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# Approximate hedging problem with transaction costs in stochastic volatility markets

Huu-Thai Nguyen · Serguei Pergamenschchikov

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**Abstract** This paper investigates the problem of hedging European call options using Leland's strategy in stochastic volatility markets with transaction costs. Introducing a new form for the enlarged volatility in Leland's algorithm, we establish a limit theorem and determine a convergence rate for the hedging error. This provides a suggestion to release the underhedging property pointed out by Kabanov and Safarian in [20]. Possibilities to improve the convergence rate and lower the option price inclusive transaction costs are also discussed.

**Keywords** Leland strategy, Delta hedging, transaction costs, quantile hedging, pricing option.

**Mathematics Subject Classification (2010)** 91G20; 60G44; 60H07

**JEL Classification** G11; G13

## 1 Introduction

In the theory of hedging options, Leland's strategy provides a way to eliminate efficiently risks caused by transaction costs. This prescription is based on the idea that transaction costs can be compensated by enlarging the volatility parameter in the *delta* Black-Scholes strategy. The pioneering work in this field was first given

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in [23], where a discrete approximation was used to study the asymptotic behavior of the *hedging error*, defined as difference between the terminal portfolio value and the payoff, as the number of transactions goes to infinity. It was then shown in [25] that the hedging error vanishes if the transaction cost percentage converges to 0 at a power rate. Unfortunately, this property does not hold for the most interesting case when the proportional cost is a constant. In [20], the authors found an explicit limit for the hedging error, but unexpectedly, it is a negative quantity. It means that the option is actually underhedged in limit if the investor follows the Leland strategy. The convergence problem was investigated in the paper [16] and then, a complete answer was provided in [27] with the corresponding limit theorem allowing to identify the asymptotic distribution of the hedging error. Recently, a modified strategy with non-uniform revisions has been suggested in [10] and it turns out that this modification considerably improves the convergence rate, even in models with general convex payoffs. For related results, see further in [9, 10, 15, 16].

Many empirical studies show that the constant volatility condition in the classical Black-Scholes is restrictive and the Black-Scholes formula constructed under this assumption has an inaccuracy to anticipate the market option prices. The discrepancy between Black-Scholes option prices and market-traded ones, known as *smile curve*, can be explained by using stochastic volatility models which have been used to describe complex markets e.g. when fat-tailed returns are taken into account. Note that modeling stochastic volatility markets contains some intrinsic difficulties since stochastic volatility quantity may not be directly observed in practice. The incompleteness property of market makes the pricing problem more challenging in both constructing an explicit formula and estimating parameters. Hence, derivatives may not be perfectly hedged with only trading the underlying assets and, in general, asymptotic analysis is an efficient tool for studying such models. See [13] and the references therein for detailed discussions.

The main objective of this paper is to study the problem of hedging European style options in stochastic volatility markets with the presence of transaction costs. Under a general non-uniform revisions setting, we establish a limit result using a new form of the *adjusted volatility* parameter. The achievement here is an extension of the ones obtained in [20] and [27].

The choice of this new form is motivated from the fact that it is not possible to use the enlarged volatility (2.5) proposed in [23] and then applied in [20, 21, 7, 8, 10], since as mentioned above, stochastic volatility is not observable in practice. Another important thing can be learned from Leland's approach is that the modified volatility should be chosen so that it goes to infinity as fast as the square radical of the number of revisions in order to release effects of transaction costs. Therefore, it is reasonable to adjust the volatility parameter in some way to keep this property for stochastic volatility markets also. This idea not only allows us to carry out an approximation much more simply but also gives possibilities to improve the convergence rate. The results obtained here can be extended to options with general convex pay-off derivatives studied in [7, 10] but we leave this direction for further research.

As shown in [27] that the option price of Leland's strategy is too high because it includes transaction costs. Another practical advantage of our method is that we propose a method in order to lower the option price as long as the seller is

willing to take risks with a given probability. This approach is inspired from the quantile hedging theory [12, 3, 5].

The remain of the paper is organized as follows. We first briefly give in the next section a general view of Leland's approach then formulate the problem and present our principal results in Section 3. The new choice of adjusted volatility allows us to propose a reasonable way in Section 4 to fix the underhedging situation (shown in [20]) and lower the option price in the presence of transaction costs. The next section is devoted to present our basic approximate tools. Before showing the proof of our main Theorems, we explicitly present the asymptotic approximations for the terms consisting of the hedging error defined in (3.9). These analyses are essential to identify the asymptotic distribution through Central Limit Theorem for martingales, see [18]. All useful estimates and convergence lemmas used in the proof can be found in Appendix.

## 2 Hedging with transaction costs: a review on the Leland approach

In a complete no-arbitrage model (i.e. there exists a unique equivalent martingale measure under which the stock price is a martingale), options can be completely replicated by a self-financing trading strategy. Option price, defined as the replication cost, is the initial capital that the investor must introduce into his portfolio to obtain a complete hedge. It can be computed as the expectation of the discounted claim under the unique equivalent martingale measure.

Let us consider a continuous time model of two-asset financial market on the time interval  $[0, 1]$ , where the bond price is a constant over the time and equals to 1. The stock price dynamics follows the stochastic differential equation

$$dS_t = \sigma_0 S_t dW_t, \quad (2.1)$$

where  $\sigma_0 > 0$  is a positive constant and  $(W_t)_{0 \leq t \leq 1}$  is a standard Wiener process. As usual we denote  $\mathcal{F}_t = \sigma\{W_u, 0 \leq u \leq t\}$ .

We recall that a financial strategy  $(\beta_t, \gamma_t)_{0 \leq t \leq 1}$  (the fractions of wealth invested in bonds and stocks respectively) is called an *admissible self-financing strategy* if it is  $(\mathcal{F}_t)$ -adapted, integrable with

$$\int_0^t (|\beta_t| + \gamma_t^2) dt < \infty \quad \text{a.s.}$$

and the portfolio value satisfies the equality

$$V_t = \beta_t + \gamma_t S_t = V_0 + \int_0^t \gamma_u dS_u$$

for any  $t \in [0, 1]$ . The classical hedging problem is to find an admissible self-financing strategy  $(\beta_t, \gamma_t)$  whose terminal portfolio value exceeds the payoff  $h(S_1) = (S_1 - K)_+$ ; that is

$$V_1 = V_0 + \int_0^1 \gamma_u dS_u \geq h(S_1) \quad \text{a. s.}$$

where  $K$  is the option strike. For this problem, Black and Scholes [2] proposed a dynamically replicating self-financing strategy with  $\gamma_t = C_x(t, S_t)$  (partial derivative with respect to the space variable), where the option price  $C(t, S_t)$  reads the famous formula

$$C(t, x) = C(t, x, \sigma_0) = x\Phi(\tilde{\mathbf{v}}(t, x)) - K\Phi(\tilde{\mathbf{v}}(t, x) - \sigma_0\sqrt{1-t}), \quad (2.2)$$

where  $\tilde{\mathbf{v}}(t, x) = \mathbf{v}(\sigma_0^2(1-t), x)$  and

$$\mathbf{v}(\lambda, x) = \frac{\ln(x/K)}{\sqrt{\lambda}} + \frac{\sqrt{\lambda}}{2}. \quad (2.3)$$

Here  $\Phi$  is the standard normal distribution function. In the sequel, we denote by  $\varphi$  the  $\mathcal{N}(0, 1)$  density, i.e.  $\varphi(z) = \Phi'(z)$ . One can check directly that

$$C_x(t, x) = \Phi(\tilde{\mathbf{v}}(t, x)) \quad \text{and} \quad C_{xx}(t, x) = \frac{\varphi(\tilde{\mathbf{v}}(t, x))}{x\sigma_0\sqrt{1-t}}. \quad (2.4)$$

Clearly, hedging via discrete strategies is specially attractive since dynamically adjusted portfolios are impossible to be carried out in practice. However, discrete time hedging, in turn, will now face with some problems in view of transaction costs. Firstly, transaction costs are random and path-dependent, so they significantly effect to the hedging error. Additionally, despite of the fact argued by Black and Scholes that the hedging error may be relatively small if trading activities take place reasonably frequently, transaction costs may increase without limit as portfolio revisions are frequent, so it may lead to an explosion. If one wishes to bound the option price by using an arbitrage argument as in the Black-Scholes model then it is necessary to know the maximum transaction costs rather than the average. To our knowledge, computation of the maximum transaction costs is not a trivial task at all and this is still an open question at the moment.

These considerations lead us to the Leland approach [23], which provides an efficient technique to compensate transaction costs in a way so that they are bounded. This method is simply based on the intuition that the option price should include transaction costs as a reasonable extra fee necessary for the seller to cover the option return. In some situations (discussed in the next two sections), this strategy successfully replicates the payoff including transaction costs by simply adjusting the volatility parameter in Black-Scholes's model.

## 2.1 Constant volatility case

Let us shortly describe the Leland approach in [23, 20]. Suppose that for each trading activity, the investor has to pay a fee directly proportional to the trading volume. Naturally, we suppose that the proportional transaction cost depends upon the number of trades  $n$ , given by the law  $\kappa_* n^{-\alpha}$ , where  $\alpha \geq 0$  and  $\kappa_* > 0$  are two fixed parameters. To compensate transaction costs the investor is suggested to enlarge the volatility as

$$\hat{\sigma}^2 = \sigma_0^2 + \varrho n^{1/2-\alpha} \quad \text{and} \quad \varrho = \kappa_* \sigma_0 \sqrt{8/\pi}. \quad (2.5)$$

We assume further that the portfolio is revised discretely at

$$t_i = \frac{i}{n}, \quad i = 1, 2, \dots, n, \quad (2.6)$$

with the trading volume determined by the piecewise process (so-called Leland's strategy)

$$\gamma_t^n = \sum_{i=1}^n \widehat{C}_x(t_{i-1}, S_{t_{i-1}}) \mathbf{1}_{(t_{i-1}, t_i]}(t), \quad (2.7)$$

where  $\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) = C_x(t_{i-1}, S_{t_{i-1}}, \widehat{\sigma})$ . It means that the number of shares holding in the interval  $(t_{i-1}, t_i]$  is nothing than the delta strategy calculated at the left bound of this interval. Then, the portfolio value takes the following form

$$V_1^n = V_0^n + \int_0^1 \gamma_u^n dS_u - \kappa_* n^{-\alpha} J_n, \quad (2.8)$$

where the total trading volume is given by

$$J_n = \sum_{i=1}^n S_{t_i} |\gamma_{t_i}^n - \gamma_{t_{i-1}}^n|. \quad (2.9)$$

The option price is now given by the initial time-value of the solution  $\widehat{C}(t, x) = C(t, x, \widehat{\sigma})$  of the Black-Scholes PDE with the adjusted volatility  $\widehat{\sigma}$

$$\begin{cases} \widehat{C}_t(t, x) + \frac{1}{2} \widehat{\sigma}^2 x^2 \widehat{C}_{xx}(t, x) = 0, & 0 \leq t < 1, \\ \widehat{C}(1, x) = h(x). \end{cases} \quad (2.10)$$

Using Itô's formula we can represent the hedging error as

$$\begin{aligned} V_1^n - h(S_1) &= \int_0^1 \left( \gamma_t^n - \widehat{C}_x(t, S_t) \right) dS_t \\ &\quad + \frac{1}{2} (\widehat{\sigma}^2 - \sigma_0^2) \int_0^1 S_t^2 \widehat{C}_{xx}(t, S_t) dt - \kappa_* n^{-\alpha} J_n. \end{aligned} \quad (2.11)$$

*Remark 1 (Leland)* The specific form (2.5) results from the following intuition: the Lebesgues integral in (2.11) is clearly well-approximated by the Riemman sum of the terms  $\sigma_0 S_{t_{i-1}}^2 \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) \Delta t$ , while

$$\begin{aligned} S_{t_i} |\gamma_{t_i}^n - \gamma_{t_{i-1}}^n| &\approx \sigma_0 S_{t_{i-1}}^2 |\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})| |W_{t_i} - W_{t_{i-1}}| \\ &\approx \sigma_0 \sqrt{2/\pi} S_{t_{i-1}}^2 \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) \sqrt{\Delta t}, \end{aligned}$$

since  $\mathbf{E}|W_{t_i} - W_{t_{i-1}}| = \sqrt{2/\pi} \sqrt{\Delta t} = \sqrt{2/(\pi n)}$ . Hence, it is reasonable to expect that choosing the modified volatility as in (2.5) would give an appropriate approximation to compensate transaction costs.

Leland [23] conjectured that if the proportional transaction cost is a constant i.e.  $\alpha = 0$  then, the portfolio value of strategy (2.7) converges in probability to the payoff  $h(S_1)$  as  $n \rightarrow \infty$ . He also gave a remark without giving a complete proof that this result is still true in the case  $\alpha = 1/2$ . The later remark is correct and was completely proved by Lott in [25], where one can find a rigorous explanation why the Leland strategy is important in practice.

