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Approximate hedging problem with transaction costs in stochastic volatility markets

Huu-Thai Nguyen* and Serguei Pergamenshchikov[†]

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Abstract

This paper investigates the problem of hedging European call options using Leland's algorithm in stochastic volatility markets with transaction costs. Introducing a new form for the enlarged volatility, we establish limit theorems and determine a convergence rate of the hedging error for both Leland [27] and the modified one proposed by Lépinette in [12]. These asymptotic properties enable us to release the underhedging property pointed out by Kabanov and Safarian in [23]. Possibilities to improve the convergence rate and lower the option price inclusive transaction costs are also discussed.

Keywords Leland strategy, Delta hedging, transaction costs, quantile hedging, pricing option.

Mathematics Subject Classification (2010) 91G20; 60G44; 60H07

JEL Classification G11; G13

1 Introduction

In the theory of hedging options, Leland's strategy provides a simple way to eliminate efficiently risks caused by transaction costs. This prescription is based on the idea

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that transaction costs can be compensated by enlarging the volatility parameter in the *delta* Black-Scholes strategy. The pioneering work in this field was first given in [27], where a discrete approximation was used to study the asymptotic behavior of the *hedging error*, defined as difference between the terminal portfolio value and the payoff, as the number of transactions goes to infinity. It was then shown in [29] that the hedging error vanishes if the transaction cost percentage converges to 0 at a power rate. Unfortunately, this property does not hold for the most interesting case when the proportional cost is a constant. In [23], the authors found an explicit limit for the hedging error but unexpectedly, it is a negative quantity. It means that the option is actually underhedged in limit if the investor follows the Leland strategy. The convergence problem was investigated in the paper [18] and then, a complete answer was provided in [31] with the corresponding limit theorem allowing to identify the asymptotic distribution of the hedging error. Recently, a modified strategy with non-uniform revisions has been suggested in [12] and it turns out that this modification considerably improves the convergence rate, even in models with general convex payoffs [8]. For related results, see further in [11, 12, 17, 18].

Many empirical studies show that the constant volatility condition in the classical Black-Scholes is restrictive and the Black-Scholes formula constructed under this assumption has an inaccuracy for anticipating the market option prices. The discrepancy between Black-Scholes option prices and market-traded ones, known as *smile curve*, can be explained by using stochastic volatility (SV) models which have been used to describe complex markets e.g. when fat-tailed returns are taken into account. Note that modeling SV markets contains some intrinsic difficulties [15]. The incompleteness property of market makes the pricing problem more challenging in both constructing an explicit formula and estimating parameters. Hence, derivatives may not be perfectly hedged with only trading the underlying assets and, in general, asymptotic analysis is an efficient tool for studying such models. See [15] and the references therein for detailed discussions.

The main objective of this paper is to study the problem of hedging European style options in stochastic volatility markets with the presence of transaction costs. Using a new form of *adjusted volatility* in Leland's spirit, we study asymptotic properties of hedging error for both Leland's strategy and Lépinette's one under a general non-uniform revision setting. The achievement here is an extension of the ones obtained in [23, 31] and [8, 12].

Let us emphasize that the enlarged volatility (2.5) proposed in [27] and then applied in [23, 24, 9, 10, 12] is no longer applicable in SV models from a practical point of view. The reason is that the quantity $\lambda_t = \int_t^1 \hat{\sigma}_u^2 du$ appearing in the Black-Scholes formula is substantially dependent on future realizations of the random process driving the volatility. Therefore, the strategy is not available for investors in this case. To surpass this issue, we suggest a simpler form (2.24) for the adjusted volatility by making it independent of the initial volatility. On other hand, it can be learned from Leland's approach that the modified volatility should be chosen

so that it goes to infinity as fast as the square radical of the number of revisions in the hope of approximately releasing effects of transaction costs. Therefore, it is reasonable to adjust the volatility parameter as in (3.2) to keep this property for stochastic volatility markets also. This idea not only allows us to carry out an approximation much more simply but also gives possibilities to improve the convergence rate. The results obtained here can be extended to options with general convex pay-off derivatives studied in [9, 12]. This extension will be left in our coming contribution.

It is shown in Section 3 that our enlarged volatility is far simpler than the one used in the previous works [23, 24, 31, 12, 9] but the same asymptotic results are obtained not only for constant volatility models but for SV contexts as well. This important fact proves our significant contribution in discrete hedging literature in the presence of proportional transaction costs. Additionally, the setting of our SV model (3.1) is enough general for practice purposes, including famous SV models such as the Hull-White and Scott models.

As discussed in [31] that the option price of Leland's strategy is too high because it includes transaction costs. Another practical advantage of our method is that we propose a simple method in order to lower the option price as long as the seller is willing to take risks with a given probability. This approach is inspired from the quantile hedging theory [14, 4, 6].

The remain of the paper is organized as follows. In Section 2, we briefly give a general view of Leland's approach then formulate the problem and present our principal results in Section 3. The new choice of adjusted volatility allows us to propose a reasonable way in Section 4 to fix the underhedging situation (shown in [23]) and reduce the option price in the presence of transaction costs. Section 5 is devoted to present our basic approximate tools. We explicitly present the asymptotic approximations for the terms consisting of the hedging error in Section 6, Section 7 and Section 8. Proofs of the main results are found in Section 9. Section 10 discusses some common SV models for which our condition on volatility (\mathbf{C}_1) is fulfilled. All useful estimates and convergence lemmas used in the proof can be found in Appendix.

2 Hedging with transaction costs: a review on the Leland approach

In a complete no-arbitrage model (i.e. there exists a unique equivalent martingale measure under which the stock price is a martingale), options can be completely replicated by a self-financing trading strategy. Option price, defined as the replication cost, is the initial capital that the investor must introduce into his portfolio to obtain a complete hedge. It can be computed as the expectation of the discounted claim under the unique equivalent martingale measure.

Let us consider a continuous time model of two-asset financial market on the time interval $[0, 1]$, where the bond price is a constant over the time and equals to 1. The stock price dynamics follows the stochastic differential equation

$$dS_t = \sigma_0 S_t dW_t, \quad (2.1)$$

where $\sigma_0 > 0$ is a positive constant and $(W_t)_{0 \leq t \leq 1}$ is a standard Wiener process. As usual we denote $\mathcal{F}_t = \sigma\{W_u, 0 \leq u \leq t\}$.

We recall that a financial strategy $(\beta_t, \gamma_t)_{0 \leq t \leq 1}$ (the fractions of wealth invested in bonds and stock respectively) is called an *admissible self-financing strategy* if it is (\mathcal{F}_t) -adapted, integrable with

$$\int_0^t (|\beta_t| + \gamma_t^2) dt < \infty \quad \text{a.s.}$$

and the portfolio value satisfies the equality

$$V_t = \beta_t + \gamma_t S_t = V_0 + \int_0^t \gamma_u dS_u$$

for any $t \in [0, 1]$. The classical hedging problem is to find an admissible self-financing strategy (β_t, γ_t) whose terminal portfolio value exceeds the payoff $h(S_1) = (S_1 - K)_+$; that is

$$V_1 = V_0 + \int_0^1 \gamma_u dS_u \geq h(S_1) \quad \text{a. s.}$$

where K is the option strike. For this problem, Black and Scholes [3] proposed a dynamically replicating self-financing strategy with $\gamma_t = C_x(t, S_t)$ (partial derivative with respect to the space variable), where the option price $C(t, S_t)$ reads the famous formula

$$C(t, x) = C(t, x, \sigma_0) = x\Phi(\tilde{\mathbf{v}}(t, x)) - K\Phi(\tilde{\mathbf{v}}(t, x) - \sigma_0\sqrt{1-t}), \quad (2.2)$$

where $\tilde{\mathbf{v}}(t, x) = \mathbf{v}(\sigma_0^2(1-t), x)$ and

$$\mathbf{v}(\lambda, x) = \frac{\ln(x/K)}{\sqrt{\lambda}} + \frac{\sqrt{\lambda}}{2}. \quad (2.3)$$

Here Φ is the standard normal distribution function. In the sequel, we denote by φ the $\mathcal{N}(0, 1)$ density, i.e. $\varphi(z) = \Phi'(z)$. One can check directly that

$$C_x(t, x) = \Phi(\tilde{\mathbf{v}}(t, x)) \quad \text{and} \quad C_{xx}(t, x) = \frac{\varphi(\tilde{\mathbf{v}}(t, x))}{x\sigma_0\sqrt{1-t}}. \quad (2.4)$$

Clearly, hedging via discrete strategies is specially attractive since dynamically adjusted portfolios are impossible to be carried out in practice. However, discrete

time hedging, in turn, will now face with some problems in view of transaction costs. Firstly, transaction costs are random and path-dependent, so they significantly effect to the hedging error. Additionally, despite of the fact argued by Black and Scholes that the hedging error may be relatively small if trading activities take place reasonably frequently, transaction costs may increase without limit as portfolio revisions are frequent, so it may lead to an explosion. If one wishes to bound the option price by using an arbitrage argument as in the Black-Scholes model then it is necessary to know the maximum transaction costs rather than the average but this task is not easy.

These considerations lead us to the Leland approach [27], which provides an efficient technique to compensate transaction costs. This method is simply based on the intuition that the option price should include transaction costs as a reasonable extra fee necessary for the seller to cover the option return. In some situations (discussed in the next two sections), this strategy successfully replicates the payoff including transaction costs by simply adjusting the volatility parameter in Black-Scholes's model.

2.1 Constant volatility case

Let us shortly describe the Leland approach in [27, 23]. Suppose that for each trading activity, the investor has to pay a fee directly proportional to the trading volume. Naturally, we suppose that the proportional transaction cost depends upon the number of trades n , given by the law $\kappa_* n^{-\alpha}$, where $\alpha \geq 0$ and $\kappa_* > 0$ are two fixed parameters. To compensate transaction costs the investor is suggested to enlarge the volatility as

$$\widehat{\sigma}^2 = \sigma_0^2 + \varrho n^{1/2-\alpha} \quad \text{and} \quad \varrho = \kappa_* \sigma_0 \sqrt{8/\pi}. \quad (2.5)$$

We assume further that the portfolio is revised discretely at

$$t_i = \frac{i}{n}, \quad i = 1, 2, \dots, n, \quad (2.6)$$

with the trading volume determined by the piecewise process (so-called Leland's strategy)

$$\gamma_t^n = \sum_{i=1}^n \widehat{C}_x(t_{i-1}, S_{t_{i-1}}) \mathbf{1}_{(t_{i-1}, t_i]}(t), \quad (2.7)$$

where $\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) = C_x(t_{i-1}, S_{t_{i-1}}, \widehat{\sigma})$. It means that the number of shares holding in the interval $(t_{i-1}, t_i]$ is nothing than the delta strategy calculated at the left bound of this interval. Then, the portfolio value takes the following form

$$V_1^n = V_0^n + \int_0^1 \gamma_u^n dS_u - \kappa_* n^{-\alpha} J_n, \quad (2.8)$$

where the total trading volume is given by

$$J_n = \sum_{i=1}^n S_{t_i} |\gamma_{t_i}^n - \gamma_{t_{i-1}}^n|. \quad (2.9)$$

The option price is now given by the initial time-value of the solution $\widehat{C}(t, x) = C(t, x, \widehat{\sigma})$ of the Black-Scholes PDE with the adjusted volatility $\widehat{\sigma}$

$$\begin{cases} \widehat{C}_t(t, x) + \frac{1}{2} \widehat{\sigma}^2 x^2 \widehat{C}_{xx}(t, x) = 0, & 0 \leq t < 1, \\ \widehat{C}(1, x) = h(x). \end{cases} \quad (2.10)$$

Using Itô's formula we can represent the hedging error as

$$\begin{aligned} V_1^n - h(S_1) &= \int_0^1 \left(\gamma_t^n - \widehat{C}_x(t, S_t) \right) dS_t \\ &\quad + \frac{1}{2} (\widehat{\sigma}^2 - \sigma_0^2) \int_0^1 S_t^2 \widehat{C}_{xx}(t, S_t) dt - \kappa_* n^{-\alpha} J_n. \end{aligned} \quad (2.11)$$

Remark 1 (Leland). *The specific form (2.5) results from the following intuition: the Lebesgue's integral in (2.11) is clearly well-approximated by the Riemann sum of the terms $\sigma_0 S_{t_{i-1}}^2 \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) \Delta t$, while*

$$\begin{aligned} S_{t_i} |\gamma_{t_i}^n - \gamma_{t_{i-1}}^n| &\approx \sigma_0 S_{t_{i-1}}^2 |\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})| |W_{t_i} - W_{t_{i-1}}| \\ &\approx \sigma_0 \sqrt{2/\pi} S_{t_{i-1}}^2 \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) \sqrt{\Delta t}, \end{aligned}$$

since $\mathbf{E}|W_{t_i} - W_{t_{i-1}}| = \sqrt{2/\pi} \sqrt{\Delta t} = \sqrt{2/(\pi n)}$. Hence, it is reasonable to expect that choosing the modified volatility as in (2.5) would give an appropriate approximation to compensate transaction costs.

Leland [27] conjectured that if the proportional transaction cost is a constant i.e. $\alpha = 0$ then, the portfolio value of strategy (2.7) converges in probability to the payoff $h(S_1)$ as $n \rightarrow \infty$. He also gave a remark without giving a complete proof that this result is still true in the case $\alpha = 1/2$. The later remark is correct and was completely proved by Lott in [29], where one can find a rigorous explanation why the Leland strategy is important in practice.

Theorem 2.1 (Leland-Lott). *For $\alpha = 1/2$, strategy (2.7) defines an approximately replicating strategy as the number of revision intervals n tends to infinity i.e.*

$$\mathbf{P} - \lim_{n \rightarrow \infty} V_1^n = h(S_1).$$

This result was then extended by Ahn *et al* in [1], where the replication problem with transaction costs was considered in the general diffusion models. Kabanov and Safarian [23] also observed that the Leland approach is still valid as long as the transaction proportion converges to zero as $n \rightarrow \infty$.

Theorem 2.2 (Kabanov-Safarian). *For any $\alpha > 0$,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} V_1^n = h(S_1).$$

It is crucial to note that the Leland approximation in Remark 1 is not mathematically accurate and so, his first conjecture is not correct. In fact, as $n \rightarrow \infty$, the trading volume J_n may be approximated by the following sum

$$\sum_{i=1}^n \lambda_{i-1}^{-1/2} S_{t_{i-1}} \tilde{\varphi}(\lambda_{i-1}, S_{t_{i-1}}) \left| \frac{\sigma_0}{\varrho} Z_i + q(\lambda_{i-1}, S_{t_{i-1}}) \right| \Delta \lambda_i, \quad (2.12)$$

where

$$\lambda_i = \lambda_{t_i} = \hat{\sigma}^2(1 - t_i), \quad Z_i = (W_{t_i} - W_{t_{i-1}}) / \sqrt{\Delta t_{i-1}}$$

and

$$\tilde{\varphi}(\lambda, x) = \varphi(\mathbf{v}(\lambda, x)), \quad q(\lambda, x) = \frac{\ln(x/K)}{2\lambda} - \frac{1}{4}. \quad (2.13)$$

In approximation procedures, one should also pay attention to the fact that $\hat{C}(\cdot, \cdot)$ and its derivatives substantially depend upon n . This property leads to the following important result: there is a *non trivial discrepancy* between the limit of the terminal portfolio value and the payoff in the most practical case $\alpha = 0$.

Theorem 2.3 (Kabanov-Safarian). *If $\alpha = 0$ then,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} V_1^n = h(S_1) + \min(S_1, K) - \kappa_* J(S_1, \varrho),$$

where

$$J(x, \varrho) = x \int_0^{+\infty} \lambda^{-1/2} \tilde{\varphi}(\lambda, x) \mathbf{E} \left| \sigma_0 \varrho^{-1} Z + q(\lambda, x) \right| d\lambda, \quad (2.14)$$

and $Z \sim \mathcal{N}(0, 1)$ independent of S_1 .

