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# LIFTING OF $\mathbb{S}^1$ -VALUED MAPS IN SUMS OF SOBOLEV SPACES

PETRU MIRONESCU

ABSTRACT. We describe, in terms of lifting, the closure of smooth  $\mathbb{S}^1$ -valued maps in the space  $W^{s,p}((-1, 1)^N; \mathbb{S}^1)$ . (Here,  $0 < s < \infty$  and  $1 \leq p < \infty$ .) This description follows from an estimate for the phase of smooth maps: let  $0 < s < 1$ , let  $\varphi \in C^\infty([-1, 1]^N; \mathbb{R})$  and set  $u = e^{i\varphi}$ . Then we may split  $\varphi = \varphi_1 + \varphi_2$ , where the smooth maps  $\varphi_1$  and  $\varphi_2$  satisfy

$$(*) \quad |\varphi_1|_{W^{s,p}} \leq C|u|_{W^{s,p}} \quad \text{and} \quad \|\nabla\varphi_2\|_{L^{sp}}^{sp} \leq C|u|_{W^{s,p}}^p.$$

(\*) was proved for  $s = 1/2$ ,  $p = 2$  and arbitrary space dimension  $N$  by Bourgain and Brezis [3] and for  $N = 1$ ,  $p > 1$  and  $s = 1/p$  by Nguyen [14].

Our proof is a sort of continuous version of the Bourgain-Brezis approach (based on paraproducts). Estimate (\*) answers (and generalizes) a question of Bourgain, Brezis, and the author [5].

## 1. INTRODUCTION

In [4], the authors addressed the problem of lifting of  $\mathbb{S}^1$ -valued maps in Sobolev spaces:

( $L_{s,p}$ ) Given an arbitrary  $u \in W^{s,p}(Q; \mathbb{S}^1)$ , is there a  $\varphi \in W^{s,p}(Q; \mathbb{R})$  such that  $u = e^{i\varphi}$ ?

Here,  $0 < s < \infty$ ,  $1 \leq p < \infty$  and  $Q = (-1, 1)^N$ . The complete answer is [4]

SPACE DIMENSION $N$	SIZE OF $s$	SIZE OF $sp$	ANSWER TO ( $L_{s,p}$ )
$N = 1$	ANY	ANY	YES
$N \geq 2$	$0 < s < 1$	$0 < sp < 1$	YES
$N \geq 2$	$0 < s < 1$	$1 \leq sp < N$	NO
$N \geq 2$	$0 < s < 1$	$sp \geq N$	YES
$N \geq 2$	$s \geq 1$	$1 \leq sp < 2$	NO
$N \geq 2$	$s \geq 1$	$sp \geq 2$	YES

The non existence results rely on two kinds of counterexamples: **topological** and **analytical**.

