

Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces

Haïm Brezis, Petru Mironescu

▶ To cite this version:

Haïm Brezis, Petru Mironescu. Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces. Journal of Evolution Equations, 2001, 1 (4), pp.387-404. 10.1007/PL00001378 . hal-00747694

HAL Id: hal-00747694 https://hal.science/hal-00747694

Submitted on 1 Nov 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

GAGLIARDO-NIRENBERG, COMPOSITION AND PRODUCTS IN FRACTIONAL SOBOLEV SPACES

Haïm $BREZIS^{(1),(2)}$ and Petru MIRONESCU⁽³⁾

Dedicated with emotion to the memory of Tosio Kato

I. Introduction

Our main result is the following: let $1 \le s < \infty$, 1 , and let

$$m = \begin{cases} s, \text{ if } s \text{ is an integer} \\ [s] + 1, \text{ otherwise.} \end{cases}$$

 Set

$$\mathcal{R} = \{ f \in C^m(\mathbb{R}) ; f(0) = 0, f, f', ..., f^{(m)} \in L^\infty(\mathbb{R}) \}.$$

Theorem 1. For every $f \in \mathcal{R}$ the map $\psi \mapsto f(\psi)$ is well-defined and continuous from $W^{s,p}(\mathbb{R}^n) \cap W^{1,sp}(\mathbb{R}^n)$ into $W^{s,p}(\mathbb{R}^n)$.

An immediate consequence of Theorem 1 is

Theorem 1'. Let Ω be a smooth bounded domain in \mathbb{R}^n and $f \in C^m$ be such that $f, f', ..., f^{(m)} \in L^{\infty}$. Then the map

$$W^{s,p}(\Omega) \cap W^{1,sp}(\Omega) \ni u \mapsto f(u) \in W^{s,p}(\Omega)$$

is well-defined and continuous.

Our original motivation in proving Theorem 1 comes from the study of properties of the space

$$X = W^{s,p}(\Omega; S^1) = \{ u \in W^{s,p}(\Omega; \mathbb{R}^2) ; |u| = 1 \text{ a.e.} \}.$$

Here, $0 < s < \infty$, $1 and <math>\Omega$ is a smooth bounded simply connected domain in \mathbb{R}^n . In particular, one may ask whether X is path-connected and whether $C^{\infty}(\overline{\Omega}; S^1)$ is dense in X. Several results concerning the first question were obtained in [10] (and subsequently in [18]) for the spaces $W^{1,p}(M; N)$, where M, N are compact oriented Riemannian manifolds. The second question was studied in [3], [4] and [18] for the spaces $W^{1,p}(M; N)$ and in [16] for the spaces $W^{s,p}(M; S^k)$. The case where $N = S^1$ is somehow special; one may attempt to answer these questions by lifting the maps $u \in X$. Here is a strategy: given $u \in W^{s,p}(\Omega; S^1)$, one may try to find some $\varphi \in W^{s,p}(\Omega; \mathbb{R})$ such that $u = e^{i\varphi}$. Then, hopefully, the path

$$t \in [0,1] \mapsto e^{it\varphi}$$

will connect continuously $u_0 \equiv 1$ to u.

Moreover, if φ_j are smooth \mathbb{R} -valued functions on $\overline{\Omega}$ such that $\varphi_j \to \varphi$ in $W^{s,p}$, then, hopefully, the smooth maps $e^{i\varphi_j}$ converge to u in $W^{s,p}(\Omega; S^1)$.

We are thus naturally led to the study of the mapping

$$W^{s,p}(\Omega) \ni \psi \mapsto f(\psi)$$

for "reasonable" functions f (e.g., $f(x) = e^{ix} - 1$), where Ω is either a smooth bounded domain or $\Omega = \mathbb{R}^n$ and $s \ge 1$.

In a forthcoming paper [12], we will apply Theorem 1 to settle the above mentioned questions about $W^{s,p}(\Omega; S^1)$ when $s \ge 1$.

Another motivation for analysing composition and products in fractional Sobolev spaces comes from the study of nonlinear evolution equations (e.g. Schrödinger equation) in H^s spaces; see e.g. T. Kato [20] and the references therein. In fact, the Appendix in [20] contains a result which is a special case of the Runst-Sickel lemma about products: it coincides with Lemma 5 below when q = 2.

Remark 1. The reader may wonder why we impose the additional condition $u \in W^{1,sp}$. At least for the case we are interested in, i.e. $f(x) = e^{ix} - 1$, this condition is also necessary in order to conclude that $f(\psi) \in W^{s,p}(\mathbb{R}^n)$.

Indeed, assume that $\psi \in W^{s,p}$ and $(e^{i\psi} - 1) \in W^{s,p}$. Then $(e^{i\psi} - 1) \in W^{s,p} \cap L^{\infty} \implies (e^{i\psi} - 1) \in W^{1,sp}$ (by Gagliardo-Nirenberg, see Corollary 2 below). Therefore, $ie^{i\psi}D\psi \in L^{sp}$, so that $D\psi \in L^{sp}$. Thus $\psi \in W^{1,sp}$.

Remark 2. There is a vast literature about composition, starting with the result of Moser [26] asserting that $f(\psi) \in W^{m,p}$ when $\psi \in W^{m,p} \cap L^{\infty}$, $f \in \mathcal{R}$ and m is an integer. (See the historical comments at the end of section V). Unfortunately, for the application we have in mind, the lifting φ of an arbitrary $u \in W^{s,p}(\Omega; S^1)$ need not belong to L^{∞} . However, if $s \geq 1$ and if the lifting φ exists in $W^{s,p}(\Omega; \mathbb{R})$, it must belong to $W^{1,sp}$, by the above remark.

Surprisingly, Theorem 1 is new, but it is closely related and implies two earlier results having a similar flavour; see Adams-Frazier [1] and Runst-Sickel [32], Theorem 1, p. 345, and Remark 1, p. 348.

Remark 3. When s is an integer, the proof of Theorem 1 is very easy via the standard Gagliardo-Nirenberg inequality [27] (e.g. $W^{k,p} \cap L^{\infty} \subset W^{\ell,q}$, with $\ell < k$, $\ell q = kp$). When s > 1, s is not an integer, our proof is quite involved. The standard form of the Gagliardo-Nirenberg inequality (e.g. $W^{s,p} \cap L^{\infty} \subset W^{\sigma,q}$, with $\sigma < s$, $\sigma q = sp$) does not suffice. We rely on a "microscopic" improvement (due to T. Runst [31]) of the Gagliardo-Nirenberg inequality, in the Triebel-Lizorkin scale, namely $W^{s,p} \cap L^{\infty} \subset \tilde{F}_{q,\nu}^{\sigma}$ for every ν . We present in Section III a more general form of the Gagliardo-Nirenberg inequality due to Oru [28];

see also P. Gérard, Y. Meyer and F. Oru [17] for a special case. We combine this with an important estimate on products of functions in the Triebel-Lizorkin spaces, due to T. Runst and W. Sickel (see [32] and Section IV).