It is important to observe that the problem of option replicating is not solved in this case. Indeed, taking into account that $\mathbf{E} |\sigma_0 \varrho^{-1} Z| = 1/(2\kappa_*)$ and

$$\min(x, K) = \frac{x}{2} \int_0^{\infty} \frac{1}{\sqrt{\lambda}} \tilde{\varphi}(\lambda, x) d\lambda, \quad (2.15)$$

we obtain (for the parameter ϱ given in (2.5))

$$\min(x, K) - \kappa_* J(x, \varrho) = x \kappa_* \int_0^{+\infty} \frac{\tilde{\varphi}(\lambda, x)}{\sqrt{\lambda}} \left(\mathbf{E} \left| \frac{\sigma_0}{\varrho} Z \right| - \mathbf{E} \left| \frac{\sigma_0}{\varrho} Z + q(\lambda, x) \right| \right) d\lambda.$$

Now, Andreson's inequality (see, for example [22], page 155) implies directly that for any $q \in \mathbb{R}$,

$$\mathbf{E} |\sigma_0 \varrho^{-1} Z + q| \geq \mathbf{E} |\sigma_0 \varrho^{-1} Z|.$$

Therefore, $\mathbf{P} - \lim_{n \rightarrow \infty} (V_1^n - h(S_1)) < 0$ i.e. the option is asymptotically underhedged in this case.

Another important point to note here is that the coefficient ϱ appearing in (2.5) can be chosen in an arbitrary way. We now state the main result in [31], which also provides the convergence rate for the hedging error.

Theorem 2.4 (Pergamenshchikov). *Consider the Leland strategy (2.7) with $\alpha = 0$ and let ϱ in (2.5) be some fixed positive constant. Then, the sequence of random variables*

$$n^{1/4}(V_1^n - h(S_1) - \min(S_1, K) + \kappa_* J(S_1, \varrho)) \quad (2.16)$$

weakly converges to a centered mixed Gaussian variable as $n \rightarrow \infty$.

This result is important because it not only gives the asymptotic information of the hedging error but also provides a reasonable way to fix the underhedging issue. More precisely, as discussed in [31], by choosing a suitable value of ϱ the investor can get a portfolio whose terminal value exceeds the option return as desired.

It is, of course, possible to study the Leland-Lott approximation in sense of L^2 -convergence. A such result¹ was established in [11, 24] for the case $\alpha = 1/2$.

Theorem 2.5 (Kabanov-Lépinette). *Let $\alpha = 1/2$. The mean-square approximation error for Leland's strategy with ϱ defined in (2.5) satisfies the following asymptotic equality*

$$\mathbf{E} (V_1^n - h(S_1))^2 = Bn^{-1} + o(n^{-1}) \quad \text{as } n \rightarrow \infty,$$

where B is some positive constant.

Darses and Lépinette [12] noted that one can modify the Leland strategy to improve the convergence rate in (2.16). In particular, one can apply a non-uniform revision times $(t_i)_{1 \leq i \leq n}$ as

$$t_i = g\left(\frac{i}{n}\right), \quad g(t) = 1 - (1-t)^\mu \quad \text{for some } \mu \geq 1 \quad (2.17)$$

and then adjust the volatility as

$$\widehat{\sigma}_t^2 = \sigma_0^2 + \kappa_* \sigma_0 \sqrt{8/\pi} \sqrt{nf'(t)}, \quad (2.18)$$

¹Seemingly, mean estimates do not contain much useful information since gains and losses have different meaning in practice. Clearly, if $\alpha = 1/2$ the modified volatility is independent of n

where f is the inverse function of g . It is also suggested in [12] to use the following modified discrete strategy to release the discrepancy appearing in Theorems 2.3 and Theorem 2.4:

$$\gamma_t^n = \sum_{i=1}^n \left(\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \int_0^{t_{i-1}} \widehat{C}_{xt}(u, S_u) du \right) \mathbf{1}_{(t_{i-1}, t_i]}(t). \quad (2.19)$$

Theorem 2.6. *Let V_1^n be the terminal portfolio value of the strategy (2.19) with $\alpha = 0$. Then, for any $1 \leq \mu < \mu_{max}$ the sequence $n^\beta(V_1^n - h(S_1))$ weakly converges to a centered mixed Gaussian variable as $n \rightarrow \infty$, where*

$$\beta = \frac{\mu}{2(\mu + 1)} \quad \text{and} \quad \mu_{max} = \frac{3 + \sqrt{57}}{8}. \quad (2.20)$$

2.2 Time-dependent volatility case

We assume in this section that the stock price is now driven by the following SDE

$$dS_t = \sigma(t)S_t dW_t. \quad (2.21)$$

Under the non-uniform rebalancing plan (2.17) the investor should modify the volatility as

$$\widehat{\sigma}_t^2 = \sigma^2(t) + \kappa_* \sigma(t) n^{1/2-\alpha} \sqrt{f'(t)8/\pi}. \quad (2.22)$$

Suppose further that the general European payoff $h(\cdot)$ is a continuous function having continuous derivatives except a finite number of points. From the Black-Scholes formula, the option price (inclusive of transaction costs) is now given by

$$\widehat{C}(t, x) = \int_{-\infty}^{\infty} h(x \exp(y\lambda_t^{1/2} - \lambda_t/2)) \varphi(y) dy, \quad (2.23)$$

where $\lambda_t = \int_t^1 \widehat{\sigma}_u^2 du$. Recall that $\widehat{C}(t, S_t)$ is the solution of the Cauchy problem (2.10). We now state the main achievement in time-dependent volatility models in [9].

Theorem 2.7 (Lépinette). *Suppose that $(\sigma(t))_{t \geq 0}$ is a strictly positive Lipschitzian and bounded function, $h(\cdot)$ is piecewise twice differentiable and there exist $x_* \geq 0$ and $\delta \geq 3/2$ such that $\sup_{x \geq x_*} x^\delta |h''(x)| < \infty$. Then, for $\alpha > 0$ the portfolio value of strategy (2.19) converges in probability to the payoff $h(S_1)$ as $n \rightarrow \infty$. If $\alpha = 0$, then*

$$\mathbf{P} - \lim_{n \rightarrow \infty} V_1^n = h(S_1) + h_1(S_1) - \kappa_* h_2(S_1),$$

where $h_1(\cdot)$ and $h_2(\cdot)$ are positive functions depending on the payoff.

Remark 2. *Theorem 2.6 still holds in the setting of Theorem 2.7 [12].*

Another way to extend the applicable range of the Leland strategy is to consider the hedging problem with transaction costs in models where volatility values not only depend on time but also on the stock price itself. For these models, well-known as *local volatility models*, the result in Theorem 2.7 still holds for any $1/4 \leq \alpha \leq 1/2$ [9]. However, carrying out a proof of this result is really complicated, since the existence of solution for the parabolic equation (2.10) is now not trivial if the adjusted volatility takes the form (2.22).

In summary, the Leland strategy plays an important role in the option pricing and hedging problems thanks to the easy implement in practice. The most interesting case $\alpha = 0$ still needs to be investigated in more general situations, for instance, where volatility depends on other external random factors or jumps in stock prices are taken into account. It is worth noticing that the methodology used in the works of Lépinette and Kabanov needs a delicate treatment and seemingly, it is difficult to apply for such models.

2.3 Forms of adjusted volatility

Recall from Remark 1 that choosing the modified volatility as in (2.5) would give an appropriate approximation to compensate transaction costs. However, it is not always the case since the option price inclusive transaction costs $\widehat{C}(t, S_t)$ is now depends intrinsically on the rebalancing number n . In more general models this specific choice can cause to technical issues. For example, in local stochastic models [9], proving the existence of solution to (2.10) requires much effort since now $\widehat{\sigma}$ is computed in term of the stock price and time. This feature makes the Cauchy problem more challenging to deal with. Nevertheless, it is interesting to point out that the term $\sigma^2(t)$ in (2.22) plays no role in the approximation procedure. In fact, all results reviewed above for the case $\alpha = 0$ can be recovered by using the form

$$\widehat{\sigma}_t^2 = \kappa_* \sigma(t) n^{1/2} \sqrt{f'(t) 8/\pi},$$

where the first term $\sigma^2(t)$ has been removed. More general, we can completely remove $\sigma(t)$ from the formula of enlarged volatility by taking the new form

$$\widehat{\sigma}_t^2 = \varrho n^{1/2} \sqrt{f'(t)}, \quad (2.24)$$

for some positive constant ϱ . Of course, the limit of transaction costs will slightly change since ϱ is no longer related to the terminal value of volatility, see Theorem 2.4. This important observation follows from the fact which can be proved similarly as Lemma 1.2.8 in [24] (page 16)

$$\int_0^1 \sigma^2(t) S_t^k \frac{\partial^k \widehat{C}}{\partial x^k}(t, S_t) dt = O(\widehat{\sigma}^{-1}) = O(n^{-1/4}) \quad \text{as } n \rightarrow \infty, \quad (2.25)$$

for all $k \geq 2$. The asymptotic representation (2.25) still holds if $\sigma = \sigma(y_t)$ for some extra random process y_t as long as $\mathbf{E} \sup_{0 \leq t \leq 1} \sigma^2(y_t) < \infty$. This motivates our assumption (\mathbf{C}_1) in our models, see more in [33].

Let us emphasize that using the new simple form (2.24) has two folds of importance. From a technical point of view, it allows us to carry out a much more simple approximation than what have done in the existing literature. More importantly, when volatility depends on some exterior factor, says y_t , so the Balck-Schole formula (2.23) is no longer available for practitioners if one uses the well-known form (2.22). The reason is that the quantity $\lambda_t = \int_t^1 \widehat{\sigma}_u^2 du$, which is substantially dependent on future realizations of y_t (from now, at time t , to the terminal date $t = 1$), is impossible to obtain from practical point of view. In contrast, in this context the simpler form still helpful since it is a deterministic function of t . Following this idea, we will make use the simple form for the adjusted volatility to study the Leland's approximation in SV models in this contribution.

3 Main results

Let $(\Omega, \mathcal{F}_1, (\mathcal{F}_t)_{0 \leq t \leq 1}, \mathbf{P})$ be the standard filtered probability space with two standard independent $(\mathcal{F}_t)_{0 \leq t \leq 1}$ adapted Wiener processes $(W_t^{(1)})$ and $(W_t^{(2)})$ taking their values in \mathbb{R} . Our financial market consists of one risky asset governed by the following equations on the time interval $[0, 1]$:

$$\begin{cases} dS_t = \sigma(y_t)S_t dW_t^{(1)}, \\ dy_t = F_1(t, y_t)dt + F_2(t, y_t)(\mathbf{r}dW_t^{(1)} + \sqrt{1 - \mathbf{r}^2}dW_t^{(2)}) \end{cases} \quad (3.1)$$

where $-1 \leq \mathbf{r} \leq 1$. We assume that the $[0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ functions $F_1(t, y)$ and $F_2(t, y)$ provide the existence of the unique solution to the last stochastic differential equation². Note that in this model the bond interest rate $r = 0$ i.e. the non-risky asset is chosen as the *numéraire*.

As discussed in the previous section, we use the adjusted volatility given by

$$\theta_t^2 = \varrho \sqrt{n f'(t)} = \frac{1}{\sqrt{\mu}} \varrho \sqrt{n} (1-t)^{\frac{1-\mu}{2\mu}}, \quad 1 \leq \mu < 2. \quad (3.2)$$

The parameter $\varrho > 0$ plays an important role in controlling the convergence rate and it will be specified later. As discussed in details below, the limit of the total trading volume J_n essentially involves in how ϱ depends upon the number of revisions n . We will formulate our dependence condition concerning the following parameter

$$\mu_* = \frac{1}{2(\mu + 2)}. \quad (3.3)$$

²Existence and uniqueness of system (3.1) are well known in the literature of SDEs, see for example [28] or [16].

Let $\widehat{C}(t, x)$ be the solution of the Cauchy problem

$$\widehat{C}_t(t, x) + \frac{1}{2}\theta_t^2 x^2 \widehat{C}_{xx}(t, x) = 0, \quad \widehat{C}(1, x) = h(x),$$

with two first derivatives given as in (2.4)

$$\widehat{C}_x(t, x) = \widetilde{\Phi}(\lambda_t, x), \quad \widehat{C}_{xx}(t, x) = \frac{1}{x\sqrt{\lambda_t}}\widetilde{\varphi}(\lambda_t, x), \quad (3.4)$$

where $\widetilde{\Phi}(\lambda, x) = \Phi(\mathbf{v}(\lambda, x))$ and

$$\lambda_t = \int_t^1 \theta_s^2 ds = \widetilde{\mu} \varrho \sqrt{n} (1-t)^{\frac{1}{4\beta}}, \quad \widetilde{\mu} = \frac{2\sqrt{\mu}}{\mu+1}.$$

Remark 3. We will also see in Section 4.1 that the underhedging situation pointed out in [23] can be improved by choosing some suitable value of the parameter ϱ .

We will make use the following condition of the volatility.

(C₁) Assume that $\sigma(y)$ is a C^2 -function which is bounded from zero and

$$\mathbf{E} \sup_{0 \leq t \leq 1} \{|\sigma(y_t)|^2 + |\sigma'(y_t)| + |\sigma''(y_t)|\} < \infty.$$

Assumption (C₁) is not too restrictive and it indeed includes almost popular SV models in the existing literature, see Section 10 and [32].

3.1 Asymptotic results for Leland's strategy

Let us consider the option hedging problem for the model (3.1) in the case of constant proportional cost via Leland's strategy (2.7)

$$\gamma_t^n = \sum_{i=1}^n \widetilde{\Phi}(\lambda_{t_{i-1}}, S_{t_{i-1}}) \mathbf{1}_{(t_{i-1}, t_i]}(t).$$

This strategy yields a portfolio whose terminal value V_1^n is defined in (2.8) where rebalancing times (t_i) given by (2.17). Now by Itô's formula we obtain

$$h(S_1) = \widehat{C}(1, S_1) = \widehat{C}(0, S_0) + \int_0^1 \widehat{C}_x(t, S_t) dS_t - \frac{1}{2} I_{1,n}, \quad (3.5)$$

where

$$I_{1,n} = \int_0^1 (\theta_t^2 - \sigma^2(y_t)) S_t^2 \widehat{C}_{xx}(t, S_t) dt. \quad (3.6)$$

Setting $V_0 = \widehat{C}(0, S_0)$ we represent the hedging error as

$$V_1^n - h(S_1) = \frac{1}{2}I_{1,n} + I_{2,n} - \kappa_* J_n, \quad (3.7)$$

where

$$I_{2,n} = \int_0^1 \left(\gamma_t^n - \widehat{C}_x(t, S_t) \right) dS_t \quad (3.8)$$

and J_n is defined as in (2.9).

The goal is to find the limit of $V_1^n - h(S_1)$ and point out the convergence rate as $n \rightarrow \infty$. To this end, we investigate the limit of the terms in the right hand side of (3.7). Note that $I_{2,n}$ is known as the discrete hedging error that appears in discrete hedging. In our setting, $I_{2,n}$ converges to zero faster than n^β while the gamma hedging error $I_{1,n}$ approaches to the value $2 \min(S_1, K)$ at the rate n^β . As shown in Section 8, the total trading volume J_n converges in probability to the random variable $J(S_1, y_1, \varrho)$ defined by

$$J(x, y, \varrho) = x \int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x) \mathbf{E} \left| \frac{\sigma(y)}{\varrho} Z + q(\lambda, x) \right| d\lambda, \quad (3.9)$$

where $Z \sim \mathcal{N}(0, 1)$ independent of S_1 and y_1 .

In order to determine the asymptotic distribution we need to find the martingale remaining part of the terms in the right hand side of (3.7). The most challenging issue in our analysis is that the rest term of total transaction costs naturally takes a discrete form whereas the one obtained by studying $I_{1,n}$ (see Section 6) has a continuous form. In order to combine these two quantities into a unified form that permits one to apply the Central Limit Theorem for martingales, we use a special discretization procedure in Section 5.

We now state our first asymptotic result for Leland's strategy.

