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# COMPLEMENTS TO THE PAPER “LIFTING, DEGREE AND THE DISTRIBUTIONAL JACOBIAN REVISITED”

JEAN BOURGAIN<sup>(1)</sup>, HAIM BREZIS<sup>(2),(3)</sup>  
AND PETRU MIRONESCU<sup>(4)</sup>

## 1. Existence of a degree and optimal estimates.

Let  $0 < s < \infty, 1 \leq p < \infty$  and set  $X = W^{s,p}(S^N; S^N)$ . We say that there is a (topological) degree in  $X$  if

- a)  $C^\infty(S^N; S^N)$  is dense in  $X$ ;
- b) the mapping  $g \mapsto \deg g$ , defined on  $C^\infty(S^N; S^N)$ , extends by continuity to  $X$ .

We recall the following result, which is part of the folklore:

**Lemma 1.1.** *There is a degree in  $X$  if and only if  $sp \geq N$ .*

*Proof.* Property a) holds for each  $s$  and  $p$ . When  $s$  is not an integer and  $sp < N$ , this was proved in [15]. When  $s = 1$  and  $p < N$ , this assertion can be found in [4]; the same argument holds when  $s \geq 2$  is an integer and  $sp < N$ .

When  $sp > N$ , property a) follows immediately from the embedding  $W^{s,p} \hookrightarrow C^0$ . Finally, property a) when  $sp = N$  is essentially established in [13].

We next turn to property b). When  $sp > N$ , it is easy to see that the usual Brouwer degree of the (continuous) maps in  $W^{s,p}$  has the required properties. When  $sp = N$ , we have  $W^{s,p} \hookrightarrow VMO$ ; in this case, the degree of VMO maps (studied in [13]) is the desired extension. Finally, we prove that b) does not hold when  $sp < N$ .

We fix a map  $g \in C^\infty(\mathbb{R}^N; S^N)$  such that  $g(x) \equiv P$  when  $|x| \geq 1$  and  $\deg g = 1$ ; here,  $P$  is the North pole of  $S^N$ . Let  $\pi : S^N \rightarrow \mathbb{R}^N$  be the stereographic projection and set  $g_k(x) = g(k\pi(x))$ ,  $x \in S^N$ . Then  $\deg g_k = 1, \forall k$ . However, it is easy to see that  $g_k \rightarrow P$  strongly in  $W^{s,p}$  and, therefore, the degree is not preserved in the strong limit.

In view of Lemma 1.1, it is natural to ask whether, for  $sp \geq N$ , there is a control of the form

$$(1.1) \quad |\deg g| \leq F(|g|_{s,p}), \forall g \in W^{s,p}(S^N; S^N).$$

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It follows from Theorem 0.6 and the Sobolev embeddings that the answer is yes. Indeed, if  $sp \geq N$  and  $(s, p) \neq (1, 1)$ , then there is some  $q > N$  such that  $W^{s,p}(S^N; S^N) \hookrightarrow W^{N/q,q}(S^N; S^N)$ . On the other hand, if  $s = p = N = 1$ , we have the estimate

$$|\deg g| = \left| \frac{1}{2i\pi} \int_{S^1} \frac{\dot{g}}{g} \right| \leq \frac{1}{2\pi} |g|_{1,1}.$$

We next examine the optimality of the estimates in Theorem 0.6, which we restate in a slightly more general form.

**Theorem 1.** *Let  $1 \leq p < \infty$  and  $g \in W^{N/p,p}(S^N; S^N)$ . Then*

$$(1.2) \quad |\deg g| \leq C_{p,N} |g|_{N/p,p}^p.$$

*Proof.* When  $p > N$ , this is the content of Theorem 0.6. When  $p = N$ , estimate (1.2) is an immediate consequence of the Kronecker formula

$$(1.3) \quad \deg g = \frac{1}{|S^N|} \int_{S^N} \det(\nabla g).$$

Finally, when  $1 \leq p < N$ , (1.2) follows from (1.3) and the Gagliardo-Nirenberg type inequality

$$(1.4) \quad |g|_{1,N} \leq |g|_{N/p,p}^{p/N} \|g\|_{L^\infty}^{1-p/N}.$$

Estimate (1.2) is optimal in the following sense:

**Lemma 1.2.** *For  $1 \leq p < \infty$ , there is a sequence  $(g_k) \subset W^{N/p,p}(S^N; S^N)$  such that  $|g_k|_{N/p,p} \xrightarrow{k} \infty$  and  $\deg g_k \geq C'_{p,N} |g_k|_{N/p,p}^p$ .*

*Proof.* Let  $h : \mathbb{R}^N \rightarrow S^N$  be such that  $h(x) \equiv P$  for  $|x| \geq 1$ . Then, clearly,  $|h \circ \pi|_{N/p,p}(S^N) \sim |h|_{N/p,p}(\mathbb{R}^N)$ . In view of this remark, it suffices to construct a sequence  $(h_k) \subset \dot{W}^{N/p,p}(\mathbb{R}^N; S^N)$  such that  $|h_k|_{N/p,p} \xrightarrow{k} \infty$ ,  $h_k \equiv P$  for  $|x| \geq 1$ ,  $\deg h_k \geq C'_{p,N} |h_k|_{N/p,p}^p$ . Fix a map  $g \in C^\infty(\mathbb{R}^N; S^N)$  such that  $\deg g = 1$  and  $g(x) \equiv P$  for  $|x| \geq 1$ . For  $k \geq 1$ , we fix  $k$  distinct points  $a_1, \dots, a_k \in B_1$ . Let

$$g_{\lambda,k}(x) = P + \sum_{j=1}^k (g - P) \left( \frac{x - a_j}{\lambda} \right), \lambda > 0.$$

It is easy to see that

$$|g_{\lambda,k}|_{N/p,p}^p \xrightarrow{\lambda \rightarrow 0} k |g|_{N/p,p}^p.$$

In addition, for sufficiently small  $\lambda$ , the map  $g_{\lambda,k}$  is  $S^N$ -valued, has degree  $k$ , and equals  $P$  for  $|x| \geq 1$ . If we set, for sufficiently large  $\lambda_k$ ,  $h_k = g_{\lambda_k,k}$ , then  $|h_k|_{N/p,p}^p \sim k$  and  $\deg h_k = k$ .

## 2. Existence of a distributional Jacobian.

As in the previous section, we discuss whether, given  $0 < s < \infty, 1 \leq p < \infty$ , there is a notion of a distributional Jacobian in  $W^{s,p}(S^{N+1}; S^N)$ . As noted in the discussion before Theorem 0.8, the answer is yes in  $W^{N/p,p}(S^{N+1}; S^N)$ , and therefore also in  $W^{s,p}(S^{N+1}; S^N)$  if  $sp \geq N$  (via the Sobolev embeddings). On the other hand, there is **no** natural notion of distribution Jacobian if  $sp < N$ . Indeed, in this case  $C^\infty(S^{N+1}; S^N)$  is dense in  $W^{s,p}(S^{N+1}; S^N)$  (this follows from [4] and [5]). Let  $g : S^{N+1} \rightarrow S^N, g(x', x_{N+1}) = \frac{x'}{|x'|}$ , for which  $\text{Det}(\nabla g) \neq 0$ . Consider a sequence  $(g_k) \subset C^\infty(S^{N+1}; S^N)$  such that  $g_k \rightarrow g$  in  $W^{s,p}$ . If a natural  $\text{Det}(\nabla)$  would exist, this would yield

$$0 = \lim_k \text{Det}(\nabla g_k) = \text{Det}(\nabla g) \neq 0, \text{ impossible.}$$

However, the answer given by Theorem 0.8 is not completely satisfactory. Indeed, the perfect analogs of a), b) in Section 1 are, for  $0 < s < \infty, 1 \leq p < \infty$  such that  $N \leq sp < N + 1$ :

a') that the class  $\mathcal{R}_{s,p} = \{g \in W^{s,p}(S^{N+1}; S^N); g \in C^\infty \text{ except a finite set, } g \in W^{1,N}\}$  is dense in  $W^{s,p}(S^{N+1}; S^N)$ ;

b') that  $\text{Det}(\nabla)$  extends by continuity from  $\mathcal{R}_{s,p}$  to  $W^{s,p}$ .

