 1)^{-\beta}(\lfloor\sqrt{j + 1}\rfloor + 3)^{-1/2}, \qquad \text{for all } j \in \mathbb{Z}_+. \tag{5.3}$$

Additionally, note that the path $\gamma = (x_{1,1};\, x_{2,1};\, x_{3,1};\, \dots)$ is a geodesic with respect to the path metric d_{σ_1}, that is, $d_{\sigma_1}(x_{1,1};\, x_{j,1}) = l_{\sigma_1}(x_{1,1};\, x_{2,1};\, \dots;\, x_{j,1})$ for all $j \in \mathbb{Z}_+$, where l_{σ_1} is as in (1.1). Since $\alpha + 2\beta > 3/2$, it follows that

$$\sum_{j=1}^{\infty} j^{-\alpha/2}(j + 1)^{-\beta}(\lfloor\sqrt{j + 1}\rfloor + 3)^{-1/2} < \infty;$$

hence, by [15, Theorem A.1] the space (V, d_{σ_1}) is not metrically complete. Let $D(x)$ be as in (1.3) corresponding to d_{σ_1}. If a vertex x is in the n-th row, using (5.3) and

$$\lfloor\sqrt{j + 1}\rfloor + 3 \leq 3\sqrt{j + 1}, \qquad \text{for all } j \in \mathbb{Z}_+,$$

we have

$$D(x) \geq \frac{1}{\sqrt{3}} \sum_{k=n}^{\infty} (j + 1)^{-\beta-\alpha/2-1/4} \geq \frac{(n + 1)^{-\beta-\alpha/2+3/4}}{\sqrt{3}(\beta + \alpha/2 - 3/4)},$$

which leads to

$$\frac{1}{2D(x)^2} \le \frac{3(4\beta+2\alpha-3)^2(n+1)^{2\beta+\alpha-3/2}}{32}, \tag{5.4}$$

for all vertices x in the n-th row, where $n \ge 1$. Define $W(x) = -n^{2\beta+\alpha-3/2}$ for all vertices x in the n-th row, where $n \ge 1$. Using (5.2) and $W_e(x_{1,1}) = b_1/2$, together with (5.4) and the assumption $0 < \beta < 3/4$, it follows that there exists a constant $C > 0$ (depending on α and β) such that (1.11) is satisfied. Thus, by Theorem 2 the operator $\Delta_{b,\mu;\theta} + W$ is essentially self-adjoint on $C_c(V)$. Clearly, Theorem 2 is also applicable in the case $W(x) = 0$ for all $x \in V$, that is, the operator $\Delta_{b,\mu;\theta}$ is essentially self-adjoint on $C_c(V)$. A calculation shows that μ and b in this example do not satisfy [19, Assumption A]; hence, we cannot use [19, Theorem 1.2].

We will now show that under more restrictive assumption $1/2 < \beta < 3/4$, we cannot apply [9, Proposition 2.2] to this example with $W(x) \equiv 0$. To see this, using (1.16) and the fact that among the $\lfloor\sqrt{j}\rfloor + 3$ edges departing from the vertex $x_{j,1}$, there are $\lfloor\sqrt{j}\rfloor + 1$ edges with weight b_j and 2 edges with weight b_{j-1}, we first note that

$$\mathrm{Deg}(x_{1,1}) = 2, \quad \mathrm{Deg}(x_{j,1}) = j^{2\beta}((\lfloor\sqrt{j}\rfloor + 1)j^{\alpha} + 2(j-1)^{\alpha}), \quad \text{for all } j \ge 2.$$

Let $\lambda \in \mathbb{R}$ be such that (1.15) is satisfied, with $W(x) \equiv 0$. Let a_n be as in (1.17) corresponding to the path $\gamma = (x_{1,1}; x_{2,1}; x_{3,1}; \dots)$, the potential $W \equiv 0$, and λ. Then $a_1 = 1$,

$$(a_2)^2 = (1 + \lambda/2)^2 = (\lambda+2)^2/4,$$

and

$$\begin{aligned}
(a_n)^2 &= \frac{(\lambda+2)^2}{4}\prod_{j=2}^{n-1}\left(1 + \frac{\lambda}{j^{\alpha+2\beta}(\lfloor\sqrt{j}\rfloor+1) + 2j^{2\beta}(j-1)^{\alpha}}\right)^2 \\
&= \frac{(\lambda+2)^2}{4}\prod_{j=2}^{n-1}\left(\frac{j^{\alpha+2\beta}(\lfloor\sqrt{j}\rfloor+1) + 2j^{2\beta}(j-1)^{\alpha} + \lambda}{j^{\alpha+2\beta}(\lfloor\sqrt{j}\rfloor+1) + 2j^{2\beta}(j-1)^{\alpha}}\right)^2, \qquad n \ge 3.
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}(a_n)^2\mu(x_{n,1}) = 1 + \frac{(\lambda+2)^2}{4(2)^{2\beta}} + \sum_{n=3}^{\infty}\frac{(a_n)^2}{n^{2\beta}}.$$

Using Raabe's test, it can be checked that the series on the right hand side of this equality converges. (Here, we used the more restrictive assumption $1/2 < \beta < 3/4$.) Hence, looking at (1.17), we see that [9, Proposition 2.2] cannot be used in this example.