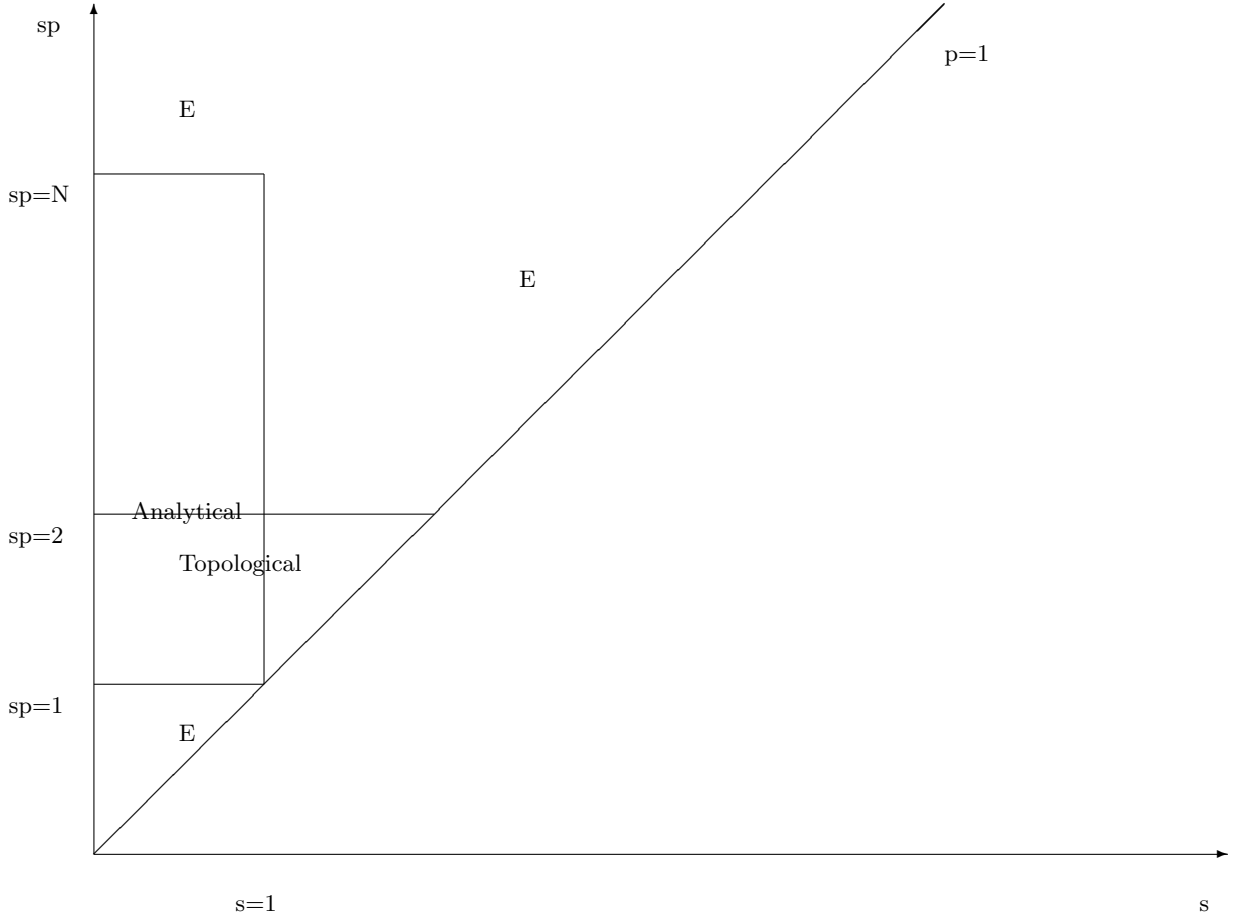
**Topological counterexamples.** One may prove (see Proposition 1) that, if there is lifting in  $W^{s,p}$ , then  $C^\infty(\overline{Q}; \mathbb{S}^1)$  is dense in  $W^{s,p}(Q; \mathbb{S}^1)$ . Thus the answer to ( $L_{s,p}$ ) is no whenever  $C^\infty(\overline{Q}; \mathbb{S}^1)$  is not dense in  $W^{s,p}(Q; \mathbb{S}^1)$ . When  $1 \leq sp < 2$ , the typical "topological counterexample" is the

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The author thanks H.-M. Nguyen for sending him the paper [14] and for stimulating discussions. He warmly thanks H. Brezis for his comments on the paper.

map  $Q \ni x \mapsto \frac{(x_1, x_2)}{|(x_1, x_2)|}$  which belongs to  $W^{s,p}(Q; \mathbb{S}^1)$  but cannot be approximated by smooth maps in the  $W^{s,p}$ -norm. (This goes back essentially to [16]; for a proof, see e. g. [9].) Such a counterexample does not exist outside the "topological region"  $1 \leq sp < 2$ . Indeed, when  $sp < 1$  or  $sp \geq 2$ ,  $C^\infty(\overline{Q}; \mathbb{S}^1)$  is dense in  $W^{s,p}(Q; \mathbb{S}^1)$  [9]. Thus, topological counterexamples are confined to the topological region.