It would be interesting to find a more elementary argument which avoids this excursion into the $\tilde{F}_{n,q}^s$ scale.

The paper is organized as follows. In Section II we recall the definition of the Triebel-Lizorkin spaces $\tilde{F}_{p,q}^s$, their connection with the classical function spaces and some results needed in the proof of Theorem 1. In Section III we recall the general form of the Gagliardo-Nirenberg inequality, due to Oru [28]. Section IV deals with the Runst-Sickel lemma. This beautiful result contains all the usual statements about products in fractional Sobolev spaces: e.g., it implies that if $u, v \in W^{s,p} \cap L^{\infty}$ then $uv \in W^{s,p} \cap L^{\infty}$, and if $s \ge 1$, then $uDv \in W^{s-1,p}$. More consequences of the Runst-Sickel lemma are presented in Section VI. Theorem 1 is proved in Section V.

Plan

Section I. Introduction Section II. Triebel-Lizorkin spaces and maximal inequalities Section III. A microscopic improvement of the Gagliardo-Nirenberg inequality Section IV. The Runst-Sickel lemma Section V. Proof of Theorem 1 Section VI. More about products

II. Triebel-Lizorkin spaces and maximal inequalities

We start by recalling the Littlewood-Paley decomposition of temperate distributions. Let $\psi_0 \in C_0^{\infty}(\mathbb{R}^n)$ be such that $0 \leq \psi_0 \leq 1$, $\psi_0(\xi) = 1$ for $|\xi| \leq 1$, $\psi_0(\xi) = 0$ for $|\xi| \geq 2$. Set $\psi_j(\xi) = \psi_0(2^{-j}\xi) - \psi_0(2^{-j+1}\xi)$, $j \geq 1$, and $\varphi_j = \mathcal{F}^{-1}(\psi_j)$, $j \geq 0$. Thus

(1)
$$\varphi_j(x) = 2^{nj}\varphi_0(2^j x) - 2^{n(j-1)}\varphi_0(2^{j-1}x), \ j \ge 1,$$

and

(2)
$$\sum_{k \le j} \varphi_k(x) = 2^{nj} \varphi_0(2^j x), \ j \ge 0.$$

For $f \in \mathcal{S}'$, set $f_j = f \star \varphi_j$. We have $f = \sum_{j \ge 0} f_j$ in \mathcal{S}' .

Definition ([34], 2.3.1). For $-\infty < s < \infty$, $0 , <math>0 < q \le \infty$, set

$$\tilde{F}_{p,q}^{s} = \{ f \in \mathcal{S}' ; \| f \|_{\tilde{F}_{p,q}^{s}} = \left\| \| 2^{sj} f_{j}(x) \|_{\ell^{q}} \right\|_{L^{p}(\mathbb{R}^{n})} < \infty \}$$

For $0 or <math>p = q = \infty$, these are the standard Triebel-Lizorkin spaces $F_{p,q}^s$ [34]. We have added the $\tilde{}$ to avoid confusions in the exceptional cases where they do not coincide.

When $0 , different choices of <math>\psi_0$ yield equivalent quasi-norms ([34], 2.3.5). The usual function spaces are special cases of these Triebel-Lizorkin spaces ([34]):

a)
$$L^{p} = \tilde{F}_{p,2}^{0}, \quad 1
b) $W^{m,p} = \tilde{F}_{p,2}^{m}, \quad m = 1, 2, ..., 1
c) $W^{s,p} = \tilde{F}_{p,p}^{s}, \quad 0 < s < \infty, s \text{ non-integer}, \ 1 \le p < \infty;$
d) $L^{s,p} = \tilde{F}_{p,2}^{s}, \ s \in \mathbb{R}, \ 1
e) $L^{\infty} \subset \tilde{F}_{\infty,\infty}^{0}, \text{ i.e.},$

$$\sup_{j,x} |f_{j}(x)| \le C ||f||_{L^{\infty}}.$$$$$$

In this list, when $1 \le p < \infty$, $0 < s < \infty$, s non-integer, the $W^{s,p}$ are the Sobolev-Slobodeckij spaces. An equivalent norm on these spaces may be obtained as follows: let $s = k + \sigma$, k integer, $0 < \sigma < 1$. Then

(4)
$$\|f\|_{W^{s,p}}^p \sim \|f\|_{L^p}^p + \|D^k f\|_{L^p}^p + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|D^k f(x) - D^k f(y)|^p}{|x - y|^{n + \sigma p}} dx \, dy$$

([34], 2.6.1). These spaces also coincide with the Besov spaces $B_{p,p}^s$ (recall that s is not an integer). We warn the reader that, for $p \neq 2$, the spaces $W^{s,p}$ do not coincide with the Bessel potential spaces $L^{s,p}$.

We will often use the trivial fact that, for fixed s and p, the space $\tilde{F}_{p,q}^s$ increases with q.

The following result is well-known:

(3)

Lemma 1 ([35]). Let $0 < s < \infty$, $1 , <math>1 < q < \infty$. For every $j \ge 0$, let $f^j \in S'$ be such that supp $\mathcal{F}(f^j) \subset B_{2^{j+2}}$. Then

(5)
$$\left\|\sum_{j} f^{j}\right\|_{\tilde{F}^{s}_{p,q}} \le C \left\|\|2^{sj} f^{j}(x)\|_{\ell^{q}}\right\|_{L^{p}(\mathbb{R}^{n})}$$

In the H^s -spaces (p = q = 2), this result is proved in [14], p. 21. We postpone the proof of Lemma 1 after the discussion of some maximal inequalities. Recall that, for any $f \in L^1_{loc}$, the maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy.$$

For t > 0, set, for $\varphi : \mathbb{R}^n \to \mathbb{R}$,

(6)
$$\varphi^t(x) = t^{-n}\varphi(x/t), \quad x \in \mathbb{R}^n.$$

We recall some classical inequalities

Lemma 2. We have:

a) ([33], p. 13) for 1 and any function <math>f,

(7)
$$||Mf||_{L^p} \sim ||f||_{L^p};$$

b) ([33], p. 55) for $1 , <math>1 < q < \infty$, and any sequence of function (f^j) ,

(8)
$$|||Mf^{j}(x)||_{\ell^{q}}||_{L^{p}(\mathbb{R}^{n})} \leq C ||||f^{j}(x)||_{\ell^{q}}||_{L^{p}(\mathbb{R}^{n})};$$

c) ([33], p. 57) for any fixed $\varphi \in S$ and any function f,

(9)
$$|f \star \varphi^t(x)| \le C M f(x), \quad \forall t > 0, \quad \forall x \in \mathbb{R}^n.$$