**Theorem 2.1 (Leland-Lott)** *For  $\alpha = 1/2$ , strategy (2.7) defines an approximately replicating strategy as the number of revision intervals  $n$  tends to infinity i.e.*

$$\mathbf{P} - \lim_{n \rightarrow \infty} V_1^n = h(S_1).$$

This result was then extended by Ahn *et al* in [1], where the replication problem with transaction costs was considered in the general diffusion models. Kabanov and Safarian [20] also observed that the Leland approach is still valid as long as the transaction proportion converges to zero as  $n \rightarrow \infty$ .

**Theorem 2.2 (Kabanov-Safarian)** *For any  $\alpha > 0$ ,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} V_1^n = h(S_1).$$

It is crucial to note that the Leland approximation in Remark 1 is not mathematically accurate and so, his first conjecture is not correct. In fact, as  $n \rightarrow \infty$ , the trading volume  $J_n$  may be approximated by the following sum

$$\sum_{i=1}^n \lambda_{i-1}^{-1/2} S_{t_{i-1}} \tilde{\varphi}(\lambda_{i-1}, S_{t_{i-1}}) \left| \frac{\sigma_0}{\varrho} Z_i + q(\lambda_{i-1}, S_{t_{i-1}}) \right| \Delta \lambda_i, \quad (2.12)$$

where

$$\lambda_i = \lambda_{t_i} = \hat{\sigma}^2(1 - t_i), \quad Z_i = (W_{t_i} - W_{t_{i-1}})/\sqrt{\Delta t_{i-1}}$$

and

$$\tilde{\varphi}(\lambda, x) = \varphi(\mathbf{v}(\lambda, x)), \quad q(\lambda, x) = \frac{\ln(x/K)}{2\lambda} - \frac{1}{4}. \quad (2.13)$$

In approximation procedures, one should also pay attention to the fact that  $\hat{C}(\cdot, \cdot)$  and its derivatives substantially depend upon  $n$ . This property leads to the following important result: there is a *non trivial discrepancy* between the limit of the terminal portfolio value and the payoff in the most practical case  $\alpha = 0$ .

**Theorem 2.3 (Kabanov-Safarian)** *If  $\alpha = 0$  then,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} V_1^n = h(S_1) + \min(S_1, K) - \kappa_* J(S_1, \varrho),$$

where

$$J(x, \varrho) = x \int_0^{+\infty} \lambda^{-1/2} \tilde{\varphi}(\lambda, x) \mathbf{E} \left| \sigma_0 \varrho^{-1} Z + q(\lambda, x) \right| d\lambda, \quad (2.14)$$

and  $Z \sim \mathcal{N}(0, 1)$  independent of  $S_1$ .

It is important to observe that the problem of option replicating is not solved in this case. Indeed, taking into account that  $\mathbf{E}|\sigma_0\varrho^{-1}Z| = 1/(2\kappa_*)$  and

$$\min(x, K) = \frac{x}{2} \int_0^\infty \frac{1}{\sqrt{\lambda}} \tilde{\varphi}(\lambda, x) d\lambda, \quad (2.15)$$

we obtain (for the parameter  $\varrho$  given in (2.5))

$$\min(x, K) - \kappa_* J(x, \varrho) = x\kappa_* \int_0^{+\infty} \frac{\tilde{\varphi}(\lambda, x)}{\sqrt{\lambda}} \left( \mathbf{E} \left| \frac{\sigma_0}{\varrho} Z \right| - \mathbf{E} \left| \frac{\sigma_0}{\varrho} Z + q(\lambda, x) \right| \right) d\lambda.$$

Now, Andreson's inequality (see, for example [19], page 155) implies directly that for any  $q \in \mathbb{R}$ ,

$$\mathbf{E}|\sigma_0\varrho^{-1}Z + q| \geq \mathbf{E}|\sigma_0\varrho^{-1}Z|.$$

Therefore,  $\mathbf{P} - \lim_{n \rightarrow \infty} (V_1^n - h(S_1)) < 0$  i.e. the option is asymptotically underhedged in this case.

Another important point to note here is that the coefficient  $\varrho$  appearing in (2.5) can be chosen in an arbitrary way. We now state the main result in [27], which also provides the convergence rate for the hedging error.

**Theorem 2.4 (Pergamenshchikov)** *Consider the Leland strategy (2.7) with  $\alpha = 0$  and let  $\varrho$  in (2.5) be some fixed positive constant. Then, the sequence of random variables*

$$n^{1/4}(V_1^n - h(S_1) - \min(S_1, K) + \kappa_* J(S_1, \varrho)) \quad (2.16)$$

*weakly converges to a centered mixed Gaussian variable as  $n \rightarrow \infty$ .*

This result is important because it not only gives the asymptotic information of the hedging error but also provides a reasonable way to fix the underhedging issue. More precisely, as discussed in [27], by choosing a suitable value of  $\varrho$  the investor can get a portfolio whose terminal value exceeds the option return as desired.

It is, of course, possible to study the Leland-Lott approximation in sense of  $L^2$ -convergence. A such result<sup>1</sup> was established in [9, 21] for the case  $\alpha = 1/2$ .

**Theorem 2.5 (Kabanov-Denis)** *Let  $\alpha = 1/2$ . The mean-square approximation error for Leland's strategy with  $\varrho$  defined in (2.5) satisfies the following asymptotic equality*

$$\mathbf{E}(V_1^n - h(S_1))^2 = Bn^{-1} + o(n^{-1}) \quad \text{as } n \rightarrow \infty,$$

*where  $B$  is some positive constant.*

Denis [10] noted that one can modify the Leland strategy to improve the convergence rate in (2.16). In particular, one can apply a non-uniform revision times  $(t_i)_{1 \leq i \leq n}$  as

$$t_i = g\left(\frac{i}{n}\right), \quad g(t) = 1 - (1-t)^\mu \quad \text{for some } \mu \geq 1 \quad (2.17)$$

<sup>1</sup> Seemingly, mean estimates do not contain much useful information since gains and losses have different meaning in practice. Clearly, if  $\alpha = 1/2$  the modified volatility is independent of  $n$

and then adjust the volatility as

$$\hat{\sigma}_t^2 = \sigma_0^2 + \kappa_* \sigma_0 \sqrt{8/\pi} \sqrt{n f'(t)}, \quad (2.18)$$

where  $f$  is the inverse function of  $g$ . He also suggested to use the following modified discrete strategy to release the discrepancy appearing in Theorems 2.3 and Theorem 2.4:

$$\gamma_t^n = \sum_{i=1}^n \left( \hat{C}_x(t_{i-1}, S_{t_{i-1}}) - \int_0^{t_{i-1}} \hat{C}_{xt}(u, S_u) du \right) \mathbf{1}_{(t_{i-1}, t_i]}(t). \quad (2.19)$$

**Theorem 2.6** *Let  $V_1^n$  be the terminal portfolio value of the strategy (2.19) with  $\alpha = 0$ . Then, for any  $1 \leq \mu < \mu_{max}$  the sequence  $n^\beta (V_1^n - h(S_1))$  weakly converges to a centered mixed Gaussian variable as  $n \rightarrow \infty$ , where*

$$\beta = \frac{\mu}{2(\mu+1)} \quad \text{and} \quad \mu_{max} = \frac{3 + \sqrt{57}}{8}. \quad (2.20)$$

## 2.2 Time-depending volatility case

We assume in this section that the stock price is now driven by the following SDE

$$dS_t = \sigma(t) S_t dW_t. \quad (2.21)$$

Under the non-uniform rebalancing plan (2.17) the investor should modify the volatility as

$$\hat{\sigma}_t^2 = \sigma^2(t) + \kappa_* \sigma(t) n^{1/2-\alpha} \sqrt{f'(t) 8/\pi}. \quad (2.22)$$

Suppose further that the general European payoff  $h(\cdot)$  is a continuous function having continuous derivatives except a finite number of points. From the Black-Scholes formula, the option price (inclusive of transaction costs) is now given by

$$\hat{C}(t, x) = \int_{-\infty}^{\infty} h(x \exp(y \lambda_t^{1/2} - \lambda_t/2)) \varphi(y) dy, \quad (2.23)$$

where  $\lambda_t = \int_t^1 \hat{\sigma}_u^2 du$ . Recall that  $\hat{C}(t, S_t)$  is the solution of the Cauchy problem (2.10). We now state the main achievement in time-depending volatility models in [7].

**Theorem 2.7 (Denis)** *Suppose that  $(\sigma(t))_{t \geq 0}$  is a strictly positive Lipschitzian function,  $h(\cdot)$  is piecewise twice differentiable and there exist  $x_* \geq 0$  and  $\delta \geq 3/2$  such that  $\sup_{x \geq x_*} x^\delta |h''(x)| < \infty$ . Then, for  $\alpha > 0$  the portfolio value of strategy (2.19) converges in probability to the payoff  $h(S_1)$  as  $n \rightarrow \infty$ . If  $\alpha = 0$ , then*

$$\mathbf{P} - \lim_{n \rightarrow \infty} V_1^n = h(S_1) + h_1(S_1) - \kappa_* h_2(S_1),$$

where  $h_1(\cdot)$  and  $h_2(\cdot)$  are positive functions depending on the payoff.

*Remark 2* Theorem 2.6 still holds in the setting of Theorem 2.7 [10].

Another way to extend the applicable rang of the Leland strategy is to consider the hedging problem with transaction costs in models where volatility values not only depend on time but also on the stock price itself. For these models, well-known as *local volatility models*, the result in Theorem 2.7 still holds for any  $1/4 \leq \alpha \leq 1/2$  [7]. However, carrying out a proof of this result is really complicated, since the existence of solution for the parabolic equation (2.10) is now not trivial if the adjusted volatility takes the form (2.22).

In summary, the Leland strategy plays an important role in the option pricing and hedging problems thanks to the easy implement in practice. The most interesting case  $\alpha = 0$  still needs to be investigated in more general situations, for instance, where volatility depends on other external random factors or jumps in stock prices are taken into account. It is worth noticing that the methodology used in the works of Denis and Kabanov needs a delicate treatment and seemingly, it is difficult to apply for such models.

### 3 Main results

Let  $(\Omega, \mathcal{F}_1, (\mathcal{F}_t)_{0 \leq t \leq 1}, \mathbf{P})$  be the standard filtered probability space with two standard independent  $(\mathcal{F}_t)_{0 \leq t \leq 1}$  adapted Wiener processes  $(W_t^{(1)})$  and  $(W_t^{(2)})$  taking their values in  $\mathbb{R}$ . Our financial market consists of one risky asset governed by the following equations on the time interval  $[0, 1]$ :

$$\begin{cases} dS_t = \sigma(y_t)S_t dW_t^{(1)}, \\ dy_t = F_1(t, y_t)dt + F_2(t, y_t)dW_t^{(3)} \end{cases} \quad (3.1)$$

where  $W_t^{(3)} = \mathbf{r}W_t^{(1)} + \sqrt{1 - \mathbf{r}^2}W_t^{(2)}$  and  $-1 \leq \mathbf{r} \leq 1$ . We assume that the  $[0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  functions  $F_1(t, y)$  and  $F_2(t, y)$  provide the existence of the unique solution to the last stochastic differential equation. Note that in this model the bond interest rate  $r = 0$  i.e. the non-risky asset is chosen as the *numéraire*. In what follows, we use the adjusted volatility given by a new form

$$\theta_t^2 = \frac{1}{\sqrt{\mu}} \varrho \sqrt{n} (1 - t)^{\frac{1-\mu}{2\mu}}, \quad 1 \leq \mu < 2. \quad (3.2)$$

The parameter  $\varrho > 0$  plays an important role in controlling the convergence rate and it will be specified later. As discussed in details below, the limit of the total trading volume  $J_n$  essentially involves in how  $\varrho$  depends upon the number of revisions  $n$ . We will formulate our dependence condition concerning the following parameter

$$\mu_* = \frac{1}{2(\mu + 2)}. \quad (3.3)$$

*Remark 3* The use of (3.2) is motivated from the following fact. Generally, the volatility and the process  $(y_t)_{0 \leq t \leq 1}$  are difficult to be observed therefore we can not use the formula (2.22). This modification also permits one to determine directly the solution of the Cauchy problem (2.10). This suggestion turns out to be important in dealing with stochastic volatility models since even in local stochastic models, proving the existence of solution to (2.10) requires much efforts [7]. We will also see in Section 4.1 that the underhedging situation pointed out in [20] can be improved by choosing some suitable value of the parameter  $\varrho$ .