In the "E" regions, there is lifting. The "topological" region is a trapezoid, the "analytical" one a rectangle

**Analytical counterexamples.** In the region  $1 < sp < N$  and  $0 < s < 1$ , one may prove non existence of lifting as follows [4]: pick some  $\psi \in W^{1,sp}(Q; \mathbb{R}) \setminus W^{s,p}(Q; \mathbb{R})$  (such a  $\psi$  exists, by the Sobolev "non embedding"  $W^{1,sp}(Q) \not\subset W^{s,p}(Q)$ ). Let  $u := e^{i\psi}$ . Then  $u \in W^{1,sp} \cap L^\infty$ , so that  $u \in W^{s,p}(Q; \mathbb{S}^1)$ , by the Gagliardo-Nirenberg embedding  $W^{1,sp} \cap L^\infty \subset W^{s,p}$ .

This  $u$  does not lift as  $u = e^{i\varphi}$  with  $\varphi \in W^{s,p}(Q; \mathbb{R})$ . Argue by contradiction: since  $e^{2(\varphi-\psi)} = 1$ , we have  $\eta := \varphi - \psi \in (W^{1,sp} + W^{s,p})(Q; 2\pi\mathbb{Z})$ . Thus  $\eta$  is constant a. e. [4] (this uses  $sp \geq 1$ ), so

that  $\psi \in W^{s,p}$ , contradiction.

When  $sp = 1$  and  $0 < s < 1$ , we still have non lifting. However, the above argument has to be slightly modified: one has to construct explicitly some  $\psi \in W^{1,1}(Q; \mathbb{R}) \setminus W^{s,p}(Q; \mathbb{R})$  and  $e^{i\psi} \in W^{s,p}(Q; \mathbb{S}^1)$  ( $\psi(x) = |x|^{-\alpha}$  for appropriate  $\alpha > 0$  will do it). Note that, when  $sp = 1$ , the property  $\psi \in W^{1,1}$  does not imply  $u \in W^{s,p}$ ; this follows from the Gagliardo-Nirenberg "non embedding"  $W^{1,1} \cap L^\infty \not\subset W^{s,p}$ . (Note the contrast with the embedding  $W^{1,sp} \cap L^\infty \subset W^{s,p}$ , valid when  $sp > 1$ .)

This argument implies non existence of lifting in the "analytical region"  $1 \leq sp < N$ ,  $0 < s < 1$ .

Unlike the topological counterexamples, the "analytical" ones belong to the subspace

$$X^{s,p} := \overline{C^\infty(\overline{Q}; \mathbb{S}^1)}^{W^{s,p}}$$

(see Proposition 2).

As we have already seen, in the region "analytical\topological" there are no topological counterexamples, since  $C^\infty(\overline{Q}; \mathbb{S}^1)$  is dense in  $W^{s,p}(Q; \mathbb{S}^1)$  when  $0 < s < 1$  and  $sp \geq 2$ . On the other hand, there are no analytical counterexamples (=maps in  $X^{s,p}$  that do not lift) in the "topological\analytical" region. Indeed, if  $s \geq 1$  and  $u \in X^{s,p}$ , then  $u$  has a lifting in  $W^{s,p}(Q; \mathbb{R})$  (Proposition 3).

The main purpose of the present paper is to prove that the analytical counterexamples we presented above are the only ones.

**Theorem 1.** *Assume that  $0 < s < 1$ ,  $sp \geq 1$  and that  $u \in X^{s,p}$ . Then  $u = e^{i\varphi}$  for some  $\varphi \in W^{s,p}(Q; \mathbb{R}) + W^{1,sp}(Q; \mathbb{R})$ .*

This relies essentially on the following estimate for the lifting of smooth maps.

