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Decomposition of \mathbb{S}^1 -valued maps in Sobolev spaces

Petru Mironescu *

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Abstract

Let $n \geq 2$, $s > 0$, $p \geq 1$ be such that $1 \leq sp < 2$. We prove that for each map $u \in W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$ one can find $\varphi \in W^{s,p}(\mathbb{S}^n; \mathbb{R})$ and $v \in W^{sp,1}(\mathbb{S}^n; \mathbb{S}^1)$ such that $u = ve^{i\varphi}$. This yields a decomposition of u into a part that has a lifting in $W^{s,p}$, $e^{i\varphi}$, and a map "smoother" than u but without lifting, namely v . Our result generalizes a previous one of Bourgain and Brezis (which corresponds to the case $s = 1/2$, $p = 2$). As a consequence, we find an intuitive proof for the existence of the distributional Jacobian Ju of maps $u \in W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$ (originally due to Bourgain, Brezis and the author). By completing a result of Bousquet, we characterize the distributions of the form Ju .

Résumé

Décomposition des applications unimodulaires dans les espaces de Sobolev. Soient $n \geq 2$, $s > 0$, $p \geq 1$ tels que $1 \leq sp < 2$. Nous montrons que, pour chaque $u \in W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$, il existe $\varphi \in W^{s,p}(\mathbb{S}^n; \mathbb{R})$ et $v \in W^{sp,1}(\mathbb{S}^n; \mathbb{S}^1)$ tels que $u = ve^{i\varphi}$. Ceci donne une décomposition de u comme produit d'un facteur qui se relève dans $W^{s,p}$, $e^{i\varphi}$, et d'un facteur "plus régulier" que u mais qui ne se relève pas, à savoir v . Notre décomposition généralise un résultat antérieur de Bourgain et Brezis (qui ont traité le cas $s = 1/2$, $p = 2$). Une conséquence de notre résultat est une preuve intuitive de l'existence du jacobien au sens des distributions Ju pour les applications $u \in W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$ (résultat dû, avec un argument différent, à Bourgain, Brezis et l'auteur). En complétant un résultat de Bousquet, nous caractérisons les distributions de la forme Ju .

1 Decomposition of \mathbb{S}^1 -valued maps

Our main result is the following

Theorem 1 *Let $n \geq 2$, $s > 0$, $p \geq 1$ be such that $1 \leq sp < 2$. Let $u \in W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$. Then there exist $\varphi \in W^{s,p}(\mathbb{S}^n; \mathbb{R})$ and $v \in W^{sp,1}(\mathbb{S}^n; \mathbb{S}^1)$ such that $u = ve^{i\varphi}$.*

In addition, we have (with $|\cdot|_{W^{r,q}}$ standing for the semi-norm given by the highest order term in $\|\cdot\|_{W^{r,q}}$)

$$|\varphi|_{W^{s,p}} \lesssim |u|_{W^{s,p}}, |v|_{W^{sp,1}} \lesssim |u|_{W^{s,p}}^p. \quad (1)$$

The special case $s = 1/2$, $p = 2$ of Theorem 1 is due to Bourgain and Brezis [4]. (In [4], u is supposed to be in the $H^{1/2}$ -closure of $C^\infty(\mathbb{S}^n; \mathbb{S}^1)$. This extra assumption was removed in [6].) In Theorem 1, \mathbb{S}^n does not play special role; one could replace, e. g., \mathbb{S}^n by any smooth bounded simply connected domain. Theorem 1 yields a satisfactory substitute to the lifting theory in $W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$, theory

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developed successively in [5], [18] and [14]. As proved in these papers, when $n \geq 2$ and $sp \notin [1, 2)$, one may characterize maps $u \in W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$ in terms of their liftings. (For a precise statement, we refer to [15], Theorem 6.1, p. 15.) However, when $1 \leq sp < 2$, there is no satisfactory description of maps in terms of their phases. A typical example is the map $\mathbb{C} \ni z \mapsto z/|z|$, which belongs to $W^{s,p}(B(0,1))$ when $sp < 2$, but does not have a phase better than $z \mapsto \arg z$, which merely belongs to BV. Our result allows to decompose u into two parts, one as smooth as u and which admits a lifting in $W^{s,p}$, the other one without lifting in $W^{s,p}$, but "smoother" than u . In Theorem 1, one cannot replace $W^{s,p}$ (for φ) or $W^{sp,1}$ (for v) by smaller Sobolev spaces.