Theorem 3.1. *If condition (C_1) holds then for any $\varrho > 0$ the sequence*

$$n^\beta (V_1^n - h(S_1) - \min(S_1, K) + \kappa_* J(S_1, y_1, \varrho))$$

with β defined in (2.20) weakly converges to some centered mixed Gaussian variable as $n \rightarrow \infty$.

Remark 4. *This theorem is a generalization including an improved convergence rate of the results in [23, 31], where the uniform revision is taken and the volatility is assumed to be a constant.*

By letting $\varrho \rightarrow \infty$ we observe that

$$\lim_{\varrho \rightarrow \infty} J(x, y, \varrho) = \int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x) |q(\lambda, x)| d\lambda := J^*(x). \quad (3.10)$$

The convergence in Theorem 3.1 can be improved if ϱ is taken as a function of n . In this context, we will use the following condition.

(C₂) *The function σ is continuous such that*

$$\mathbf{E} \sup_{0 \leq t \leq 1} \sigma^2(y_t) < \infty.$$

Note that condition (C₂) is weaker than (C₁) with the differentiability condition of σ has been removed. Our next result is established under the following condition on ϱ .

(C₃) *The parameter $\varrho = \varrho(n)$ is a function of n such that*

$$\lim_{n \rightarrow \infty} \frac{\varrho}{n^{\mu_*}} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\sqrt{\ln n}}{\varrho} = 0,$$

where μ_* is given in (3.3).

Theorem 3.2. *Under conditions (C₂), (C₃), the sequence*

$$n^\beta \varrho^{2\beta} (V_1^n - h(S_1) - \min(S_1, K) + \kappa_* J^*(S_1))$$

weakly converges to a centered mixed Gaussian variable as $n \rightarrow \infty$.

Remark 5. *The asymptotic distributions in both Theorem 3.1 and Theorem 3.2 are explicitly defined in Section 9.*

Remark 6. *In case ϱ depends on n as in Theorem 3.2 we only need the integrability of $\sigma(y_t)$ for convergence results. In other words, it is possible to relax the C^2 -condition of the volatility function σ . This relaxation is needed for Heston's models, see Section 10.*

Remark 7. *Theorem 3.1 and Theorem 3.2 still hold if the enlarged volatility of the form (2.22)*

$$\theta_t^2 = \sigma^2(y_t) + \varrho \sigma(y_t) \sqrt{nf'(t)}, \quad (3.11)$$

is used but now the limit of transaction costs is given by

$$J'(x, y, \varrho) = x \int_0^{+\infty} \lambda^{-1/2} \tilde{\varphi}(\lambda, x) \mathbf{E} \left| \frac{Z}{\varrho} + q(\lambda, x) \right| d\lambda. \quad (3.12)$$

However, this is far away from practical significance, see Section 2.3.

3.2 Asymptotic result for Lépinette's strategy

Let us recall the modified strategy (2.19) proposed by Lépinette [12]

$$\bar{\gamma}_t^n = \sum_{i=1}^n \left(\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \int_0^{t_{i-1}} \widehat{C}_{xt}(u, S_u) du \right) \mathbf{1}_{(t_{i-1}, t_i]}(t).$$

Strategy $\bar{\gamma}_t^n$ produces a portfolio whose terminal values defined by

$$\bar{V}_1^n = \bar{V}_0^n + \int_0^1 \bar{\gamma}_t^n dS_t - \kappa_* \bar{J}_n,$$

where

$$\bar{J}_n = \sum_{i=1}^n S_{t_i} |\bar{\gamma}_{t_i}^n - \bar{\gamma}_{t_{i-1}}^n|. \quad (3.13)$$

Now by Itô's formula one presents the hedging error as

$$\bar{V}_1^n - h(S_1) = \frac{1}{2} I_{1,n} + \bar{I}_{2,n} - \kappa_* \bar{J}_n, \quad (3.14)$$

where

$$\bar{I}_{2,n} = I_{2,n} + \sum_{i \geq 1} (S_{t_i} - S_{t_{i-1}}) \int_0^{t_{i-1}} \widehat{C}_{xt}(u, S_u) du. \quad (3.15)$$

We obtain the following result using the form (3.2) for enlarged volatility.

Theorem 3.3. *Let \bar{V}_1^n be the terminal portfolio terminal value of strategy (3.2) and suppose that (\mathbf{C}_1) is fulfilled. Then, for any $\varrho > 0$, the sequence*

$$n^\beta (\bar{V}_1^n - h(S_1) - \eta \min(S_1, K))$$

with

$$\eta = 1 - \kappa_* \frac{\sigma(y_1)}{\varrho} \sqrt{8/\pi}$$

weakly converges to some centered mixed Gaussian variable as $n \rightarrow \infty$.

It is interesting to see that Theorem 2.6 can be recovered from Theorem 3.3 with $\varrho = \kappa_* \sigma \sqrt{8/\pi}$ in the context of constant volatility. Also see that in our model, the parameter μ takes its values in the interval $[1, 2)$, that is slightly more general than the condition imposed in Theorem 2.6.

Remark 8. *If (3.11) is used for Lépinette's strategy $\bar{\gamma}_t^n$ then the option can be completely replicated by taking $\varrho = \kappa_* \sqrt{8/\pi}$, even in the SV models. In other words, if $\theta_t^2 = \sigma^2(y_t) + \kappa_* \sigma(y_t) \sqrt{8/\pi} \sqrt{n f'(t)}$ is applied for Lépinette strategy then the sequence*

$$n^\beta (\bar{V}_1^n - h(S_1))$$

weakly converges to some centered mixed Gaussian variable as $n \rightarrow \infty$.

In the context of condition (\mathbf{C}_3) , the cumulative transaction costs $\kappa_* \bar{J}_n$ converges to 0. Hence, the option is over replicated.

Corollary 3.1. *If $\varrho \rightarrow \infty$ under condition (\mathbf{C}_3) and condition (\mathbf{C}_2) is satisfied then the portfolio \bar{V}_1^n converges to $h(S_1) + \min(S_1, K)$ in probability.*

Note that no convergence improved version of Theorem 3.3 is obtained as for Leland's strategy since $\kappa_* \bar{J}_n$ converges to 0 at order of ϱ .

Let us emphasize the essential fact that our enlarged volatility is far simpler than the one used in the previous works [23, 24, 31, 12, 9] but the same asymptotic results are obtained not only for constant volatility models but for SV contexts as well. This important fact proves our significant contribution in discrete hedging literature in the presence of transaction costs. Additionally, the setting of our SV model 3.1 is enough general for practice purposes since it includes famous SV models, see Section 10.

4 Applications for pricing problems

In this section we focus on how to apply our main results to option pricing with transaction costs. We first emphasize that it is impossible to obtain a *non-trivial* perfect hedge with the presence of transaction costs even in constant volatility models. In other words, to cover completely the option return, the seller can take the *buy-and-hold* strategy, but this makes the option price too expensive. However, once the investor accepts to take a risk in his hedging problem, the option price can be lowered in a way so that the payoff will be covered with a given probability using Leland's strategy.

4.1 Superhedging with transaction costs

To stand on the safe side, the investor will search for strategies providing the terminal value that exceeds the payoff. Such strategy usually concern solutions to dynamic optimization problems. More precisely, let H be a general contingent claim and denote by $\mathcal{A}(x)$ the set of all *admissible strategies* π with the initial capital x and $V_T^{\pi, x}$ the terminal value of strategy π . Then, the super-replication cost of H is determined as

$$U_0 = \inf \left\{ x \in \mathbb{R} : \exists \pi \in \mathcal{A}(x) : V_T^{\pi, x} \geq H \quad \text{a.s.} \right\}, \quad (4.1)$$

see [26] and the references therein.

In the presence of transaction costs, Cvitanic [7] shows that the *buy-and-hold strategy* is the unique choice if one wishes to successfully replicate the option and then S_0 is the super-replication price. In this section, we will show that this property still holds in the sense of approximate superhedging via Leland's spirit. The following

observation is just a direct consequence of Theorem 3.2 when ϱ is used as a function of n .

Proposition 4.1. *Under conditions (\mathbf{C}_2) and (\mathbf{C}_3) , $\mathbf{P} - \lim_{n \rightarrow \infty} V_1^n \geq h(S_1)$. The same property holds for Lépinette's strategy*

Proof. Note first that $J^*(x) \leq \min(x, K)$, for all $x > 0$. Hence, by Theorem 3.2

$$\mathbf{P} - \lim_{n \rightarrow \infty} (V_1^n - h(S_1)) \geq (1 - \kappa_*) \min(S_1, K).$$

The term in the left hand side is obviously non negative since $\kappa_* < 1$ hence the conclusion follows. The conclusion for Lépinette strategy directly follows from Theorem 3.3. \square

Remark 9. *This is not a big surprise because the option is now sold at high price. In fact, from the Black-Scholes formula, one observes that*

$$C(0, S_0, \hat{\sigma}) \rightarrow S_0 \quad \text{as} \quad \hat{\sigma} \rightarrow \infty.$$

It leads to the well-known "buy and hold" strategy in the superhedging problem [26] i.e. to cover the option the seller just takes the trivial strategy: buy a stock share at time $t = 0$ at price S_0 and keep it until the expiry.

4.2 Asymptotic quantile pricing

As seen in the previous section, the superhedging cost is too high from the buyer's point of view though it indeed gives the seller a successful hedge with probability 1. More practically, one can ask that how much initial capital can be reduced by accepting a shortfall probability. For this aim, the seller may take a risk and look for hedges with the minimal initial cost defined by

$$\inf \{x \in \mathbb{R} : \exists \pi \in A(x) : P(V_T^{\pi, x} \geq H) \geq 1 - \varepsilon\},$$

with a given significant level $0 \leq \varepsilon \leq 1$. This is the key problem of quantile hedging theory which was first studied by Föllmer and Leukert [14] then developed by Novikov [30]. Baran [4] considered the situation where transaction costs are taken into account but in a different approach. Recently, Bratyk and Mishura [6] have studied quantile hedging inclusive dependence structure driven by fractional Brownian motions. See also Barski [5], where a lot of specific examples are shown.

Let us return to the quantile hedging problem in the presence of transaction costs. Thanks to Theorem 3.1, we will suggest a reasonable way adapted from [31] to lower the option price for the Leland strategy to obtain a hedge with a given probability $1 - \varepsilon$.

More precisely, we propose to sell the option at the price δS_0 , (where $0 < \delta < 1$ will be properly chosen) and the remaining part $(1 - \delta)S_0$ will be paid by the

option seller himself. He must include this fee in the hedging problem; that is, the asymptotic terminal portfolio value must exceed the sum of the payoff $h(S_1)$ and the rest fee $(1 - \delta)S_0$. By Proposition 4.1 this purpose can be achieved with a positive probability. To determine the option price it now remains to choose value δ . We suggest to define it as

$$\delta_\varepsilon = \inf \{a > 0 : \Upsilon(a) \geq 1 - \varepsilon\}, \quad (4.2)$$

where $\Upsilon(a) = \mathbf{P}((1 - \kappa_*) \min(S_1, K) > (1 - a)S_0)$.

The price of Leland strategy can be substantially lowered compared with the power function of parameter ε with an arbitrary small power.

Proposition 4.2. *Let δ_ε be Leland price defined by (4.2). Assume that*

$$\sigma_{max} = \sup_{y \in \mathbb{R}} \sigma(y) < \infty,$$

then for any $r > 0$,

$$\lim_{\varepsilon \rightarrow 0} \frac{1 - \delta_\varepsilon}{\varepsilon^r} = +\infty. \quad (4.3)$$

Proof. Observe that $0 < \delta_\varepsilon \leq 1$ and δ_ε tends to 1 as $\varepsilon \rightarrow 0$. Setting $b = 1 - \kappa_*$, then for sufficiently small ε such that $\delta_\varepsilon > a > 1 - bK/S_0$ one has

$$\begin{aligned} 1 - \varepsilon &> \mathbf{P}(b \min(S_1, K) > (1 - a)S_0) \\ &= \mathbf{P}(\min(S_1/S_0, K/S_0) > (1 - a)/b) \\ &= 1 - \mathbf{P}(S_1/S_0 \leq (1 - a)/b). \end{aligned}$$

Therefore,

$$\varepsilon < \mathbf{P}(S_1/S_0 \leq (1 - a)/b) \leq \mathbf{P}(X_1 \leq -z_a),$$

where $X_t = \int_0^t \sigma(y_t) dW_t^{(1)}$ and $z_a = \ln(b/(1 - a)) - \sigma_{max}^2/2$. To estimate this probability we note that for any integer $m \geq 1$,

$$\mathbf{E}(X_1)^{2m} \leq \sigma_{max}^{2m} (2m - 1)!!$$

(see, for example, [28, Lemma 4.11, page 130]). Setting now $R(v) = 2v\sigma_{max}^2$, we obtain that for any $0 < v < 1/2\sigma_{max}^2$,

$$\mathbf{E} e^{vX_1^2} = \sum_{m=0}^{\infty} \frac{v^m}{m!} \mathbf{E}(X_1)^{2m} \leq \sum_{m=0}^{\infty} \frac{v^m}{m!} \sigma_{max}^{2m} (2m - 1)!! \leq \frac{1}{1 - R(v)}.$$

Therefore, for $\varepsilon > 0$ sufficiently small one has

$$\varepsilon \leq \mathbf{P}(X_1 \leq -z_a) = \mathbf{P}(-X_1 \geq z_a) \leq e^{-vz_a^2} \mathbf{E} e^{vX_1^2} \leq \frac{e^{-vz_a^2}}{1 - R(v)},$$

i.e.

$$1 - a \geq b \exp \left\{ -\sqrt{\frac{1}{v} |\ln \varepsilon(1 - R(v))| - \sigma_{max}^2/2} \right\}.$$

Letting $a \rightarrow \delta_\varepsilon$ one obtains

$$1 - \delta_\varepsilon \geq b \exp \left\{ -\sqrt{\frac{1}{v} |\ln \varepsilon(1 - R(v))| - \sigma_{max}^2/2} \right\}.$$

This inequality implies immediately the asymptotic bound (4.3). \square

The boundedness of volatility function is essential for the above comparison proposition. If one wishes to relax this assumption, the price reduction is now less free than in Proposition 4.2.

Proposition 4.3. *Suppose that there exists a positive constant $\alpha > 1/2$ such that $\mathbf{E} \exp\{\alpha \int_0^1 \sigma^2(y_s) ds\} < \infty$. Then*

$$\liminf_{\varepsilon \rightarrow 0} \frac{1 - \delta_\varepsilon}{\varepsilon^{r_\alpha}} > 0, \quad r_\alpha = (2\sqrt{2\alpha} + 1)/2\alpha. \quad (4.4)$$

Proof. For any positive constant L , let us introduce the stopping time τ_L defined by

$$\tau_L = \inf \left\{ t > 0 : \int_0^1 \sigma^2(y_s) ds \geq L \right\} \wedge 1, \quad (4.5)$$

which is understood as the first time the log-price's variance passes the level L .