For the sake of simplicity, we restrict our attention only to the standard European call options, although it is possible to extend the results here to cases of general convex payoff functions in [7]. Let us consider the option hedging problem for the model (3.1) in the case  $\alpha = 0$  by using the modified Leland's strategy (2.7)

$$\gamma_t^n = \sum_{i=1}^n \tilde{\Phi}(\lambda_{t_{i-1}}, S_{t_{i-1}}) \mathbf{1}_{(t_{i-1}, t_i]}(t), \quad (3.4)$$

where  $\tilde{\Phi}(\lambda, x) = \Phi(\mathbf{v}(\lambda, x))$ ,

$$\lambda_t = \int_t^1 \theta_s^2 ds = \tilde{\mu} \varrho \sqrt{n} (1-t)^{\frac{1}{4\beta}} \quad \text{and} \quad \tilde{\mu} = \frac{2\sqrt{\mu}}{\mu+1}.$$

We recall that the parameter  $\beta$  is defined in (2.20). This strategy yields the portfolio value (2.8), which is discretely adjusted at times given by (2.17). Now by Itô's formula we obtain

$$h(S_1) = \hat{C}(1, S_1) = \hat{C}(0, S_0) + \int_0^1 \hat{C}_x(t, S_t) dS_t - \frac{1}{2} I_{1,n}, \quad (3.5)$$

where

$$I_{1,n} = \int_0^1 \left( \theta_t^2 - \sigma^2(y_t) \right) S_t^2 \hat{C}_{xx}(t, S_t) dt. \quad (3.6)$$

Recall that  $\hat{C}(t, x)$  is the solution of the Cauchy problem

$$\hat{C}_t(t, x) + \frac{1}{2} \theta_t^2 x^2 \hat{C}_{xx}(t, x) = 0, \quad \hat{C}(1, x) = h(x),$$

with two first derivatives given as in (2.4):

$$\hat{C}_x(t, x) = \tilde{\Phi}(\lambda_t, x) \quad \text{and} \quad \hat{C}_{xx}(t, x) = \frac{1}{x\sqrt{\lambda_t}} \tilde{\varphi}(\lambda_t, x).$$

Setting  $V_0 = \hat{C}(0, S_0)$  we represent the hedging error as

$$V_1^n - h(S_1) = \frac{1}{2} I_{1,n} + I_{2,n} - \kappa_* J_n, \quad (3.7)$$

where

$$I_{2,n} = \int_0^1 \left( \gamma_t^n - \hat{C}_x(t, S_t) \right) dS_t \quad (3.8)$$

and  $J_n$  is defined as in (2.9).

The goal now is to find the limit of  $V_1^n - h(S_1)$  and point out the convergence rate as  $n \rightarrow \infty$ . To this end, we investigate the limit of the terms in the right hand side of (3.7). Note that  $I_{2,n}$  is known as the discrete hedging error that appears in discretizing hedging. In our setting with the rebalancing times given by (2.17),  $I_{2,n}$  converges to zero faster than  $n^\beta$  while the gamma hedging error  $I_{1,n}$  approaches to  $2 \min(S_1, K)$  with the same rate. Finally, as shown in Section 7, the total trading volume  $J_n$  converges in probability (as  $n \rightarrow \infty$ ) to the random variable  $J(S_1, y_1, \varrho)$  defined as follows

$$J(x, y, \varrho) = x \int_0^{+\infty} \lambda^{-1/2} \tilde{\varphi}(\lambda, x) \mathbf{E} \left| \frac{\sigma(y)}{\varrho} Z + q(\lambda, x) \right| d\lambda, \quad (3.9)$$

where  $Z \sim \mathcal{N}(0, 1)$  independent of  $\mathcal{F}_1$ .

In order to determine the asymptotic distribution we need to find the principal part of these terms. The most challenging issue in our analyses is that the rest term of total transaction costs naturally takes a discrete form whereas the one obtained by studying  $I_{1,n}$  (see Section 6) has a continuous form. In order to combine these two quantities into a unified form that permits one to apply the Central Limit Theorem for martingales, we use a special discretization procedure. The following condition of the volatility will be necessary throughout the paper:

**C<sub>1</sub>)** Assume that  $\sigma(y)$  is a  $\mathbb{R} \rightarrow \mathbb{R}$  twice differentiable function such that for all  $y \in \mathbb{R}$

$$0 < \sigma_1 \leq \sigma(y) \leq \sigma_2 < \infty$$

for some fixed bounds  $\sigma_1$  and  $\sigma_2$ .

**Theorem 3.1** For any  $\varrho > 0$  the sequence

$$n^\beta (V_1^n - h(S_1) - \min(S_1, K) + \kappa_* J(S_1, y_1, \varrho))$$

weakly converges to some centered mixed Gaussian variable as  $n \rightarrow \infty$ .

*Remark 4* This theorem is a generalization including an improved convergence rate of the results in [20, 27], where the uniform revision is taken and the volatility is assumed to be a constant. Also see that in our model, the parameter  $\mu$  takes its values in the interval  $[1, 2)$ , that is slightly more general than the condition imposed in Theorem 2.6.

To treat the underhedging issue, intuitively, the option writer will sell the option at a higher price to cover the hedging expenses made by the presence of transaction costs, but this is equivalent to the situation where a reasonably big value of the volatility is applied in the Black-Scholes formula. In particular, to obtain a non-negative terminal value for the hedging error, the modified volatility must to be taken with some large enough value of  $\varrho$ . By letting  $\varrho \rightarrow \infty$  we observe that

$$\lim_{\varrho \rightarrow \infty} J(x, y, \varrho) = \int_0^{+\infty} \lambda^{-1/2} \tilde{\varphi}(\lambda, x) |q(\lambda, x)| d\lambda := J^*(x). \quad (3.10)$$

Our next result is established under the following condition on  $\varrho$ .

**C<sub>2</sub>)** The parameter  $\varrho = \varrho(n)$  is a function of  $n$  such that

$$\lim_{n \rightarrow \infty} \frac{\varrho}{n^{\mu_*}} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\sqrt{\ln n}}{\varrho} = 0,$$

where  $\mu_*$  is given in (3.3).

**Theorem 3.2** If  $\varrho$  satisfies condition (C<sub>2</sub>) then the sequence

$$n^\beta \varrho^{2\beta} (V_1^n - h(S_1) - \min(S_1, K) + \kappa_* J^*(S_1))$$

weakly converges to a centered mixed Gaussian variable as  $n \rightarrow \infty$ .

*Remark 5* It turns out that the convergence rate is improved if  $\varrho$  is chosen as a function of the revision number  $n$ . The asymptotic distributions in both Theorems 3.1 and Theorem 3.2 are explicitly defined in Section 8.

#### 4 Applications for pricing problems

In this section we focus on how to apply our main results to option pricing with transaction costs. We first emphasize that it is impossible to obtain a *non-trivial* perfect hedge with the presence of transaction costs even in constant volatility models. In other words, to cover completely the option return, the seller can take the *buy-and-hold* strategy, but this makes the option price too high. However, once the investor accepts to take a risk in his hedging problem, the price option of Leland's strategy can be lowered in a way so that the payoff is covered with a given probability.

##### 4.1 Superhedging with transaction costs

To stand on the safe side, the investor will search for strategies providing the terminal value that exceeds the payoff. Such strategy usually concern solutions to dynamic optimization problems. More precisely, let  $H$  be a general contingent claim and denote by  $\mathcal{A}(x)$  the set of all *admissible strategies*  $\pi$  with the initial capital  $x$  and  $V_T^{\pi,x}$  the terminal value of strategy  $\pi$ . Then, the super-replication cost of  $H$  is determined as

$$U_0 = \inf \{x \in \mathbb{R} : \exists \pi \in \mathcal{A}(x) : V_T^{\pi,x} \geq H \quad \text{a.s.} \quad \}, \quad (4.1)$$

see [22] and the references therein.

In the presence of transaction costs, Cvitanic [6] shows that the *buy-and-hold strategy* is the unique choice if one wishes to successfully replicate the option and then  $S_0$  is the super-replication price. In this section, we will show that this property still holds in the sense of robust superhedging via the Leland strategy.

Recall first that the option is asymptotically underhedged if the parameter  $\varrho$  takes values smaller than  $\kappa_* \sigma(y_1) \sqrt{8/\pi}$ . The problem is challenging because the process driving volatility  $y_t$  is, in general, not observable. In our setting with boundedness volatility, this problem can be fixed by taking a suitable value of  $\varrho$ . The following analysis is based on our first main Theorem 3.1 where  $\varrho$  is a fixed positive constant independent of  $n$ . By (3.10) we have  $\lim_{\varrho \rightarrow \infty} J(S_1, y_1, \varrho) = J^*(S_1)$ . It is easy to see that

$$J^*(S_1) \leq \min(S_1, K) \quad (4.2)$$

and

$$|J(S_1, y_1, \varrho) - J^*(S_1)| \leq \frac{1}{\varrho} \sqrt{2/\pi} S_1 \sigma(y_1) \int_0^{+\infty} \lambda^{-1/2} \tilde{\varphi}(\lambda, S_1) d\lambda.$$

By virtue of (2.15), one deduces that

$$|J(S_1, y_1, \varrho) - J^*(S_1)| \leq 2 \sqrt{\frac{2}{\pi}} \frac{\sigma(y_1)}{\varrho} \min(S_1, K). \quad (4.3)$$

Now, (4.2)–(4.3) imply directly that

$$\min(S_1, K) - \kappa_* J(S_1, y_1, \varrho) \geq \tilde{b} \min(S_1, K) > 0 \quad (4.4)$$

almost surely for

$$\varrho > \varrho^* = \sigma_2 \sqrt{8/\pi \kappa_*} / (1 - \kappa_*),$$

where

$$\tilde{b} = 1 - \kappa_* - 2\kappa_* \sqrt{\frac{2}{\pi} \frac{\sigma(y_1)}{\varrho}}.$$

Thus, asymptotically the option is successfully replicated via Leland's strategy for some value  $\varrho > \varrho^*$ . But this is not a big surprise because the option is now sold at high price. In fact, from the Black-Scholes formula, one observes that

$$C(0, S_0, \hat{\sigma}) \rightarrow S_0 \quad \text{as} \quad \hat{\sigma} \rightarrow \infty.$$

It leads to the well-known strategy in the superhedging problem [22] i.e. to cover the option the seller just takes the trivial strategy: buy a stock share at time  $t = 0$  at price  $S_0$  and keep it until the expiry.

#### 4.2 Asymptotic quantile pricing

As seen in the previous section, the superhedging cost is too high for the investor though it indeed gives him a successful hedge with probability 1. One can ask that how much initial capital the investor can save by accepting a shortfall probability. From an applied point of view, the investor may take a risk and look for hedges with the minimal initial cost defined by

$$\inf \{x \in \mathbb{R} : \exists \pi \in A(x) : P(V_T^{\pi, x} \geq H) \geq 1 - \varepsilon\},$$

with a given significant level  $0 \leq \varepsilon \leq 1$ . This is one key problem of quantile hedging theory which was first studied by Föllmer and Leukert [12] then developed by Novikov [26]. Baran [3] considered the situation where transaction costs are taken into account but in a different approach. Recently, Bratyk and Mishura [5] have studied quantile hedging inclusive dependence structure driven by fractional Brownian motions. See also Barski [4], where a lot of specific examples are shown.

Let us return to the quantile hedging problem in the presence of transaction costs. Thanks to Theorem 3.1, we will suggest a reasonable way adapted from [27] to lower the option price for the Leland strategy to obtain a hedge with a given probability  $1 - \varepsilon$ .

More precisely, we propose to sell the option at the price  $\delta S_0$ , (where  $0 < \delta < 1$  will be properly chosen) and the remaining part  $(1 - \delta)S_0$  will be paid by the option seller himself. He must include this fee in the hedging problem; that is, the asymptotic terminal portfolio value must exceed the sum of the payoff  $h(S_1)$  and the rest fee  $(1 - \delta)S_0$ . By (4.4) this purpose can be achieved with a positive probability for some value  $\delta \in (0, 1)$ . To determine the option price it now remains to choose value  $\delta$ . We suggest to define it as

$$\delta_\varepsilon = \inf \{a > 0 : \Upsilon(a) \geq 1 - \varepsilon\}, \quad (4.5)$$

where  $\Upsilon(a) = \mathbf{P}(\min(S_1, K) - \kappa_* J(S_1, y_1, \varrho) > (1 - a)S_0)$ . Then, the price of Leland strategy can be substantially lowered as compared with the power function of parameter  $\varepsilon$  with an arbitrary small power.