By (1), (2) and (9) we obtain the following

Corollary 1. For every $f \in L^1_{loc}$ we have

(10)
$$|f_j(x)| \le C M f(x), \quad j \ge 0, \quad x \in \mathbb{R}^n,$$

(11)
$$|\sum_{j \le k} f_j(x)| \le C M f(x), \quad k \ge 0, \quad x \in \mathbb{R}^n.$$

We now return to the

Proof of Lemma 1. With $f = \sum_{j} f^{j}$, we have

$$f_k = \left(\sum_j f^j\right)_k = \left(\sum_{j \ge k-3} f^j\right)_k = \sum_{j \ge k-3} (f^j)_k.$$

Therefore

$$\begin{split} \|f\|_{\tilde{F}^{s}_{p,q}} &= \left\| \left\| 2^{sk} \sum_{j \ge k-3} (f^{j})_{k}(x) \right\|_{\ell^{q}} \right\|_{L^{p}(\mathbb{R}^{n})} \\ &= \left\| \left(\sum_{k} 2^{sqk} \left| \sum_{j \ge k-3} (f^{j})_{k}(x) \right|^{q} \right)^{1/q} \right\|_{L^{p}(\mathbb{R}^{n})} \\ &\leq C \left\| \left(\sum_{k} 2^{sqk} \sum_{j \ge k-3} \left| (f^{j})_{k}(x) \right|^{q} (j-k+4)^{2q} \right)^{1/q} \right\|_{L^{p}(\mathbb{R}^{n})}, \end{split}$$

by the Hölder inequality with exponents q and $q' = \frac{q}{q-1}$ applied to the inner sum. We obtain, using (10), that

(12)
$$\|f\|_{\tilde{F}^{s}_{p,q}} \leq C \left\| \left(\sum_{j} \sum_{k \leq j+3} 2^{sqk} (j-k+4)^{2q} |Mf^{j}(x)|^{q} \right)^{1/q} \right\|_{L^{p}(\mathbb{R}^{n})}$$
$$\leq C \left\| \left(\sum_{j} 2^{sqj} |Mf^{j}(x)|^{q} \right)^{1/q} \right\|_{L^{p}(\mathbb{R}^{n})}$$
$$= C \left\| \|2^{sj} Mf^{j}(x)\|_{\ell^{q}} \right\|_{L^{p}(\mathbb{R}^{n})}.$$

The desired conclusion is a consequence of (8) and (12).

III. A "microscopic" improvement of the Gagliardo-Nirenberg inequality

The main result of this section is that, in the Gagliardo-Nirenberg type inequalities for the spaces $\tilde{F}_{p,q}^s$, there is a gain in the "microscopic" parameter q; this gain is also called sometimes "precised" or "improved" Sobolev inequalities. Let us explain what we mean. In the context of Besov spaces, a typical Gagliardo-Nirenberg inequality asserts that

$$B^s_{p,r} \cap L^\infty \subset B^{s/2}_{2p,2r}, \text{ for } 0 < s < \infty, \ 0 < p < \infty, \ 0 < r \le \infty$$

(see, e.g. [31], Lemma 2, p. 331).

Here, the value 2r of the microscopic parameter is optimal in general. By contrast, in the scale of \tilde{F} -spaces we have, given $0 < s < \infty$, $0 , <math>0 < r \le \infty$,

$$ilde{F}^s_{p,r} \cap L^\infty \subset ilde{F}^{s/2}_{2p,q} \quad \text{for every } 0 < q \leq \infty$$

([31], Lemma 1, p. 329).

A more general version of this phenomenon, due to Oru [28], is the following. Let $-\infty < s_1 < s_2 < \infty$, $0 < q_1, q_2 \le \infty$, $0 < p_1, p_2 \le \infty$, $0 < \theta < 1$, and define

$$s = \theta s_1 + (1 - \theta) s_2$$
$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2}.$$

Lemma 3. Under the above hypotheses we have, for every $0 < q \leq \infty$,

(13)
$$\|f\|_{\tilde{F}^{s}_{p,q}} \le C \|f\|^{\theta}_{\tilde{F}^{s_{1}}_{p_{1},q_{1}}} \|f\|^{1-\theta}_{\tilde{F}^{s_{2}}_{p_{2},q_{2}}},$$

where C depends on s_i , p_i , θ and q.

For the convenience of the reader, we reproduce the proof of Oru, since it is not yet published.

Before proving Lemma 3, we state some interesting consequences:

Corollary 2. We have

a) for $0 \le s_1 < s_2 < \infty$, $1 < p_1 < \infty$, $1 < p_2 < \infty$,

$$s = \theta s_1 + (1 - \theta) s_2, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2},$$

(14)
$$||f||_{W^{s,p}} \le C ||f||_{W^{s_1,p_1}}^{\theta} ||f||_{W^{s_2,p_2}}^{1-\theta};$$

b) ([31], Lemma 1, p. 329) for $0 < s < \infty$, $1 , <math>0 < q \le \infty$,

(15)
$$\|f\|_{\tilde{F}^{\theta_s}_{p/\theta,q}} \le C \|f\|^{\theta}_{W^{s,p}} \|f\|^{1-\theta}_{L^{\infty}}$$

In particular, we have

c) for $0 < s < \infty$, $1 , <math>0 < \theta < 1$,

(16)
$$||f||_{W^{\theta_{s,p}/\theta}} \le C ||f||_{W^{s,p}}^{\theta} ||f||_{L^{\infty}}^{1-\theta}.$$

Remark 4. Inequality (14) is a special case of (13), with q = 2 when s is an integer, q = p otherwise, and similarly for q_1 and q_2 . Inequality (15) is a consequence of (13) for $s_1 = 0$ θ replaced by $1 - \theta$, $p_1 = q_1 = \infty$, $s_2 = s$, $q_2 = 2$ if s is an integer, $q_2 = p$ otherwise. Here one uses in addition the fact that $||f||_{\tilde{F}^0_{\infty,\infty}} \leq C||f||_{L^{\infty}}$ (inequality (3) above). Finally, (16) is a special case of (15).

Remark 5. There is something intriguing about inequality (16). It is trivial when s < 1 (with C = 1) if one takes the usual Gagliardo norm (4). It is also straightforward when both s and θs are integers. We do not know any elementary (i.e., without the Littlewood-Paley machinery) proof when s = 1. It would be of interest to establish (16) with control of the constant C, in particular when $s \nearrow 1$. In view of the results in [8], one may expect an inequality of the form

$$||f||_{W^{s/2,2p}} \le C(p)(1-s)^{1/2p} ||f||_{W^{s,p}}^{1/2} ||f||_{L^{\infty}}^{1/2}$$
as $s \nearrow 1$,

if we take the Gagliardo norms (4).