By a similar manner as in the previous demonstration we obtain that

$$\begin{aligned} \varepsilon \leq & \mathbf{P} \left(\exp \left\{ -\int_0^1 \sigma(y_s) dW_s^{(1)} + \frac{1}{2} \int_0^1 \sigma^2(y_s) ds \right\} \geq u_a, \int_0^1 \sigma^2(y_s) ds \leq L \right) \\ & + \mathbf{P} \left(\int_0^1 \sigma^2(y_s) ds \geq L \right), \end{aligned} \quad (4.6)$$

where

$$u_a = \frac{1 - \kappa_*}{1 - a} \quad \text{and} \quad \delta_\varepsilon > a > 1 - bK/S_0.$$

For $p^2 \leq 2\alpha$, the stopped process

$$\chi_t = \exp \left\{ -p \int_0^{\tau_L \wedge t} \sigma(y_s) dW_s^{(1)} - \frac{p^2}{2} \int_0^{\tau_L \wedge t} \sigma^2(y_s) ds \right\}$$

is a martingale by Novikov's condition and hence $\mathbf{E}\chi_t = 1$. Now, by Chebysev's inequality, the probability in the right side of (4.6) can be bounded by

$$(u_a)^{-p} \mathbf{E} \exp \left\{ -p \int_0^{\tau_L \wedge t} \sigma(y_s) dW_s^{(1)} - \frac{p}{2} \int_0^{\tau_L \wedge t} \sigma^2(y_s) ds \right\} \leq (u_a)^{-p} e^{\frac{p^2+p}{2}L}.$$

By hypothesis and Chebysev's inequality and one obtains

$$\mathbf{P} \left(\int_0^1 \sigma^2(y_s) ds \geq L \right) \leq C e^{-\alpha L}$$

for some constant $C > 0$. Hence,

$$\varepsilon \leq (u_a)^{-p} e^{\frac{p^2+p}{2}L} + C e^{-\alpha L}.$$

Choosing

$$L = \frac{1}{\alpha} \ln(2C/\varepsilon) \Leftrightarrow C e^{-\alpha L} = \varepsilon/2,$$

one deduces that

$$1 - a \geq \tilde{C} \varepsilon^{\frac{p+1}{2\alpha} + \frac{1}{p}},$$

where \tilde{C} is some positive constant independent of ε and a . Thus,

$$1 - \delta_\varepsilon \geq \tilde{C} \varepsilon^{\frac{p+1}{2\alpha} + \frac{1}{p}},$$

The quantity $\frac{p+1}{2\alpha} + \frac{1}{p}$ attains the minimal value $r_\alpha = (2\sqrt{2\alpha} + 1)/2\alpha$ as $p = \sqrt{2\alpha}$ with $\alpha > 1/2$ and the proof is completed. \square

Remark 10. *It is clear that $r_\alpha < 1$ if $\alpha > 3/2 + \sqrt{2}$. Condition **E** $\exp\{\alpha \int_0^1 \sigma^2(y_s) ds\} < \infty$ for such values of α is fulfilled when σ is linear bounded and y_t follows an Orstein-Uhlenbeck process, see Lemma D.1.*

5 Basic asymptotic tools

In this section we construct a special procedure in order to combine the gamma error $I_{1,n}$ with the transaction costs naturally being a discrete sum. Firstly, for n -functions

$$l_* = 1/\ln^3 n \quad \text{and} \quad l^* = \ln^3 n$$

we set

$$M_1 = \left[n \left(\frac{l_*}{\lambda_0} \right)^{2/(\mu+1)} \right] \quad \text{and} \quad M_2 = \left[n \left(\frac{l^*}{\lambda_0} \right)^{2/(\mu+1)} \right] - 1, \quad (5.1)$$

where $\lambda_0 = \tilde{\mu}\varrho\sqrt{n}$ ($[x]$ is the integer part of a number x).

Our approximation needs the following two sequences (u_m) and (λ_m) defined as

$$u_m = 1 - \left(\frac{m}{n} \right)^\mu, \quad \lambda_m = \int_{u_m}^1 \theta_s^2 ds = \lambda_0 \left(\frac{m}{n} \right)^{\frac{\mu+1}{2}}, \quad M_1 \leq m \leq M_2. \quad (5.2)$$

Note that (u_m) is a decreasing sequence with values in $[u^*, u_*]$, where $u_* = 1 - (l_*/\lambda_0)^{4\beta}$ and $u^* = 1 - (l^*/\lambda_0)^{4\beta}$.

The discretization steps require their “invert” sequences (t_j) and $(\tilde{\lambda}_j)$ defined as

$$t_j = 1 - (1 - j/n)^\mu \quad \text{and} \quad \tilde{\lambda}_j = \lambda_0(1 - t_j)^{\frac{1}{4\beta}}, \quad N_2 \leq j \leq N_1 \quad (5.3)$$

where $N_1 = n - M_1$ and $N_2 = n - M_2$.

In the sequel, Itô integrals will be discretized through the following sequences of independent normal random variables

$$\tilde{Z}_{1,j} = \frac{W_{t_j}^{(1)} - W_{t_{j-1}}^{(1)}}{\sqrt{t_j - t_{j-1}}}, \quad \tilde{Z}_{2,j} = \frac{W_{t_j}^{(2)} - W_{t_{j-1}}^{(2)}}{\sqrt{t_j - t_{j-1}}}$$

and

$$Z_{1,m} = \frac{W_{u_m}^{(1)} - W_{u_{m+1}}^{(1)}}{\sqrt{u_m - u_{m+1}}}, \quad Z_{2,m} = \frac{W_{u_m}^{(2)} - W_{u_{m+1}}^{(2)}}{\sqrt{u_m - u_{m+1}}}.$$

With the sequence of rebalancing times (t_j) in hand we set

$$\tilde{Z}_{3,j} = |\tilde{Z}_{1,j} + \tilde{p}_{j-1}| - \mathbf{E} \left(|\tilde{Z}_{1,j} + \tilde{p}_{j-1}| \mid \mathcal{F}_{j-1} \right), \quad (5.4)$$

$$\tilde{Z}_{4,j} = |\tilde{Z}_{1,j}| - \mathbf{E} \left(|\tilde{Z}_{1,j}| \mid \mathcal{F}_{j-1} \right) = |\tilde{Z}_{1,j}| - \sqrt{2/\pi}, \quad (5.5)$$

where $\tilde{p}_{j-1} = p(\tilde{\lambda}_{j-1}, S_{t_{j-1}}, y_{t_{j-1}})$ and

$$p(\lambda, x, y) = \frac{\varrho}{\sigma(y)} \left(\frac{\ln(x/K)}{2\lambda} - \frac{1}{4} \right). \quad (5.6)$$

For any $q \in \mathbb{R}$ and $Z \sim \mathcal{N}(0, 1)$ let

$$G(q) = \mathbf{E} (|Z + q|) = 2\varphi(q) + q(2\Phi(q) - 1). \quad (5.7)$$

We also write $o_{\mathbf{P}}(n^{-r})$ for any sequence of random variables $(X_n)_{n \geq 1}$ satisfying

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^r X_n = 0.$$

(H₁) f is a $\mathbb{R}_+ \rightarrow \mathbb{R}$ continuously differentiable function having absolutely integrable derivative on the $[0, +\infty)$ and

$$\lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \left(\int_0^{l_*} |f(x)| dx + \int_{l_*}^{+\infty} |f(x)| dx \right) = 0.$$

Lemma 5.1. *Let $A(\cdot)$ be a function satisfying condition (\mathbf{H}_1) . If ϱ either is constant or satisfies condition (\mathbf{C}_3) , then*

$$\lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \left| \sum_{j=N_2}^{N_1} A(\tilde{\lambda}_{j-1}) \Delta \tilde{\lambda}_{j-1} - \int_0^{+\infty} A(\lambda) d\lambda \right| = 0.$$

Proof. It suffices to show that

$$\lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \left| \sum_{j=N_2}^{N_1} A(\tilde{\lambda}_{j-1}) \Delta \tilde{\lambda}_{j-1} - \int_{l_*}^{l^*} A(\lambda) d\lambda \right| = 0.$$

Indeed, this term can be rewritten as $\sum_{j=N_2}^{N_1} \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} A_1(z) dz$, with

$$A_1(z) = A(\tilde{\lambda}_{j-1}) - A(z) = \int_z^{\tilde{\lambda}_{j-1}} \dot{A}(u) du.$$

We then deduce that

$$\sum_{j=N_2}^{N_1} \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} A_1(z) dz \leq \sum_{j=N_2}^{N_1} \Delta \tilde{\lambda}_j \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} |\dot{A}(u)| du \leq \max_{N_2 \leq j \leq N_1} |\Delta \tilde{\lambda}_j| \cdot \int_0^{+\infty} |\dot{A}(u)| du.$$

On the other hand,

$$\max_{N_2 \leq j \leq N_1} |\Delta \tilde{\lambda}_j| \leq C n^{-2\beta} \varrho^{\frac{2}{\mu+1}} (l^*)^{\frac{\mu-1}{\mu+1}}. \quad (5.8)$$

So,

$$n^\beta \varrho^{2\beta} \sum_{j=N_2}^{N_1} \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} A_1(z) dz \leq C n^{-\frac{\mu}{2(\mu+1)}} \varrho^{\frac{\mu+2}{\mu+1}} (l^*)^{\frac{\mu-1}{\mu+1}} \int_0^{+\infty} |\dot{A}(u)| du.$$

The left hand side clearly converges to zero by condition (\mathbf{C}_3) and the lemma is proved. \square

By the same manner, we obtain the following approximation.

Lemma 5.2. *Let $A(\cdot)$ be a function satisfying condition (\mathbf{H}_1) and assume that ϱ either is constant or satisfies condition (\mathbf{C}_3) . Then*

$$\lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \sup_{a>0} \left| \sum_{j=N_2}^{N_1} \mathbf{1}_{\{\tilde{\lambda}_{j-1} \geq a\}} A(\tilde{\lambda}_{j-1}) \Delta \tilde{\lambda}_j - \int_a^\infty A(\lambda) d\lambda \right| = 0$$

Proof. It suffices to consider the case when a takes values in the interval $[l_*, l^*]$. Then there exists $N_2 \leq k \leq N_1$ such that $a \in [\tilde{\lambda}_k, \tilde{\lambda}_{k-1}]$. After a direct computation we can write the difference in the absolute sign as

$$\sum_{j=N_2}^{N_1} \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} \left(A(\tilde{\lambda}_{j-1}) - A(\lambda) \right) d\lambda - \int_a^{\tilde{\lambda}_{k-1}} A(\lambda) d\lambda.$$

Then, the limit follows in view of estimate (5.8). \square

In the sequel we will use the following limit theorem for martingales.

Theorem 5.1. *Let $\mathbf{M}_n = \sum_{i=1}^n X_i$ be a zero-mean, square integrable martingale and ς be an a.s. finite random variable. Assume that the following convergences are satisfied in probability: for any $\delta > 0$,*

$$\sum_{i=1}^n \mathbf{E} \left(X_i^2 \mathbf{1}_{\{|X_i| > \delta\}} | \mathcal{F}_{i-1} \right) \longrightarrow 0 \quad \text{and} \quad \sum_{i=1}^n \mathbf{E} \left(X_i^2 | \mathcal{F}_{i-1} \right) \longrightarrow \varsigma^2.$$

Then, the sequence (\mathbf{M}_n) converges in law to Y whose characteristic function is $\mathbf{E} \exp(-\frac{1}{2}\varsigma^2 t^2)$ i.e. Y is a mixed Gaussian variable.

Proof. See Theorem 3.2 and Corollary 3.1, p.58 in [20]. \square

The following technical condition will be imposed to estimate moment of a $\mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ function \mathbf{A} holding condition (\mathbf{H}_1) .

(\mathbf{H}_2) *For any $\gamma > 0$, there exist functions $U_\gamma(\cdot) > 0$ such that*

$$\sup_{x>0} \left(\sup_{\lambda>0} \frac{|\mathbf{A}(\lambda, x)|}{\min(\lambda^\gamma, \lambda^{-\gamma})} - U_\gamma(x) \right) \leq 0$$

and for any $m \geq 1$,

$$\mathbf{E} \sup_{0 \leq t \leq 1} (U_\gamma(S_t))^m < \infty.$$

We are looking for the limit of the following sequence (\mathbf{M}_k) defined as

$$\mathbf{M}_k = \sum_{j=N_2}^k v_j, \quad N_2 \leq k \leq N_1 \tag{5.9}$$

where $v_j = \sum_{i=1}^3 \tilde{\mathbf{A}}_{i,j-1} \tilde{Z}_{i,j} \Delta \tilde{\lambda}_j$ and $\tilde{\mathbf{A}}_{i,j} = \mathbf{A}_i(\tilde{\lambda}_j, S_{t_j})$. Set

$$\begin{aligned} \mathbf{L}(\lambda, x, y) &= \mathbf{A}_1^2(\lambda, x) + 2\mathbf{A}_1(\lambda, x)\mathbf{A}_3(\lambda, x)(2\Phi(\tilde{p}) - 1) \\ &\quad + \mathbf{A}_3^2(\lambda, x)\Lambda(\tilde{p}) + \mathbf{A}_2^2(\lambda, x), \end{aligned} \tag{5.10}$$

where $\Lambda(q) = 1 + q^2 - G^2(q)$ and the function \tilde{p} is defined in (5.6).

Proposition 5.1. *Assume that for $1 \leq i \leq 3$ the functions $\mathbf{A}_i(\cdot, \cdot)$ satisfy conditions (\mathbf{H}_1) and (\mathbf{H}_2) . Then, for any fixed $\varrho > 0$ the sequence $(n^\beta \mathbf{M}_{N_1})_{n \geq 1}$ weakly converges to a mixed Gaussian variable with mean zero and variance ς^2 defined as*

$$\varsigma^2 = \check{\mu} \varrho^{\frac{2}{\mu+1}} \int_0^{+\infty} \lambda^{\frac{\mu-1}{\mu+1}} \mathbf{L}(\lambda, S_1, y_1) d\lambda \quad \text{with} \quad \check{\mu} = \frac{1}{2}(\mu+1) \tilde{\mu}^{\frac{2}{\mu+1}}.$$

Proof. Putting $v_{i,j} = \tilde{\mathbf{A}}_{i,j-1} \tilde{Z}_{i,j} \Delta \tilde{\lambda}_j$ we can rewrite \mathbf{M}_k as follows

$$\mathbf{M}_k = \sum_{j=N_2}^k v_j = \sum_{i=1}^3 \sum_{j=N_2}^k v_{i,j}.$$

Using Cauchy-Schwartz's inequality then Markov's inequality, we get for any $\mathbf{a} > 0$

$$\mathbf{E} v_{1,j}^2 \mathbf{1}_{\{|v_{1,j}| > \mathbf{a}\}} \leq \sqrt{\mathbf{E} v_{1,j}^4} \sqrt{\mathbf{P}(|v_{1,j}| > \mathbf{a})} \leq \frac{1}{\mathbf{a}^2} \mathbf{E} v_{1,j}^4.$$

On the other hand, from condition (\mathbf{H}_2) there exists $r > 0$ such that

$$\mathbf{E} v_{1,j}^4 \leq \mathbf{E} \tilde{\mathbf{A}}_{1,j-1}^4 \tilde{Z}_{1,j}^2 (\Delta \tilde{\lambda}_j)^4 \leq C(\tilde{\lambda}_{j-1}^r + \tilde{\lambda}_{j-1}^{-r}) (\Delta \tilde{\lambda}_j)^4.$$

Thanks to (5.8)

$$n^{2\beta} \sum_{j=N_2}^{N_1} \mathbf{E} \left(v_{1,j}^2 \mathbf{1}_{\{|v_{1,j}| > \mathbf{a}\}} | \mathcal{F}_{j-1} \right) \longrightarrow 0 \quad \text{in } L^1.$$

Remark that the random variable $\tilde{Z}_{3,j}$ admits moments bounded by a constant independent of n . Hence, there exists $r > 0$ such that

$$\mathbf{E}(v_{2,j}^4) \leq C(\Delta \tilde{\lambda}_{j-1})^4 \left(\mathbf{E} \tilde{\mathbf{A}}_{2,j-1}^8 \mathbf{E} \left(\tilde{Z}_{3,j} \right)^8 \right)^{1/2} \leq C \left(\tilde{\lambda}_{j-1}^r + \tilde{\lambda}_{j-1}^{-r} \right) (\Delta \tilde{\lambda}_j)^4,$$

which implies the convergence to zero in L^1 of the sum

$$n^{2\beta} \sum_{j=N_2}^{N_1} \mathbf{E} \left(v_{2,j}^2 \mathbf{1}_{\{|v_{2,j}| > \mathbf{a}\}} | \mathcal{F}_{j-1} \right).$$

Similarly,

$$n^{2\beta} \sum_{j=N_2}^{N_1} \mathbf{E} \left(v_{3,j}^2 \mathbf{1}_{\{|v_{3,j}| > \mathbf{a}\}} | \mathcal{F}_{j-1} \right) \longrightarrow 0 \quad \text{in } L^1.$$