Example 5.2. Consider the graph whose vertices are arranged in a "triangular" pattern so that $x_{1,1}$ is in the first row, $x_{2,1}$ and $x_{2,2}$ are in the second row, $x_{3,1}$, $x_{3,2}$, and $x_{3,3}$ are in the third row, and so on. The vertex $x_{1,1}$ is connected to $x_{2,1}$ and $x_{2,2}$. The vertex $x_{2,i}$, where $i = 1, 2$, is connected to every vertex $x_{3,j}$, where $j = 1, 2, 3$. The pattern continues so that each of k vertices in the k-th row is connected to each of $k+1$ vertices in the $(k+1)$-st row. Note that for all $k \ge 1$ and $j \ge 1$ we have $\deg(x_{k,j}) = 2k$, where $\deg(x)$ is as in (1.2). Let $\mu(x) = n^{-1/2}$

for every vertex x in the n-th row, and let $b(x,y) \equiv 1$ for all vertices $x \sim y$. Following (1.2), for every vertex x in the n-th row and every vertex y in the $(n+1)$-st row, define

$$\sigma(x,y) := \min\left\{\frac{n^{-1/2}}{2n}, \frac{(n+1)^{-1/2}}{2(n+1)}\right\}^{1/2} = 2^{-1/2}(n+1)^{-3/4}.$$

For all vertices x in the n-th row, define $W(x) = -n^{1/2}$ and $q(x) = n^{1/2}$. Clearly, the inequality (1.13) is satisfied. With this choice of q, following (1.14), for every vertex x in the n-th row and every vertex y in the $(n+1)$-st row, define

$$\sigma_q(x,y) := \min\{n^{-1/4}, (n+1)^{-1/4}\} \cdot \sigma(x,y) = 2^{-1/2}(n+1)^{-1}.$$

Since

$$\sum_{j=1}^{\infty} 2^{-1/2}(j+1)^{-1} = \infty,$$

by [15, Theorem A.1] it follows that the space (V, d_{σ_q}) is metrically complete. Additionally, it is easily checked that (1.12) is satisfied with $K = \sqrt{2}$. Therefore, by Theorem 3 the operator $\Delta_{b,\mu} + W$ is essentially self-adjoint on $C_c(V)$. Furthermore, it is is easy to see that for every $k \in \mathbb{R}$, there exists a function $u \in C_c(V)$ such that the inequality

$$((\Delta_{b,\mu} + W)u, u) \geq k\|u\|^2$$

is not satisfied. Thus, the operator $\Delta_{b,\mu} + W$ is not semi-bounded from below, and we cannot use [19, Theorem 1.2].

It turns out that [9, Proposition 2.2] is not applicable in this example. To see this, using (1.16) we first note that $\mathrm{Deg}(x_{k,j}) = 2k^{3/2}$, for all $k \geq 1$ and all $j \geq 1$. Let $\lambda \in \mathbb{R}$ be such that (1.15) is satisfied, with W as in this example. Let a_n be as in (1.17) corresponding to the path $\gamma = (x_{1,1};\, x_{2,1};\, x_{3,1}; \dots)$, the potential $W(x_{k,1}) = -k^{1/2}$, and λ. Then $a_1 = 1$,

$$(a_2)^2 = (1 + (\lambda - 1)/2)^2 = (\lambda + 1)^2/4,$$

and for $n \geq 3$ we have

$$(a_n)^2 = \frac{(\lambda+1)^2}{4}\prod_{k=2}^{n-1}\left(1 + \frac{\lambda - k^{1/2}}{2k^{3/2}}\right)^2 = \frac{(\lambda+1)^2}{4}\prod_{k=2}^{n-1}\left(\frac{2k^{3/2} - k^{1/2} + \lambda}{2k^{3/2}}\right)^2.$$

Therefore,

$$\sum_{n=1}^{\infty}(a_n)^2\mu(x_{n,1}) = 1 + \frac{(\lambda+1)^2}{4\sqrt{2}} + \sum_{n=3}^{\infty}\frac{(a_n)^2}{\sqrt{n}}.$$

Using Raabe's test, it can be checked that the series on the right hand side of this equality converges. Hence, looking at (1.17), we see that [9, Proposition 2.2] cannot be used in this example.