The proof of Theorem 0.8 combined with the Sobolev embeddings shows that b') holds, provided a') holds. However, we established a') only for  $0 < s < 1$ ; when  $s = 1$ , a') holds also, see [4]. It is plausible that a') holds for any  $s$ .

Concerning the estimate

$$(2.1) \quad \|\text{Det}(\nabla g)\|_{(W^{1,\infty})^*} \leq C|g|_{N/p,p}^p, \quad g \in W^{N/p,p}(S^{N+1}; S^N), \quad p > N,$$

Theorem 2.4 implies its optimality.

### 3. The closure of $C^\infty(S^{N+1}; S^N)$ .

As we have already noted,  $C^\infty(S^{N+1}; S^N)$  is dense in  $W^{s,p}(S^{N+1}; S^N)$  if  $sp < N$  or  $sp \geq N + 1$ . It is easy to see that this is **not** true if  $N \leq sp < N + 1$ .

We mention the following straightforward generalization of a result due to Bethuel [19] when  $p = N$ .

**Theorem 2.** *Let  $N < p < \infty$ . For  $g \in W^{N/p,p}(S^{N+1}; S^N)$ , the following are equivalent:*

- a)  $g \in \overline{C^\infty(S^{N+1}; S^N)}^{W^{N/p,p}}$ ;
- b)  $\text{Det}(\nabla g) = 0$ .

A proof is presented in [20].

It is plausible that the assumptions  $s < 1, sp = N$  are irrelevant here.

**Question.** Let  $0 < s < \infty, 1 \leq p < \infty$  be such that  $N \leq sp < N + 1$ . It is true that

$$g \in \overline{C^\infty(S^{N+1}; S^N)}^{W^{N/p, p}} \Leftrightarrow \text{Det}(\nabla g) = 0?$$

#### 4. An Alternative proof of Theorem 0.1.

In this section, we present another argument that yields the estimate

$$(4.1) \quad |\varphi|_{BMO(I)} \leq C_p(|e^{i\varphi}|_{1/p, p(I)}^p + |e^{i\varphi}|_{1/p, p(I)}), \quad 1 < p < \infty, \varphi \in W^{1/p, p}(I).$$

We start with some preliminary results.

##### 4.1. Basic estimates.

If  $\varphi \in L^1(I)$ , with  $I \subset \mathbb{R}$  interval, we set  $\varphi_I = \frac{1}{|I|} \int_I \varphi$ .

**Lemma 4.1.** Let  $\varphi \in C^0((-\rho, \rho))$ . Assume that

$$(4.2) \quad |e^{i\varphi}|_{1/p, p(-\rho, \rho)} \leq C_1$$

and

$$(4.3) \quad |\varphi|_{BMO(-\rho, 0)} + |\varphi|_{BMO(0, \rho)} \leq C_2.$$

Then

$$(4.4) \quad |\varphi_{(-\rho, 0)} - \varphi_{(0, \rho)}| \leq C(1 + C_1^p)(1 + C_2).$$

*Proof.* We start by introducing some notations. For  $0 < l_1 < l_2 \leq \rho$ , set

$$\begin{aligned} f(l_1) &= \varphi_{(0, l_1)} - \varphi_{(-l_1, 0)}, \\ h(l_1, l_2) &= \varphi_{(l_1, l_2)} - \varphi_{(-l_2, -l_1)}. \end{aligned}$$

Let  $C_3 = 2 + 2C_2$ . If  $|f(\rho)| \leq 10^3 C_3$ , there is nothing to prove. Otherwise, assume, e.g.,  $f(\rho) > 0$  and set  $t = \frac{f(\rho)}{C_3} > 10^3$ . Let  $J = \left\lfloor \frac{t}{2} \right\rfloor - 1$ . For  $j = 1, \dots, J$ , we will construct inductively  $0 < \rho_1 < \dots < \rho_J < \rho$  such that, for  $j = 1, \dots, J - 1$ ,

$$(4.5) \quad f(\rho_{j+1}) - f(\rho_j) \in [C_3, 2C_3];$$

$$(4.6) \quad \rho_{j+1} \geq 2\rho_j;$$

$$(4.7) \quad \text{dist}(h(\rho_j, \rho_{j+1}), 2\pi\mathbb{Z}) > \frac{1}{2}.$$

Assume the  $\rho_j$ 's constructed, for the moment. By Corollary A.5, it follows that

$$|e^{i\varphi}|_{1/p, p((\rho_j, \rho_{j+1}) \cup (-\rho_{j+1}, -\rho_j))}^p \geq C, \quad j = 1, \dots, J-1,$$

and thus

$$|e^{i\varphi}|_{1/p, p(-\rho, \rho)}^p \geq C(J-1) \geq C' f(\rho),$$

from which the conclusion of the lemma follows.

It remains to construct the  $\rho_j$ 's. Let  $\rho_1$  be the first  $l > 0$  such that  $f(l) = C_3$ . Assuming  $\rho_1, \dots, \rho_j$  constructed ( $j < J$ ), let  $a$  be the largest  $l > 0$  such that  $f(l) = f(\rho_j) + C_3$  and let  $b$  be the smallest  $l > a$  such that  $f(l) = f(\rho_j) + 2C_3$ .

We claim that

$$(4.8) \quad a \geq 4\rho_j$$

$$(4.9) \quad \text{for at least one } l \in [a, b], \text{ it holds that } \text{dist}(h(\rho_j, l), 2\pi\mathbb{Z}) > \frac{1}{2}.$$

Properties (4.5) - (4.7) follow immediately from (4.8) - (4.9); it suffices to take  $\rho_{j+1} = l$ , where  $l \in [a, b]$  is such that (4.9) holds. It is also clear from our construction that the  $\rho_j$ 's exist up to  $j = J$ .

*Proof of (4.8).*

By Lemma A.1, we have

$$\begin{aligned} C_3 &= |f(a) - f(\rho_j)| \leq |\varphi(0, a) - \varphi(0, \rho_j)| + |\varphi(-a, 0) - \varphi(-\rho_j, 0)| \\ &\leq \frac{\rho_j}{a} (|\varphi|_{\text{BMO}(-\rho, 0)} + |\varphi|_{\text{BMO}(0, \rho)}) \leq \frac{\rho_j}{a} C_2, \end{aligned}$$

and (4.8) follows from our choice of  $C_3$ .

*Proof of (4.9).* Argue by contradiction and assume that

$$\text{dist}(h(\rho_j, l), 2\pi\mathbb{Z}) \leq \frac{1}{2}, \quad \forall l \in [a, b].$$

Since  $l \mapsto h(\rho_j, l)$  is continuous, there is some fixed  $d \in \mathbb{Z}$  such that

$$|h(\rho_j, l) - 2\pi d| \leq \frac{1}{2}, \quad \forall l \in [a, b].$$

In particular,  $|h(\rho_j, a) - h(\rho_j, b)| \leq 1$ . By Lemma A.2, we have

$$h(\rho_j, a) = \frac{a}{a - \rho_j} f(a) - \frac{\rho_j}{a - \rho_j} f(\rho_j) = f(\rho_j) + C_3 \frac{a}{a - \rho_j},$$

and similarly

$$h(\rho_j, b) = f(\rho_j) + 2C_3 \frac{b}{b - \rho_j}.$$

Thus

$$C_3 \left| 2 \frac{b}{b - \rho_j} - \frac{a}{a - \rho_j} \right| \leq 1.$$

Since  $2 \frac{b}{b - \rho_j} - \frac{a}{a - \rho_j} \geq \frac{1}{2} \frac{a}{a - \rho_j}$ , we find that  $C_3 \frac{a}{a - \rho_j} \leq 2$ , which is impossible, since  $C_3 \geq 2$ .