Thus, for any $\mathbf{a} > 0$

$$n^{2\beta} \sum_{j=N_2}^{N_1} \mathbf{E} \left(v_j^2 \mathbf{1}_{\{|v_j| > \mathbf{a}\}} | \mathcal{F}_{j-1} \right) \longrightarrow 0 \quad \text{in } L^1. \quad (5.11)$$

Let us establish the limit of the sum of conditional variances $\mathbf{E}(v_j^2|\mathcal{F}_{j-1})$. Since $\tilde{Z}_{1,j}$ and $\tilde{Z}_{2,j}$ are independent,

$$\mathbf{E}(v_{1,j}v_{3,j}|\mathcal{F}_{j-1}) = \mathbf{E}(v_{2,j}v_{3,j}|\mathcal{F}_{j-1}) = 0.$$

It follows that

$$\begin{aligned} \mathbf{E}(v_j^2|\mathcal{F}_{j-1}) &= \mathbf{E}(v_{1,j}^2|\mathcal{F}_{j-1}) + \mathbf{E}(v_{2,j}^2|\mathcal{F}_{j-1}) + \mathbf{E}(v_{3,j}^2|\mathcal{F}_{j-1}) \\ &\quad + 2\mathbf{E}(v_{1,j}v_{2,j}|\mathcal{F}_{j-1}). \end{aligned}$$

Observe that for $Z \sim N(0, 1)$ and some constant q ,

$$\mathbf{E}(Z|Z+q) = 2\Phi(q) - 1 \quad \text{and} \quad \mathbf{E}(Z+q)^2 - (\mathbf{E}|Z+q|)^2 = \Lambda(q).$$

So, setting $\hat{\mu} = (\mu - 1)/(\mu + 1)$ one can represent $\mathbf{E}(v_j^2|\mathcal{F}_{j-1})$ as

$$n^{2\beta}\mathbf{E}(v_j^2|\mathcal{F}_{j-1}) = (1 + o(1))\check{\mu} \varrho^{\frac{2}{\mu+1}} (\tilde{\lambda}_{j-1})^{\hat{\mu}} \mathbf{L}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}, y_{t_{j-1}}) \Delta \tilde{\lambda}_j.$$

Due to Lemma 5.1, the sum $n^{2\beta} \sum_{j=N_2}^{N_1} \mathbf{E}(v_j^2|\mathcal{F}_{j-1})$ converges almost surely to ζ^2 as $n \rightarrow \infty$ and through Theorem 5.1 we obtain the needed conclusion. \square

Assume for the moment that $\varrho = \varrho(n) \rightarrow \infty$ as $n \rightarrow \infty$. The following result characterizes limit distribution of sequences $(\check{\mathbf{M}}_k)$ defined by

$$\check{\mathbf{M}}_k = \sum_{j=N_2}^k \check{v}_j, \quad \check{v}_j = \left(\tilde{\mathbf{A}}_{1,j-1} \tilde{Z}_{1,j} + \tilde{\mathbf{A}}_{3,j-1} \tilde{Z}_{3,j} \right) \Delta \tilde{\lambda}_j, \quad (5.12)$$

with \mathbf{A}_1 and \mathbf{A}_3 be functions satisfying conditions (\mathbf{H}_1) and (\mathbf{H}_2) .

Proposition 5.2. *Let \mathbf{A}_1 and \mathbf{A}_3 be functions satisfying conditions (\mathbf{H}_1) and (\mathbf{H}_2) . Then, under condition (\mathbf{C}_3) the sequence of random variables*

$$\left(n^\beta \varrho^{\frac{-1}{\mu+1}} \check{\mathbf{M}}_{N_1} \right)_{n \geq 1}$$

weakly converges as $n \rightarrow \infty$ to a mixed Gaussian variable with mean zero and variance defined as

$$\zeta^2 = \check{\mu} \int_0^{+\infty} \lambda^{\frac{\mu-1}{\mu+1}} \check{\mathbf{L}}(\lambda, S_1) d\lambda,$$

where $\check{\mathbf{L}}(\lambda, x) = \mathbf{A}_1^2(\lambda, x) + 2\mathbf{A}_1(\lambda, x)\mathbf{A}_3(\lambda, x) + \mathbf{A}_3^2(\lambda, x)$.

Proof. By the same method used in Lemma 5.1 we have for any $\mathbf{a} > 0$,

$$\lim_{n \rightarrow \infty} \sum_{j=N_2}^{N_1} \mathbf{E} \left(\check{v}_j^2 \mathbf{1}_{\{|\check{v}_j| > \mathbf{a}\}} | \mathcal{F}_{j-1} \right) = 0 \quad \text{in } L^1.$$

To determine limit of the sum of conditional variances, we observe that

$$n^{2\beta} \varrho^{\frac{-2}{\mu+1}} \mathbf{E}(\check{v}_j^2 | \mathcal{F}_{j-1}) = \check{\mu}(1 + o(1)) \tilde{\lambda}_{j-1}^{\hat{\mu}} \check{\mathbf{K}}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) \Delta \tilde{\lambda}_j, \quad (5.13)$$

with $\check{\mathbf{K}}(\lambda, x) = \mathbf{A}_1^2(\lambda, x) + \mathbf{A}_3^2(\lambda, x) \Lambda(\tilde{p}) + 2\mathbf{A}_1(\lambda, x) \mathbf{A}_3(\lambda, x) (2\Phi(|\tilde{p}|) - 1)$.

One can check directly that for any $q \in \mathbb{R}$ the function $G(\cdot)$ defined in (5.7) satisfies the following inequalities:

$$|q| \leq G(q) \leq |q| + 2\varphi(q). \quad (5.14)$$

This implies that $|\Lambda(q) - 1| \leq 4|q|\varphi(q) + \varphi^2(q)$ and hence, $\sup_{q \in \mathbb{R}} |\Lambda(q)| < \infty$.

Note also that $\check{\mathbf{K}} \rightarrow \check{\mathbf{L}}$ as $n \rightarrow \infty$ since $\tilde{p}(\lambda, x) \rightarrow \infty$ as $\varrho = \varrho(n) \rightarrow \infty$ for any $x > 0$ and $\lambda \neq \mathbf{a}_0 = 2 \ln(x/K)$. Using now Lemma 5.1 and the dominated convergence theorem we obtain that the sum in the right hand side of (5.13) converges to ζ^2 as $n \rightarrow \infty$ and the proof is completed through Theorem 5.1. \square

The following limit result is similar to Proposition 5.1.

Proposition 5.3. *Under conditions in Proposition 5.1, Then, for any fixed $\varrho > 0$ the sequence $(n^\beta \bar{\mathbf{M}}_{N_1})_{n \geq 1}$ defined by*

$$\bar{\mathbf{M}}_k = \sum_{j=N_2}^k \left(\tilde{\mathbf{A}}_{1,j-1} \tilde{\mathbf{Z}}_{1,j} + \tilde{\mathbf{A}}_{2,j-1} \tilde{\mathbf{Z}}_{2,j} + \tilde{\mathbf{A}}_{4,j-1} \tilde{\mathbf{Z}}_{4,j} \right) \Delta \tilde{\lambda}_j$$

weakly converges to a mixed Gaussian variable with mean zero and variance ζ^2 given by

$$\zeta^2 = \check{\mu} \varrho^{\frac{2}{\mu+1}} \int_0^{+\infty} \lambda^{\frac{\mu-1}{\mu+1}} \bar{\mathbf{L}}(\lambda, S_1, y_1) d\lambda,$$

with

$$\bar{\mathbf{L}}(\lambda, x, y) = \mathbf{A}_1^2(\lambda, x) + \mathbf{A}_2^2(\lambda, x) + (1 - 2/\pi) \mathbf{A}_4^2(\lambda, x).$$

Proof. Taking into account that $\mathbf{E} \tilde{\mathbf{Z}}_{4,j}^2 = 1 - 2/\pi$, we can conclude using arguments in Proposition 5.1. \square

6 Asymptotic properties of $I_{1,n}$

Set

$$\tau_n = \int_0^1 \lambda_t^{-1/2} \theta_t^2 (S_t \tilde{\varphi}(\lambda_t, S_t) - S_1 \tilde{\varphi}(\lambda_t, S_1)) dt. \quad (6.1)$$

Lemma 6.1. *If ϱ either is constant or satisfies condition (\mathbf{C}_3) then,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} |I_{1,n} - 2 \min(S_1, K) - \tau_n| = 0.$$

Proof of Lemma 6.1 is given in Appendix B.1.

Let us find a smart approximation of τ_n . By Itô's formula we obtain

$$\begin{aligned} S_1 \tilde{\varphi}(\lambda_t, S_1) - S_t \tilde{\varphi}(\lambda_t, S_t) &= \int_t^1 H_1(\lambda_t, S_u) \sigma(y_u) S_u dW_u^{(1)} \\ &+ \frac{1}{2} \int_t^1 H_2(\lambda_t, S_u) \sigma^2(y_u) S_u du, \end{aligned} \quad (6.2)$$

where $H_1(v, x) = \tilde{\varphi}(v, x) (1 - \mathbf{v}(v, x)/\sqrt{v})$, and

$$H_2(v, x) = \tilde{\varphi}(v, x) \left(\frac{\mathbf{v}^2(v, x)}{v} - \frac{\mathbf{v}(v, x)}{\sqrt{v}} - \frac{1}{v} \right).$$

Setting $\bar{H}_1(t, x) = \int_t^{\lambda_0} v^{-1/2} H_1(v, x) dv$ and $\bar{H}_2(t, x) = \int_t^{\lambda_0} v^{-1/2} H_2(v, x) dv$, we can represent τ_n as

$$\begin{aligned} \tau_n &= - \int_0^1 \theta_t^2 \sigma(y_u) S_u \bar{H}_1(\lambda_u, S_u) dW_u^{(1)} - \frac{1}{2} \int_0^1 \theta_t^2 \sigma^2(y_u) S_u \bar{H}_2(\lambda_u, S_u) du \\ &:= -\tau_{1,n} - \frac{1}{2} \tau_{2,n}. \end{aligned} \quad (6.3)$$

Again, the term $\tau_{2,n}$ is negligible. In order to be adapted with the approximation of the total trading volume obtained in the next section, we will construct a discrete approximation for $\tau_{1,n}$. This objective can be done by discretizing the Itô integral at trading times (t_j) .

Lemma 6.2. *If ϱ either is constant or satisfies condition (\mathbf{C}_3) then,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} |\tau_{1,n} - \mathbf{U}_{N_1}| = \mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} |\tau_{2,n}| = 0, \quad (6.4)$$

where

$$\mathbf{U}_k = \varrho^{-1} \sum_{j=N_2}^k \sigma(y_{t_{j-1}}) S_{t_{j-1}} \check{H}_1(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) \tilde{Z}_{1,j} \Delta \tilde{\lambda}_j$$

and $\check{H}_1(t, x) = \int_t^\infty v^{-1/2} H_1(v, x) dv$.

The proof can be found in Appendix B.2.

Now combining Lemma 6.1 and Lemma 6.2 gives us the final asymptotic form of $I_{1,n}$.

Proposition 6.1. *If ϱ either is constant or satisfies condition (\mathbf{C}_3) then,*

$$I_{1,n} = 2 \min(S_1, K) - \mathbf{U}_{N_1} + o_{\mathbf{P}}(n^{-\beta} \varrho^{-2\beta}). \quad (6.5)$$

7 Approximation of $I_{2,n}$ and $\bar{I}_{2,n}$

First we state that the term $I_{2,n}$ is $n^\beta \varrho^{2\beta}$ - negligible.

Proposition 7.1. *If ϱ either is constant or satisfies condition (\mathbf{C}_3) , then*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} I_{2,n} = 0.$$

For a proof, see Appendix B.3.

Recall from (3.15) that

$$\bar{I}_{2,n} = I_{2,n} + \sum_{i \geq 1} (S_{t_i} - S_{t_{i-1}}) \int_0^{t_{i-1}} \hat{C}_{xt}(u, S_u) du.$$

The latter sum can be approximated by a discrete martingale as in Lemma 6.2.

Proposition 7.2. *If ϱ either is constant or satisfies condition (\mathbf{C}_3) , then*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} |\bar{I}_{2,n} - \bar{\mathbf{U}}_{N_1}| = 0,$$

where

$$\bar{\mathbf{U}}_k = \varrho^{-1} \sum_{j=N_2}^k \sigma(y_{t_{j-1}}) S_{t_{j-1}} Y(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) \tilde{Z}_{1,j} \Delta \tilde{\lambda}_j$$

and $Y(\lambda, x) = \int_\lambda^\infty z^{-3/2} \ln(x/K) \tilde{\varphi}(z, x) dz$.

Proof. By Proposition 7.1 it suffices to make a smart martingale approximation for the sum

$$\sum_{i \geq 1} (S_{t_i} - S_{t_{i-1}}) \int_0^{t_{i-1}} \hat{C}_{xt}(u, S_u) du,$$

but this can be done by adapting the manner used in proof of Lemma 6.2 in Appendix B.2 \square

8 Approximations of trading volume

8.1 Trading volume of Leland's strategy

Firstly, we approximate J_n by

$$J_{1,n} = \sum_{m=M_1}^{M_2} S_{u_{m+1}} |\Phi(\mathbf{v}_m) - \Phi(\mathbf{v}_{m+1})|, \quad (8.1)$$

where $\mathbf{v}_m = \mathbf{v}(\lambda_m, S_{u_m})$ defined in (2.2).

Lemma 8.1. *If ϱ either is constant or satisfies condition (\mathbf{C}_3) then,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} (n\varrho^2)^\beta |J_n - J_{1,n}| = 0.$$

Next, by making use of Itô's formula we can replace the term $J_{1,n}$ by (up to $n^\beta \varrho^{2\beta}$ -negligible) the following sum

$$J_{2,n} = \sum_{m=M_1}^{M_2} \lambda_{m+1}^{-1/2} S_{u_{m+1}} \tilde{\varphi}(\lambda_{m+1}, S_{u_{m+1}}) |\varkappa_m| \Delta \lambda_{m+1} \quad (8.2)$$

and $\varkappa_m = \varrho^{-1} \sigma(y_{u_{m+1}}) Z_{1,m} + q(\lambda_{m+1}, S_{u_{m+1}})$.

Lemma 8.2. *If ϱ either is constant or satisfies condition (\mathbf{C}_3) then,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} (n\varrho^2)^\beta |J_{1,n} - J_{2,n}| = 0.$$

Remark that from (5.3) the decreasing sequence $(\tilde{\lambda}_j)$ taking value in $[l_*, l^*]$ can be computed through (t_j) via the relation $\tilde{\lambda}_j = \lambda_0(1 - t_j)^{4\beta}$. Now, setting

$$\tilde{\varkappa}_j = \varrho^{-1} \sigma(y_{t_{j-1}}) \tilde{Z}_{1,j} + q(\tilde{\lambda}_{j-1}, S_{t_{j-1}})$$

we can represent $J_{2,n}$ as $J_{2,n} = \sum_{j=N_2}^{N_1} \tilde{\zeta}_j$, where

$$\tilde{\zeta}_j = \tilde{\lambda}_{j-1}^{-1/2} S_{t_{j-1}} \tilde{\varphi}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) |\tilde{\varkappa}_j| \Delta \tilde{\lambda}_j, \quad (8.3)$$

We need the Doob's decomposition of $J_{2,n}$ w.r.t. the filtration $(\mathcal{F}_j)_{N_2 \leq j \leq N_1}$. To this end, note that

$$\mathbf{E}(\tilde{\zeta}_j | \mathcal{F}_{j-1}) = \tilde{\lambda}_{j-1}^{-1/2} S_{t_{j-1}} \tilde{\varphi}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) \Delta \tilde{\lambda}_j \mathbf{E}(|\tilde{\varkappa}_j| | \mathcal{F}_{j-1}). \quad (8.4)$$

and hence,

$$\mathbf{E}(|\tilde{\varkappa}_j| | \mathcal{F}_{j-1}) = \varrho^{-1} \sigma(y_{t_{j-1}}) G(\tilde{p}_{j-1}) := \mathbf{G}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}, y_{t_{j-1}}), \quad (8.5)$$

where $G(\tilde{p})$ given in (5.7). Setting

$$J_{3,n} = \sum_{j=N_2}^{N_1} \tilde{\lambda}_{j-1}^{-1/2} S_{t_{j-1}} \tilde{\varphi}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) \mathbf{G}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}, y_{t_{j-1}}) \Delta \tilde{\lambda}_j$$

and $\tilde{\varkappa}_j = |\tilde{\varkappa}_j| - \mathbf{G}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}, y_{t_{j-1}})$, we observe directly that

$$J_{2,n} = J_{3,n} + \mathbf{m}'_{N_1}, \quad (8.6)$$

where

$$\mathbf{m}'_k = \sum_{j=N_2}^k \tilde{\lambda}_{j-1}^{-1/2} S_{t_{j-1}} \tilde{\varphi}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) \tilde{\varkappa}_j \Delta \tilde{\lambda}_j.$$

Now we need to distinguish the following two cases: $\varrho = \varrho(n)$ goes to infinity and ϱ is a fixed positive constant.