Remark 6. Inequality (15) may be viewed as an improvement of (16), since for $0 < q < \min\{2, p/\theta\}$ we have $\tilde{F}_{p/\theta,q}^{\theta_s} \subset W^{\theta_s,p/\theta}$, $\tilde{F}_{p/\theta,q}^{\theta_s} \neq W^{\theta_s,p/\theta}$. This improvement seems microscopic, however in our situation it is magnified and it plays a central role. A similar (microscopic) improvement of the Sobolev embeddings in the framework of Lorentz spaces which is magnified by the Trudinger inequality is presented in [13], [9].

Remark 7. We call the attention of the reader to the fact that some inequalities à la Gagliardo-Nirenberg are wrong, e.g., $W^{1,1} \cap L^{\infty}$ is not contained in $W^{\theta,1/\theta}$ for $0 < \theta < 1$; see [7], Remark D.1.

We now turn to the proof of Lemma 3. It relies on the following inequality:

Lemma 4. Let $-\infty < s_1 < s_2 < \infty$, $0 < q < \infty$, $0 < \theta < 1$, and set $s = \theta s_1 + (1 - \theta)s_2$. Then for every sequence (a_j) we have

(17)
$$\|2^{sj}a_j\|_{\ell^q} \le C \|2^{s_1j}a_j\|_{\ell^\infty}^{\theta} \|2^{s_2j}a_j\|_{\ell^\infty}^{1-\theta}.$$

Remark 8. A special case of (17) is implicit in the proof of Theorem 1, p. 328, in [31]. For similar inequalities, see also [34], Theorem 2.7.1 or [19].

Proof of Lemma 4. Let $C_1 = \sup 2^{s_1j} |a_j|$, $C_2 = \sup 2^{s_2j} |a_j|$, so that $C_1 \leq C_2$. We may assume $C_1 > 0$. Since $s_1 < s_2$, there is some $j_0 > 0$ such that

$$\min\left\{\frac{C_1}{2^{s_1j}}, \frac{C_2}{s^{s_2j}}\right\} = \begin{cases} \frac{C_1}{2^{s_1j}}, \ j \le j_0\\ \frac{C_2}{2^{s_2j}}, \ j > j_0. \end{cases}$$

Since $\frac{C_1}{2^{s_1 j_0}} \le \frac{C_2}{2^{s_2 j_0}}$ and $\frac{C_2}{2^{s_1 (j_0+1)}} \le \frac{C_1}{2^{s_1 (j_0+1)}}$ we find that

(18)
$$C_2 \sim C_1 2^{(s_2 - s_1)j_0}$$

Therefore

(19)
$$\|2^{s_1 j} a_j\|_{\ell^{\infty}}^{\theta} \|2^{s_2 j} a_j\|_{\ell^{\infty}}^{1-\theta} \sim C_1 2^{(s_2-s_1)j_0(1-\theta)}.$$

On the other hand, we have $a_j \leq \min\left\{\frac{C_1}{2^{s_1j}}, \frac{C_2}{s^{s_2j}}\right\}$, so that

(20)
$$a_j \leq \frac{C_1}{2^{s_1 j}} \text{ for } 0 \leq j \leq j_0, \quad a_j \leq \frac{C_2}{2^{s_2 j}} \text{ for } j > j_0.$$

It then follows that

$$\|2^{sj}a_j\|_{\ell^q} \le \left(\sum_{j\le j_0} C_1^q 2^{(s-s_1)jq} + \sum_{j>j_0} C_2^q 2^{(s-s_2)jq}\right)^{1/q}$$
$$\le C \left(\sum_{j\le j_0} C_1^q 2^{(s-s_1)jq} + \sum_{j>j_0} C_1^q 2^{-\theta(s_2-s_1)jq+(s_2-s_1)j_0q}\right)^{1/q}$$

so that

$$\|2^{sj}a_j\|_{\ell^q} \le C C_1 2^{(s_2-s_1)j_0(1-\theta)} \left(\sum_{j\le j_0} 2^{-(1-\theta)(s_2-s_1)(j_0-j)q} + \sum_{j>j_0} 2^{-\theta(s_2-s_1)(j-j_0)q}\right)^{1/q}$$

Finally, we find that

(21)
$$\|2^{sj}a_j\|_{\ell^q} \le C C_1 2^{(s_2-s_1)j_0(1-\theta)},$$

and (17) follows from (19) and (21).

Proof of Lemma 3. Since $||a_j||_{\ell^{\infty}} \leq ||a_j||_{\ell^q}$, $0 < q \leq \infty$, we find that the r.h.s. of (13) is

$$\geq C \|f\|_{\tilde{F}^{s_1}_{p_1,\infty}}^{\theta} \|f\|_{\tilde{F}^{s_2}_{p_2,\infty}}^{1-\theta}.$$

On the other hand, $||f||_{\tilde{F}^s_{p,\infty}} \leq ||f||_{\tilde{F}^s_{p,q}}, 0 < q < \infty$. It therefore suffices to prove (13) in the special case $0 < q < \infty, q_1 = q_2 = \infty$.

In this case, we have

(22)
$$\|f\|_{\tilde{F}^{s}_{p,q}} = \left\| \|2^{s_{j}}f_{j}(x)\|_{\ell^{q}} \right\|_{L^{p}(\mathbb{R}^{n})} \leq \text{ (by (17))}$$
$$\leq C \left\| \|2^{s_{1}j}f_{j}(x)\|_{\ell^{\infty}}^{\theta} \|2^{s_{2}j}f_{j}(x)\|_{\ell^{\infty}}^{1-\theta} \right\|_{L^{p}(\mathbb{R}^{n})}.$$

Using the Hölder inequality, (22) yields

$$\begin{aligned} \|f\|_{\tilde{F}^{s}_{p,q}} &\leq C \left\| \|2^{s_{1}j}f_{j}(x)\|_{\ell^{\infty}} \right\|_{L^{p_{1}}(\mathbb{R}^{n})}^{\theta} \left\| \|2^{s_{2}j}f_{j}(x)\|_{\ell^{\infty}} \right\|_{L^{p_{2}}(\mathbb{R}^{n})}^{1-\theta} \\ &= C \|f\|_{\tilde{F}^{s_{1}}_{p_{1},\infty}}^{\theta} \|f\|_{\tilde{F}^{s_{2}}_{p_{2},\infty}}^{1-\theta}. \end{aligned}$$

The proof of Lemma 3 is complete.

Remark 9. While talking about microscoping improvements in the \tilde{F} -scale, we call the attention of the reader to the following "improved" Sobolev embedding:

$$W^{s,p} \hookrightarrow \tilde{F}^{\sigma}_{r,q}$$
 for every $0 < q \le \infty$

if $0 \le \sigma < s$ and $\frac{1}{r} = \frac{1}{p} - \frac{s - \sigma}{n} > 0$ (see ([19] or [32], p. 31).