**Corollary 4.2.** *Let  $J, K$  be two adjacent intervals and  $\varphi \in C^0(J \cup K)$ . Assume that*

$$(4.10) \quad |e^{i\varphi}|_{1/p, p(J \cup K)} \leq C_1$$

and

$$(4.11) \quad |\varphi|_{\text{BMO}(J)} + |\varphi|_{\text{BMO}(K)} \leq C_2.$$

Then

$$(4.12) \quad |\varphi|_{\text{BMO}(J \cup K)} \leq C(1 + C_1^p)(1 + C_2).$$

*Proof.* Let  $L \subset J \cup K$  be an interval. We have to prove that

$$\frac{1}{|L|} \int_L |\varphi - \varphi_L| \leq C(1 + C_1^p)(1 + C_2).$$

If  $L \subset J$  or  $L \subset K$ , this is clear. Otherwise, assume, e. g.,  $L = (-a, b)$ , with  $(-a, 0) \subset J$  and  $(0, b) \subset K$ . By Lemma A.3, we have

$$\frac{1}{|L|} \int_L |\varphi - \varphi_L| \leq 3(|\varphi|_{\text{BMO}(-a, 0)} + |\varphi|_{\text{BMO}(0, b)} + |\varphi_{(-a, 0)} - \varphi_{(0, a)}|),$$

and the conclusion follows from Lemma 4.1.

We will also need the following variant of Lemma 4.1

**Lemma 4.3.** Let  $0 < \rho' \leq \frac{1}{4}\rho$  and  $\varphi \in C^0((\rho', \rho) \cup (-\rho, -\rho'))$ .

Assume that

$$(4.13) \quad |e^{i\varphi}|_{1/p, p((4\rho', \rho) \cup (-\rho, -4\rho'))} \leq C_1$$

and

$$(4.14) \quad |\varphi|_{\text{BMO}((\rho', \rho) \cup (-\rho, -\rho'))} \leq C.$$

Then

$$(4.15) \quad |(\varphi_{(\rho', \rho)} - \varphi_{(-\rho, -\rho')}) - (\varphi_{(\rho', 4\rho')} - \varphi_{(-4\rho', -\rho')})| \leq C(1 + C_1^p)(1 + C_2).$$

*Proof.* Let  $C_3 = 2 + 3C_2$ . We may assume, e. g., that

$$tC_3 = (\varphi_{(\rho', \rho)} - \varphi_{(-\rho, -\rho')}) - (\varphi_{(\rho', 4\rho')} - \varphi_{(-4\rho', -\rho')}) \geq 10^3 C_3.$$

Let  $J = \left\lceil \frac{t}{4} \right\rceil - 1$ . We construct inductively  $\rho_j, j = 1, \dots, J$ , as follows: set  $\rho_1 = 4\rho'$ . Assume  $\rho_1, \dots, \rho_j$  already constructed such that

$$(4.16) \quad (\varphi_{(\rho', \rho_k)} - \varphi_{(-\rho_k, -\rho')}) - (\varphi_{(\rho', \rho_{k-1})} - \varphi_{(-\rho_{k-1}, -\rho')}) \in [C_3, 2C_3];$$

$$(4.17) \quad \rho_k \geq 2\rho_{k-1};$$

$$(4.18) \quad \text{dist}(\varphi_{(\rho_{k-1}, \rho_k)} - \varphi_{(-\rho_k, -\rho_{k-1})}, 2\pi\mathbb{Z}) > \frac{1}{2},$$

$k = 1, \dots, j$ .

Let  $a$  be the largest  $l > \rho_j$  such that

$$(\varphi_{(\rho', l)} - \varphi_{(-l, -\rho')}) - (\varphi_{(\rho', \rho_j)} - \varphi_{(-\rho_j, -\rho')}) = C_3,$$

and let  $b$  be the smallest  $l > a$  such that

$$(\varphi_{(\rho', l)} - \varphi_{(-l, -\rho')}) - (\varphi_{(\rho', \rho_j)} - \varphi_{(-\rho_j, -\rho')}) = 2C_3.$$

As in the proof of Lemma 4.1, we claim that

$$(4.19) \quad a \geq 2\rho_j;$$



(4.20) there is some  $l \in [a, b]$  such that  $\text{dist}(\varphi_{(\rho_j, l)} - \varphi_{(-l, \rho_j)}, 2\pi\mathbb{Z}) > \frac{1}{2}$ .

*Proof of (4.19).* We have, by Lemma A.1,

$$\begin{aligned} C_3 &= \left| (\varphi_{(\rho', a)} - \varphi_{(\rho', \rho_j)}) - (\varphi_{(-a, -\rho')} - \varphi_{(-\rho_j, -\rho')}) \right| \\ &\leq \left| \varphi_{(\rho', a)} - \varphi_{(\rho', \rho_j)} \right| + \left| \varphi_{(-a, -\rho')} - \varphi_{(-\rho_j, -\rho')} \right| \\ &\leq \frac{a - \rho'}{\rho_j - \rho'} |\varphi|_{\text{BMO}(\rho', \rho)} + \frac{a - \rho'}{\rho_j - \rho'} |\varphi|_{\text{BMO}(-\rho, -\rho')} \leq \frac{a - \rho'}{\rho_j - \rho} C_2, \end{aligned}$$

so that

$$a \geq \frac{C_3}{C_2} \rho_j - \frac{C_3 + C_2}{C_2} \rho' \geq 2\rho_j;$$

the last inequality follows from the inequalities  $C_3 \geq 3C_2$  and  $\rho_j \geq 4\rho'$ .

*Proof of (4.20).* Argue by contradiction. As in the proof of Lemma 4.1, it follows that

$$\left| (\varphi_{(\rho_j, a)} - \varphi_{(-a, -\rho_j)}) - (\varphi_{(\rho_j, b)} - \varphi_{(-b, -\rho_j)}) \right| \leq 1.$$

Starting from the identity

$$\varphi_{(\rho_j, a)} = \frac{a - \rho'}{a - \rho_j} \varphi_{(\rho', a)} - \frac{\rho_j - \rho'}{a - \rho_j} \varphi_{(\rho', \rho_j)},$$

we obtain, as in the proof of Lemma 4.1, that

$$C_3 \left| 2 \frac{b - \rho'}{b - \rho_j} - \frac{a - \rho'}{a - \rho_j} \right| \leq 1.$$

As in the proof of Lemma 4.1, this implies that  $C_3 \frac{a - \rho'}{a - \rho_j} \leq 2$ , which is impossible, since  $C_3 \geq 2$ .

The remaining part of the proof of Lemma 4.3 is identical to the one of Lemma 4.1 and will be omitted.

*Proof of Theorem 0.1.* We may assume that  $\varphi \in C^0 \cap W^{1/p, p}$ . As explained in the main paper, when  $|e^{i\varphi}|_{1/p, p}$  is sufficiently small, (4.1) follows from the inequality  $|g|_{\text{BMO}(I)} \leq C|g|_{1/p, p(I)}$  combined with

**Lemma 4.4 ([14]).** *Let  $\varphi \in VMO(I)$ . There are constants  $C > 0, \delta > 0$  such that*

$$(4.21) \quad |\varphi|_{BMO(I)} \leq C|e^{i\varphi}|_{BMO(I)} \quad \text{if } |e^{i\varphi}|_{BMO(I)} \leq \delta.$$

Let  $\gamma = \min(\delta^{1/p}, \delta)$ . It suffices to establish (4.1) when  $|e^{i\varphi}|_{1/p, p(I)}^p \geq \delta$ . Let  $N$  be the smallest integer  $\geq \frac{|g|_{1/p, p}^p}{\gamma}$ . We consider a partition of  $I$  with  $N$  successive intervals  $I_1, \dots, I_N$  chosen such that  $|e^{i\varphi}|_{1/p(I_1 \cup \dots \cup I_j)}^p = j\gamma, j = 1, \dots, N-1$ . Thus  $|e^{i\varphi}|_{1/p, p(I_j)}^p \leq \gamma, \forall j$ .