**Proposition 1** *Let  $\delta_\varepsilon$  be Leland price defined by (4.5) and  $\sigma(\cdot)$  be a bounded function. Then, for any  $r > 0$ ,*

$$\lim_{\varepsilon \rightarrow 0} \frac{1 - \delta_\varepsilon}{\varepsilon^r} = +\infty. \quad (4.6)$$

*Proof* By definition,

$$\mathbf{P}(\min(S_1, K) - \kappa_* J(S_1, y_1, \varrho) > (1 - \delta_\varepsilon) S_0) \geq 1 - \varepsilon. \quad (4.7)$$

By (4.4),  $\Upsilon(1) = 1$  so,  $\delta_\varepsilon < 1$ . Moreover, for  $0 < \varepsilon < P(S_1 \leq S_0)$ , we have

$$\Upsilon(0) \leq \mathbf{P}(\min(S_1, K) > S_0) \leq \mathbf{P}(S_1 > S_0) \leq 1 - \varepsilon.$$

So,  $0 < \delta_\varepsilon \leq 1$  and  $\delta_\varepsilon$  tends to 1 as  $\varepsilon \rightarrow 0$ . Setting  $b = 1 - \kappa_* - 2\sqrt{2}\kappa_*\sigma_2/(\sqrt{\pi}\varrho)$ , one now deduces from (4.4) and (4.7) that, for sufficiently small  $\varepsilon$  (such that for which  $\delta_\varepsilon > 1 - bK/S_0$ )

$$\begin{aligned} 1 - \varepsilon &> \mathbf{P}(\min(S_1, K) - \kappa_0 J(S_1, y_1, \varrho) > (1 - \delta_\varepsilon) S_0) \\ &\geq \mathbf{P}(b \min(S_1, K) > (1 - \delta_\varepsilon) S_0) \\ &= \mathbf{P}(\min(S_1/S_0, K/S_0) > (1 - \delta_\varepsilon)/b) \\ &= 1 - \mathbf{P}(S_1/S_0 \leq (1 - \delta_\varepsilon)/b). \end{aligned}$$

Therefore,

$$\varepsilon \leq \mathbf{P}(S_1/S_0 \leq (1 - \delta_\varepsilon)/b) \leq \mathbf{P}(X_1 \leq -z_\varepsilon),$$

where  $X_t = \int_0^t \sigma(y_t) dW_t^{(1)}$  and  $z_\varepsilon = \ln b/(1 - \delta_\varepsilon) - \sigma_2^2/2$ . To estimate this probability we note that for any integer  $m \geq 1$ ,

$$\mathbf{E}(X_1)^{2m} \leq \sigma_2^{2m} (2m - 1)!!$$

(see, for example, [24, Lemma 4.11, page 130]). Setting now  $R(\mathbf{a}) = 2\mathbf{a}\sigma_2^2$ , we obtain that for any  $0 < \mathbf{a} < 1/2\sigma_2^2$ ,

$$\mathbf{E} e^{\mathbf{a}X_1^2} = \sum_{m=0}^{\infty} \frac{\mathbf{a}^m}{m!} \mathbf{E}(X_1)^{2m} \leq \sum_{m=0}^{\infty} \frac{\mathbf{a}^m}{m!} \sigma_2^{2m} (2m - 1)!! \leq \frac{1}{1 - R(\mathbf{a})}.$$

Therefore, for  $\varepsilon > 0$  sufficiently small so that  $z_\varepsilon > 0$  and  $\delta_\varepsilon > 1 - bK/S_0$  one has

$$\varepsilon \leq \mathbf{P}(X_1 \leq -z_\varepsilon) = \mathbf{P}(-X_1 \geq z_\varepsilon) \leq \frac{e^{-\mathbf{a}z_\varepsilon^2}}{1 - R(\mathbf{a})},$$

i.e.

$$1 - \delta_\varepsilon \geq b \exp \left\{ -\sqrt{\frac{1}{\mathbf{a}}} |\ln \varepsilon (1 - R(\mathbf{a}))| - \sigma_2^2/2 \right\}.$$

This inequality implies immediately the asymptotic bound (4.6).  $\square$

## 5 Basic asymptotic tools

In this section we construct a special procedure in order to combine the gamma error  $I_{1,n}$  with the transaction costs naturally being a discrete sum. Firstly, for  $n$ -functions

$$l_* = 1/\ln^3 n \quad \text{and} \quad l^* = \ln^3 n$$

we set

$$M_1 = \left\lceil n \left( \frac{l_*}{\lambda_0} \right)^{2/(\mu+1)} \right\rceil \quad \text{and} \quad M_2 = \left\lceil n \left( \frac{l^*}{\lambda_0} \right)^{2/(\mu+1)} \right\rceil - 1, \quad (5.1)$$

where  $\lambda_0 = \tilde{\mu} \varrho \sqrt{n}$  ( $[x]$  is the integer part of a number  $x$ ). In order to estimate the hedging error we need the following two sequences  $(u_m)_{M_1 \leq m \leq M_2}$  and  $(\lambda_m)_{M_1 \leq m \leq M_2}$  defined as

$$u_m = 1 - (\tilde{t}_m)^\mu, \quad \lambda_m = \lambda_0 (\tilde{t}_m)^{\frac{\mu+1}{2}} \quad \text{and} \quad \tilde{t}_m = m/n. \quad (5.2)$$

Note that  $(u_m)$  is a decreasing sequence with values in  $[u^*, u_*]$ , where  $u_* = 1 - (l_*/\lambda_0)^{4\beta}$  and  $u^* = 1 - (l^*/\lambda_0)^{4\beta}$ . We also use their invert sequences  $(\tilde{u}_j)_{N_2 \leq j \leq N_1}$  and  $(\tilde{\lambda}_j)_{N_2 \leq j \leq N_1}$  defined as

$$\tilde{u}_j = 1 - (1 - \tilde{t}_j)^\mu \quad \text{and} \quad \tilde{\lambda}_j = \lambda_0 (1 - \tilde{t}_j)^{\frac{\mu+1}{2}} = \lambda_0 (1 - \tilde{u}_j)^{\frac{1}{4\beta}}, \quad (5.3)$$

where  $N_1 = n - M_1$  and  $N_2 = n - M_2$ . To approximate Itô integrals, we will use the following sequences of independent normal random variables

$$\tilde{Z}_{1,j} = \frac{W_{\tilde{u}_j}^{(1)} - W_{\tilde{u}_{j-1}}^{(1)}}{\sqrt{\tilde{u}_j - \tilde{u}_{j-1}}}, \quad \tilde{Z}_{2,j} = \frac{W_{\tilde{u}_j}^{(2)} - W_{\tilde{u}_{j-1}}^{(2)}}{\sqrt{\tilde{u}_j - \tilde{u}_{j-1}}}$$

and

$$Z_{1,m} = \frac{W_{u_m}^{(1)} - W_{u_{m+1}}^{(1)}}{\sqrt{u_m - u_{m+1}}}, \quad Z_{2,m} = \frac{W_{u_m}^{(2)} - W_{u_{m+1}}^{(2)}}{\sqrt{u_m - u_{m+1}}}.$$

With the sequence of rebalancing times  $(\tilde{u}_j)$  we denote by  $(\mathcal{G}_j)$  the corresponding  $\sigma$ -fields defined as  $\mathcal{G}_j = \sigma(W_s^{(1)}, W_s^{(2)} : 0 \leq s \leq \tilde{u}_j)$  and set

$$\tilde{Z}_{3,j} = |\tilde{Z}_{1,j} + \tilde{q}_{j-1}| - \mathbf{E} \left( |\tilde{Z}_{1,j} + \tilde{q}_{j-1}| \mid \mathcal{G}_{j-1} \right),$$

where  $\tilde{q}_{j-1} = \tilde{q}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}, y_{\tilde{u}_{j-1}})$  and

$$\tilde{q}(\lambda, x, y) = \frac{\varrho}{\sigma(y)} \left( \frac{\ln(x/K)}{2\lambda} - \frac{1}{4} \right). \quad (5.4)$$

For any  $q \in \mathbb{R}$  and  $Z \sim \mathcal{N}(0, 1)$  let

$$G(q) = \mathbf{E} (|Z + q|) = 2\varphi(q) + q(2\Phi(q) - 1). \quad (5.5)$$

For convenience we write  $\mathbf{o}_{\mathbf{P}}(n^{-r})$  for any sequence of random variables  $(X_n)_{n \geq 1}$  satisfying

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^r X_n = 0.$$

**(C<sub>3</sub>)**  $f$  is a  $\mathbb{R}_+ \rightarrow \mathbb{R}$  continuously differentiable function having absolutely integrable derivative on the  $[0, +\infty)$  and

$$\lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \left( \int_0^{l_*} |f(x)| dx + \int_{l_*}^{+\infty} |f(x)| dx \right) = 0.$$

**Lemma 1** Let  $A(\cdot)$  be a function satisfying condition **(C<sub>3</sub>)**. If  $\varrho$  either is constant or satisfies condition **(C<sub>2</sub>)**, then

$$\lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \left| \sum_{j=N_2}^{N_1} A(\tilde{\lambda}_{j-1}) \Delta \tilde{\lambda}_{j-1} - \int_0^{+\infty} A(\lambda) d\lambda \right| = 0.$$

*Proof* It suffices to show that

$$\lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \left| \sum_{j=N_2}^{N_1} A(\tilde{\lambda}_{j-1}) \Delta \tilde{\lambda}_{j-1} - \int_{l_*}^{l_*^*} A(\lambda) d\lambda \right| = 0.$$

Indeed, this term can be rewritten as  $\sum_{j=N_2}^{N_1} \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} A_1(z) dz$ , with

$$A_1(z) = A(\tilde{\lambda}_{j-1}) - A(z) = \int_z^{\tilde{\lambda}_{j-1}} \dot{A}(u) du.$$

We then deduce that

$$\sum_{j=N_2}^{N_1} \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} A_1(z) dz \leq \sum_{j=N_2}^{N_1} \Delta \tilde{\lambda}_j \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} |\dot{A}(u)| du \leq \max_{N_2 \leq j \leq N_1} |\Delta \tilde{\lambda}_j| \cdot \int_0^{+\infty} |\dot{A}(u)| du.$$

On the other hand,

$$\max_{N_2 \leq j \leq N_1} |\Delta \tilde{\lambda}_j| \leq C n^{-2\beta} \varrho^{\frac{2}{\mu+1}} (l^*)^{\frac{\mu-1}{\mu+1}}. \quad (5.6)$$

So,

$$n^\beta \varrho^{2\beta} \sum_{j=N_2}^{N_1} \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} A_1(z) dz \leq C n^{-\frac{\mu}{2(\mu+1)}} \varrho^{\frac{\mu+2}{\mu+1}} (l^*)^{\frac{\mu-1}{\mu+1}} \int_0^{+\infty} |\dot{A}(u)| du.$$

The left hand side clearly converges to zero by condition **(C<sub>2</sub>)** and the lemma is proved.  $\square$

By the same way, we also deduce the following approximation.

**Lemma 2** Let  $A(\cdot)$  be a function satisfying condition **(C<sub>3</sub>)** and assume that  $\varrho$  either is constant or satisfies condition **(C<sub>2</sub>)**. Then

$$\lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \sup_{a>0} \left| \sum_{j=N_2}^{N_1} \mathbf{1}_{\{\tilde{\lambda}_{j-1} \geq a\}} A(\tilde{\lambda}_{j-1}) \Delta \tilde{\lambda}_j - \int_a^\infty A(\lambda) d\lambda \right| = 0$$

*Proof* It suffices to consider the case when  $a$  takes values in the interval  $[l_*, l^*]$ . Then there exists  $N_2 \leq k \leq N_1$  such that  $a \in [\tilde{\lambda}_k, \tilde{\lambda}_{k-1}]$ . After some computation we can write the difference in the absolute sign as

$$\sum_{j=N_2}^{N_1} \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} (A(\tilde{\lambda}_{j-1}) - A(\lambda)) d\lambda - \int_a^{\tilde{\lambda}_{k-1}} A(\lambda) d\lambda.$$

Then, the limit follows in view of estimate (5.6).  $\square$

In the sequel we will use the following limit theorem for martingales from [18] (Theorem 3.2 and Corollary 3.1, p.58).

**Theorem 5.1** *Let  $\mathbf{M}_n = \sum_{i=1}^n X_i$  be a zero-mean, square integrable martingale and  $\varsigma$  be an a.s. finite random variable. Assume that the following convergences are satisfied in probability: for any  $\delta > 0$ ,*

$$\sum_{i=1}^n \mathbf{E} \left( X_i^2 \mathbf{1}_{\{|X_i| > \delta\}} | \mathcal{F}_{i-1} \right) \longrightarrow 0 \quad \text{and} \quad \sum_{i=1}^n \mathbf{E} \left( X_i^2 | \mathcal{F}_{i-1} \right) \longrightarrow \varsigma^2.$$

*Then, the sequence  $(\mathbf{M}_n)$  converges in law to  $Y$  whose characteristic function is  $\mathbf{E} \exp(-\frac{1}{2} \varsigma^2 t^2)$  i.e.  $Y$  is a mixed Gaussian variable.*

The following technical condition will be imposed to estimate moment of a  $\mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$  function  $\mathbf{A}$  holding condition  $(\mathbf{C}_3)$ .