IV. The Runst-Sickel lemma

For the convenience of the reader, we split the statement into two parts; the first one contains the fundamental estimate, the other one deals with the continuity of the product.

Let $0 < s < \infty$, $1 < q < \infty$, $1 < p_1 \le \infty$, $1 < p_2 \le \infty$, $1 < r_1 \le \infty$, $1 < r_2 \le \infty$ be such that

(23)
$$0 < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{r_1} < 1.$$

Lemma 5 ([32], p. 345). We have, for $f \in \tilde{F}^s_{p_1,q} \cap L^{r_1}$ and $g \in \tilde{F}^s_{p_2,q} \cap L^{r_2}$,

(24)
$$||fg||_{\tilde{F}^{s}_{p,q}} \leq C\left(||Mf(x)||2^{sj}g_{j}(x)||_{\ell^{q}} ||_{L^{p}(\mathbb{R}^{n})} + ||Mg(x)||2^{sj}f_{j}(x)||_{\ell^{q}} ||_{L^{p}(\mathbb{R}^{n})} \right)$$

and

(25)
$$\|fg\|_{\tilde{F}^{s}_{p,q}} \leq C \left(\|f\|_{\tilde{F}^{s}_{p_{1},q}} \|g\|_{L^{r_{2}}} + \|g\|_{\tilde{F}^{s}_{p_{2},q}} \|f\|_{L^{r_{1}}} \right).$$

Proof. We start by noting that (25) follows from (24). Indeed, using the Hölder inequality we find

$$\begin{split} \left\| Mf(x) \| 2^{sj} g_j(x) \|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)} + \left\| Mg(x) \| 2^{sj} f_j(x) \|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)} \\ &\leq \left\| \| 2^{sj} g_j(x) \|_{\ell^q} \right\|_{L^{p_2}(\mathbb{R}^n)} \| Mf(x) \|_{L^{r_1}(\mathbb{R}^n)} + \left\| \| 2^{sj} f_j(x) \|_{\ell^q} \right\|_{L^{p_1}(\mathbb{R}^n)} \| Mg(x) \|_{L^{r_2}(\mathbb{R}^n)} \\ &\leq C(\|f\|_{\tilde{F}^s_{p_1,q}} \|g\|_{L^{r_2}} + \|g\|_{\tilde{F}^s_{p_2,q}} \|f\|_{L^{r_1}}), \end{split}$$

by (7).

We turn to the proof of (24). It relies on Lemma 1 which is valid since $1 and <math>1 < q < \infty$. We have

$$fg = \sum_{k} G_k + \sum_{j} F_j,$$

where $G_k = \left(\sum_{j \leq k} f_j\right) g_k$, $F_j = \left(\sum_{k < j} g_k\right) f_j$. Since supp $\mathcal{F}(F_j) \subset B_{2^{j+2}}$ and supp $\mathcal{F}(G_k) \subset B_{2^{k+2}}$, Lemma 1 yields

(26)
$$\|fg\|_{\tilde{F}^s_{p,q}} \le C(A+B),$$

with

$$A = \left\| \|2^{sk} G_k(x)\|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)},$$

$$B = \left\| \|2^{sk} F_j(x)\|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)}.$$

We estimate, e.g., A:

(27)
$$A = \left\| \| 2^{sk} \left(\sum_{j \le k} f_j(x) \right) g_k(x) \|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)} \le \text{ by (11)}$$
$$C \left\| M f_j(x) \| 2^{sk} g_k(x) \|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)}.$$

We obtain (24) by combining (26), (27) and the similar estimate for B.

We state the continuity part of this result in the three possible situations:

Corollary 3. We have that:
a) for
$$1 < q < \infty$$
, $0 < s < \infty$, $1 < p_1 < \infty$, $1 < p_2 < \infty$, $1 < r_1 < \infty$, $1 < r_2 < \infty$,
 $0 < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{r_1} < 1$, the map
 $\left(\tilde{F}_{p_1,q}^s \cap L^{r_1}\right) \times \left(\tilde{F}_{p_2,q}^s \cap L^{r_2}\right) \ni (f,g) \mapsto fg \in \tilde{F}_{p,q}^s$

is continuous;

b) for $1 < q < \infty$, $0 < s < \infty$, 1 , if $<math display="block">\begin{cases}
f^{\ell} \to f \text{ in } \tilde{F}_{p,q}^{s}, & \|f^{\ell}\|_{L^{\infty}} \leq C \\
g^{\ell} \to g \text{ in } \tilde{F}_{p,q}^{s}, & \|g^{\ell}\|_{L^{\infty}} \leq C
\end{cases}$ then $f^{\ell}g^{\ell} \to fg$ in $\tilde{F}_{p,q}^{s}$;

c) for $1 < q < \infty$, $0 < s < \infty$, $1 < p_1 < \infty$, $1 < r < \infty$, $1 such that <math>\frac{1}{p} = \frac{1}{p_1} + \frac{1}{r}$, if $\begin{cases} f^{\ell} \to f \text{ in } \tilde{F}^s_{p_1,q}, & \|f^{\ell}\|_{L^{\infty}} \le C \\ g^{\ell} \to g \text{ in } \tilde{F}^s_{p,q} \cap L^r, \end{cases}$

then $f^{\ell}g^{\ell} \to fg$ in $\tilde{F}^s_{p,q}$.

Proof. a) follows directly from (25).

Some care is needed when one of the $r'_j s$ is ∞ . We treat, e.g., case c). It clearly suffices to prove the following two assertions:

(i) if $f^{\ell} \to 0$ in $\tilde{F}^s_{p_1,q}$ and $\|f^{\ell}\|_{L^{\infty}} \leq C$, then $f^{\ell}g \to 0$ for each $g \in \tilde{F}^s_{p,q} \cap L^r$. (ii) if $g^{\ell} \to 0$ in $\tilde{F}^s_{p,q} \cap L^r$, $\|f^{\ell}\|_{\tilde{F}^s_{p_1,q}} \leq C$, $\|f^{\ell}\|_{L^{\infty}} \leq C$, then $f^{\ell}g^{\ell} \to 0$.

Assertion (ii) is clear from (25). We prove (i) using (24). We have

(28)
$$\|f^{\ell}g\|_{\tilde{F}^{s}_{p,q}} \leq C \left(\|f^{\ell}\|_{\tilde{F}^{s}_{p_{1},q}} \|g\|_{L^{r}} + \left\|Mf^{\ell}(x)\|2^{sj}g_{j}(x)\|_{\ell^{q}}\right\|_{L^{p}(\mathbb{R}^{n})} \right)$$
$$\leq o(1) + C \left\|Mf^{\ell}(x)\|2^{sj}g_{j}(x)\|_{\ell^{q}}\right\|_{L^{p}(\mathbb{R}^{n})}.$$

 Set

$$F^{\ell}(x) = M f^{\ell}(x) \| 2^{sj} g_j(x) \|_{\ell^q}$$

Then clearly

(29)
$$|F^{\ell}(x)| \le C ||2^{sj}g_j(x)||_{\ell^q} \in L^p.$$

On the other hand, $\tilde{F}^s_{p_1,q} \hookrightarrow L^{p_1}$ (see, e.g., [34], 2.3.2, or [32], Proposition 2.2.1, p. 29). It follows from the maximal inequality (7) that $Mf^{\ell} \to 0$ in L^{p_1} and, up to a subsequence, that $Mf^{\ell} \to 0$ a.e. Then (i) follows from (28) and (29) by dominated convergence.