The proof of Theorem 1 is constructive: there is an explicit formula giving φ . Part of the proof is inspired by similar constructions of Bourgain and Brezis [4] and of the author [14]. We describe the main lines of the proof when $s < 1$ and $1 \leq sp < 2$, and when \mathbb{S}^n is replaced by B , the unit ball in \mathbb{R}^n . We extend $u \in W^{s,p}(B; \mathbb{S}^1)$ to \mathbb{R}^n by reflections and cutoff. We let $\Pi \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ such that $\Pi(z) = z/|z|$ when $|z| \geq 1/2$ and let ρ be a suitable mollifier. With $w(x, \varepsilon) := u * \rho_\varepsilon(x)$, $x \in \mathbb{R}^n$, $\varepsilon > 0$, we set, inspired by [14],

$$\varphi_1(x) := - \int_0^\infty \Pi \circ w(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ w)(x, \varepsilon) d\varepsilon.$$

This φ_1 satisfies $\varphi_1 \in W^{s,p}(B)$ $U := ue^{-i\varphi_1} \in W^{1,sp}(B)$. If $sp = 1$, then we may take $\varphi = \varphi_1$. When $1 < sp < 2$, two more steps are needed. We extend U to \mathbb{R}^n by reflections and cutoff and define $\varphi_2 := \sum_k \sum_{j < k} U_j \wedge U_k$. Here, $U = \sum U_j$ is a Littlewood-Paley decomposition of U . The idea of improving

the regularity of a map with the help of this phase originates in the paper [4] of Bourgain and Brezis. This φ_2 satisfies $\varphi_2 \in W^{1,sp}$ and $Ue^{-i\varphi_2} \in W^{sp,1}(B)$.

Third step: since $\varphi_2 \in W^{1,sp}$, we have $\varphi_2 = \varphi_3 + \varphi_4$, where $\varphi_3 \in W^{s,p}$ and $\varphi_4 \in W^{sp,1} \cap W^{1,sp}$. The regularity of φ_4 implies that $e^{i\varphi_4} \in W^{sp,1}$ [10], [13]. Thus $u = e^{i\varphi}v$, where $\varphi := \varphi_1 + \varphi_3 \in W^{s,p}$ and $v := Ue^{i\varphi_4} \in W^{sp,1}$.

2 The distributional Jacobian revisited

We recall the definition of the distributional Jacobian for \mathbb{S}^1 -valued maps [17], [19],[2], [9],[12], [1], [6], [7]. If $u = (u_1, u_2) \in W^{1,1}(\mathbb{S}^2; \mathbb{S}^1)$, then $Ju := \frac{1}{2}d(u_1du_2 - u_2du_1)$. This distribution (current) coincides with the usual Jacobian 2-form $du_1 \wedge du_2$ if u is sufficiently smooth, say $u \in H^1$. In the latter case, $Ju = 0$ for \mathbb{S}^1 -valued maps u . As a distribution, Ju is defined by

$$\langle Ju, \zeta \rangle = \frac{1}{2} \int_{\mathbb{S}^2} (u_1du_2 - u_2du_1) \wedge d\zeta, \quad \forall \zeta \in C^\infty(\mathbb{S}^2; \mathbb{R}). \quad (2)$$

More generally, when $u \in W^{1,1}(\mathbb{S}^n; \mathbb{S}^1)$, Ju is defined as an $(n-2)$ -current through the formula

$$\langle Ju, \zeta \rangle = \frac{1}{2} \int_{\mathbb{S}^n} (u_1du_2 - u_2du_1) \wedge d\zeta, \quad \forall \zeta \in \Lambda^{n-2}(\mathbb{S}^n). \quad (3)$$

The following result was proved in [6].

Theorem 2 ([6]) *Let $n \geq 2$, $s > 0$, $p \geq 1$ be such that $1 \leq sp < 2$. Then $W^{s,p} \cap W^{1,1}(\mathbb{S}^n; \mathbb{S}^1)$ is dense in $W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$. In addition, the map $u \mapsto Ju$ extends by continuity from $W^{s,p} \cap W^{1,1}(\mathbb{S}^n; \mathbb{S}^1)$ to $W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$.*

Denoting by $u \mapsto Ju$ this extension, Theorem 1 sheds a new light on Theorem 2 via the following