It suffices to establish the estimate

$$(4.22) \quad \frac{1}{|I|} \int_I |\varphi - \varphi_I| \leq C|e^{i\varphi}|_{1/p, p(I)}^p.$$

In view of lemmas A.6 and 4.4, we have

$$\frac{1}{|I|} \int_I |\varphi - \varphi_I| \leq C\gamma + \frac{1}{|I|^2} \sum_{j,k} |I_j| |I_k| |\varphi_{I_j} - \varphi_{I_k}|,$$

so that (4.22) bounds to proving

$$(4.23) \quad \frac{1}{|I|^2} \sum_{j,k} |I_j| |I_k| |\varphi_{I_j} - \varphi_{I_k}| \leq C|e^{i\varphi}|_{1/p, p(I)}^p.$$

The remaining part of the proof is devoted to estimating the differences  $|\varphi_{I_j} - \varphi_{I_k}|$ . Without any loss of generality, we will assume  $j = 1$ .

**Lemma 4.5.** *Assume that  $|I_1| = |I_k| \geq \frac{1}{2} \max |I_l|$ . Let  $l_0$  be such that  $|I_{l_0}| = \max\{|I_l|; 2 \leq l \leq k-1\}$  and set  $J = I_2 \cup \dots \cup I_{k-1}$ . Assume that  $|I_1| \leq 4|J|$ , and consider the following intervals*

*picture*  
(with  $|\bar{I}_1| = |\bar{I}_0| = |\tilde{I}_0| = |\tilde{I}_k| = \frac{1}{2}|I_{l_0}|$ ). Then

$$(4.24) \quad |\varphi_{I_1} - \varphi_{I_k}| \leq |\varphi_{\bar{I}_1} - \varphi_{\bar{I}_0}| + |\varphi_{\tilde{I}_0} - \varphi_{\tilde{I}_k}| + C \left( 1 + \log \frac{|I|}{|I_{l_0}|} \right)$$

*Proof.* We have

$$(4.25) \quad |\varphi_{I_1} - \varphi_{I_k}| \leq |\varphi_{I_1} - \varphi_{\bar{I}_1}| + |\varphi_{\bar{I}_1} - \varphi_{\bar{I}_0}| + |\varphi_{\bar{I}_0} - \varphi_{\tilde{I}_0}| + |\varphi_{\tilde{I}_0} - \varphi_{\tilde{I}_k}| + |\varphi_{\tilde{I}_k} - \varphi_{I_k}|.$$

By Lemma A.7, we have

$$(4.26) \quad |\varphi_{I_1} - \varphi_{\bar{I}_1}| + |\varphi_{\bar{I}_0} - \varphi_{\tilde{I}_0}| + |\varphi_{\tilde{I}_k} - \varphi_{I_k}| \leq C \left( 1 + \log \frac{|I|}{|I_{l_0}|} \right),$$

and the conclusion follows.

**Lemma 4.6.** *Same hypotheses as above, except that we assume  $|I_1| > 4|J|$ . Let  $I_1^*, I_k^*$  be as below.*

*picture*  
(with  $|I_1^*| = |I_k^*| = 4|J|$ ). Then

$$(4.27) \quad |\varphi_{I_1} - \varphi_{I_k}| \leq |\varphi_{I_1} - \varphi_{\bar{I}_{l_0}}| + |\varphi_{\bar{I}_{l_0}} - \varphi_{\bar{I}_k}| + C \left( 1 + \log \frac{4|J|}{|I_{l_0}|} + |e^{i\varphi}|_{1/p, p((I_1 \setminus I_1^*) \cup (I_k \setminus I_k^*))}^p \right).$$

*Proof.* We have

$$(4.28) \quad |\varphi_{I_1} - \varphi_{I_k}| \leq |\varphi_{I_1^*} - \varphi_{I_k^*}| + |(\varphi_{I_1} - \varphi_{I_1^*}) - (\varphi_{I_k} - \varphi_{I_k^*})|.$$

By Lemma 4.5, we have

$$(4.29) \quad |\varphi_{I_1^*} - \varphi_{I_k^*}| \leq |\varphi_{\bar{I}_1} - \varphi_{\bar{I}_k}| + |\varphi_{\bar{I}_{l_0}} - \varphi_{\bar{I}_k}| + C \left( 1 + \log \frac{4|J|}{|I_{l_0}|} \right).$$

On the other hand, Lemma 4.3 (with  $(-\rho', \rho')$  replaced by  $J$  and  $(-\rho, \rho)$  replaced by  $I_1 \cup J \cup I_k$ ) yields

$$(4.30) \quad |(\varphi_{I_1} - \varphi_{I_1^*}) - (\varphi_{I_k} - \varphi_{I_k^*})| \leq C \left( 1 + |e^{i\varphi}|_{1/p, p((I_1 \setminus I_1^*) \cup (I_k \setminus I_k^*))}^p \right),$$

and the conclusion follows.

**Corollary 4.7.** *If  $|I_1| = |I_k| \geq \frac{1}{2} \max_{2 \leq l \leq k-1} |J_l|$ , then (with  $l_0$  as above)*

$$(4.31) \quad |\varphi_{I_1} - \varphi_{I_k}| \leq |\varphi_{\bar{I}_1} - \varphi_{\bar{I}_{l_0}}| + |\varphi_{\bar{I}_{l_0}} - \varphi_{\bar{I}_k}| + C \left( 1 + \log \frac{\min\{4|J|, |I_1|\}}{|I_{l_0}|} + |e^{i\varphi}|_{1/p, p((I_1 \setminus I_1^*) \cup (I_k \setminus I_k^*))}^p \right).$$

(Here,  $I_1 \setminus I_1^*$  and  $I_k \setminus I_k^*$  could be empty).

**Lemma 4.8.** *Assume that  $|I_1| = |I_k| \geq \frac{1}{2} \max_{2 \leq l \leq k-1} |J_l|$ . Then*

$$(4.32) \quad |\varphi_{I_1} - \varphi_{I_k}| \leq C(k + |e^{i\varphi}|_{1/p, p(I)}^p).$$

*Proof.* We start by applying Corollary 4.7. We note that, by construction, we may apply again Corollary 4.7 to the consecutive intervals  $\bar{I}_1, I_2, \dots, I_{l_0-1}, \bar{I}_{l_0}$ , respectively to  $\bar{I}_{l_0}, I_{l_0+1}, \dots, I_{k-1}, \bar{I}_k$ ; next we iterate this procedure.

We find that

$$(4.33) \quad |\varphi_{I_1} - \varphi_{I_k}| \leq \sum_{l=1}^{2(k-1)} |\varphi_{J_l} - \varphi_{K_l}| + 2C(k-1) + \Sigma_1 + \Sigma_2.$$

Here,  $J_l, K_l$  are adjacent intervals of equal length, each one contained into one of the original  $I_j$ 's;  $\Sigma_1$  is the sum of the logarithmic terms, while  $\Sigma_2$  is the sum of the  $|\cdot|_{1/p,p}^p$  terms. Lemma 4.1 implies that

$$(4.34) \quad \sum_{l=1}^{2(k-1)} |\varphi_{J_l} - \varphi_{K_l}| \leq C(k-1) \leq Ck.$$

On the other hand, the  $|\cdot|_{1/p,p}^p$  terms we consider appear on disjoint intervals, and thus

$$(4.35) \quad \Sigma_2 \leq C|e^{i\varphi}|_{1/p,p(I)}^p.$$

Therefore,

$$(4.36) \quad |\varphi_{I_1} - \varphi_{I_k}| \leq Ck + C|e^{i\varphi}|_{1/p,p(I)}^p + \Sigma_1.$$

**Claim** We have  $\Sigma_1 \leq Ck$ . In order to prove the claim, we give a formal description of how  $\Sigma_1$  is computed.