$\mathbf{C}_4$ ) *For any  $\gamma > 0$ , there exist functions  $U_\gamma(\cdot) > 0$  such that*

$$\sup_{x>0} \left( \sup_{\lambda>0} \frac{|\mathbf{A}(\lambda, x)|}{\min(\lambda^\gamma, \lambda^{-\gamma})} - U_\gamma(x) \right) \leq 0$$

*and for any  $m \geq 1$ ,*

$$\mathbf{E} \sup_{0 \leq t \leq 1} (U_\gamma(S_t))^m < \infty.$$

We are looking for the limit of the following sequence  $(\mathbf{M}_k)_{N_2 \leq k \leq N_1}$  defined as

$$\mathbf{M}_k = \sum_{j=N_2}^k v_j, \tag{5.7}$$

where  $v_j = \sum_{i=1}^3 \tilde{\mathbf{A}}_{i,j-1} \tilde{Z}_{i,j} \Delta \tilde{\lambda}_j$  and  $\tilde{\mathbf{A}}_{i,j} = \mathbf{A}_i(\tilde{\lambda}_j, S_{\tilde{u}_j})$ . Set

$$\begin{aligned} \mathbf{L}(\lambda, x, y) &= \mathbf{A}_1^2(\lambda, x) + 2\mathbf{A}_1(\lambda, x)\mathbf{A}_3(\lambda, x)(2\Phi(\tilde{q}) - 1) \\ &\quad + \mathbf{A}_3^2(\lambda, x)\Lambda(\tilde{q}) + \mathbf{A}_2^2(\lambda, x), \end{aligned} \tag{5.8}$$

where  $\Lambda(q) = 1 + q^2 - G^2(q)$  and the function  $\tilde{q} = \tilde{q}(\lambda, x, y)$  is defined in (5.4).

**Proposition 2** *Assume that for  $1 \leq i \leq 3$  the functions  $\mathbf{A}_i(\cdot, \cdot)$  satisfy conditions  $(\mathbf{C}_3)$  and  $(\mathbf{C}_4)$ . Then, for any fixed  $\varrho > 0$  the sequence  $(n^\beta \mathbf{M}_{N_1})_{n \geq 1}$  weakly converges to a mixed Gaussian variable with mean zero and variance  $\varsigma^2$  defined as*

$$\varsigma^2 = \check{\mu} \varrho^{\frac{2}{\mu+1}} \int_0^{+\infty} \lambda^{\frac{\mu-1}{\mu+1}} \mathbf{L}(\lambda, S_1, y_1) d\lambda \quad \text{with} \quad \check{\mu} = \frac{1}{2}(\mu+1) \tilde{\mu}^{\frac{2}{\mu+1}}.$$

*Proof* Putting  $v_{i,j} = \tilde{\mathbf{A}}_{i,j-1} \tilde{Z}_{i,j} \Delta \tilde{\lambda}_j$  we can rewrite  $\mathbf{M}_k$  as follows

$$\mathbf{M}_k = \sum_{j=N_2}^k v_j = \sum_{i=1}^3 \sum_{j=N_2}^k v_{i,j}.$$

Using Cauchy-Schwartz's inequality then Markov's inequality, we get for any  $\mathbf{a} > 0$

$$\mathbf{E} v_{1,j}^2 \mathbf{1}_{\{|v_{1,j}| > \mathbf{a}\}} \leq \sqrt{\mathbf{E} v_{1,j}^4} \sqrt{\mathbf{P}(|v_{1,j}| > \mathbf{a})} \leq \frac{1}{\mathbf{a}^2} \mathbf{E} v_{1,j}^4.$$

On the other hand, from condition  $(\mathbf{C}_4)$  there exists  $r > 0$  such that

$$\mathbf{E} v_{1,j}^4 \leq \mathbf{E} \tilde{\mathbf{A}}_{1,j-1}^4 \tilde{Z}_{1,j}^2 (\Delta \tilde{\lambda}_j)^4 \leq C(\tilde{\lambda}_{j-1}^r + \tilde{\lambda}_{j-1}^{-r}) (\Delta \tilde{\lambda}_j)^4.$$

Thanks to (5.6)

$$n^{2\beta} \sum_{j=N_2}^{N_1} \mathbf{E} \left( v_{1,j}^2 \mathbf{1}_{\{|v_{1,j}| > \mathbf{a}\}} | \mathcal{G}_{j-1} \right) \longrightarrow 0 \text{ in } L^1.$$

Remark that the random variable  $\tilde{Z}_{3,j}$  admits moments bounded by a constant independent of  $n$ . Hence, there exists  $r > 0$  such that

$$\mathbf{E}(v_{2,j}^4) \leq C(\Delta \tilde{\lambda}_{j-1})^4 \left( \mathbf{E} \tilde{\mathbf{A}}_{2,j-1}^8 \mathbf{E} \left( \tilde{Z}_{3,j} \right)^8 \right)^{1/2} \leq C \left( \tilde{\lambda}_{j-1}^r + \tilde{\lambda}_{j-1}^{-r} \right) (\Delta \tilde{\lambda}_j)^4,$$

which implies the convergence to zero in  $L^1$  of the sum

$$n^{2\beta} \sum_{j=N_2}^{N_1} \mathbf{E} \left( v_{2,j}^2 \mathbf{1}_{\{|v_{2,j}| > \mathbf{a}\}} | \mathcal{G}_{j-1} \right).$$

Similarly,

$$n^{2\beta} \sum_{j=N_2}^{N_1} \mathbf{E} \left( v_{3,j}^2 \mathbf{1}_{\{|v_{3,j}| > \mathbf{a}\}} | \mathcal{G}_{j-1} \right) \longrightarrow 0 \text{ in } L^1.$$

Thus, for any  $\mathbf{a} > 0$

$$n^{2\beta} \sum_{j=N_2}^{N_1} \mathbf{E} \left( v_j^2 \mathbf{1}_{\{|v_j| > \mathbf{a}\}} | \mathcal{G}_{j-1} \right) \longrightarrow 0 \text{ in } L^1. \quad (5.9)$$

Let us establish the limit of the sum of conditional variances  $\mathbf{E}(v_j^2 | \mathcal{G}_{j-1})$ . Since  $\tilde{Z}_{1,j}$  and  $\tilde{Z}_{2,j}$  are independent,

$$\mathbf{E}(v_{1,j} v_{3,j} | \mathcal{G}_{j-1}) = \mathbf{E}(v_{2,j} v_{3,j} | \mathcal{G}_{j-1}) = 0.$$

It follows that

$$\begin{aligned} \mathbf{E}(v_j^2 | \mathcal{G}_{j-1}) &= \mathbf{E}(v_{1,j}^2 | \mathcal{G}_{j-1}) + \mathbf{E}(v_{2,j}^2 | \mathcal{G}_{j-1}) + \mathbf{E}(v_{3,j}^2 | \mathcal{G}_{j-1}) \\ &\quad + 2\mathbf{E}(v_{1,j} v_{2,j} | \mathcal{G}_{j-1}). \end{aligned}$$

Observe that for  $Z \sim N(0, 1)$  and some constant  $q$ ,

$$\mathbf{E}(Z|Z+q) = 2\Phi(q) - 1 \quad \text{and} \quad \mathbf{E}(Z+q)^2 - (\mathbf{E}|Z+q|)^2 = \Lambda(q).$$

So, setting  $\hat{\mu} = (\mu - 1)/(\mu + 1)$  one can represent  $\mathbf{E}(v_j^2|\mathcal{G}_{j-1})$  as

$$n^{2\beta} \mathbf{E}(v_j^2|\mathcal{G}_{j-1}) = (1 + o(1)) \check{\mu} \varrho^{\frac{2}{\mu+1}} (\tilde{\lambda}_{j-1})^{\hat{\mu}} \mathbf{L}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}, y_{\tilde{u}_{j-1}}) \Delta \tilde{\lambda}_j.$$

Due to Lemma 1, the sum  $n^{2\beta} \sum_{j=N_2}^{N_1} \mathbf{E}(v_j^2|\mathcal{G}_{j-1})$  converges almost surely to  $\varsigma^2$  as  $n \rightarrow \infty$  and through Theorem 5.1 we obtain the needed conclusion.  $\square$

Consider the case where  $\varrho \rightarrow \infty$  as  $n \rightarrow \infty$ . Put

$$\check{\mathbf{M}}_k = \sum_{j=N_2}^k \check{v}_j, \quad (5.10)$$

where  $\check{v}_j = \left( \tilde{\mathbf{A}}_{1,j-1} \tilde{Z}_{1,j} + \tilde{\mathbf{A}}_{3,j-1} \tilde{Z}_{3,j} \right) \Delta \tilde{\lambda}_j$ .

**Proposition 3** *Let  $\mathbf{A}_1$  and  $\mathbf{A}_3$  be functions satisfying conditions  $(\mathbf{C}_3)$  and  $(\mathbf{C}_4)$ . Then, under condition  $(\mathbf{C}_2)$  the sequence of random variables*

$$\left( n^\beta \varrho^{\frac{-1}{\mu+1}} \check{\mathbf{M}}_{N_1} \right)_{n \geq 1}$$

*weakly converges as  $n \rightarrow \infty$  to a mixed Gaussian variable with mean zero and variance defined as*

$$\varsigma^2 = \check{\mu} \int_0^{+\infty} \lambda^{\frac{\mu-1}{\mu+1}} \check{\mathbf{L}}(\lambda, S_1) d\lambda,$$

where  $\check{\mathbf{L}}(\lambda, x) = \mathbf{A}_1^2(\lambda, x) + 2\mathbf{A}_1(\lambda, x)\mathbf{A}_3(\lambda, x) + \mathbf{A}_3^2(\lambda, x)$ .

*Proof* By the same way used in Lemma 2 we have for any  $\mathbf{a} > 0$ ,

$$\lim_{n \rightarrow \infty} \sum_{j=N_2}^{N_1} \mathbf{E} \left( \check{v}_j^2 \mathbf{1}_{\{|\check{v}_j| > \mathbf{a}\}} | \mathcal{G}_{j-1} \right) = 0 \quad \text{in } L^1.$$

To determine limit of the sum of conditional variances, we observe that

$$n^{2\beta} \varrho^{\frac{-2}{\mu+1}} \mathbf{E}(\check{v}_j^2 | \mathcal{G}_{j-1}) = \check{\mu} (1 + o(1)) \tilde{\lambda}_{j-1}^{\hat{\mu}} \check{\mathbf{K}}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}) \Delta \tilde{\lambda}_j, \quad (5.11)$$

with  $\check{\mathbf{K}}(\lambda, x) = \mathbf{A}_1^2(\lambda, x) + \mathbf{A}_3^2(\lambda, x) \Lambda(\tilde{q}) + 2\mathbf{A}_1(\lambda, x)\mathbf{A}_3(\lambda, x) (2\Phi(|\tilde{q}|) - 1)$ .

One can check directly that for any  $q \in \mathbb{R}$  the function  $G(\cdot)$  defined in (5.5) satisfies the following inequalities:

$$|q| \leq G(q) \leq |q| + 2\varphi(q). \quad (5.12)$$

This implies  $|\Lambda(q) - 1| \leq 4|q|\varphi(q) + \varphi^2(q)$  and hence,  $\sup_{q \in \mathbb{R}} |\Lambda(q)| < \infty$ .

Note also that  $\check{\mathbf{K}} \rightarrow \check{\mathbf{L}}$  as  $n \rightarrow \infty$  since  $\tilde{q}(\lambda, x) \rightarrow \infty$  as  $\varrho = \varrho(n) \rightarrow \infty$  for any  $x > 0$  and  $\lambda \neq \mathbf{a}_0 = 2 \ln(x/K)$ . Using now Lemma 1 and the dominated convergence theorem we obtain that the sum in the right hand side of (5.11) converges to  $\varsigma^2$  as  $n \rightarrow \infty$  and the proof is completed through Theorem 5.1.  $\square$

## 6 Asymptotic properties of $I_{1,n}$ and $I_{2,n}$

First we state that the term  $I_{2,n}$  is  $n^\beta \varrho^{2\beta}$  - negligible, i.e.