V. Proof of Theorem 1

The conclusion is well-known when s is an integer (this uses the standard Gagliardo-Nirenberg inequalities).

Assume s non integer. Clearly, the map

$$W^{s,p} \cap W^{1,sp} \ni u \mapsto f(u) \in L^p$$

is well-defined and continuous, since f(0) = 0, f is Lipschitz and $W^{s,p} \hookrightarrow L^p$.

Thus it suffices to prove that the map

$$W^{s,p} \cap W^{1,sp} \ni u \mapsto D(f(u)) = f'(u)Du \in W^{s-1,p}$$

is well-defined and continuous.

With $m = [s] + 1 \ge 2$, we obtain, using (14), that the inclusion

(30)
$$W^{s,p} \cap W^{1,sp} \hookrightarrow W^{m-1,\frac{sp}{m-1}} \cap W^{1,sp}$$

is continuous. Applying Theorem 1 to the integer $s = m - 1 \ge 1$, we find that

(31)
 if
$$u^{\ell} \to u$$
 in $W^{s,p} \cap W^{1,sp}$, then $f'(u^{\ell}) \to f'(u)$ in $\tilde{F}^{m-1}_{\frac{sp}{m-1},2} = W^{m-1,\frac{sp}{m-1}}_{\frac{sp}{m-1},2}$
and $\|f'(u^{\ell})\|_{L^{\infty}} \leq C$.

On the other hand, we clearly have that

(32) if
$$u^{\ell} \to u$$
 in $W^{s,p} \cap W^{1,sp}$, then $Du^{\ell} \to Du$ in $W^{s-1,p} \cap L^{sp} = \tilde{F}_{p,p}^{s-1} \cap L^{sp}$.

Using (31) and the Gagliardo-Nirenberg type inequality (15) (with q = p, s = m - 1, $\theta = \frac{s-1}{m-1}, p = \frac{sp}{m-1}$), we obtain

(33) if
$$u^{\ell} \to u$$
 in $W^{s,p} \cap W^{1,sp}$, then $f'(u^{\ell}) \to f'(u)$ in $\tilde{F}^{s-1}_{\frac{sp}{s-1},p}$ and $\|f'(u^{\ell})\|_{L^{\infty}} \le C$.

Finally, by (32), (33), the Runst-Sickel Lemma 5 and Corollary 3c), we obtain that $f'(u)Du \in \tilde{F}_{p,p}^{s-1} = W^{s-1,p}$ and that

if $u^{\ell} \to u$ in $W^{s,p} \cap W^{1,sp}$, then $f'(u^{\ell})Du^{\ell} \to f'(u)Du$ in $W^{s-1,p}$.

Remark 10. The same proof yields the following variant of Theorem 1.

Theorem 1". Assume $1 < s < \infty$, s non integer, $1 , <math>1 < q < \infty$. Then, for every $f \in \mathcal{R}$, the map

$$\tilde{F}^s_{p,q} \cap W^{1,sp} \ni u \mapsto f(u) \in \tilde{F}^s_{p,q}$$

is well-defined and continuous.

Remark 11. There is a natural strategy for proving Theorem 1: assume, e.g., that 1 < s < 2 and try to prove that $f'(u)Du \in W^{s-1,p}$. Set $s = 1 + \sigma$. On the one hand, we have $Du \in W^{\sigma,p} \cap L^{(1+\sigma)p}$. On the other hand, since $u \in W^{1,(1+\sigma)p}$, we find that $f'(u) \in W^{1,(1+\sigma)p} \cap L^{\infty}$. By the "standard" Gagliardo-Nirenberg inequality, we obtain $f'(u) \in W^{\sigma,\frac{1+\sigma}{\sigma}p} \cap L^{\infty}$. The conclusion of Theorem 1 would follow if we can prove that

(34)
$$\begin{array}{c} U \in W^{\sigma,p} \cap L^{(1+\sigma)p} \\ V \in W^{\sigma,\frac{1+\sigma}{\sigma}p} \cap L^{\infty} \end{array} \end{array} \} \Longrightarrow UV \in W^{\sigma,p}$$

Using the Gagliardo norm (4), we have to estimate

(35)
$$\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{|U(x+h)V(x+h) - U(x)V(x)|^{p}}{|h|^{n+\sigma p}} dx dh \\ \leq C \left(\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{|V(x)|^{p}|U(x+h) - U(x)|^{p}}{|h|^{n+\sigma p}} dx dh \\ + \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{|U(x)|^{p}|V(x+h) - V(x)|^{p}}{|h|^{n+\sigma p}} dx dh \right) \\ \leq C \left(\|V\|_{L^{\infty}}^{p} \|U\|_{W^{\sigma,p}}^{p} + \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{|U(x)|^{p}|V(x+h) - V(x)|^{p}}{|h|^{n+\sigma p}} dx dh \right).$$

It is natural to estimate the last integral in (34) using the Hölder inequality with exponents $1 + \sigma$ and $\frac{1 + \sigma}{\sigma}$. We find

$$\|UV\|_{W^{\sigma,p}}^{p} \leq C\left(\|V\|_{L^{\infty}}^{p}\|U\|_{W^{\sigma,p}}^{p} + \|V\|_{W^{\sigma,\frac{1+\sigma}{\sigma}p}}^{p}\left(\int_{\mathbb{R}^{n}}\int_{\mathbb{R}^{n}}\frac{|U(x)|^{(1+\sigma)p}}{|h|^{n}}dx\,dh\right)^{\frac{1}{1+\sigma}}\right).$$

Unfortunately, the last integral diverges, but we are "close" to convergence. In fact, we suspect that (34) is wrong.

It is here that the microscopic improvement of the Gagliardo-Nirenberg inequality Lemma 3, combined with the Runst-Sickel Lemma 5, magically saves the proof. We make use, in an essential way, of the additional information that $V = f'(u) \in F^{\sigma}_{\frac{1+\sigma}{\sigma}p,p}$.