Case 1: $\lim_{n \rightarrow \infty} \varrho = \infty$.

Note that $\mathbf{E} |aZ + b|$ may be approximated by $b(2\Phi(b/a) - 1)$ as $a \rightarrow 0$. Therefore, taking into account that $\sigma(\cdot)$ is bounded we replace $J_{3,n}$ with the sum

$$\hat{J}_{3,n} = \sum_{j=N_2}^{N_1} \hat{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) \Delta \tilde{\lambda}_j, \quad (8.7)$$

where

$$\hat{\mathbf{B}}(\lambda, x) = \lambda^{-1/2} x \tilde{\varphi}(\lambda, x) q(\lambda, x) (2\Phi(\varrho q(\lambda, x)) - 1)$$

and $q(\lambda, x)$ defined in (2.13). Put $\hat{J}_{4,n} = \sum_{j=N_2}^{N_1} \hat{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_1) \Delta \tilde{\lambda}_j$ and

$$\iota_n = \hat{J}_{3,n} - \hat{J}_{4,n} = \sum_{j=N_2}^{N_1} \left(\hat{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) - \hat{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_1) \right) \Delta \tilde{\lambda}_j. \quad (8.8)$$

Using Lemma 5.1 we can show directly that the sum $\hat{J}_{4,n} \rightarrow J^*(S_1)$ in probability as $n \rightarrow \infty$ faster than $n^\beta \varrho^{2\beta}$.

Lemma 8.3. *Under the condition (\mathbf{C}_3) we have*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \left| \hat{J}_{4,n} - J^*(S_1) \right| = 0.$$

Furthermore, by Itô's formula we obtain

$$\hat{\mathbf{B}}(\tilde{\lambda}_j, S_1) - \hat{\mathbf{B}}(\tilde{\lambda}_j, S_{t_j}) = \hat{\mathbf{B}}_{1,j} + \frac{1}{2} \hat{\mathbf{B}}_{2,j}, \quad (8.9)$$

where

$$\widehat{\mathbf{B}}_{1,j} = \int_{t_j}^1 \frac{\partial \widehat{\mathbf{B}}}{\partial x}(\tilde{\lambda}_j, S_u) dS_u, \quad \widehat{\mathbf{B}}_{2,j} = \int_{t_j}^1 \frac{\partial^2 \widehat{\mathbf{B}}}{\partial x^2}(\tilde{\lambda}_j, S_u) \sigma^2(y_u) S_u^2 du.$$

Now we represent ι_n as

$$\iota_n = - \sum_{j=N_2}^{N_1} \widehat{\mathbf{B}}_{1,j-1} \Delta \tilde{\lambda}_j - \frac{1}{2} \sum_{j=N_2}^{N_1} \widehat{\mathbf{B}}_{2,j-1} \Delta \tilde{\lambda}_j. := -\iota_{1,n} - \frac{1}{2} \iota_{2,n}. \quad (8.10)$$

A direct calculation gives

$$\begin{aligned} \frac{\partial \widehat{\mathbf{B}}}{\partial x} &= \lambda^{-1/2} \tilde{\varphi}(\lambda, x) [-2q^2(\lambda, x)(2\Phi(\varrho q(\lambda, x)) - 1) \\ &\quad + \frac{1}{2\lambda} (2\Phi(\varrho q(\lambda, x)) - 1) + \frac{\varrho}{\lambda} \varphi(\varrho q(\lambda, x))], \\ \frac{\partial^2 \widehat{\mathbf{B}}}{\partial x^2} &= \frac{\tilde{\varphi}(\lambda, x)}{x} \left\{ \left(-2q^2(\lambda, x)(2\Phi(\varrho q(\lambda, x)) - 1) + \frac{1}{2\lambda} (2\Phi(\varrho q(\lambda, x)) - 1) \right. \right. \\ &\quad \left. \left. + \frac{\varrho}{\lambda} \varphi(\varrho q(\lambda, x)) \right) \frac{\mathbf{v}(\lambda, x)}{\sqrt{\lambda}} - \frac{2q(\lambda, x)}{\lambda} (2\Phi(\varrho q(\lambda, x)) - 1) \right. \\ &\quad \left. - \frac{2q^2(\lambda, x)\varrho}{\lambda} \varphi(\varrho q(\lambda, x)) + \frac{\varrho}{4\lambda^2} \varphi(\varrho q(\lambda, x)) - \frac{q\varrho^3}{2\lambda^2} \varphi(\varrho q(\lambda, x)) \right\}. \end{aligned}$$

Note that $2\Phi(\varrho q) - 1 \rightarrow \text{sign}(q)$ as $n \rightarrow \infty$. Now putting

$$\widehat{\mathbf{V}}(\lambda, x) = \int_{\lambda}^{+\infty} \frac{\partial \widehat{\mathbf{B}}}{\partial x}(v, x) dv,$$

and using Lemma 5.8 we can approximate the term $\iota_{1,n}$ by

$$\widehat{\iota}_{1,n} = \sum_{j=N_2}^{N_1} \widehat{\mathbf{V}}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) \int_{t_{j-1}}^{t_j} \sigma(y_t) S_t dW_t^{(1)}. \quad (8.11)$$

Note that as $n \rightarrow \infty$

$$\widehat{\mathbf{V}}(\lambda, x) \approx \mathbf{V}(\lambda, x) = \int_{\lambda}^{+\infty} \frac{\tilde{\varphi}(z, x)}{\sqrt{z}} \left(-2q^2(z, x) + \frac{1}{2z} \right) \text{sign}(q(z, x)) dz.$$

So, asymptotically one can replace $\widehat{\iota}_{1,n}$ by \mathbf{m}_{N_1}'' , where

$$\mathbf{m}_k'' = \frac{1}{\varrho} \sum_{j=N_2}^k \sigma(y_{t_{j-1}}) S_{t_{j-1}} \tilde{\mathbf{V}}_{j-1} \tilde{Z}_{1,j} \Delta \tilde{\lambda}_j \quad (8.12)$$

and $\tilde{\mathbf{V}}_j = \mathbf{V}(\tilde{\lambda}_j, S_{t_j})$.

Lemma 8.4. *If $\varrho = \varrho(n)$ satisfies condition (\mathbf{C}_3) then,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \iota_{2,n} = \mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \left| \widehat{\iota}_{1,n} - \mathbf{m}_{N_1}'' \right| = 0.$$

It is clear that both sequences $(\mathbf{m}'_k)_{N_2 \leq k \leq N_1}$ and $(\mathbf{m}''_k)_{N_2 \leq k \leq N_1}$ are martingales w.r.t. $(\mathcal{F}_k)_{N_2 \leq k \leq N_1}$. Thus, if $\varrho(n) \rightarrow \infty$ under condition (\mathbf{C}_3) the final asymptotic representation of J_n is given in the following proposition.

Proposition 8.1. *Under condition (\mathbf{C}_3) the total trading volume J_n admits the following asymptotic form*

$$J_n = J^*(S_1) + \mathbf{m}'_{N_1} - \mathbf{m}''_{N_1} + o_{\mathbf{P}}(n^{-\beta} \varrho^{-2\beta}).$$

Case 2: ϱ is a fixed positive constant.

Similarly, we replace $J_{3,n}$ with

$$\widetilde{J}_{3,n} = \sum_{j=N_2}^{N_1} \widetilde{\mathbf{B}}(\widetilde{\lambda}_{j-1}, S_1, y_1) \Delta \widetilde{\lambda}_j,$$

and write the difference, which is not a negligible, as

$$\widetilde{J}_{4,n} = J_{3,n} - \widetilde{J}_{3,n} = \sum_{j=N_2}^{N_1} \left(\widetilde{\mathbf{B}}(\widetilde{\lambda}_{j-1}, S_{t_{j-1}}, y_{t_{j-1}}) - \widetilde{\mathbf{B}}(\widetilde{\lambda}_{j-1}, S_1, y_1) \right) \Delta \widetilde{\lambda}_j,$$

where $\widetilde{\mathbf{B}}(\lambda, x, y) = \lambda^{-1/2} x \widetilde{\varphi}(\lambda, x) \mathbf{G}(\lambda, x, y)$.

The term $\widetilde{J}_{3,n}$ gives us the limit of the total trading volume by using Lemma 5.1.

To study the term $\widetilde{J}_{4,n}$ we define the following functions

$$\mathbf{K}_1(\lambda, x, y) = x \sigma(y) \int_{\lambda}^{+\infty} \frac{\partial \widetilde{\mathbf{B}}}{\partial x}(u, x, y) du + \mathbf{r} F_2(t(\lambda), y) \int_{\lambda}^{+\infty} \frac{\partial \widetilde{\mathbf{B}}}{\partial y}(u, x, y) du,$$

$$\mathbf{K}_2(\lambda, x, y) = \sqrt{1 - \mathbf{r}^2} F_2(t(\lambda), y) \int_{\lambda}^{+\infty} \frac{\partial \widetilde{\mathbf{B}}}{\partial y}(u, x, y) du,$$

where $t(\lambda) = 1 - (\lambda/\lambda_0)^{1/4\beta}$. Consider the martingale $(\widetilde{\mathbf{m}}'_k)_{k \geq 1}$ defined as

$$\widetilde{\mathbf{m}}'_k = \frac{1}{\varrho} \sum_{j=N_2}^k (\widetilde{\mathbf{K}}_{1,j-1} \widetilde{Z}_{1,j} + \widetilde{\mathbf{K}}_{2,j-1} \widetilde{Z}_{2,j}) \Delta \widetilde{\lambda}_j, \quad (8.13)$$

where $\widetilde{\mathbf{K}}_{1,j} = \mathbf{K}_1(\widetilde{\lambda}_j, S_{t_j}, y_{t_j})$ and $\widetilde{\mathbf{K}}_{2,j} = \mathbf{K}_2(\widetilde{\lambda}_j, S_{t_j}, y_{t_j})$. Again, by virtue of Lemma 5.8 one states that $\widetilde{J}_{4,n}$ can be approximated by $\widetilde{\mathbf{m}}'_{N_1}$.

Lemma 8.5. For any fixed $\varrho > 0$,

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \left| \tilde{J}_{3,n} - J(S_1, y_1, \varrho) \right| = \mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \left| \tilde{J}_{4,n} - \tilde{\mathbf{m}}'_{N_1} \right| = 0.$$

Finally, the asymptotic form of J^n is summarized in the following.

Proposition 8.2. For any fixed $\varrho > 0$ the total trading volume J^n admits the following asymptotic form

$$J_n = J(S_1, y_1, \varrho) + \mathbf{m}'_{N_1} - \tilde{\mathbf{m}}'_{N_1} + o_{\mathbf{P}}(n^{-\beta}).$$

8.2 Trading volume of Lépinette's strategy

From definition and Itô's Lemma one has

$$\bar{\gamma}_{t_i} - \bar{\gamma}_{t_{i-1}} = \int_{t_{i-1}}^{t_i} \hat{C}_{xx}(u, S_u) dS_u + \frac{1}{2} \int_{t_{i-1}}^{t_i} \hat{C}_{xxx}(u, S_u) \sigma^2(y_u) S_u^2 du.$$

Applying again the arguments in Lemmas 8.1 and 8.2 we can approximate \bar{J}_n by

$$\bar{J}_{1,n} = \varrho^{-1} \sum_{j=N_2}^{N_1} \tilde{\lambda}_{j-1}^{-1/2} \sigma(y_{t_{j-1}}) S_{t_{j-1}} \tilde{\varphi}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) \left| \tilde{Z}_{1,j} \right| \Delta \tilde{\lambda}_j. \quad (8.14)$$

Taking into account the elementary property $\mathbf{E}|Z| = \sqrt{2/\pi}$, for $Z \sim \mathcal{N}(0, 1)$, we obtain the Dood' decomposition of $\bar{J}_{1,n}$ as

$$\bar{J}_{1,n} = \bar{J}_{2,n} + \bar{J}_{3,n} + \varrho^{-1} \sum_{j=N_2}^{N_1} \tilde{\lambda}_{j-1}^{-1/2} \sigma(y_{t_{j-1}}) S_{t_{j-1}} \tilde{\varphi}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) \tilde{Z}_{4,j} \Delta \tilde{\lambda}_j,$$

where

$$\bar{J}_{2,n} = \varrho^{-1} \sqrt{2/\pi} \sum_{j=N_2}^{N_1} \tilde{\lambda}_{j-1}^{-1/2} \bar{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_1, y_1) \Delta \tilde{\lambda}_j, \quad (8.15)$$

$$\bar{J}_{3,n} = \varrho^{-1} \sqrt{2/\pi} \sum_{j=N_2}^{N_1} \tilde{\lambda}_{j-1}^{-1/2} \left(\bar{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}, y_{t_{j-1}}) - \bar{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_1, y_1) \right) \Delta \tilde{\lambda}_j \quad (8.16)$$

and $\bar{\mathbf{B}}(\lambda, x, y) = \sigma(y) x \tilde{\varphi}(\lambda, x)$. Note that the sum

$$\bar{J}_{2,n} = \varrho^{-1} \sqrt{2/\pi} \sigma(y_1) S_1 \sum_{j=N_2}^{N_1} \tilde{\lambda}_{j-1}^{-1/2} \tilde{\varphi}(\tilde{\lambda}_{j-1}, S_1) \Delta \tilde{\lambda}_j$$

converges to $\eta \min(S_1, K)$ by Lemma 5.1 and (2.15). The arguments in the approximation of $\tilde{J}_{4,n}$ can be applied to replace $\bar{J}_{3,n}$ by $\bar{\mathbf{m}}'_{N_2}$, where

$$\bar{\mathbf{m}}'_k = \frac{1}{\varrho} \sum_{j=N_2}^k (\tilde{\mathbf{D}}_{1,j-1} \tilde{Z}_{1,j} + \tilde{\mathbf{D}}_{2,j-1} \tilde{Z}_{2,j}) \Delta \tilde{\lambda}_j, \quad (8.17)$$

where $\tilde{\mathbf{D}}_{1,j} = \mathbf{D}_1(\tilde{\lambda}_j, S_{t_j}, y_{t_j})$ and $\tilde{\mathbf{D}}_{2,j} = \mathbf{D}_2(\tilde{\lambda}_j, S_{t_j}, y_{t_j})$.

In summary, the asymptotic representation of \bar{J}_n is given in the following proposition.