**Proposition 4** *If  $\varrho$  either is constant or satisfies condition  $(C_2)$ , then*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} I_{2,n} = 0.$$

Now let us study the term  $I_{1,n}$  defined in (3.5). Set

$$\tau_n = \int_0^1 \lambda_t^{-1/2} \theta_t^2 (S_t \tilde{\varphi}(\lambda_t, S_t) - S_1 \tilde{\varphi}(\lambda_t, S_1)) dt. \quad (6.1)$$

**Lemma 3** *If  $\varrho$  either is constant or satisfies condition  $(C_2)$  then,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} |I_{1,n} - 2 \min(S_1, K) - \tau_n| = 0.$$

We are then looking for a reasonable approximation of  $\tau_n$ . By Itô's formula we obtain

$$\begin{aligned} S_1 \tilde{\varphi}(\lambda_t, S_1) - S_t \tilde{\varphi}(\lambda_t, S_t) &= \int_t^1 H_1(\lambda_t, S_u) \sigma(y_u) S_u dW_u^{(1)} \\ &\quad + \frac{1}{2} \int_t^1 H_2(\lambda_t, S_u) \sigma^2(y_u) S_u du, \end{aligned} \quad (6.2)$$

where  $H_1(v, x) = \tilde{\varphi}(v, x) (1 - \mathbf{v}(v, x)/\sqrt{v})$ , and

$$H_2(v, x) = \tilde{\varphi}(v, x) \left( \frac{\mathbf{v}^2(v, x)}{v} - \frac{\mathbf{v}(v, x)}{\sqrt{v}} - \frac{1}{v} \right).$$

Setting  $\bar{H}_1(t, x) = \int_t^{\lambda_0} v^{-1/2} H_1(v, x) dv$  and  $\bar{H}_2(t, x) = \int_t^{\lambda_0} v^{-1/2} H_2(v, x) dv$ , we can represent  $\tau_n$  as

$$\begin{aligned} \tau_n &= - \int_0^1 \theta_t^2 \sigma(y_u) S_u \bar{H}_1(\lambda_u, S_u) dW_u^{(1)} - \frac{1}{2} \int_0^1 \theta_t^2 \sigma^2(y_u) S_u \bar{H}_2(\lambda_u, S_u) du \\ &:= -\tau_{1,n} - \frac{1}{2} \tau_{2,n}. \end{aligned} \quad (6.3)$$

Again, the term  $\tau_{2,n}$  is negligible. In order to be adapted with the approximation of the total trading volume obtained in the next section, we will construct a discrete approximation for  $\tau_{1,n}$ . This objective can be done by discretizing the Itô integral at trading times  $(\tilde{u}_j)$ .

**Lemma 4** *If  $\varrho$  either is constant or satisfies condition  $(C_2)$  then,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} |\tau_{1,n} - \mathbf{U}_{N_1}| = \mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} |\tau_{2,n}| = 0, \quad (6.4)$$

where

$$\mathbf{U}_k = \varrho^{-1} \sum_{j=N_2}^k \sigma(y_{\tilde{u}_{j-1}}) S_{\tilde{u}_{j-1}} \check{H}_1(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}) \tilde{Z}_{1,j} \Delta \tilde{\lambda}_j$$

and  $\check{H}_1(t, x) = \int_t^\infty v^{-1/2} H_1(v, x) dv$ .

Now combining Lemma 3 and Lemma 4 gives us the final asymptotic form of  $I_{1,n}$ .

**Proposition 5** *If  $\varrho$  either is constant or satisfies condition  $(\mathbf{C}_2)$  then,*

$$I_{1,n} = 2 \min(S_1, K) - \mathbf{U}_{N_1} + o_{\mathbf{P}}(n^{-\beta} \varrho^{-2\beta}). \quad (6.5)$$

## 7 Approximations for trading volume $J_n$

Firstly, we approximate  $J_n$  by

$$J_{1,n} = \sum_{m=M_1}^{M_2} S_{u_{m+1}} |\Phi(\mathbf{v}_m) - \Phi(\mathbf{v}_{m+1})|, \quad (7.1)$$

where  $\mathbf{v}_m = \mathbf{v}(\lambda_m, S_{u_m})$  defined in (2.2).

**Lemma 5** *If  $\varrho$  either is constant or satisfies condition  $(\mathbf{C}_2)$  then,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} (n\varrho^2)^\beta |J_n - J_{1,n}| = 0.$$

Next, by making use of Itô's formula we can replace the term  $J_{1,n}$  by (up to  $n^\beta \varrho^{2\beta}$ -negligible) the following sum

$$J_{2,n} = \sum_{m=M_1}^{M_2} \lambda_{m+1}^{-1/2} S_{u_{m+1}} \tilde{\varphi}(\lambda_{m+1}, S_{u_{m+1}}) |\varkappa_m| \Delta \lambda_{m+1} \quad (7.2)$$

and  $\varkappa_m = \varrho^{-1} \sigma(y_{u_{m+1}}) Z_{1,m} + q(\lambda_{m+1}, S_{u_{m+1}})$ .

**Lemma 6** *If  $\varrho$  either is constant or satisfies condition  $(\mathbf{C}_2)$  then,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} (n\varrho^2)^\beta |J_{1,n} - J_{2,n}| = 0.$$

Remark that from (5.3) the decreasing sequence  $(\tilde{\lambda}_j)$  taking value in  $[l_*, l^*]$  can be computed through  $(\tilde{u}_j)$  via the relation  $\tilde{\lambda}_j = \lambda_0(1 - \tilde{u}_j)^{4\beta}$ . Now, setting

$$\tilde{\varkappa}_j = \varrho^{-1} \sigma(y_{\tilde{u}_{j-1}}) \tilde{Z}_{1,j} + q(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}})$$

we can represent  $J_{2,n}$  as  $J_{2,n} = \sum_{j=N_2}^{N_1} \tilde{\zeta}_j$ , where

$$\tilde{\zeta}_j = \tilde{\lambda}_{j-1}^{-1/2} S_{\tilde{u}_{j-1}} \tilde{\varphi}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}) |\tilde{\varkappa}_j| \Delta \tilde{\lambda}_j, \quad (7.3)$$

We need the Doob's decomposition for  $J_{2,n}$  w.r.t. the filtration  $(\mathcal{G}_j)_{N_2 \leq j \leq N_1}$ . To this end, note that

$$\mathbf{E}(\tilde{\zeta}_j | \mathcal{G}_{j-1}) = \tilde{\lambda}_{j-1}^{-1/2} S_{\tilde{u}_{j-1}} \tilde{\varphi}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}) \Delta \tilde{\lambda}_j \mathbf{E}(|\tilde{\varkappa}_j| | \mathcal{G}_{j-1}). \quad (7.4)$$

and hence,

$$\mathbf{E}(|\tilde{\varkappa}_j| | \mathcal{G}_{j-1}) = \varrho^{-1} \sigma(y_{\tilde{u}_{j-1}}) G(\tilde{q}_{j-1}) := \mathbf{G}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}, y_{\tilde{u}_{j-1}}), \quad (7.5)$$

where  $G(\tilde{q})$  given in (5.5). Setting

$$J_{3,n} = \sum_{j=N_2}^{N_1} \tilde{\lambda}_{j-1}^{-1/2} S_{\tilde{u}_{j-1}} \tilde{\varphi}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}) \mathbf{G}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}, y_{\tilde{u}_{j-1}}) \Delta \tilde{\lambda}_j$$

$\bar{\kappa}_j = |\tilde{\kappa}_j| - \mathbf{G}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}, y_{\tilde{u}_{j-1}})$  we observe directly that

$$J_{2,n} = J_{3,n} + \mathbf{m}'_{N_1}, \quad (7.6)$$

where

$$\mathbf{m}'_k = \sum_{j=N_2}^k \tilde{\lambda}_{j-1}^{-1/2} S_{\tilde{u}_{j-1}} \tilde{\varphi}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}) \bar{\kappa}_j \Delta \tilde{\lambda}_j.$$

Now we need to distinguish the following two cases:  $\varrho = \varrho(n)$  goes to infinity and  $\varrho$  is a fixed positive constant.

**Case 1:**  $\lim_{n \rightarrow \infty} \varrho = \infty$ .

Note that  $\mathbf{E}|aZ + b|$  may be approximated by  $b(\Phi(b/a) - 1)$  as  $a \rightarrow 0$ . Therefore, taking into account that  $\sigma(\cdot)$  is bounded we replace  $J_{3,n}$  with the sum

$$\hat{J}_{3,n} = \sum_{j=N_2}^{N_1} \hat{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}) \Delta \tilde{\lambda}_j, \quad (7.7)$$

where

$$\hat{\mathbf{B}}(\lambda, x) = \lambda^{-1/2} x \tilde{\varphi}(\lambda, x) q(\lambda, x) (2\Phi(\varrho q(\lambda, x)) - 1)$$

and  $q(\lambda, x)$  defined in (2.13). Put  $\hat{J}_{4,n} = \sum_{j=N_2}^{N_1} \hat{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_1) \Delta \tilde{\lambda}_j$  and

$$\iota_n = \hat{J}_{3,n} - \hat{J}_{4,n} = \sum_{j=N_2}^{N_1} \left( \hat{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}) - \hat{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_1) \right) \Delta \tilde{\lambda}_j. \quad (7.8)$$

Using Lemma 1 we can show directly that the sum  $\hat{J}_{4,n} \rightarrow J^*(S_1)$  in probability as  $n \rightarrow \infty$  faster than  $n^\beta \varrho^{2\beta}$ .

**Lemma 7** *Under the condition (C<sub>2</sub>) we have*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \left| \hat{J}_{4,n} - J^*(S_1) \right| = 0.$$

Furthermore, by Itô's formula we obtain

$$\hat{\mathbf{B}}(\tilde{\lambda}_j, S_1) - \hat{\mathbf{B}}(\tilde{\lambda}_j, S_{\tilde{u}_j}) = \hat{\mathbf{B}}_{1,j} + \frac{1}{2} \hat{\mathbf{B}}_{2,j}, \quad (7.9)$$

where

$$\hat{\mathbf{B}}_{1,j} = \int_{\tilde{u}_j}^1 \frac{\partial \hat{\mathbf{B}}}{\partial x}(\tilde{\lambda}_j, S_u) dS_u, \quad \hat{\mathbf{B}}_{2,j} = \int_{\tilde{u}_j}^1 \frac{\partial^2 \hat{\mathbf{B}}}{\partial x^2}(\tilde{\lambda}_j, S_u) \sigma^2(y_u) S_u^2 du.$$

Now we represent  $\iota_n$  as

$$\iota_n = - \sum_{j=N_2}^{N_1} \widehat{\mathbf{B}}_{1,j-1} \Delta \tilde{\lambda}_j - \frac{1}{2} \sum_{j=N_2}^{N_1} \widehat{\mathbf{B}}_{2,j-1} \Delta \tilde{\lambda}_j := -\iota_{1,n} - \frac{1}{2} \iota_{2,n}. \quad (7.10)$$

A direct calculation gives

$$\begin{aligned} \frac{\partial \widehat{\mathbf{B}}}{\partial x} &= \lambda^{-1/2} \tilde{\varphi}(\lambda, x) [-2q^2(\lambda, x)(2\Phi(\varrho q(\lambda, x)) - 1) \\ &\quad + \frac{1}{2\lambda}(2\Phi(\varrho q(\lambda, x)) - 1) + \frac{\varrho}{\lambda} \varphi(\varrho q(\lambda, x))], \\ \frac{\partial^2 \widehat{\mathbf{B}}}{\partial x^2} &= \frac{\tilde{\varphi}(\lambda, x)}{x} \left\{ \left( -2q^2(\lambda, x)(2\Phi(\varrho q(\lambda, x)) - 1) + \frac{1}{2\lambda}(2\Phi(\varrho q(\lambda, x)) - 1) \right. \right. \\ &\quad \left. \left. + \frac{\varrho}{\lambda} \varphi(\varrho q(\lambda, x)) \right) \frac{\mathbf{v}(\lambda, x)}{\sqrt{\lambda}} - \frac{2q(\lambda, x)}{\lambda} (2\Phi(\varrho q(\lambda, x)) - 1) \right. \\ &\quad \left. - \frac{2q^2(\lambda, x)\varrho}{\lambda} \varphi(\varrho q(\lambda, x)) + \frac{\varrho}{4\lambda^2} \varphi(\varrho q(\lambda, x)) - \frac{q\varrho^3}{2\lambda^2} \varphi(\varrho q(\lambda, x)) \right\}. \end{aligned}$$

Note that  $2\Phi(\varrho q) - 1 \rightarrow \text{sign}(q)$  as  $n \rightarrow \infty$ . Now putting

$$\widehat{\mathbf{V}}(\lambda, x) = \int_{\lambda}^{+\infty} \frac{\partial \widehat{\mathbf{B}}}{\partial x}(v, x) dv,$$

and using Lemma 5.6 we can approximate the term  $\iota_{1,n}$  by

$$\widehat{\iota}_{1,n} = \sum_{j=N_2}^{N_1} \widehat{\mathbf{V}}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}) \int_{\tilde{u}_{j-1}}^{\tilde{u}_j} \sigma(y_t) S_t dW_t^{(1)}. \quad (7.11)$$

Note that as  $n \rightarrow \infty$

$$\widehat{\mathbf{V}}(\lambda, x) \approx \mathbf{V}(\lambda, x) = \int_{\lambda}^{+\infty} \frac{\tilde{\varphi}(z, x)}{\sqrt{z}} \left( -2q^2(z, x) + \frac{1}{2z} \right) \text{sign}(q(z, x)) dz.$$

So, asymptotically one can replace  $\widehat{\iota}_{1,n}$  by  $\mathbf{m}_{N_1}''$ , where

$$\mathbf{m}_k'' = \frac{1}{\varrho} \sum_{j=N_2}^k \sigma(y_{\tilde{u}_{j-1}}) S_{\tilde{u}_{j-1}} \tilde{\mathbf{V}}_{j-1} \tilde{Z}_{1,j} \Delta \tilde{\lambda}_j \quad (7.12)$$

and  $\tilde{\mathbf{V}}_j = \mathbf{V}(\tilde{\lambda}_j, S_{\tilde{u}_j})$ .