We conclude this section with a brief survey of earlier results dealing with composition. a) if $0 < s \le 1$, 1 , <math>f(0) = 0, f Lipschitz, then

$$u \in W^{s,p} \Longrightarrow f(u) \in W^{s,p}$$
 (trivial for $s < 1$; see [21] and [22] for $s = 1$);

b) if $s = n/p, 1 , where <math>m = \begin{cases} s, \text{ if } s \text{ is an integer} \\ [s] + 1, \text{ otherwise} \end{cases}$,

then $u \in W^{s,p} \Longrightarrow f(u) \in W^{s,p}$.

This result is explicitly stated in [11]; G. Bourdaud has pointed out that it may also be derived from a result of T. Runst and W. Sickel, see p. 345 in [32], combined with a result in [19] which asserts that, when s = n/p, $W^{s,p} \hookrightarrow \tilde{F}^{\theta s}_{p/\theta,q}$ for $0 < \theta < 1$ and every $0 < q < \infty$ (see Remark 9 above);

c) if $s > n/p, 1 and <math>f \in C^m$, then $u \in W^{s,p} \Longrightarrow f(u) \in W^{s,p}$; see [25] for p = 2 and [29] for the general case;

d) if 1 < s < n/p, we have to impose additional restrictions on u. Indeed, if 1 + 1/p < s < 1n/p, the only $C^2 f$'s that act on $W^{s,p}$ are of the form f(t) = ct; see [15] for s integer and [31], Theorem 3.2, p. 319, for a general s. For 1 < s < n/p, it follows from Remark 1 in the Introduction that \mathcal{R} does not act on $W^{s,p}$, since $W^{s,p} \not\subset W^{1,sp}$. A standard additional condition on u is $u \in L^{\infty}$: if f(0) = 0 and $f \in C^m$, then $u \in W^{s,p} \cap L^{\infty} \Longrightarrow f(u) \in W^{s,p}$; see [29], [16];

e) an improvement is that, for f as above and $0 < \sigma < 1$ we have $u \in W^{s,p} \cap W^{\sigma,sp/\sigma} \Longrightarrow$ $f(u) \in W^{s,p}$; see [11]. This result implies the previous one, since $W^{s,p} \cap L^{\infty} \hookrightarrow W^{\sigma,sp/\sigma}$ (by Corollary 2);

f) a finer result asserts that, for f as above, we have $u \in W^{s,p} \cap \tilde{F}^1_{sp,q}$ (with $q \leq 2$ sufficiently small depending on s and $p \implies f(u) \in W^{s,p}$; see [32], Theorem 1, p. 345. This hypothesis on u is weaker than the previous one, since $W^{s,p} \cap W^{\sigma,sp/\sigma} \hookrightarrow \tilde{F}^1_{sp,q}$ for all q > 0, by Lemma 3. This result is contained in Theorem 1, since $\tilde{F}^1_{sp,q} \hookrightarrow W^{1,sp} = \tilde{F}^1_{sp,2}$ as soon as $q \leq 2$ (recall that $\tilde{F}_{p,q}^s$ increases with q). However, when $p \leq 2$ or 1 < s < 2, Runst and Sickel point out in Remark 1, p. 348 that the above smallness condition on q is precisely $q \leq 2$. This means that Runst and Sickel had established Theorem 1 when $p \leq 2$ or 1 < s < 2;

g) in the framework of Bessel potential spaces

$$L^{s,p} = \{ f = G_s \star g \; ; \; g \in L^p, \hat{G}_s(\xi) = (1 + |\xi|^2)^{-s/2} \} = \tilde{F}^s_{p,2},$$

there are various similar results about composition, starting with [23], [24] when s > n/p, [30], [2] and [14] for $H^s \cap L^\infty$ when $s \ge 1$. The ultimate result for $s \ge 1$ was obtained by Adams-Frazier in [1]: if $1 \leq s < \infty$, $1 , <math>f \in \mathcal{R}$, then $u \in L^{s,p} \cap L^{1,sp} \Longrightarrow f(u) \in \mathcal{R}$ $L^{s,p}$. This is a special case (q=2) of Theorem 1" since $L^{1,sp} = W^{1,sp}$.

h) Other questions concerning composition in Sobolev spaces have been investigated e.g. in [5], [6], [32].

VI. More about products

In this last Section, we state some natural results about products which may be derived from the Runst-Sickel lemma.

Let $1 , <math>0 < s < \infty$, $1 < r < \infty$, $0 < \theta < 1$, $1 < t < \infty$ be such that

$$\frac{1}{r} + \frac{\theta}{t} = \frac{1}{p}$$

Lemma 6. For $f \in W^{s,t} \cap L^{\infty}$, $g \in W^{\theta s,p} \cap L^r$, we have $fg \in W^{\theta s,p}$ and

(36)
$$\|fg\|_{W^{\theta_{s,p}}} \leq C \left(\|f\|_{L^{\infty}} \|g\|_{W^{\theta_{s,p}}} + \|g\|_{L^{r}} \|f\|_{W^{s,t}}^{\theta} \|f\|_{L^{\infty}}^{1-\theta} \right).$$

In the special case s > 1, $\theta = \frac{s-1}{s}$, we have r = sp and we obtain the following

Corollary 4. If $1 < s < \infty$, $1 and <math>f \in W^{s,p} \cap L^{\infty}$, $g \in W^{s-1,p} \cap L^{sp}$, then $fg \in W^{s-1,p}$ and

(37)
$$\|fg\|_{W^{s-1,p}} \leq C\left(\|f\|_{L^{\infty}}\|g\|_{W^{s-1,p}} + \|g\|_{L^{sp}}\|f\|_{W^{s,p}}^{1-1/s}\|f\|_{L^{\infty}}^{1/s}\right).$$

In particular, if $f, g \in W^{s,p} \cap L^{\infty}$, then $Dg \in W^{s-1,p} \cap L^{sp}$, so that Corollary 4 contains as a special case the following result

Corollary 5 ([7], Lemma 2). If $1 < s < \infty$, $1 and <math>f, g \in W^{s,p} \cap L^{\infty}$, then $f Dg \in W^{s-1,p}$.

Remark 12. Clearly, Corollary 5 implies the well-known assertion that $W^{s,p} \cap L^{\infty}$ is an algebra.