Proposition 3 *Let $n \geq 2$, $s > 0$, $p \geq 1$ be such that $1 \leq sp < 2$. Let $u \in W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$ and write $u = ve^{i\varphi}$, with $\varphi \in W^{s,p}$ and $v \in W^{sp,1}$. Then, for each choice of φ and v , we have*

$$\langle Ju, \zeta \rangle = \frac{1}{2} \int_{\mathbb{S}^n} (v_1 dv_2 - v_2 dv_1) \wedge d\zeta, \quad \forall \zeta \in \Lambda^{n-2}(\mathbb{S}^n). \quad (4)$$

3 Existence of maps with prescribed singularities. The two dimensional case

Set $\mathcal{R} := \{u \in W^{1,1}(\mathbb{S}^2; \mathbb{S}^1); u \text{ is smooth outside some finite set } A = A(u)\}$. When $u \in \mathcal{R}$, we have $\langle Ju, \zeta \rangle = \pi \sum_{a \in A} d_a \zeta(a)$, where the integers d_a are the degrees of u on suitably oriented small circles around $a \in A$ and satisfy $\sum d_a = 0$ [12]. Thus $Ju = \pi \sum_{a \in A} d_a \delta_a$. Since \mathcal{R} is dense in $W^{1,1}(\mathbb{S}^2; \mathbb{S}^1)$ [3], one obtains that $\{Ju; u \in W^{1,1}(\mathbb{S}^2; \mathbb{S}^1)\} \subset E_{1,1}$, where $E_{1,1} := \pi \overline{\{\sum (\delta_{P_j} - \delta_{N_j})\}}^{(W^{1,\infty})^*}$. The reversed inclusion is true.

Theorem 4 ([1], [11]) *We have $\{Ju; u \in W^{1,1}(\mathbb{S}^2; \mathbb{S}^1)\} = E_{1,1}$.*

Bousquet [7] partially completed this result.

Theorem 5 ([7]) *Assume that $s \geq 1$ and $1 \leq sp < 2$. Then $\{Ju; u \in W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)\} = E_{s,p}$, where $E_{s,p} := \pi \overline{\{\sum (\delta_{P_j} - \delta_{N_j})\}}^{(W^{1,sp/(sp-1)})^* \cap (W^{2-s,p/(p-1)})^*}$.*

Note that the definition of $E_{s,p}$ suggests that different values of s and p yield different $E_{s,p}$'s. Our first result in this direction is somewhat surprising.

Theorem 6 *Assume that $s \geq 1$ and $1 \leq sp < 2$. Then $E_{s,p} = E_{1,sp}$. In particular, if a (possible infinite) sum of the form $\sum (\delta_{P_j} - \delta_{N_j})$, with $\sum |P_j - N_j| < \infty$, acts on $W^{1,r}$ for some $r \in (2, \infty)$, then it also acts on the Hölder space $C^{2-r/(r-1)}$.*

As a byproduct, the proof of the above theorem yields the following curious estimate

$$\left\| \sum (\delta_{P_j} - \delta_{N_j}) \right\|_{(C^\alpha)^*} \leq K_\alpha \left\| \sum (\delta_{P_j} - \delta_{N_j}) \right\|_{(W^{1,(2-\alpha)/(1-\alpha)})^*}^{2-\alpha}, \quad (5)$$

with K_α depending on $0 < \alpha < 1$ but independent of the P_j 's and N_j 's.

Our next result completes Theorem 5.

Theorem 7 *Assume that $1 \leq sp < 2$. Then $\{Ju; u \in W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)\} = E_{1,sp}$.*

4 Existence of maps with prescribed singularities. The higher dimensional case

In dimension 3 or higher, the class \mathcal{R} is defined as

$\mathcal{R} := \{u \in W^{1,1}(\mathbb{S}^n; \mathbb{S}^1); u \text{ is smooth outside some } (n-2)\text{-submanifold without boundary } A = A(u) \text{ of } \mathbb{S}^n\}$.