**Lemma 8** *If  $\varrho = \varrho(n)$  satisfies condition  $(\mathbf{C}_2)$  then,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \iota_{2,n} = \mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \left| \widehat{\iota}_{1,n} - \mathbf{m}_{N_1}'' \right| = 0.$$

It is clear that both sequences  $(\mathbf{m}_k')_{N_2 \leq k \leq N_1}$  and  $(\mathbf{m}_k'')_{N_2 \leq k \leq N_1}$  are martingales w.r.t.  $(\mathcal{G}_k)_{N_2 \leq k \leq N_1}$ . Thus, if  $\varrho(n) \rightarrow \infty$  under condition  $(\mathbf{C}_2)$  the final asymptotic representation of  $J_n$  is given in the following proposition.

**Proposition 6** *Under condition (C<sub>2</sub>) the total trading volume  $J_n$  admits the following asymptotic form*

$$J_n = J^*(S_1) + \mathbf{m}'_{N_1} - \mathbf{m}''_{N_1} + o_{\mathbf{P}}(n^{-\beta} \varrho^{-2\beta}).$$

**Case 2:  $\varrho$  is a fixed positive constant.**

Similarly, we replace  $J_{3,n}$  with

$$\tilde{J}_{3,n} = \sum_{j=N_2}^{N_1} \tilde{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_1, y_1) \Delta \tilde{\lambda}_j,$$

and write the difference, which is not a negligible, as

$$\tilde{J}_{4,n} = J_{3,n} - \tilde{J}_{3,n} = \sum_{j=N_2}^{N_1} \left( \tilde{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_{\tilde{u}_{j-1}}, y_{\tilde{u}_{j-1}}) - \tilde{\mathbf{B}}(\tilde{\lambda}_{j-1}, S_1, y_1) \right) \Delta \tilde{\lambda}_j,$$

where  $\tilde{\mathbf{B}}(\lambda, x, y) = \lambda^{-1/2} x \tilde{\varphi}(\lambda, x) \mathbf{G}(\lambda, x, y)$ .

The term  $\tilde{J}_{3,n}$  gives us the limit of the total trading volume by using Lemma 1.

To study the term  $\tilde{J}_{4,n}$  we define the following functions

$$\begin{aligned} \mathbf{K}_1(\lambda, x, y) &= \int_{\lambda}^{+\infty} \frac{\partial \tilde{\mathbf{B}}}{\partial x}(x, y, z) dz + \mathbf{r} F_2(t(\lambda), y) \int_{\lambda}^{+\infty} \frac{\partial \tilde{\mathbf{B}}}{\partial y}(x, y, z) dz, \\ \mathbf{K}_2(\lambda, x, y) &= \sqrt{1 - \mathbf{r}^2} F_2(t(\lambda), y) \int_{\lambda}^{+\infty} \frac{\partial \tilde{\mathbf{B}}}{\partial y}(x, y, z) dz, \end{aligned}$$

where  $t(\lambda) = 1 - (\lambda/\lambda_0)^{1/4\beta}$ . Consider the martingale  $(\tilde{\mathbf{m}}'_k)_{k \geq 1}$  defined as

$$\tilde{\mathbf{m}}'_k = \frac{1}{\varrho} \sum_{j=N_2}^k \sigma(y_{\tilde{u}_{j-1}}) S_{\tilde{u}_{j-1}} (\tilde{\mathbf{K}}_{1,j-1} \tilde{Z}_{1,j} + \tilde{\mathbf{K}}_{2,j-1} \tilde{Z}_{2,j}) \Delta \tilde{\lambda}_j, \quad (7.13)$$

where  $\tilde{\mathbf{K}}_{1,j} = \mathbf{K}_1(\tilde{\lambda}_j, S_{\tilde{u}_j}, y_{\tilde{u}_j})$  and  $\tilde{\mathbf{K}}_{2,j} = \mathbf{K}_2(\tilde{\lambda}_j, S_{\tilde{u}_j}, y_{\tilde{u}_j})$ . Again, by virtue of Lemma 5.6 one states that  $\tilde{J}_{4,n}$  can be approximated by  $\tilde{\mathbf{m}}'_{N_1}$ .

**Lemma 9** *For any fixed  $\varrho > 0$ ,*

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^{\beta} \left| \tilde{J}_{3,n} - J(S_1, y_1, \varrho) \right| = \mathbf{P} - \lim_{n \rightarrow \infty} n^{\beta} \left| \tilde{J}_{4,n} - \tilde{\mathbf{m}}'_{N_1} \right| = 0.$$

**Proposition 7** *For any fixed  $\varrho > 0$  the total trading volume  $J^n$  admits the following asymptotic form*

$$J_n = J(S_1, y_1, \varrho) + \mathbf{m}'_{N_1} - \tilde{\mathbf{m}}'_{N_1} + o_{\mathbf{P}}(n^{-\beta}).$$

## 8 Proofs

### 8.1 Proof of Proposition 4

Setting  $\varepsilon_n = n^{-2\beta} \varrho^{-4\beta} l_*$ , we can represent  $I_{2,n}$  as

$$I_{2,n} = \int_0^{1-\varepsilon_n} \left( \gamma_t^n - \widehat{C}_x(t, S_t) \right) \sigma(y_t) S_t dW_t^{(1)} + \varpi_n,$$

where  $\varpi_n = \int_{1-\varepsilon_n}^1 \left( \gamma_t^n - \widehat{C}_x(t, S_t) \right) \sigma(y_t) S_t dW_t^{(1)}$ . Taking into account that  $\left| \gamma_t^n - \widehat{C}_x(t, S_t) \right| \leq 1$ , we obtain  $\lim_{n \rightarrow \infty} n^{2\beta} \varrho^{4\beta} \mathbf{E} \varpi_n^2 = 0$ . Thus, it remains to prove that

$$\lim_{n \rightarrow \infty} n^{2\beta} \varrho^{4\beta} \sum_{j=1}^n \int_{\widehat{t}_{j-1}}^{\widehat{t}_j} \mathbf{E} \left( \gamma_t^n - \widehat{C}_x(t, S_t) \right)^2 dt = 0, \quad (8.1)$$

where  $\widehat{t}_j = \min(t_j, 1 - \varepsilon_n)$ . First, we introduce the following functions

$$\begin{aligned} G_1(t) &= \sum_{j=1}^n \frac{1}{\lambda_t} \left( x_t - x_{\widehat{t}_{j-1}} \right)^2 \mathbf{1}_{\{\widehat{t}_{j-1} < t \leq \widehat{t}_j\}}, \\ G_2(t) &= \sum_{j=1}^n x_t^2 \left( \lambda_t^{-1/2} - \lambda_{\widehat{t}_{j-1}}^{-1/2} \right)^2 \mathbf{1}_{\{\widehat{t}_{j-1} < t \leq \widehat{t}_j\}}, \\ G_3(t) &= \sum_{j=1}^n \left( \lambda_t^{1/2} - \lambda_{\widehat{t}_{j-1}}^{1/2} \right)^2 \mathbf{1}_{\{\widehat{t}_{j-1} < t \leq \widehat{t}_j\}}, \end{aligned}$$

where  $x_t = \ln(S_t/K)$ . Clearly,

$$\begin{aligned} \left| \gamma_t^n - \widehat{C}_x(t, S_t) \right|^2 &= \sum_{j=1}^n \left| \widetilde{\Phi}(\lambda_t, S_t) - \widetilde{\Phi}(\lambda_{\widehat{t}_{j-1}}, S_{\widehat{t}_{j-1}}) \right|^2 \mathbf{1}_{\{\widehat{t}_{j-1} < t \leq \widehat{t}_j\}} \\ &\leq G_1(t) + G_2(t) + G_3(t). \end{aligned}$$

We can show directly in view of condition  $(\mathbf{C}_2)$  that

$$n^{2\beta} \varrho^{4\beta} \mathbf{E} \int_0^{1-\varepsilon_n} G_1(t) dt \leq C n^{2\beta-3/2} \varrho^{4\beta-1} \rightarrow 0$$

since  $\sup_{0 \leq t \leq 1} \mathbf{E} x_t^2 < \infty$  and

$$\sup_{n \geq 1, 1 \leq j \leq n} \left( n \sup_{0 \leq t \leq 1} \mathbf{E} \left( x_t - x_{\widehat{t}_{j-1}} \right)^2 \mathbf{1}_{\{\widehat{t}_{j-1} < t \leq \widehat{t}_j\}} \right) < \infty.$$

The particular choice of  $\varepsilon$  ensures

$$n^{2\beta} \varrho^{4\beta} \mathbf{E} \int_0^{1-\varepsilon_n} G_2(t) dt \leq C \frac{n^{2\beta} \varrho^{4\beta}}{n^2 \lambda_0} (\varepsilon_n)^{-(4\beta+1)/4\beta} \rightarrow 0,$$

while the convergence of  $G_3(t)$  can be shown in a similar way.  $\square$

## 8.2 Proof of Theorem 3.1

It is clear from the previous results that the asymptotic distribution strongly depends on how the parameter  $\varrho$  is chosen. If  $\varrho$  is a fixed positive constant then the martingale part of the hedging error is

$$\mathbf{M}_k = -\frac{1}{2}\mathbf{U}_k - \kappa_*(\mathbf{m}'_k - \tilde{\mathbf{m}}'_k).$$

This can be represented as

$$\mathbf{M}_k = \frac{1}{\varrho} \sum_{j=N_2}^k \sigma(y_{\tilde{u}_{j-1}}) S_{\tilde{u}_{j-1}} (\tilde{\mathbf{A}}_{1,j-1} \tilde{Z}_{1,j} + \tilde{\mathbf{A}}_{3,j} \tilde{Z}_{3,j-1} + \tilde{\mathbf{A}}_{2,j-1} \tilde{Z}_{2,j}) \Delta \tilde{\lambda}_j,$$

where  $\tilde{\mathbf{A}}_{i,j} = \mathbf{A}_i(\tilde{\lambda}_j, S_{\tilde{u}_j})$  and

$$\begin{aligned} \mathbf{A}_1(\lambda, x, y) &= -\frac{1}{2} \check{H}(\lambda, x) + \kappa_* \mathbf{K}_1(\lambda, x, y), & \mathbf{A}_2(\lambda, x, y) &= \kappa_* \mathbf{K}_2(\lambda, x, y), \\ \mathbf{A}_3(\lambda, x) &= -\kappa_* \lambda^{-1/2} \tilde{\varphi}(\lambda, x). \end{aligned}$$

Then, the sequence  $(n^\beta \mathbf{M}_{N_1})_{n \geq 1}$  converges in law to a mixed Gaussian variable by Proposition 2 and hence, Theorem 3.1 is proved.  $\square$

## 8.3 Proof of Theorem 3.2

If  $\varrho = \varrho(n)$  diverges to infinity under condition  $(\mathbf{C}_2)$  then, by virtue of Proposition (5), the hedging error admits the following asymptotic form

$$\min(S_1, K) - \kappa_* J^*(S_1) + \frac{1}{\varrho} \sum_{j=N_2}^{N_1} \check{v}_j + o_{\mathbf{P}}(n^{-\beta} \varrho^{-2\beta})$$

where

$$\check{v}_j = \sigma(y_{\tilde{u}_{j-1}}) S_{\tilde{t}_{j-1}} (\check{\mathbf{A}}_{1,j-1} \tilde{Z}_{1,j} + \check{\mathbf{A}}_{3,j-1} \tilde{Z}_{3,j}) \Delta \tilde{\lambda}_j, \quad (8.2)$$

with  $\check{\mathbf{A}}_{i,j} = \check{\mathbf{A}}_i(\tilde{\lambda}_j, S_{\tilde{u}_j})$  and

$$\check{\mathbf{A}}_1(\lambda, x) = -\frac{1}{2} \check{H}_1(\lambda, x) + \kappa_* \mathbf{V}(\lambda, x), \quad \check{\mathbf{A}}_3 \equiv \mathbf{A}_3.$$