Let  $Z_\phi = \{2, \dots, k-1\}$  and let  $s_\phi \in Z_\phi$  be such that  $|I_{s_\phi}| = \max |I_s|$  (with the notations used up to now, we have  $s_\phi = l_0$ ). Let  $Z_{(0)} = \{2, \dots, s_\phi - 1\}$ ,  $Z_{(1)} = \{s_\phi + 1, \dots, k-1\}$  if  $s_\phi$  is closer to 2 than to  $k-1$ ; otherwise, let  $Z_{(0)} = \{s_\phi + 1, \dots, k-1\}$ ,  $Z_{(1)} = \{2, \dots, s_\phi - 1\}$ . We have

$$|Z_\phi| = 1 + |Z_{(0)}| + |Z_{(1)}|, |Z_{(0)}| \leq |Z_{(1)}|.$$

Assuming  $Z_c$  constructed, we proceed to constructing  $Z_{(c,0)}$  and  $Z_{(c,1)}$  as above. More specifically, if  $Z_c \neq \phi$ , we pick  $s_c \in Z_c$  such that  $|I_{s_c}| = \max_{s \in Z_c} |I_s|$ . We next write  $Z_c \setminus \{s_c\} = Z_{(c,0)} \sqcup Z_{(c,1)}$ , with  $Z_{(c,0)}, Z_{(c,1)}$  intervals of integers and  $|Z_{(c,0)}| \leq |Z_{(c,1)}|$ . If  $Z_c = \emptyset$ , we stop.

If  $Z_c = \{m, m+1, \dots, n\}$ , then the corresponding term in  $\Sigma_1$  is of the form

$$(4.37) \quad \log \frac{\min\{|K|, 4 \sum_{s \in Z_c} |I_s|\}}{|I_{s_c}|};$$

here,  $K$  is an interval contained in  $I_{m-1}$  and of length  $\leq \min\{|I_{m-1}|, |I_{n+1}|\}$ .

Assume  $c \neq \emptyset$ . If  $\hat{c}$  is the predecessor of  $c$ , we have either  $m - 1 \in Z_{\hat{c}}$ , or  $n + 1 \in Z_{\hat{c}}$ , and thus  $|K| \leq |I_{s_{\hat{c}}}|$ .

In conclusion,

$$(4.38) \quad \Sigma_1 \leq C \log \frac{\min\{4|J|, |I_1|\}}{|I_{l_0}|} + C \sum_{c \neq \emptyset} \log 4 \underbrace{\frac{\min\{|I_{s_{\hat{c}}}|, \sum_{s \in Z_c} |I_s|\}}{|I_{s_c}|}}_{R_c}.$$

Setting  $R_{\emptyset} = \frac{\min\{|J|, |I_1|\}}{|I_{l_0}|}$ , the claim amounts to proving that

$$(4.39) \quad \sum_c \log(4R_c) \leq Ck.$$

This is an immediate consequence of the two following

**Lemma 4.9.** *We have*

$$(4.40) \quad \sum_c \log R_c \leq Ck + \sum_c \log |Z_{(c,0)}|.$$

**Lemma 4.10.** *We have*

$$(4.41) \quad \sum_c \log |Z_{(c,0)}| \leq Ck.$$

*Proof of Lemma 4.9.* Let  $t$  be the largest integer such that  $Z_{\underbrace{(1,1,\dots,1)}_t, 0} \neq \phi$ .

Set  $a_0 = Z_{(0)}$ ,  $a_1 = Z_{(1,0)}$ ,  $\dots$ ,  $a_t = Z_{\underbrace{(1,1,\dots,1)}_t, 0}$ ,  $x_0 = R_{\phi}$ ,  $x_1 = R_{(1)}$ ,  $\dots$ ,  $x_t =$

$R_{\underbrace{(1,\dots,1)}_t}$ .

$t$  times

Writing

$$Z_{\phi} = (Z_{(0)} \cup \{s_{\phi}\}) \cup (Z_{(1,0)} \cup \{s_{(1)}\}) \cup (Z_{(1,1,0)} \cup \{s_{(1,1)}\}) \cup \dots,$$

we find that

$$R_{\phi} \leq \frac{\sum_{s \in Z_{\phi}} |I_s|}{|I_{s_{\phi}}|} = \sum_{s \in Z_{(0)} \cup \{s_{\phi}\}} \frac{|I_s|}{|I_{s_{\phi}}|} + \sum_{s \in Z_{(1,0)} \cup \{s_1\}} \frac{|I_s|}{|I_{s_{\phi}}|} + \sum_{s \in Z_{(1,1,0)} \cup \{s_{(1,1,0)}\}} \frac{|I_s|}{|I_{s_{\phi}}|} + \dots,$$

so that

$$R_\phi \leq 2|Z_{(0)}| + 2|Z_{(1,0)}| \frac{|I_{s(1)}|}{|I_{s_\phi}|} + 2|Z_{(1,1,0)}| \frac{|I_{s(1,1)}|}{|I_{s_\phi}|} + \dots$$

Since

$$\frac{|I_{s(1)}|}{|I_{s_\phi}|} \leq \frac{1}{R_{(1)}}, \frac{|I_{s(1,1)}|}{|I_{s_\phi}|} = \frac{|I_{s(1,1)}|}{|I_{s(1)}|} \cdot \frac{|I_{s(1)}|}{|I_{s_\phi}|} \leq \frac{1}{R_{(1)}R_{(1,1)}}, \dots,$$

we obtain

$$(4.42) \quad \frac{1}{2}x_0 \leq a_0 + \frac{a_1}{x_1} + \frac{a_2}{x_1x_2} + \dots + \frac{a_t}{x_1x_2 \dots x_t}$$

and similarly

$$(4.43) \quad \begin{cases} \frac{1}{2}x_1 \leq a_1 + \frac{a_2}{x_2} + \dots + \frac{a_t}{x_2 \dots x_t} \\ \vdots \\ \frac{1}{2}x_{t-1} \leq a_{t-1} + \frac{a_t}{x_t}. \end{cases}$$

Noting that  $x_t = 1$ , we find from (4.42) - (4.43) by backward induction on  $j$  that

$$x_j \dots x_t \leq \sum_{m=1}^{t-j} 2^m \sum_{\substack{J \subset \{j, \dots, t\} \\ |J|=m}} \prod_{l \in J} a_l.$$

In particular, since  $a_j \geq 1, \forall j$ , we obtain

$$(4.44) \quad x_0 \dots x_t \leq \sum_{m=1}^t 2^m \sum_{J \subset \{0, \dots, t\}, |J|=m} \prod_{l \in J} a_l \leq \sum_{m=1}^t 2^m \binom{t+1}{m} \prod_0^t a_l \leq 3^{t+1} \prod_0^t a_l.$$

Similarly, for any fixed  $\bar{c}$  we have

$$(4.45) \quad \prod_{\substack{c \text{ contains} \\ \text{only 1's}}} R_{(\bar{c}, c)} \leq \prod_{\substack{c \text{ contains} \\ \text{only 1's}}} (3|Z_{(\bar{c}, c, 0)}|).$$

Since each  $c$  can be uniquely written as  $c = (\bar{c}, 0, \bar{c})$  where  $\bar{c}$  contains only 1's, by multiplying the inequalities of type (4.45) we find that

$$\prod_c R_c \leq 3^{2k} \prod_c |Z_{(c, 0)}|,$$

from which the conclusion of the lemma follows.