If $u \in \mathcal{R}$, then we may identify Ju with the $(n-2)$ -current $\pi \sum d_j \int_{\Gamma_j}$, where Γ_j are the (orientable, without boundary) connected components of A and the integers d_j are the degrees of u

on suitably oriented small circles linking to the Γ_j 's [12], [1], [7]. We then define, for $1 < q < 2$,

$E_{1,q} := \pi \overbrace{\left\{ \sum d_j \int_{\Gamma_j} \right\}}^{(W^{1,q/(q-1)})^*}$. For $q = 1$, the suitable higher dimensional analog of $E_{1,1}$ was pointed out by Alberti, Baldo, Orlandi [1] and is given by $E_{1,1} := \pi \{ \partial M; M \text{ is a rectifiable } (n-1) \text{-current} \}$. With these notations, we have

Theorem 8 *Assume that $n \geq 3$ and $1 \leq sp < 2$. Then $\{Ju; u \in W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)\} = E_{1,sp}$.*

The case $s = 1, p = 1$ was known before [1]. The case $sp = 1$ was obtained jointly with Bousquet [8]. The case $1 < sp < 2$ relies on Theorem 1 and on techniques from [7]. Finally, the analog of (5) is given by

$$\left\| \sum d_j \int_{\Gamma_j} \right\|_{(C^\alpha)^*} \leq K_\alpha \left\| \sum d_j \int_{\Gamma_j} \right\|_{(W^{1,(2-\alpha)/(1-\alpha)})^*}^{2-\alpha}, \quad 0 < \alpha < 1. \quad (6)$$

Detailed proofs will appear in [16].

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References

- [1] G. Alberti, S. Baldo, G. Orlandi, Functions with prescribed singularities, J. Eur. Math. Soc. 5 (2003), no. 3, 275–311.
- [2] J. M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Arch. Ration. Mech. Anal. 63 (1977), 337–403.
- [3] F. Bethuel, X. Zheng, Density of Smooth Functions between Two Manifolds in Sobolev Spaces, J. Funct. Anal. 80 (1988), 60–75.
- [4] J. Bourgain, H. Brezis, On the equation $\operatorname{div} Y = f$ and application to control of phases, J. Amer. Math. Soc. 16 (2003), 393–426.
- [5] J. Bourgain, H. Brezis, P. Mironescu, Lifting in Sobolev spaces, J. Anal. Math. 80 (2000), 37–86.
- [6] J. Bourgain, H. Brezis, P. Mironescu, $H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation, Publ. Math. Inst. Hautes Études Sci. 99 (2004), 1–115.
- [7] P. Bousquet, Topological singularities in $W^{s,p}(\mathbb{S}^N, \mathbb{S}^1)$, Journal d'Analyse Mathématique 102 (2007), 311–346.
- [8] P. Bousquet, P. Mironescu, in preparation.
- [9] H. Brezis, J.-M. Coron, E. Lieb, Harmonic maps with defects, Comm. Math. Phys. 107 (1986), no. 4, 649–705.
- [10] H. Brezis, P. Mironescu, Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces, J. Evol. Equ. 1 (2001), 387–404.
- [11] H. Brezis, P. Mironescu, A. Ponce, $W^{1,1}$ -maps with values into \mathbb{S}^1 , in Geometric analysis of PDE and several complex variables (S. Chanillo, P. D. Cordaro, N. Hanges, J. Hounie, A. Meziani eds), Contemporary Mathematics, Amer. Math. Soc., Providence, RI, 368 (2005), 69–100.

- [12] R. L. Jerrard, H. M. Soner, Functions of bounded higher variation, *Indiana Univ. Math. J.* 51 (2002), 645–677.
- [13] V. Maz'ya, T. Shaposhnikova, An elementary proof of the Brezis and Mironescu theorem on the composition operator in fractional Sobolev spaces, *J. Evol. Equ.* 2 (2002), no. 1, 113–125.
- [14] P. Mironescu, Lifting default for \mathbb{S}^1 -valued maps, *C. R. Acad. Sci. Paris, Ser. I* 346 (2008), nos. 19–20, 1039–1044.
- [15] P. Mironescu, \mathbb{S}^1 -valued Sobolev maps, <http://math.univ-lyon1.fr/~mironescu/7.pdf>
- [16] P. Mironescu, Sobolev spaces of circle-valued maps, in preparation.
- [17] C. B. Morrey, Multiple integrals in the calculus of variations, *Die Grundlehren der mathematischen Wissenschaften*, vol. 130, Springer-Verlag New York, Inc., New York, 1966.
- [18] H.-M. Nguyen, Inequalities related to liftings and applications, *C. R. Acad. Sci. Paris, Ser. I* 346 (2008), nos. 17–18, 957–962.
- [19] Y. G. Reshetnyak, The weak convergence of completely additive vector-valued set functions, *Sibirsk. Mat. Z.* 9 (1968), 1386–1394.