Proposition 8.3. *If ϱ is a constant independent of n then,*

$$\bar{J}_n = \eta \min(S_1, K) + \bar{\mathbf{m}}_{N_2} - \bar{\mathbf{m}}'_{N_2} + o(n^{-\beta}),$$

where the discrete martingale $\bar{J}_{3,n}$ defined by

$$\bar{\mathbf{m}}'_k = \varrho^{-1} \sum_{j=N_2}^k \tilde{\lambda}_{j-1}^{-1/2} \sigma(y_{t_{j-1}}) S_{t_{j-1}} \tilde{\varphi}(\tilde{\lambda}_{j-1}, S_{t_{j-1}}) \tilde{Z}_{4,j} \Delta \tilde{\lambda}_j.$$

Proof. It can be shown by adapting the manner in Proposition 8.1. \square

Remark 11. *When $\varrho \rightarrow \infty$, the sum $\bar{J}_{2,n}$ converges to 0 slowly than $\varrho^{2\beta} n^\beta$. Therefore one has no result on convergence rate improvement as in Theorem 3.2 for Leland's strategy.*

9 Proof of Main Theorems

9.1 Proof of Theorem 3.1

It is clear from the previous results that the asymptotic distribution strongly depends on how the parameter ϱ is chosen. If ϱ is a fixed positive constant then the martingale part of the hedging error is

$$\mathbf{M}_k = -\frac{1}{2} \mathbf{U}_k - \kappa_*(\mathbf{m}'_k - \tilde{\mathbf{m}}'_k).$$

This can be represented as

$$\mathbf{M}_k = \frac{1}{\varrho} \sum_{j=N_2}^k (\tilde{\mathbf{A}}_{1,j-1} \tilde{Z}_{1,j} + \tilde{\mathbf{A}}_{3,j} \tilde{Z}_{3,j-1} + \tilde{\mathbf{A}}_{2,j-1} \tilde{Z}_{2,j}) \Delta \tilde{\lambda}_j,$$

where $\tilde{\mathbf{A}}_{i,j} = \mathbf{A}_i(\tilde{\lambda}_j, S_{t_j})$ and

$$\begin{aligned}\mathbf{A}_1(\lambda, x, y) &= -\frac{1}{2}\sigma(y)x\check{H}(\lambda, x) + \kappa_*\mathbf{K}_1(\lambda, x, y), \\ \mathbf{A}_3(\lambda, x) &= -\kappa_*\lambda^{-1/2}\tilde{\varphi}(\lambda, x), \quad \mathbf{A}_2(\lambda, x, y) = \kappa_*\mathbf{K}_2(\lambda, x, y).\end{aligned}$$

Then, the sequence $(n^\beta \mathbf{M}_{N_1})_{n \geq 1}$ converges in law to a mixed Gaussian variable by Proposition 5.1 and hence, Theorem 3.1 is proved. \square

9.2 Proof of Theorem 3.2

If $\varrho = \varrho(n)$ diverges to infinity under condition (\mathbf{C}_3) then, by virtue of Proposition (6.1), the hedging error admits the following asymptotic form

$$\min(S_1, K) - \kappa_* J^*(S_1) + \frac{1}{\varrho} \sum_{j=N_2}^{N_1} \check{v}_j + o_{\mathbf{P}}(n^{-\beta} \varrho^{-2\beta})$$

where

$$\check{v}_j = \sigma(y_{t_{j-1}}) S_{\tilde{t}_{j-1}} (\check{\mathbf{A}}_{1,j-1} \tilde{Z}_{1,j} + \check{\mathbf{A}}_{3,j-1} \tilde{Z}_{3,j}) \Delta \tilde{\lambda}_j, \quad (9.1)$$

with $\check{\mathbf{A}}_{i,j} = \check{\mathbf{A}}_i(\tilde{\lambda}_j, S_{t_j})$ and

$$\check{\mathbf{A}}_1(\lambda, x) = -\frac{1}{2}\check{H}_1(\lambda, x) + \kappa_*\mathbf{V}(\lambda, x), \quad \check{\mathbf{A}}_3 \equiv \mathbf{A}_3.$$

Put $\check{\mathbf{M}}_k = \sum_{j=N_2}^k \check{v}_j$ and remark that

$$n^\beta \varrho^{2\beta} \frac{1}{\varrho} \sum_{j=N_2}^{N_1} \check{v}_j = n^\beta \varrho^{-\frac{1}{\mu+1}} \check{\mathbf{M}}_{N_1}.$$

Then, Theorem 3.2 is proved through Proposition 6.1 \square

9.3 Proof of Theorem 3.3

By Proposition 8.2 and Proposition 7.2, the martingale part of the hedging error for Lépinette's strategy is given by

$$\bar{\mathbf{M}}_k = -\frac{1}{2}\mathbf{U}_k + \bar{\mathbf{U}}_k - \kappa_*(\bar{\mathbf{m}}_k - \bar{\mathbf{m}}'_k).$$

This can be represented as

$$\bar{\mathbf{M}}_k = \frac{1}{\varrho} \sum_{j=N_2}^k (\tilde{\mathbf{A}}_{1,j-1} \tilde{Z}_{1,j} + \tilde{\mathbf{A}}_{4,j} \tilde{Z}_{4,j-1} + \tilde{\mathbf{A}}_{2,j-1} \tilde{Z}_{2,j}) \Delta \tilde{\lambda}_j,$$

where $\tilde{\mathbf{A}}_{i,j} = \mathbf{A}_i(\tilde{\lambda}_j, S_{t_j})$ and

$$\begin{aligned}\mathbf{A}_1(\lambda, x, y) &= -\frac{1}{2}\sigma(y)x\check{H}(\lambda, x) + \sigma(y)xY(\lambda, x) + \kappa_*\mathbf{D}_1(\lambda, x, y), \\ \mathbf{A}_3(\lambda, x) &= -\kappa_*\lambda^{-1/2}\tilde{\varphi}(\lambda, x), \quad \mathbf{A}_2(\lambda, x, y) = \kappa_*\mathbf{D}_2(\lambda, x, y).\end{aligned}$$

Then, the sequence $(n^\beta \bar{\mathbf{M}}_{N_1})_{n \geq 1}$ converges in law to a mixed Gaussian variable by Proposition 5.3 and hence, Theorem 3.1 is proved. \square

10 Examples

In this section we explicitly justify some well-known SV models for which condition (\mathbf{C}_1) is fulfilled. For this aim we will need some moment bounds of solutions to non-linear SDE

$$dy_t = F_1(t, y_t)dt + F_2(t, y_t)dZ_t, \quad y(0) = y_0, \quad (10.1)$$

with Z is a standard Wiener process and F_i are two smooth functions.

We recall here the well-known estimate for solutions in theory of SDEs.

Theorem 10.1. *Suppose that $F_1(t, y)$, $F_2(t, y)$ are measurable in $(t, y) \in [0, T] \times \mathbb{R}$ and*

$$|A(t, y)| \leq K(1 + |y|), \quad |B(t, y)| \leq K(1 + |y|).$$

Assume further that for any $N > 0$ there exists a positive constant C_N such that

$$|F_1(t, y) - F_1(t, y')| \leq K_N|y - y'|, \quad |F_2(t, y) - F_2(t, y')| \leq K_N|y - y'| \quad (10.2)$$

if $|y| \leq N$, $|y'| \leq N$ and $\mathbf{E}|y_0|^{2m} < \infty$ for some positive m . Then there exists a unique solution y_t of SDE (10.7) and

$$\mathbf{E}|y_t|^{2m} < (1 + \mathbf{E}|y_0|^{2m})e^{Ct}, \quad \mathbf{E} \sup_{0 \leq s \leq t} |y_s|^{2m} < M(1 + \mathbf{E}|y_0|^{2m}),$$

where C, M are positive constants depending only on K, t, m .

Proof. See, for example [16], Th.2.3, p.107. \square

We will see that in the context of the previous theorem, condition (\mathbf{C}_1) holds if the volatility function σ satisfies polynomial growth assumption $|\sigma(y)| \leq C(1 + |y|^m)$ for some positive constant C and $m \geq 1$.

Hull-White models: Consider the case where y_t follows a geometric Brownian motion

$$\begin{cases} dS_t = (y_t + \sigma_{min})S_t dW_t, & S_0 > 0, \\ dy_t = y_t(ad + b dZ_t), & y_0 > 0 \end{cases} \quad (10.3)$$

where a, b and $\sigma_{min} > 0$ are some constants and Z is a standard Brownian motion correlated to W_t .³ Putting $y^* = \sup_{0 \leq t \leq 1} |y_t|$ then, by Theorem 10.1 one has

$$\mathbf{E}(y^*)^{2m} \leq C(1 + \mathbf{E}|y_0|^{2m}) < \infty$$

as long as $\mathbf{E}|y_0|^{2m} < \infty$. Therefore, condition (\mathbf{C}_1) is clearly fulfilled.

Uniform Elliptic Volatility models: Consider the case where volatility is driven by a Orstein-Uhlenbeck process which admits the mean-reverting property

$$\begin{cases} dS_t = (y_t^2 + \sigma_{min})S_t dW_t, \\ dy_t = (a - by_t)dt + dZ \end{cases} \quad (10.4)$$

In this case $\sigma(y) = y^2 + \sigma_{min}$ and condition (\mathbf{C}_1) is obviously verified throughout Theorem 10.1.

Stein-Stein models:

$$\begin{cases} dS_t = \sqrt{y_t^2 + \sigma_{min}} S_t dW_t, \\ dy_t = (a - by_t)dt + dZ_t \end{cases} \quad (10.5)$$

In this case $\sigma(y) = \sqrt{y^2 + \sigma_{min}}$ and by Theorem 10.1 condition (\mathbf{C}_1) is verified.

Heston models: Heston [21] proposed a SV model where volatility is driven by a CIR process which also called squared root process. We will justify that this kind of model can be used in our context.

Assume now that the price dynamics is given by the following

$$\begin{cases} dS_t = \sqrt{y_t + \sigma_{min}} S_t dW_t, \\ dy_t = a(b_0 - b_1 y_t)dt + v\sqrt{y_t} dZ_t, \quad y_0 > 0. \end{cases} \quad (10.6)$$

For $a > 0, b_0 > 0, b_1 > 0$ there exists a unique strong solution y_t . Using stopping times method, we can directly show that $\mathbf{E} y^* < \infty$ (see Appendix C) hence condition (\mathbf{C}_1) is satisfied.

Similarly, one can verify that (\mathbf{C}_1) also holds for Ball-Roma's models [2]. In fact, condition (\mathbf{C}_1) satisfies for a wide class of process y_t with bounded diffusion. In what follows we need the condition on the coefficients the dynamics of y_t

(\mathbf{C}^*) *There exist positive constants a, b, M such that*

$$yF_1(t, y) \leq a - by^2, \quad \text{for all } t > 0, y \in \mathbb{R}$$

and $|F_2(t, y)| \leq M$.

For simplicity, assume now on that $y_0 = 0$.

³If $y_0 > 0$ then y_t is almost surely positive and hence we can consider $\sigma(y) = y + \sigma_{min}$.

Proposition 10.1. *Under condition (\mathbf{C}^*) , there exists a constant $\alpha > 0$ such that $\mathbf{E} e^{\alpha \|y\|_1^2} < \infty$, where $\|\cdot\|_1^2$ stands for $\sup_{0 \leq t \leq 1} y_t^2$.*

Proof. A proof can be made using the method in Proposition 1.1.2 in [25]. \square

Scott models: Let us consider the situation where volatility follows an Orstein-Uhlenbeck as in Stein-Stein models but now the function σ takes the exponential form

$$\begin{cases} dS_t = (e^{\delta y_t} + \sigma_{min})S_t dW_t^{(1)}, \\ dy_t = (a - by_t)dt + dZ_t \end{cases} \quad (10.7)$$

where a, b and $\sigma_{min} > 0$ are constants and $\delta > 0$ is chosen such that $2\delta \leq \alpha$ defined as in Proposition 10.1. Here $\sigma(y) = e^{\delta y} + \sigma_{min}$ and then condition (\mathbf{C}_1) is fulfilled since

$$\mathbf{E} \sup_{0 \leq t \leq 1} |\sigma(y)|^2 \leq 2\sigma_{min}^2 + 2\mathbf{E} (e^{2\delta} \mathbf{1}_{\{|y_t| \leq 1\}} + e^{2\delta \|y\|_1^2} \mathbf{1}_{\{|y_t| > 1\}}) < \infty.$$

In summary, condition (\mathbf{C}_1) is justified for most of SV models in practice.

A Auxiliary Lemmas

Lemma A.1. *For any $a > 0$ and $0 < t \leq 1$,*

$$\mathbf{P}(S_t = a) = 0.$$

Proof. Let t be a fixed point in $[0, 1]$. We will prove that $\mathbf{P}(x_t = a) = 0$ for any $a \in \mathbb{R}$, where

$$x_t = \int_0^t \sigma(y_s) dW_s^{(1)} - \int_0^t \sigma^2(y_s) ds.$$

For some sequence $(t_n) \uparrow t$ we define

$$x_t^{(n)} = \int_0^{t_n} \sigma(y_s) dW_s^{(1)} - \int_0^{t_n} \sigma^2(y_s) ds + \sigma(y_{t_n})(W_t^{(1)} - W_{t_n}^{(1)}).$$

It is clear that conditionally with respect to \mathcal{F}_{t_n} the random variable $x_t^{(n)}$ is Gaussian with the parameters $(x_{t_n}, \sigma^2(y_{t_n}))$. Hence, one can check directly that

$$\lim_{\epsilon \rightarrow 0} \sup_{n \geq 1} \sup_{a > 0} \mathbf{P}(x_t^{(n)} \in [a - \epsilon, a + \epsilon]) = 0.$$

Moreover, taking into account that $x_t^{(n)} \rightarrow x_t$ we get

$$\begin{aligned} \mathbf{P}(x_t = a) &\leq \mathbf{E}(\mathbf{1}_{\{x_t = a\}} \mathbf{1}_{\{|x_t - x_t^{(n)}| < \epsilon\}}) + \mathbf{P}(|x_t - x_t^{(n)}| \geq \epsilon) \\ &\leq \mathbf{P}(x_t^{(n)} \in [a - \epsilon, a + \epsilon]) + o(1) \end{aligned}$$

and Lemma A.1 is proved. \square

Lemma A.2. *Let ι be a continuously decreasing $[0, 1] \rightarrow [0, 1]$ function with $\iota(0) = 1$. Then for any $\mathbb{R}_+^2 \rightarrow \mathbb{R}$ function \mathbf{A} satisfying condition (\mathbf{H}_2) we have*

$$\limsup_{n \rightarrow \infty} \int_0^n |\check{A}(u, S_{\iota(u/n)})| du < \infty \quad a.s. \quad (\text{A.1})$$

and

$$\limsup_{n \rightarrow \infty} \int_0^n \int_v^n \check{A}_n(u, v) du dv < \infty \quad a.s., \quad (\text{A.2})$$

where $\check{A}_n(u, v) = |A(u, S_{\iota(v/n)})| \tilde{\varphi}(u, S_{\iota(v/n)})$.