*Proof of Lemma 4.10.* Let, for  $l \geq 0$ ,  $S_l = \{c ; |Z_{(c,0)}| \in [2^l, 2^{l+1})\}$ . We claim that

$$[c \neq c', c, c' \in S_l] \Rightarrow Z_{(c,0)} \cap Z_{(c',0)} = \phi.$$

Argue by contradiction and assume that  $Z_{(c,0)} \cap Z_{(c',0)} \neq \phi$ .

Then, for example, we have  $Z_{(c,0)} \subsetneq Z_{(c',0)}$ , so that  $Z_c \subset Z_{(c',0)}$ , by construction.

Thus  $|Z_{(c,0)}| \leq \frac{1}{2}|Z_c| \leq \frac{1}{2}|Z_{(c',0)}|$ , which is impossible if  $c, c' \in S_l$ . Therefore,

$$\prod |Z_{(c,0)}| = \prod_{l=1}^{[\log_2 k]+1} \prod_{c \in S_l} |Z_{(c,0)}| \leq \prod_{l=1}^{[\log_2 k]+1} 2^{l|S_l|} \leq \prod_{l \geq 1} 2^{lk/2^l} = 2^{Ak},$$

where  $A = \sum_{l \geq 1} l2^{-l}$ .

**Lemma 4.11.** *Assume that  $|I_1| \geq |I_l|, l = 2, \dots, k$ . Then*

$$(4.46) \quad |\varphi_{I_1} - \varphi_{I_k}| \leq C \left( k + |e^{i\varphi}|_{1/p, p(I_1 \cup \dots \cup I_k)}^p + \log \frac{|I_1|}{|I_k|} \right).$$

*Proof.* Let  $l_0 = 1$  and define inductively  $l_j$  such that  $|I_{l_j}| = \max_{l_{j-1} < l \leq k} |I_l|$ . Then

$$|\varphi_{I_1} - \varphi_{I_k}| \leq \sum_{j \geq 1} |\varphi_{I_{s_{j-1}}} - \varphi_{I_{s_j}}|.$$

Let  $\bar{I}_{s_{j-1}}$  be as follows:

picture  
(such that  $|\bar{I}_{s_{j-1}}| = |I_{s_j}|$ ). We may apply Lemma 4.7 to the sequence of intervals  $\bar{I}_{s_{j-1}}, \dots, I_{s_j}$ , and find that

$$(4.47) \quad |\varphi_{\bar{I}_{s_{j-1}}} - \varphi_{I_{s_j}}| \leq C(s_j - s_{j-1} + |e^{i\varphi}|_{1/p, p(I_{s_{j-1}} \cup \dots \cup I_{s_j})}^p).$$

On the other hand, Lemma A.7 yields

$$(4.48) \quad |\varphi_{I_{s_{j-1}}} - \varphi_{\bar{I}_{s_{j-1}}}| \leq C \left( 1 + \log \frac{|I_{s_{j-1}}|}{|I_{s_j}|} \right).$$

By summing up all the inequalities of type (4.47)-(4.48), we find that (4.46) holds.

**Lemma 4.12.** *For each  $j, k$  we have*

$$(4.49) \quad |\varphi_{I_j} - \varphi_{I_k}| \leq C \left( |k - j| + |e^{i\varphi}|_{1/p, p(I)}^p + \log \frac{|I|^2}{|I_j||I_k|} \right).$$

*Proof.* Assume  $j = 1$ . If  $I_1$  (or  $I_k$ ) is the largest among the intervals  $I_1, \dots, I_k$ , the conclusion follows from Lemma 4.11. Otherwise, let  $l \in \{2, \dots, k-1\}$  be such that  $|I_l| \geq |I_t|, t = 1, \dots, k$ . By Lemma 4.11, we have

$$(4.50) \quad |\varphi_{I_1} - \varphi_{I_l}| \leq C \left( (l-1) + |e^{i\varphi}|_{1/p, p(I_1 \cup \dots \cup I_l)}^p + \log \frac{|I_l|}{|I_1|} \right)$$

and

$$(4.51) \quad |\varphi_{I_k} - \varphi_{I_l}| \leq C \left( (k-l) + |e^{i\varphi}|_{1/p, p(I_l \cup \dots \cup I_k)}^p + \log \frac{|I_l|}{|I_k|} \right),$$

from which the lemma follows.

**Corollary 4.13.** *We have*

$$(4.51) \quad |\varphi_{I_j} - \varphi_{I_k}| \leq C \left( N + |e^{i\varphi}|_{1/p, p(I)}^p + \frac{|I|}{\sqrt{|I_j||I_k|}} \right).$$

*Proof of Theorem 0.1.* We have to estimate the r.h.s. of (4.23). In view of Corollary 4.13, we have

$$\begin{aligned} \frac{1}{|I|^2} \sum_{j,k} |I_j||I_k| |\varphi_{I_j} - \varphi_{I_k}| &\leq C \left( N + |e^{i\varphi}|_{1/p, p(I)}^p \right) + \frac{C}{|I|} \sum_{j,k} |I_j|^{1/2} |I_k|^{1/2} \\ &\leq C \left( N + |e^{i\varphi}|_{1/p, p(I)}^p \right) \leq C |e^{i\varphi}|_{1/p, p(I)}^p, \end{aligned}$$

since  $N \leq C |e^{i\varphi}|_{1/p, p(I)}^p$ .

## 5. An improvement of Theorem 0.1 and the answer to OP2 when $N = 1$ .

If  $I \subset \mathbb{R}$  is an interval and  $g : I \rightarrow \mathbb{C}$ , we set, for  $\delta > 0$ ,

$$J(g, \delta, I) = \iint_{\{(x,y) \in I^2; |g(x) - g(y)| \geq \delta\}} \frac{1}{|x - y|^2}.$$

In this section, we prove the following generalization of Theorem 0.1.



**Theorem 3.** For sufficiently small  $\delta > 0$ , we have

$$(5.1) \quad |\varphi|_{\text{BMO}(I)} \leq C(\delta + J(e^{i\varphi}, \delta, I)), \quad \forall \varphi \in C^0(I; \mathbb{R}).$$

An immediate consequence is the following

**Theorem 4.** Let  $g \in C^0(S^1; S^1)$ . Then, for sufficiently small  $\delta > 0$ , we have

$$(5.2) \quad |\deg g| \leq CJ(g, \delta, S^1).$$

This answer OP 2 when  $N = 1$ .

*Proof of Theorem 4.* By Lemma A.8, we have  $|g|_{\text{BMO}(S^1)} \leq \delta + 2J(g, \delta, S^1)$ . Recall that  $\deg g = 0$  provided  $|g|_{\text{BMO}(S^1)}$  is sufficiently small (see [13]). Thus (5.2) holds (for small  $\delta > 0$ ) provided  $J(g, \delta, S^1)$  is sufficiently small.

When  $J(g, \delta, S^1)$  is not too small, estimate (5.2) is obtained from (5.1) in the same way (0.6) follows from Theorem 0.1.

*Proof of Theorem 3.* The proof is the same as the one of Theorem 0.1, except that  $|\cdot|_{1/p,p}^p$  has to be replaced by  $J(g, \delta, I)$ . The only two places where  $|\cdot|_{1/p,p}^p$  comes into the picture are the inequality

$$(5.3) \quad |g|_{\text{BMO}(I)} \leq |g|_{1/p,p(I)}$$

and Lemma A.4 (together with Corollary A.5). The substitute of (5.3) is Lemma A.8. The analog of Lemma A.4/Corollary A.5 are Lemma A.9/Corollary A.10 presented into the appendix.