**Theorem 2.** *Let  $0 < s < 1$  and  $1 \leq p < \infty$ . Let  $\varphi \in C^\infty(\overline{Q}; \mathbb{R})$  and set  $u := e^{i\varphi}$ . Then we may split  $\varphi = \varphi_1 + \varphi_2$ , where the maps  $\varphi_j \in C^\infty(Q; \mathbb{R})$ ,  $j = 1, 2$ , satisfy*

$$(1) \quad |\varphi_1|_{W^{s,p}} \leq C|u|_{W^{s,p}}$$

and

$$(2) \quad \|\nabla \varphi_2\|_{L^{sp}}^{sp} \leq C|u|_{W^{s,p}}^p.$$

Here,  $|\cdot|_{W^{s,p}}$  stands for the Gagliardo sem-norm

$$\begin{aligned} |u|_{W^{s,p}} &= \left( \iint_{Q \times Q} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p} \\ &\sim \left( \sum_{j=1}^N \int_{(-1,1)^{N+1}} \frac{\left| u\left( \sum_{k \neq j} x_k e_k + t e_j \right) - u\left( \sum_{k \neq j} x_k e_k + s e_j \right) \right|^p}{|t - s|^{1+sp}} \otimes_{k \neq j} dx_k dt ds \right)^{1/p}. \end{aligned}$$

Two special cases of Theorem 2 were already known. In [3], Bourgain and Brezis established Theorem 2 when  $s = 1/2$ ,  $p = 2$  and  $N$  is arbitrary. Their proof adapts to the case  $1 < p \leq 2$ ,

$s = 1/p$  (and arbitrary  $N$ ).

In [14], Nguyen proved Theorem 2 when  $N = 1$ ,  $p > 1$  and  $s = 1/p$  (without the restriction  $p \leq 2$ ).

The argument there adapts to the case where  $N$  is arbitrary, provided  $sp > 1$ .

Thus the really new cases are a)  $sp < 1$  and b)  $N \geq 2$ ,  $sp = 1$ ,  $p > 2$ .

Theorem 1 was conjectured in [5], [12]. The results presented here were announced in [13].

The paper is organized as follows: in Section 2, we explain how lifting and density are related. In Section 3, we explain the proof of Theorem 2 and why this proof is a cousin of the Bourgain-Brezis argument. In Section 4, we establish the main estimates needed in the proof of Theorem 2. The proofs of Theorems 1 and 2 are presented in Section 5. Finally, in Section 6, we characterize  $X^{s,p}$  in terms of lifting.

## 2. DENSITY VS LIFTING

In this section, we discuss the connection between density of  $C^\infty(\overline{Q}; \mathbb{S}^1)$  in  $W^{s,p}(Q; \mathbb{S}^1)$  and existence of lifting.

**Proposition 1.** *Assume that the answer to  $(L_{s,p})$  is yes. Then  $C^\infty(\overline{Q}; \mathbb{S}^1)$  is dense in  $W^{s,p}(Q; \mathbb{S}^1)$ .*

*Proof.* Let first  $s \leq 1$ . Write an arbitrary  $u \in W^{s,p}(Q; \mathbb{S}^1)$  as  $u = e^{i\varphi}$  with  $\varphi \in W^{s,p}(Q; \mathbb{R})$ . Since the map  $W^{s,p}(Q; \mathbb{R}) \ni \varphi \mapsto e^{i\varphi} \in W^{s,p}(Q; \mathbb{S}^1)$  is clearly continuous, the conclusion follows by approximating  $\varphi$  with smooth maps.

Let now  $s > 1$ . If  $u \in W^{s,p}(Q; \mathbb{S}^1)$  and if  $\varphi \in W^{s,p}(Q; \mathbb{R})$  is a lifting of  $u$ , then actually  $\varphi$  belongs also to  $W^{1,sp}(Q; \mathbb{R})$  [4]. We conclude by using the fact that the map  $W^{1,sp}(Q; \mathbb{R}) \cap W^{s,p}(Q; \mathbb{R}) \ni \varphi \mapsto e^{i\varphi} \in W^{s,p}(Q; \mathbb{S}^1)$  is continuous [7, 11].  $\square$

As a byproduct of the proof, we obtain the following

**Corollary 1.** *Assume that  $u \in W^{s,p}(Q; \mathbb{S}^1)$  has a lifting in  $W^{s,p}(Q; \mathbb{R})$ . Then  $u$  is in  $X^{s,p}$ .*