Proof. We can represent $\tilde{\varphi}(u, x)$ as

$$\tilde{\varphi}(u, x) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{K}{x}} \phi_*(u, x), \quad \phi_*(u, x) = e^{-\frac{\ln^2(x/K)}{2u} - \frac{u}{8}}.$$

Now, by condition (\mathbf{C}_4) there exists $\gamma > 0$ and a function $U_\gamma > 0$ such that

$$\begin{aligned} \int_0^n |\check{A}(u, S_{\iota(u/n)})| du &\leq CU_\gamma^* \int_1^\infty (u^\gamma + 1) e^{-\frac{u}{8}} du \\ &\quad + CU_\gamma^* \int_0^1 (1 + u^{-\gamma}) e^{-\frac{\ln^2(S_{\iota(u/n)/K})}{2u}} du \end{aligned}$$

where $U_\gamma^* = \sup_{0 \leq t \leq 1} U_\gamma(S_t) / \sqrt{S_t}$. Taking into account that $\iota(u/n) \rightarrow 1$ uniformly in $0 \leq u \leq 1$ and that the function $(S_t)_{0 \leq t \leq 1}$ is uniformly continuous we get, that for sufficiently large n

$$\sup_{0 \leq u \leq 1} e^{-\frac{\ln^2(S_{\iota(u/n)/K})}{2u}} \leq e^{-\frac{\ln^2(S_1/K)}{8u}} \quad a.s.$$

This inequality clearly implies (A.1). To show the inequality (A.2) we note that

$$\int_0^n \int_v^n \check{A}_n(u, v) du dv = \int_0^n \int_0^u \check{A}_n(u, v) dv du \leq \int_0^n u \sup_{0 \leq v \leq u} \check{A}_n(u, v) du.$$

A same argument can be run again to obtain (A.2) and hence Lemma A.2 is completely proved. \square

B Proof of convergence lemmas

B.1 Proof of Lemma 6.1

Changing variable $v = \lambda_t$ and using Lemma A.2 we can show that

$$\lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \int_0^1 S_t \lambda_t^{-1/2} \tilde{\varphi}(\lambda_t, S_t) dt = 0 \quad \text{a.s.} \quad (\text{B.1})$$

Now we can represent $I_{1,n}$ as

$$\begin{aligned} I_{1,n} &= \int_0^1 \theta_t^2 \lambda_t^{-1/2} S_t \tilde{\varphi}(\lambda_t, S_t) dt + o_{\mathbf{P}}(n^{-\beta} \varrho^{-2\beta}) \\ &= S_1 \int_0^1 \theta_t^2 \lambda_t^{-1/2} \tilde{\varphi}(\lambda_t, S_1) dt + \tau_n + o_{\mathbf{P}}(n^{-\beta} \varrho^{-2\beta}) \\ &= S_1 \int_0^{\lambda_0} v^{-1/2} \tilde{\varphi}(v, S_1) dv + \tau_n + o_{\mathbf{P}}(n^{-\beta} \varrho^{-2\beta}). \end{aligned}$$

Taking into account here (2.15) we obtain Lemma 6.1. \square

B.2 Proof of Lemma 6.2

We first check the first equality in (6.4). To do this we rewrite $\tau_{1,n}$ as

$$\tau_{1,n} = \tau_{3,n} + \delta_{1,n} + \delta_{2,n}, \quad (\text{B.2})$$

where $\tau_{3,n} = \int_{u^*}^{u_*} \sigma(y_u) S_u \overline{H}_1(\lambda_u, S_u) dW_u^{(1)}$, $\delta_{1,n} = \int_0^{u^*} \sigma(y_u) S_u \overline{H}_1(\lambda_u, S_u) dW_u^{(1)}$ and $\delta_{2,n} = \int_{u_*}^1 \sigma(y_u) S_u \overline{H}_1(\lambda_u, S_u) dW_u^{(1)}$.

Thanks to condition (\mathbf{C}_3) , $\varrho = o\left(n^{\frac{3-\mu}{2(3\mu-1)}}\right)$ as $n \rightarrow \infty$ and

$$\sup_{n \geq 1} \max_{M_1 \leq m \leq M_2} \left(n \mathbf{E} \left(S_{u_m} - S_{u_{m-1}} \right)^2 + n^{1/2} \varrho^{-1} \Delta \lambda_m \right) < \infty.$$

This allows to show directly that

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} |\tau_{3,n} - \mathbf{U}_{N_1}| = 0.$$

Next, through Lemma A.2 one can show that for any $r > 0$

$$\lim_{n \rightarrow \infty} n^r \int_{l^*}^{\infty} \tilde{\varphi}(v, S_{u(v)}) dv = \lim_{n \rightarrow \infty} n^r \int_0^{l^*} \tilde{\varphi}(v, S_{u(v)}) dv = 0, \quad (\text{B.3})$$

where $u = u(v) = 1 - (v/\lambda_0)^{4\beta}$. This directly implies that both $\delta_{j,n}$, $j = 1, 2$ are $n^\beta \varrho^{2\beta}$ -negligible and by virtue of Lemma A.2 we obtain the same property for the Lebesgue integral $\tau_{2,n}$. Hence the proof is completed. \square

B.3 Proof of Proposition 7.1

Setting $\varepsilon_n = n^{-2\beta} \varrho^{-4\beta} l_*$, we can represent $I_{2,n}$ as

$$I_{2,n} = \int_0^{1-\varepsilon_n} \left(\gamma_t^n - \widehat{C}_x(t, S_t) \right) \sigma(y_t) S_t dW_t^{(1)} + \varpi_n,$$

where $\varpi_n = \int_{1-\varepsilon_n}^1 \left(\gamma_t^n - \widehat{C}_x(t, S_t) \right) \sigma(y_t) S_t dW_t^{(1)}$. Taking into account that

$$\left| \gamma_t^n - \widehat{C}_x(t, S_t) \right| \leq 1,$$

we obtain $\lim_{n \rightarrow \infty} n^{2\beta} \varrho^{4\beta} \mathbf{E} \varpi_n^2 = 0$. Thus, it remains to prove that

$$\lim_{n \rightarrow \infty} n^{2\beta} \varrho^{4\beta} \sum_{j=1}^n \int_{\widehat{t}_{j-1}}^{\widehat{t}_j} \mathbf{E} \left(\gamma_t^n - \widehat{C}_x(t, S_t) \right)^2 dt = 0, \quad (\text{B.4})$$

where $\widehat{t}_j = \min(t_j, 1 - \varepsilon_n)$. First, we introduce the following functions

$$\begin{aligned} G_1(t) &= \sum_{j=1}^n \frac{1}{\lambda_t} \left(x_t - x_{\widehat{t}_{j-1}} \right)^2 \mathbf{1}_{\{\widehat{t}_{j-1} < t \leq \widehat{t}_j\}}, \\ G_2(t) &= \sum_{j=1}^n x_t^2 \left(\lambda_t^{-1/2} - \lambda_{\widehat{t}_{j-1}}^{-1/2} \right)^2 \mathbf{1}_{\{\widehat{t}_{j-1} < t \leq \widehat{t}_j\}}, \\ G_3(t) &= \sum_{j=1}^n \left(\lambda_t^{1/2} - \lambda_{\widehat{t}_{j-1}}^{1/2} \right)^2 \mathbf{1}_{\{\widehat{t}_{j-1} < t \leq \widehat{t}_j\}}, \end{aligned}$$

where $x_t = \ln(S_t/K)$. Clearly,

$$\begin{aligned} \left| \gamma_t^n - \widehat{C}_x(t, S_t) \right|^2 &= \sum_{j=1}^n \left| \widetilde{\Phi}(\lambda_t, S_t) - \widetilde{\Phi}(\lambda_{\widehat{t}_{j-1}}, S_{\widehat{t}_{j-1}}) \right|^2 \mathbf{1}_{\{\widehat{t}_{j-1} < t \leq \widehat{t}_j\}} \\ &\leq G_1(t) + G_2(t) + G_3(t). \end{aligned}$$

We can show directly in view of condition (\mathbf{C}_3) that

$$n^{2\beta} \varrho^{4\beta} \mathbf{E} \int_0^{1-\varepsilon_n} G_1(t) dt \leq C n^{2\beta-3/2} \varrho^{4\beta-1} \rightarrow 0$$

since $\sup_{0 \leq t \leq 1} \mathbf{E} x_t^2 < \infty$ and

$$\sup_{n \geq 1, 1 \leq j \leq n} \left(n \sup_{0 \leq t \leq 1} \mathbf{E} \left(x_t - x_{\widehat{t}_{j-1}} \right)^2 \mathbf{1}_{\{\widehat{t}_{j-1} < t \leq \widehat{t}_j\}} \right) < \infty.$$

The particular choice of ε ensures

$$n^{2\beta} \varrho^{4\beta} \mathbf{E} \int_0^{1-\varepsilon_n} G_2(t) dt \leq C \frac{n^{2\beta} \varrho^{4\beta}}{n^2 \lambda_0} (\varepsilon_n)^{-(4\beta+1)/4\beta} \rightarrow 0,$$

while the convergence of $G_3(t)$ can be shown in a similar way. \square

B.4 Proof of Lemma 8.1

The proof directly follows from Lemma A.2. \square

B.5 Proof of Lemma 8.2

First, we represent $J_{1,n}$ as

$$J_{1,n} = \sum_{m=M_1}^{M_2} S_{u_{m+1}} |\Delta\Phi(\mathbf{v}_{m+1})| + \varepsilon_{1,n}, \quad (\text{B.5})$$

where $\varepsilon_{1,n} = \sum_{m=M_1}^{M_2} (S_{u_{m-1}} - S_{u_{m+1}}) |\Delta\Phi(\mathbf{v}_{m+1})|$ and $\Delta\Phi(\mathbf{v}_m) = \Phi(\mathbf{v}_m) - \Phi(\mathbf{v}_{m-1})$. Moreover, setting $r_m = |\Delta\Phi(\mathbf{v}_m)| - \varphi(\mathbf{v}_m) |\Delta\mathbf{v}_m|$, we can rewrite (B.5) as

$$J_{1,n} = T_{1,n} + \varepsilon_{1,n} + \varepsilon_{2,n}, \quad (\text{B.6})$$

where $T_{1,n} = \sum_{m=M_1}^{M_2} S_{u_{m+1}} \varphi(\mathbf{v}_{m+1}) |\Delta\mathbf{v}_{m+1}|$ and $\varepsilon_{2,n} = \sum_{m=M_1}^{M_2} S_{u_{m+1}} r_{m+1}$. The first term in (B.6) can be represented as $T_{1,n} = T_{2,n} + \varepsilon_{3,n}$, where

$$T_{2,n} = \sum_{m=M_1}^{M_2} \frac{1}{\sqrt{\lambda_{m+1}}} S_{u_{m+1}} \varphi(\mathbf{v}_{m+1}) |z_m| \Delta\lambda_{m+1},$$

$\varepsilon_{3,n} = T_{1,n} - T_{2,n}$ and $z_m = \lambda_m^{-1} \int_{u_m}^{u_{m-1}} \sigma(y_u) dW_u^{(1)} + q(\lambda_m, S_{u_m})$. Moreover, by replacing z_m in $T_{2,n}$ with

$$\widehat{z}_m = \frac{\sigma(y_{u_m})}{\Delta\lambda_m} \Delta W_{u_m} + q(\lambda_m, S_{u_m}),$$

we obtain

$$J_{1,n} = J_{2,n} + \sum_{\iota=1}^4 \varepsilon_{\iota,n}, \quad \text{where } \varepsilon_{4,n} = J_{2,n} - T_{2,n}.$$

In view of inequality (5.8) and condition (\mathbf{C}_3)

$$\lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \max_{M_1 \leq m \leq M_2} \Delta \lambda_m = 0, \quad (\text{B.7})$$

we can show directly that the term $\varepsilon_{1,n}, \varepsilon_{3,n}$ converges to zero at rate of $n^\beta \varrho^{2\beta}$. Furthermore, using the Tailor expansion we obtain

$$|\varepsilon_{2,n}| \leq CS^* \sum_{m=M_1}^{M_2} |\mathbf{v}_{m+1} - \mathbf{v}_m|^2, \quad S^* = \sup_{0 \leq t \leq 1} S_t$$

and for some constant $C > 0$,

$$\mathbf{E} |\mathbf{v}_{m+1} - \mathbf{v}_m|^2 \leq C \left(\frac{1}{n\lambda_{m+1}} + \left(\lambda_{m+1}^{1/2} - \lambda_m^{1/2} \right)^2 + \left(\lambda_{m+1}^{-1/2} - \lambda_m^{-1/2} \right)^2 \right).$$

Through condition (\mathbf{C}_3) and the inequality (5.8) we get

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \varepsilon_{2,n} = 0.$$

To finish, it remains to prove that

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \sum_{m=M_1}^{M_2} S_{u_{m+1}} \varphi(\mathbf{v}_{m+1}) |\varkappa_m - \widehat{z}_m| \Delta \lambda_{m+1} = 0$$

but this follows directly from limit equality (B.7). \square

C Justification for Heston's models

We will prove directly that $\mathbf{E} y^* < \infty$. In the sequel, we will make use c_1, c_2, \dots for positive constants. For this aim, introduce the process $\psi_t = y_{t \wedge \tau_N}$, where τ_N is a stopping time defined by

$$\tau_N = \inf\{t \geq 0 : |y_t| \geq N\} \wedge 1$$

for $N > 0$. Note that ψ_t is the solution of the following SDE $d\psi_t = A_t dt + v B_t dZ_t$ with

$$A_t = a(b - \alpha y_t) \mathbf{1}_{t \leq \tau_N}, \quad B_t = \sqrt{y_t} \mathbf{1}_{t \leq \tau_N}.$$

It is easy to see that $|A_s| \leq c_1 + c_2 \psi_s^*$, where the maximal process $\psi_t^* = \sup_{0 \leq s \leq t} |\psi_s|$. Consequently, for $0 \leq u \leq t \leq 1$,

$$|\psi_u| \leq c_1 + c_2 \int_0^t \psi_s^* ds + v \sup_{0 \leq u \leq t} \left| \int_0^u B_s dZ_s \right|.$$

Therefore,

$$\mathbf{E}|\psi_t^*| \leq c_1 + c_2 \int_0^t \mathbf{E}\psi_s^* ds + v \sqrt{\mathbf{E} \sup_{0 \leq u \leq t} \left| \int_0^u B_s dZ_s \right|^2}.$$

Taking into account that $B_s^2 = \psi_s^2 \mathbf{1}_{s \leq \tau_N} \leq \psi_s^*$ and using Dood's inequality we get

$$\mathbf{E}\psi_t^* \leq c_1 + c_2 \int_0^t \mathbf{E}\psi_s^* ds + c_3 \sqrt{\int_0^t \mathbf{E}\psi_s^* ds} \leq c_4 + c_5 \int_0^t \mathbf{E}\psi_s^* ds.$$

Thanks to Gronwall-Bellman's inequality one obtains $\mathbf{E}\psi_t^* \leq c_6$, where c_6 is some positive constant independent of N . Hence

$$\mathbf{E} \sup_{0 \leq t \leq 1} |y_{t \wedge \tau_N}| = \mathbf{E}\psi^* \leq c_6.$$

Note that $\sup_{0 \leq t \leq 1} |y_{t \wedge \tau_N}|$ converges to $\sup_{0 \leq t \leq 1} |y_t| = y^*$ as $N \rightarrow \infty$. Then Fatou's Lemma allows us to conclude that

$$\mathbf{E}y^* \leq \liminf_{N \rightarrow \infty} \mathbf{E} \sup_{0 \leq t \leq 1} |y_{t \wedge \tau_N}| < \infty$$

and model (10.6) enjoys condition (\mathbf{C}_1) . \square

D Orstein-Uhlenbeck's processes

Lemma D.1. *Suppose that $\sigma(z) \leq M(1 + |z|)$ for all z with some constant $M > 0$ and let y_t be an Orstein-Uhlenbeck process*

$$dy_t = (a - by_t)dt + dZ_t, \quad y_0 = 0$$

with some constants $a, b > 0$. Then, for all

$$0 < \alpha < \frac{b^2}{2M^2(2b + a^2)}$$

we have

$$\mathbf{E} \exp \left\{ \alpha \int_0^1 \sigma^2(y_s) ds \right\} < \infty.$$

Proof. Because of the linear growth condition, it suffices to prove that

$$\mathbf{E} \exp \left\{ 2\alpha M^2 \int_0^1 y_s^2 ds \right\} < \infty$$

for suitable values of $\alpha > 0$. Remark that for all y ,

$$(a - by)y \leq \frac{a^2}{2b} - \frac{b}{2}y^2.$$

Then, by adapting Proposition 1.1.5 in [25], p. 24 we can show that

$$\mathbf{E}|y_t|^{2m} \leq m! \left(\frac{2}{b} + \frac{a^2}{b^2} \right)^m$$

It follows that

$$\begin{aligned} \mathbf{E} \exp \left\{ 2\alpha M^2 \int_0^1 y_s^2 ds \right\} &\leq \sum_{m=0}^{\infty} \frac{(\alpha 2M^2)^m}{m!} \mathbf{E}|y_t|^{2m} \\ &\leq \sum_{m=0}^{\infty} \left(\frac{2}{b} + \frac{a^2}{b^2} \right)^m (\alpha 2M^2)^m < \infty \end{aligned}$$

if

$$0 < \alpha < \frac{b^2}{2M^2(2b + a^2)}.$$

Note that if y_t admits mean-reverting property then b takes very big values. Therefore, in this context, it is possible to choose $\alpha > 3/2 + \sqrt{2}$ as requirement of Proposition 4.3. \square

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