### Appendix. Elementary properties of averages.

**Lemma A.1.** Let  $J \subset K$ . Then  $|\varphi_J - \varphi_K| \leq \frac{|K|}{|J|} |\varphi|_{\text{BMO}(K)}$ .

*Proof.* We have

$$|\varphi_J - \varphi_K| = \frac{1}{|J|} \left| \int_J (\varphi - \varphi_K) \right| \leq \frac{1}{|J|} \int_K |\varphi - \varphi_K| = \frac{|K|}{|J|} \frac{1}{|K|} \int_K |\varphi - \varphi_K| \leq \frac{|K|}{|J|} |\varphi|_{\text{BMO}(K)}.$$

The following identities are trivial:

**Lemma A.2.** Let  $J, K$  be two adjacent intervals. Then

$$(A.1) \quad \varphi_J - \varphi_{J \cup K} = \frac{|K|}{|J| + |K|} (\varphi_J - \varphi_K), \quad \varphi_J = \frac{|J| + |K|}{|J|} \varphi_{J \cup K} - \frac{|K|}{|J|} \varphi_K$$

and

$$(A.2) \quad \varphi_{J \cup K} = \frac{|K|}{|J| + |K|} \varphi_K + \frac{|J|}{|J| + |K|} \varphi_J.$$

**Lemma A.3.** *Let  $0 < a \leq b$ . Then*

$$\frac{1}{a+b} \int_{-a}^b |\varphi - \varphi_{(-a,b)}| \leq 3 \left( |\varphi|_{\text{BMO}(-a,0)} + |\varphi|_{\text{BMO}(0,b)} \right) + |\varphi_{(-a,0)} - \varphi_{(0,a)}|.$$

*Proof.* We have

$$\begin{aligned} M = \int_{-a}^b |\varphi - \varphi_{(-a,b)}| &\leq \int_{-a}^0 |\varphi - \varphi_{(-a,0)}| + \int_{-a}^0 |\varphi_{(-a,0)} - \varphi_{(-a,b)}| + \\ &\quad \int_0^b |\varphi - \varphi_{(0,b)}| + \int_0^b |\varphi_{(0,b)} - \varphi_{(-a,b)}| \\ &\leq a|\varphi|_{\text{BMO}(-a,0)} + b|\varphi|_{\text{BMO}(0,b)} + a|\varphi_{(-a,0)} - \varphi_{(-a,b)}| + b|\varphi_{(0,b)} - \varphi_{(-a,b)}|. \end{aligned}$$

By Lemma A.2, we further obtain

$$\begin{aligned} M &\leq \underbrace{(a+b) \left( |\varphi|_{\text{BMO}(-a,0)} + |\varphi|_{\text{BMO}(0,b)} \right)}_N + \frac{2ab}{a+b} |\varphi_{(-a,0)} - \varphi_{(0,b)}| \\ &\leq N + \frac{2ab}{a+b} |\varphi_{(-a,0)} - \varphi_{(0,a)}| + \frac{2ab}{a+b} |\varphi_{(0,a)} - \varphi_{(0,b)}|, \end{aligned}$$

and Lemma A.1 implies that

$$(A.3) \quad M \leq N + \frac{2ab}{a+b} |\varphi_{(-a,0)} - \varphi_{(0,a)}| + \frac{2b^2}{a+b} |\varphi|_{\text{BMO}(0,b)}.$$

Dividing (A.3) by  $a+b$ , we find that

$$\begin{aligned} \frac{1}{a+b} \int_{-a}^b |\varphi - \varphi_{(-a,b)}| &\leq |\varphi|_{\text{BMO}(-a,0)} + |\varphi|_{\text{BMO}(0,b)} + \\ &\quad \frac{2ab}{(a+b)^2} |\varphi_{(-a,0)} - \varphi_{(0,a)}| + \frac{2b^2}{(a+b)^2} |\varphi|_{\text{BMO}(0,b)} \\ &\leq 3 \left( |\varphi|_{\text{BMO}(-a,0)} + |\varphi|_{\text{BMO}(0,b)} \right) + |\varphi_{(-a,0)} - \varphi_{(0,a)}|. \end{aligned}$$

**Lemma A.4.** *Let  $L \geq 2l > 0$  and  $\varphi \in C^0((l, L) \cup (-L, -l))$ . There is some  $\gamma > 0$  such that*

$$\left[ \text{dist}(\varphi_{(l,L)} - \varphi_{(-L,-l)}, 2\pi\mathbb{Z}) > \frac{1}{2} \text{ and } |\varphi|_{\text{BMO}(l,L)} + |\varphi|_{\text{BMO}(-L,-l)} < \gamma \right] \Rightarrow$$

$$\int_l^L \int_{-L}^{-l} \frac{|e^{i\varphi(x)} - e^{i\varphi(y)}|^p}{(x-y)^2} \geq C;$$

here,  $\gamma$  and  $C$  depend only on  $l, L, \varphi$ .

*Proof.* We have, with  $I = (l, L), J = (-L, -l)$ ,

$$\frac{1}{L-l} \int_I |\varphi - \varphi_I|^p \leq C |\varphi|_{\text{BMO}(I)}^p,$$

with  $C$  independent of  $I$  (this is the scale invariant form of the John-Nirenberg inequality). Thus

$$(A.4) \quad \frac{1}{L} \int_l^L |e^{i\varphi} - e^{i\varphi_I}|^p \leq \frac{1}{L-l} \int_I |\varphi - \varphi_I|^p \leq C |\varphi|_{\text{BMO}(I)}^p;$$

a similar inequality holds for  $J$ .

Since, for  $x \in I, y \in J$ , we have

$$(A.5) \quad |e^{i\varphi_I} - e^{i\varphi_J}|^p \leq 3^p \left( |e^{i\varphi(x)} - e^{i\varphi_I}|^p + |e^{i\varphi(x)} - e^{i\varphi(y)}|^p + |e^{i\varphi(y)} - e^{i\varphi_J}|^p \right),$$

we find that

$$\int_I \int_J |e^{i\varphi_I} - e^{i\varphi_J}|^p \leq 3^p \int_I \int_J \left( |e^{i\varphi(x)} - e^{i\varphi_I}|^p + |e^{i\varphi(x)} - e^{i\varphi(y)}|^p + |e^{i\varphi(y)} - e^{i\varphi_J}|^p \right),$$

so that

$$(A.6) \quad \begin{aligned} |e^{i\varphi_I} - e^{i\varphi_J}|^p &\leq \frac{C}{L} \left( \int_I |e^{i\varphi(x)} - e^{i\varphi_I}|^p + \int_J |e^{i\varphi(y)} - e^{i\varphi_J}|^p \right) + \\ &\frac{C}{L^2} \int_I \int_J |e^{i\varphi(x)} - e^{i\varphi(y)}|^p \\ &\leq C' \left( |\varphi|_{\text{BMO}(I)} + |\varphi|_{\text{BMO}(J)}^p \right) + C'' \int_I \int_J \frac{|e^{i\varphi(x)} - e^{i\varphi(y)}|^p}{(x-y)^2}. \end{aligned}$$

Thus

$$(A.7) \quad \int_I \int_J \frac{|e^{i\varphi(x)} - e^{i\varphi(y)}|^p}{(x-y)^2} \geq C''' \left( |e^{i\varphi_I} - e^{i\varphi_J}|^p - |\varphi|_{\text{BMO}(I)}^p - |\varphi|_{\text{BMO}(J)}^p \right),$$

from which the lemma follows immediately.

**Corollary A.5.** Let  $L \geq 2l > 0$  and  $\varphi \in C^0((l, L) \cup (-L, -l))$ .

Then

$$\text{dist}(\varphi_{(l,L)} - \varphi_{(-L,-l)}, 2\pi\mathbb{Z}) > \frac{1}{2} \Rightarrow |e^{i\varphi}|_{1/p,p((l,L) \cup (-L,-l))} \geq C,$$

for some  $C$  independent of  $l, L, \varphi$ .

*Proof.* If  $|\varphi|_{\text{BMO}(l,L)} + |\varphi|_{\text{BMO}(-L,-l)} < \gamma$ , the conclusion follows from Lemma A.4. Otherwise, Lemma 4.4 combined with the embedding  $W^{1/p,p} \subset \text{VMO}$  implies that  $|e^{i\varphi}|_{1/p,p(l,L)} + |e^{i\varphi}|_{1/p,p(-L,-l)} \geq C$  for some  $C$  depending only on  $\gamma$ , and the conclusion follows again.