Proof of Lemma 6. Let q = 2 if θs is an integer, q = p otherwise. By (15), we find that $f \in \tilde{F}_{t/\theta,q}^{\theta s}$ and

(38)
$$\|f\|_{\tilde{F}^{\theta_s}_{t/\theta,g}} \leq C \|f\|^{\theta}_{W^{s,t}} \|f\|^{1-\theta}_{L^{\infty}}.$$

From the Runst-Sickel lemma, we deduce that $fg \in \tilde{F}_{p,q}^{\theta s}$ and

$$\begin{aligned} \|fg\|_{W^{\theta_{s,p}}} &= \|fg\|_{\tilde{F}_{p}^{\theta_{s}},q} \leq C\left(\|f\|_{L^{\infty}}\|g\|_{\tilde{F}_{p,q}^{\theta_{s}}} + \|g\|_{L^{r}}\|f\|_{\tilde{F}_{t/\theta,q}^{\theta_{s}}}\right) \\ &\leq C\left(\|f\|_{L^{\infty}}\|g\|_{W^{\theta_{s,p}}} + \|g\|_{L^{r}}\|f\|_{W^{s,t}}^{\theta}\|f\|_{L^{\infty}}^{1-\theta}\right). \end{aligned}$$

Acknowledgements. The authors thank G. Bourdaud and F.Planchon for interesting discussions. The first author (H.B.) is partially supported by a European Grant ERB FMRX CT98 0201. He is also a member of the Institut Universitaire de France. Part of this work was done when the second author (P.M.) was visiting Rutgers University; he thanks the Mathematics Department for its invitation and hospitality.

References

- D.R. Adams and M. Frazier, Composition operators on potential spaces, Proc. Amer. Math. Soc. 114 (1992), 155–165.
- [2] S. Alinhac and P. Gérard, Opérateurs pseudo-différentiels et théorème de Nash-Moser, Interéditions, 1991.
- [3] F. Bethuel, The approximation problem for Sobolev maps between two manifolds, Acta Math. 167 (1991), 153–206.
- [4] F. Bethuel and X. Zheng, Density of smooth functions between two manifolds in Sobolev spaces, J. Funct. Anal. 80 (1988), 60–75.
- [5] G. Bourdaud, Le calcul fonctionnel dans les espaces de Sobolev, Invent. Math. 104 (1991), 435–446.
- [6] G. Bourdaud and Y. Meyer, Fonctions qui opèrent sur les espaces de Sobolev, J. Funct. Anal. 97 (1991), 351–360.
- [7] J. Bourgain, H. Brezis and P. Mironescu, Lifting in Sobolev spaces, J. d'Analyse Mathématique 80 (2000), 37–86.
- [8] J. Bourgain, H. Brezis and P. Mironescu, Another look at Sobolev spaces, in Optimal Control and Partial Equations, in honor of Professor Alain Bensoussan's 60th birthday, J.L. Menaldi, E. Rofman and A. Sulem (eds), IOS Press, Amsterdam, 2001, 439–455.
- H. Brezis, Laser beams and limiting cases of Sobolev inequality, in Nonlinear Partial Differential Equations and Their Applications, Collège de France, Sem. Vol. II (H. Brezis, J.-L. Lions eds), Pitman, 1982, 86–97.
- [10] H. Brezis and Y. Li, Topology and Sobolev spaces, J. Funct. Anal. (to appear).
- [11] H. Brezis and P. Mironescu, Composition in fractional Sobolev spaces, Discrete and Continuous Dynamical Systems 7 (2001), 241–246.
- [12] H. Brezis and P. Mironescu, in preparation.
- [13] H. Brezis and S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, Comm. in PDE 5 (1980), 773–789.
- [14] J.-Y. Chemin, Fluides parfaits incompressibles, 230, Astérisque, 1995.
- [15] B.E.J. Dahlberg, A note on Sobolev spaces, in *Harmonic Analysis in Euclidean Spaces*, Proc. Symp. in Pure Mathematics Part I, American Math. Soc., **35** (1979), 183–185.
- [16] M. Escobedo, Some remarks on the density of regular mappings in Sobolev classes of S^{M} -valued functions, Rev. Mat. Univ. Complut. Madrid 1 (1988), 127–144.
- [17] P. Gérard, Y. Meyer, F. Oru, Inégalités de Sobolev précisées, in Séminaire sur les Equations aux Dérivées Partielles, Ecole Polytechnique, 1996–1997, 11 pp.
- [18] F.B. Hang and F.H. Lin, Topology of Sobolev mappings (to appear).
- [19] B. Jawerth, Some observations on Besov and Lizorkin-Triebel spaces, Math. Scand. 40 (1977), 94–104.
- [20] T. Kato, On nonlinear Schrödinger equations. II. H^s solutions and unconditional well-posedness, J. d'Analyse Mathématique 67 (1995), 281–306.
- [21] M. Marcus and V.J. Mizel, Nemytskij operators on Sobolev spaces, Arch. Rat. Mech. Anal. 51 (1973), 347–370.
- [22] M. Marcus and V.J. Mizel, Complete characterization of functions which act via superposition on Sobolev spaces, Trans. Amer. Math. Soc. 251 (1979), 187–218.

- [23] Y. Meyer, Régularité des solutions des équations aux dérivées partielles non linéaires, Séminaire Bourbaki n^o 560, 1979-1980.
- [24] Y. Meyer, Remarques sur un théorème de J.-M. Bony, Suppl. Rendiconti Circ. Mat. Palermo, II, 1 (1981), 1–20.
- [25] S. Mizohata, Lectures on the Cauchy problem, Tata Institute, Bombay, 1965.
- [26] J. Moser, A rapidly convergent iteration method and nonlinear differential equations, Ann. Sc. Norm. Sup. Pisa 20 (1966), 265–315.
- [27] L. Nirenberg, On elliptic partial differential equations, Ann. Sc. Norm. Sup. Pisa 13 (1959), 115–162.
- [28] F. Oru, Rôle des oscillations dans quelques problèmes d'analyse non-linéaire, Doctorat de Ecole Normale Supérieure de Cachan, 1998.
- [29] J. Peetre, Interpolation of Lipschitz operators and metric spaces, Mathematica (Cluj) 12 (1970), 1–20.
- [30] J. Rauch and M. Reed, Nonlinear microlocal analysis of semilinear hyperbolic systems in one space dimension, Duke Math. J. 49 (1982), 397–475.
- [31] T. Runst, Mapping properties of nonlinear operators in spaces of Triebel-Lizorkin and Besov type, Analysis Mathematica 12 (1986), 313–346.
- [32] T. Runst and W. Sickel, Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations, Walter de Gruyter, Berlin and New York, 1996.
- [33] E. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, 1993.
- [34] H. Triebel, Theory of function spaces, Birkhäuser, Basel and Boston, 1983.
- [35] M. Yamazaki, A quasi-homogeneous verson of paradifferential operators, I: Boundedness on spaces of Besov type, J. Fac. Sci. Univ. Tokyo Sect. I A Math. 33 (1986), 131–174.

Analyse Numérique
 Université P. et M. Curie, BC 187
 4 Pl. Jussieu
 75252 Paris CEDEX 05

(2) Rutgers University
Dept. of Math., Hill Center, Busch Campus
110 Frelinghuysen Rd, Piscataway, NJ 08854
E-mail address: brezis@ccr.jussieu.fr; brezis@math.rutgers.edu

(3) Département de Mathématiques
Université Paris-Sud
91405 Orsay
E-mail address: Petru.Mironescu@math.u-psud.fr