Acknowledgment. The second author is grateful to Daniel Lenz for fruitful discussions.

References

[1] Bauer, F., Keller, M., Wojciechowski, R. K.: Cheeger inequalities for unbounded graph Laplacians. arXiv:1209.4911

[2] Colin de Verdière, Y., Torki-Hamza, N., Truc, F.: Essential self-adjointness for combinatorial Schrödinger operators II. Math. Phys. Anal. and Geom. **14** (2011) 21–38

[3] Colin de Verdière, Y., Torki-Hamza, N., Truc, F.: Essential self-adjointness for combinatorial Schrödinger operators III- Magnetic fields. Ann. Fac. Sci. Toulouse Math. (6) **20** (2011) 599–611

[4] Colin de Verdière, Y., Truc, F: Confining quantum particles with a purely magnetic field. Ann. Inst. Fourier (Grenoble) **60** (7) (2010) 2333–2356

[5] Folz, M.: Gaussian upper bounds for heat kernels of continuous time simple random walks. Electron. J. Probab. **16** (2011) 1693-1722

[6] Folz, M.: Volume growth and stochastic completeness of graphs. arXiv:1201.5908. To appear in: Trans. Amer. Math. Soc.

[7] Folz, M.: Volume growth and spectrum for general graph Laplacians. arXiv:1204.4770

[8] Frank, R. L., Lenz, D., Wingert, D.: Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory. arXiv:1012.5050

[9] Golénia, S.: Hardy inequality and Weyl asymptotic for discrete Laplacians. arXiv:1106.0658

[10] Grigor'yan, A., Huang, X., Masamune, J.: On stochastic completeness of jump processes. Math. Z. **271** (2012) 1211-1239

[11] Haeseler, S., Keller, M., Lenz, D., Wojciechowski, R. K.: Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions, J. Spectr. Theory **2** (2012) 397-432

[12] Haeseler, S., Keller, M., Wojciechowski, R. K.: Volume growth and bounds for the essential spectrum for Dirichlet forms. arXiv:1205.4985

[13] Huang, X.: On uniqueness class for a heat equation on graphs. J. Math. Anal. Appl. **393** (2012) 377-388

[14] Huang, X.: A note on the volume growth criterion for stochastic completeness of weighted graphs. arXiv:1209.2069

[15] Huang, X., Keller, M., Masamune, J., Wojciechowski, R. K.: A note on self-adjoint extensions of the Laplacian on weighted graphs. arXiv:1208.6358

[16] Kato, T.: Perturbation Theory for Linear Operators. Springer-Verlag, Berlin (1980)

[17] Masamune, J.: A Liouville property and its application to the Laplacian of an infinite graph. In: Contemp. Math. vol. 484, Amer. Math. Soc., Providence (2009) pp. 103–115

[18] Masamune, J., Uemura, T.: Conservation property of symmetric jump processes. Ann. Inst. Henri Poincaré Probab. Stat. **47** (2011) 650-662

[19] Milatovic, O.: Essential self-adjointness of magnetic Schrödinger operators on locally finite graphs. Integr. Equ. Oper. Theory **71** (2011) 13–27

[20] Milatovic, O.: A Sears-type self-adjointness result for discrete magnetic Schrödinger operators. J. Math. Anal. Appl. **396** (2012) 801-809

[21] Nenciu, G., Nenciu, I.: On confining potentials and essential self-adjointness for Schrödinger operators on bounded domains in $\mathbb{R}^n$. Ann. Henri Poincaré **10** (2009) 377–394

[22] Reed, M., Simon, B.: Methods of Modern Mathematical Physics II: Fourier analysis, self-adjointness. Academic Press, New York (1975)

[23] Shubin, M. A.: Essential self-adjointness for magnetic Schrödinger operators on non-compact manifolds. In: Séminaire Équations aux Dérivées Partielles (Polytechnique) (1998-1999), Exp. No. XV, Palaiseau (1999) pp. XV-1–XV-22

[24] Torki-Hamza, N.: Laplaciens de graphes infinis I Graphes métriquement complets. Confluentes Math. **2** (2010) 333–350

Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224, USA.

E-mail address: omilatov@unf.edu

Grenoble University, Institut Fourier, Unité mixte de recherche CNRS-UJF 5582, BP 74, 38402-Saint Martin d'Hères Cedex, France.

E-mail address: francoise.truc@ujf-grenoble.fr