Put  $\check{\mathbf{M}}_k = \sum_{j=N_2}^k \check{v}_j$  and remark that

$$n^\beta \varrho^{2\beta} \frac{1}{\varrho} \sum_{j=N_2}^{N_1} \check{v}_j = n^\beta \varrho^{-\frac{1}{\mu+1}} \check{\mathbf{M}}_{N_1}.$$

Then, Theorem 3.2 is proved through Proposition 5  $\square$

## A Appendix

### A.1 Auxiliary Lemmas

**Lemma 10** For any  $a > 0$  and  $0 < t \leq 1$ ,

$$\mathbf{P}(S_t = a) = 0.$$

*Proof* Let  $t$  be a fixed point in  $[0, 1]$ . We will prove that  $\mathbf{P}(x_t = a) = 0$  for any  $a \in \mathbb{R}$ , where

$$x_t = \int_0^t \sigma(y_s) dW_s^{(1)} - \int_0^t \sigma^2(y_s) ds.$$

For some sequence  $(t_n) \uparrow t$  we define

$$x_t^{(n)} = \int_0^{t_n} \sigma(y_s) dW_s^{(1)} - \int_0^{t_n} \sigma^2(y_s) ds + \sigma(y_{t_n})(W_t^{(1)} - W_{t_n}^{(1)}).$$

It is clear that conditionally with respect to  $\mathcal{F}_{t_n}$  the random variable  $x_t^{(n)}$  is Gaussian with the parameters  $(x_{t_n}, \sigma^2(y_{t_n}))$ . Hence, one can check directly that

$$\lim_{\epsilon \rightarrow 0} \sup_{n \geq 1} \sup_{a > 0} \mathbf{P}(x_t^{(n)} \in [a - \epsilon, a + \epsilon]) = 0.$$

Moreover, taking into account that  $x_t^{(n)} \rightarrow x_t$  we get

$$\begin{aligned} \mathbf{P}(x_t = a) &\leq \mathbf{E}(\mathbf{1}_{\{x_t = a\}} \mathbf{1}_{\{|x_t - x_t^{(n)}| < \epsilon\}}) + \mathbf{P}(|x_t - x_t^{(n)}| \geq \epsilon) \\ &\leq \mathbf{P}(x_t^{(n)} \in [a - \epsilon, a + \epsilon]) + o(1) \end{aligned}$$

and Lemma 10 is proved.  $\square$

**Lemma 11** Let  $\iota$  be a continuously decreasing  $[0, 1] \rightarrow [0, 1]$  function with  $\iota(0) = 1$ . Then for any  $\mathbb{R}_+^2 \rightarrow \mathbb{R}$  function  $\mathbf{A}$  satisfying condition  $(\mathbf{C}_4)$  we have

$$\limsup_{n \rightarrow \infty} \int_0^n |\check{A}(u, S_{\iota(u/n)})| du < \infty \quad a.s. \quad (\text{A.1})$$

and

$$\limsup_{n \rightarrow \infty} \int_0^n \int_v^n \check{A}_n(u, v) du dv < \infty \quad a.s., \quad (\text{A.2})$$

where  $\check{A}_n(u, v) = |A(u, S_{\iota(v/n)})| \tilde{\varphi}(u, S_{\iota(v/n)})$ .

*Proof* We can represent  $\tilde{\varphi}(u, x)$  as

$$\tilde{\varphi}(u, x) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{K}{x}} \phi_*(u, x), \quad \phi_*(u, x) = e^{-\frac{\ln^2(x/K)}{2u} - \frac{u}{8}}.$$

Now, by condition  $(\mathbf{C}_4)$  there exists  $\gamma > 0$  and a function  $U_\gamma > 0$  such that

$$\begin{aligned} \int_0^n |\check{A}(u, S_{\iota(u/n)})| du &\leq C U_\gamma^* \int_1^\infty (u^\gamma + 1) e^{-\frac{u}{8}} du \\ &\quad + C U_\gamma^* \int_0^1 (1 + u^{-\gamma}) e^{-\frac{\ln^2(S_{\iota(u/n)}/K)}{2u}} du \end{aligned}$$

where  $U_\gamma^* = \sup_{0 \leq t \leq 1} U_\gamma(S_t)/\sqrt{S_t}$ . Taking into account that  $\iota(u/n) \rightarrow 1$  uniformly in  $0 \leq u \leq 1$  and that the function  $(S_t)_{0 \leq t \leq 1}$  is uniformly continuous we get, that for sufficiently large  $n$

$$\sup_{0 \leq u \leq 1} e^{-\frac{\ln^2(S_{\iota(u/n)/K})}{2u}} \leq e^{-\frac{\ln^2(S_1/K)}{8u}} \quad \text{a.s.}$$

This inequality clearly implies (A.1). To show the inequality (A.2) we note that

$$\int_0^n \int_v^n \check{A}_n(u, v) \, du \, dv = \int_0^n \int_0^u \check{A}_n(u, v) \, dv \, du \leq \int_0^n u \sup_{0 \leq v \leq u} \check{A}_n(u, v) \, du.$$

A same argument can be run again to obtain (A.2) and hence Lemma 11 is completely proved.  $\square$

### A.2 Proof of Lemma 3

Changing variable  $v = \lambda_t$  and using Lemma 11 we can show that

$$\lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \int_0^1 S_t \lambda_t^{-1/2} \tilde{\varphi}(\lambda_t, S_t) dt = 0 \quad \text{a.s.} \quad (\text{A.3})$$

Now we can represent  $I_{1,n}$  as

$$\begin{aligned} I_{1,n} &= \int_0^1 \theta_t^2 \lambda_t^{-1/2} S_t \tilde{\varphi}(\lambda_t, S_t) dt + o_{\mathbf{P}}(n^{-\beta} \varrho^{-2\beta}) \\ &= S_1 \int_0^1 \theta_t^2 \lambda_t^{-1/2} \tilde{\varphi}(\lambda_t, S_1) dt + \tau_n + o_{\mathbf{P}}(n^{-\beta} \varrho^{-2\beta}) \\ &= S_1 \int_0^{\lambda_0} v^{-1/2} \tilde{\varphi}(v, S_1) \, dv + \tau_n + o_{\mathbf{P}}(n^{-\beta} \varrho^{-2\beta}). \end{aligned}$$

Taking into account here (2.15) we obtain Lemma 3.  $\square$

### A.3 Proof of Lemma 4

We first check the first equality in (6.4). To do this we rewrite  $\tau_{1,n}$  as

$$\tau_{1,n} = \tau_{3,n} + \delta_{1,n} + \delta_{2,n}, \quad (\text{A.4})$$

where  $\tau_{3,n} = \int_{u_*}^{u_*} \sigma(y_u) S_u \bar{H}_1(\lambda_u, S_u) dW_u^{(1)}$ ,  $\delta_{1,n} = \int_0^{u_*} \sigma(y_u) S_u \bar{H}_1(\lambda_u, S_u) dW_u^{(1)}$  and  $\delta_{2,n} = \int_{u_*}^1 \sigma(y_u) S_u \bar{H}_1(\lambda_u, S_u) dW_u^{(1)}$ .

Thanks to condition (C<sub>2</sub>),  $\varrho = o\left(n^{\frac{3-\mu}{2(3\mu-1)}}\right)$  as  $n \rightarrow \infty$  and

$$\sup_{n \geq 1} \max_{M_1 \leq m \leq M_2} \left( n \mathbf{E} \left( S_{u_m} - S_{u_{m-1}} \right)^2 + n^{1/2} \varrho^{-1} \Delta \lambda_m \right) < \infty.$$

This allows to show directly that

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} |\tau_{3,n} - \mathbf{U}_{N_1}| = 0.$$

Next, through Lemma 11 one can show that for any  $r > 0$

$$\lim_{n \rightarrow \infty} n^r \int_{l^*}^{\infty} \tilde{\varphi}(v, S_{u(v)}) dv = \lim_{n \rightarrow \infty} n^r \int_0^{l^*} \tilde{\varphi}(v, S_{u(v)}) dv = 0, \quad (\text{A.5})$$

where  $u = u(v) = 1 - (v/\lambda_0)^{4\beta}$ . This directly implies that both  $\delta_{j,n}$ ,  $j = 1, 2$  are  $n^\beta \varrho^{2\beta}$ -negligible and by virtue of Lemma 11 we obtain the same property for the Lebesgue integral  $\tau_{2,n}$ . Hence the proof is completed.  $\square$

#### A.4 Proof of Lemma 5

The proof directly follows from Lemma 11.  $\square$

#### A.5 Proof of Lemma 6

First, we represent  $J_{1,n}$  as

$$J_{1,n} = \sum_{m=M_1}^{M_2} S_{u_{m+1}} |\Delta\Phi(\mathbf{v}_{m+1})| + \varepsilon_{1,n}, \quad (\text{A.6})$$

where  $\varepsilon_{1,n} = \sum_{m=M_1}^{M_2} (S_{u_{m-1}} - S_{u_{m+1}}) |\Delta\Phi(\mathbf{v}_{m+1})|$  and  $\Delta\Phi(\mathbf{v}_m) = \Phi(\mathbf{v}_m) - \Phi(\mathbf{v}_{m-1})$ . Moreover, setting  $r_m = |\Delta\Phi(\mathbf{v}_m)| - \varphi(\mathbf{v}_m) |\Delta\mathbf{v}_m|$ , we can rewrite (A.6) as

$$J_{1,n} = T_{1,n} + \varepsilon_{1,n} + \varepsilon_{2,n}, \quad (\text{A.7})$$

where  $T_{1,n} = \sum_{m=M_1}^{M_2} S_{u_{m+1}} \varphi(\mathbf{v}_{m+1}) |\Delta\mathbf{v}_{m+1}|$  and  $\varepsilon_{2,n} = \sum_{m=M_1}^{M_2} S_{u_{m+1}} r_{m+1}$ . The first term in (A.7) can be represented as  $T_{1,n} = T_{2,n} + \varepsilon_{3,n}$ , where

$$T_{2,n} = \sum_{m=M_1}^{M_2} \frac{1}{\sqrt{\lambda_{m+1}}} S_{u_{m+1}} \varphi(\mathbf{v}_{m+1}) |z_m| \Delta\lambda_{m+1},$$

$\varepsilon_{3,n} = T_{1,n} - T_{2,n}$  and  $z_m = \lambda_m^{-1} \int_{u_m}^{u_{m-1}} \sigma(y_u) dW_u^{(1)} + q(\lambda_m, S_{u_m})$ . Moreover, by replacing  $z_m$  in  $T_{2,n}$  with

$$\hat{z}_m = \frac{\sigma(y_{u_m})}{\Delta\lambda_m} \Delta W_{u_m} + q(\lambda_m, S_{u_m}),$$

we obtain

$$J_{1,n} = J_{2,n} + \sum_{\iota=1}^4 \varepsilon_{\iota,n}, \quad \text{where } \varepsilon_{4,n} = J_{2,n} - T_{2,n}.$$

In view of inequality (5.6) and condition (C<sub>2</sub>)

$$\lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \max_{M_1 \leq m \leq M_2} \Delta\lambda_m = 0, \quad (\text{A.8})$$

we can show directly that the term  $\varepsilon_{1,n}, \varepsilon_{3,n}$  converges to zero at rate of  $n^\beta \varrho^{2\beta}$ . Furthermore, using the Taylor expansion we obtain

$$|\varepsilon_{2,n}| \leq C \sup_{0 \leq t \leq 1} S_t \sum_{m=M_1}^{M_2} |\mathbf{v}_{m+1} - \mathbf{v}_m|^2$$

and for some constant  $C > 0$ ,

$$\mathbf{E} |\mathbf{v}_{m+1} - \mathbf{v}_m|^2 \leq C \left( \frac{1}{n\lambda_{m+1}} + \left( \lambda_{m+1}^{1/2} - \lambda_m^{1/2} \right)^2 + \left( \lambda_{m+1}^{-1/2} - \lambda_m^{-1/2} \right)^2 \right).$$

Through condition  $(C_2)$  and the inequality (5.6) we get

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \varepsilon_{2,n} = 0.$$

To finish, it remains to prove that

$$\mathbf{P} - \lim_{n \rightarrow \infty} n^\beta \varrho^{2\beta} \sum_{m=M_1}^{M_2} S_{u_{m+1}} \varphi(\mathbf{v}_{m+1}) |\mathbf{z}_m - \hat{\mathbf{z}}_m| \Delta\lambda_{m+1} = 0$$

but this follows directly from limit equality (A.8).  $\square$

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