**Lemma A.6.** Let  $\varphi \in \text{BMO}(I)$  and consider a partition  $I = \bigcup_j I_j$  of  $I$  with intervals. If  $|\varphi|_{\text{BMO}(I_j)} \leq C_2, \forall j$ , then

$$\frac{1}{|I|} \int_I |\varphi - \varphi_I| \leq C_2 + \frac{1}{|I|^2} \sum_{j,k} |I_j| |I_k| |\varphi_{I_j} - \varphi_{I_k}|.$$

*Proof.* We have

$$\begin{aligned} \frac{1}{|I|} \int_I |\varphi - \varphi_I| &= \frac{1}{|I|} \sum_j \int_{I_j} |\varphi - \varphi_I| \leq \frac{1}{|I|} \sum_j \left( \int_{I_j} |\varphi - \varphi_{I_j}| + \int_{I_j} |\varphi_{I_j} - \varphi_I| \right) \\ &= \frac{1}{|I|} \sum_j |I_j| \frac{1}{|I_j|} \int_{I_j} |\varphi - \varphi_{I_j}| + \frac{1}{|I|} \sum_j \int_{I_j} \left| \sum_k \frac{|I_k|}{|I|} \varphi_{I_k} - \varphi_{I_j} \right| \\ &\leq \frac{C_2}{|I|} \sum_j |I_j| + \frac{1}{|I|^2} \sum_{j,k} \int_{I_j} |I_k| |\varphi_{I_k} - \varphi_{I_j}| \leq C_2 + \frac{1}{|I|^2} \sum_{j,k} |I_j| |I_k| |\varphi_{I_j} - \varphi_{I_k}|. \end{aligned}$$

**Lemma A.7.** Let  $J \subset K$  be intervals. Then

$$|\varphi_J - \varphi_K| \leq C \left( 1 + \log \frac{|K|}{|J|} \right) |\varphi|_{\text{BMO}(K)}.$$

*Proof.* If  $|J| \geq \frac{1}{2}|K|$ , the conclusion follows from Lemma A.1. Otherwise, let  $l \in \mathbb{N}$  be such that  $\frac{|J|}{|K|} \in [2^{-l-1}, 2^{-l})$  and consider a sequence of intervals  $J_1, \dots, J_{l+2}$ , such that  $J_1 = J, J_k \subset J_{k+1}, J_{l+2} = K, |J_k| = 2^{k-l-2}, k = 2, \dots, l+1$ . Then

$$\begin{aligned} |\varphi_J - \varphi_K| &\leq \sum_{j=1}^{l+1} |\varphi_{J_{j+1}} - \varphi_{J_j}| \leq |\varphi|_{\text{BMO}(K)} \sum_{j=1}^{l+1} \frac{|J_{j+1}|}{|J_j|} \\ &\leq 2(l+1) |\varphi|_{\text{BMO}(K)} \leq C \left( 1 + \log \frac{|K|}{|J|} \right) |\varphi|_{\text{BMO}(K)}; \end{aligned}$$

here, we use again Lemma A.1.

**Lemma A.8.** *We have, for  $g \in C^0(I; S^1)$ ,  $|g|_{\text{BMO}(I)} \leq \delta + 2J(g, \delta, I)$ .*

*Proof.* Let  $K \subset I$  be an interval. Then

$$\begin{aligned} \frac{1}{|K|^2} \int_K \int_K |g(x) - g(y)| &\leq \frac{\delta}{|K|^2} \iint_{\{(x,y) \in K^2; |g(x)-g(y)| < \delta\}} dx dy + \\ &\quad \frac{2}{|K|^2} \iint_{\{(x,y) \in K^2; |g(x)-g(y)| \geq \delta\}} dx dy \\ &\leq \delta + 2J(g, \delta, K) \leq \delta + 2J(g, \delta, I). \end{aligned}$$

**Lemma A.9.** *Let  $L \geq 2l > 0$  and  $\varphi \in C^0((l, L) \cup (-L, -l))$ . There is some  $\gamma > 0$  such that*

$$\left[ \text{dist}(\varphi_{(l,L)} - \varphi_{(-L,-l)}, 2\pi\mathbb{Z}) > \frac{1}{2} \text{ and } |\varphi|_{\text{BMO}(l,L)} + |\varphi|_{\text{BMO}(-L,-l)} < \gamma \right] \Rightarrow$$

$$\iint_{\{x \in (l,L), y \in (-L,-l); |g(x)-g(y)| \geq \delta\}} \frac{1}{(x-y)^2} \geq C;$$

here,  $\gamma$  and  $C$  depend only on  $l, L, \varphi$  and  $\delta$  is small.

*Proof.* We start from (A.6). With  $g = e^{i\varphi}$ , we have

$$\begin{aligned} C_1 \leq |e^{i\varphi_I} - e^{i\varphi_J}|^p &\leq C'(|\varphi|_{\text{BMO}(I)}^p + |\varphi|_{\text{BMO}(J)}^p) + \frac{C}{L^2} \int_I \int_J |e^{i\varphi(x)} - e^{i\varphi(y)}|^p \\ &\leq C'(|\varphi|_{\text{BMO}(I)}^p + |\varphi|_{\text{BMO}(J)}^p) + \\ &\quad \frac{C\delta}{L^2} \iint_{\{|g(x)-g(y)| \leq \delta\}} dx dy + \frac{2C}{L^2} \iint_{\{|g(x)-g(y)| > \delta\}} dx dy, \end{aligned}$$

so that

$$(A.8) \quad C_1 \leq C'(|\varphi|_{\text{BMO}(I)}^p + |\varphi|_{\text{BMO}(J)}^p) + C\delta + C'' \iint_{\{|g(x)-g(y)| > \delta\}} \frac{1}{(x-y)^2} dx dy,$$

and the lemma follows.

**Corollary A.10.** *Let  $\delta > 0$  be sufficiently small,  $L \geq 2l > 0$  and  $\varphi \in C^0((l, L) \cup (-L, -l))$ . Then*

$$\text{dist}(\varphi_{(l,L)} - \varphi_{(-L,-l)}, 2\pi\mathbb{Z}) > \frac{1}{2} \Rightarrow \iint_{\{l \leq |x|, |y| \leq L; |g(x)-g(y)| > \delta\}} \frac{1}{(x-y)^2} dx dy \geq C,$$

for some  $C$  independent of  $l, L, \varphi$ .

*Proof.* If  $|\varphi|_{\text{BMO}(l,L)} + |\varphi|_{\text{BMO}(-L,-l)} < \gamma$ , the conclusion follows from Lemma A.9. Otherwise, Lemma A.8 combined with Lemma 4.4 imply that

$$J(e^{i\varphi}, \delta, (l, L)) + J(e^{i\varphi}, \delta, (-L, -l)) \geq C$$

for  $C$  independent of  $l, L, \varphi$ , and the lemma follows.

## Additional References

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(1) INSTITUTE FOR ADVANCED STUDY  
PRINCETON, NJ 08540  
*E-mail address:* bourgain@math.ias.edu

(2) LABORATOIRE J. -L. LIONS  
UNIVERSITÉ P. ET M. CURIE, B.C. 187  
4 PL. JUSSIEU  
75252 PARIS CEDEX 05  
*E-mail address:* brezis@ccr.jussieu.fr

(3) RUTGERS UNIVERSITY  
DEPT. OF MATH., HILL CENTER, BUSCH CAMPUS  
110 FRELINGHUYSEN RD, PISCATAWAY, NJ 08854  
*E-mail address:* brezis@math.rutgers.edu

(4) DÉPARTEMENT DE MATHÉMATIQUES  
UNIVERSITÉ PARIS-SUD  
91405 ORSAY  
*E-mail address:* Petru.Mironescu@math.u-psud.fr