**Proposition 2.** *Assume that  $0 < s < 1$  and  $sp \geq 1$ . Let  $\psi \in W^{1,sp}(Q; \mathbb{R})$  and set  $u = e^{i\psi}$ . Then*

a) *When  $sp > 1$ , we have  $u \in X^{s,p}$ .*

b) *When  $sp = 1$  and  $u \in W^{s,p}(Q; \mathbb{S}^1)$ , we have  $u \in X^{s,p}$ .*

*Proof.* a) The mapping  $W^{1,sp}(Q; \mathbb{R}) \ni \psi \mapsto e^{i\psi} \in W^{s,p}(Q; \mathbb{S}^1)$  being continuous (this is an easy consequence of the Gagliardo-Nirenberg inclusion  $W^{1,sp} \cap L^\infty \subset W^{s,p}$ ), the conclusion is immediate.

b) We may assume that  $N \geq 2$ , for otherwise  $X^{s,p} = W^{s,p}(Q; \mathbb{S}^1)$ . The main ingredient we use in the proof is the approximation technique for  $W^{s,p}$ -maps (when  $0 < s < 1$  and  $sp \leq N$ ) in [9], which is recalled below.

We first extend  $u$  by reflections across  $\partial Q$ . Since  $0 < s < 1$ , this procedure will yield a map in  $W^{s,p}((-2, 2)^N; \mathbb{S}^1)$ . We next extend this map to a map in  $W^{s,p}(\mathbb{R}^N; \mathbb{R}^2)$ . We finally obtain a map, still denoted  $u$ , which is in  $W^{s,p}(\mathbb{R}^N)$  and is, in addition,  $\mathbb{S}^1$ -valued in a neighborhood of  $\overline{Q}$ . Moreover,  $u$  has, in a neighborhood of  $\overline{Q}$ , a  $W^{1,1}$ -lifting (still denoted  $\psi$ ).

We next explain how to approximate maps as above (which are  $W^{s,p}$ ,  $\mathbb{S}^1$ -valued and with a  $W^{1,1}$ -lifting near  $\overline{Q}$ ) by maps with a simple structure. To each  $\varepsilon > 0$  and  $T \in \mathbb{R}^N$  we may associate

a unique grid  $\mathcal{G} = \mathcal{G}_{T,\varepsilon}$  of size  $2\varepsilon$  passing through  $T$ , namely  $\mathcal{G} = \bigcup_{m \in \mathbb{Z}^N} (T + (0, 2\varepsilon)^N)$ . For  $j = 0, \dots, N$ , one may define the  $j^{\text{th}}$ -dimensional skeleton  $\mathcal{C}^j = \mathcal{C}_{T,\varepsilon}^j$  of  $\mathcal{G}$  as follows:  $\mathcal{C}^N$  is  $\mathbb{R}^N$ , i. e., the union of the cubes in  $\mathcal{G}$ .  $\mathcal{C}^{N-1}$  is the union of the faces of the cubes in  $\mathcal{G}$ . By backward induction,  $\mathcal{C}^j$  is the union of the (geometrical) boundaries of the  $(j+1)$ -dimensional faces that form  $\mathcal{C}^{j+1}$ .

Let  $u : \mathbb{R}^N \rightarrow \mathbb{R}$ . To each  $\varepsilon$ ,  $T$  and  $j$  we may associate a map  $u_{\varepsilon,T,j} : \mathbb{R}^N \rightarrow \mathbb{R}^M$  as follows: let  $g = g_{T,\varepsilon,j}$  be the restriction of  $u$  to  $\mathcal{C}^j = \mathcal{C}_{T,\varepsilon}^j$ . We extend  $g$  to  $\mathcal{C}^{j+1}$  "homogenously". More specifically, let  $C$  be the center of some face  $F$  of  $\mathcal{C}^{j+1}$ . Then the homogeneous extension of  $g$  from  $\partial F$  to  $F$  is the map which is constant on segments joining  $C$  to points of  $\partial F$ . Equivalently, the desired extension is  $H_{j+1}g$ , where  $H_{j+1}g(x) = g\left(\frac{\varepsilon}{\|x - C\|_\infty}(x - C)\right)$  if  $x \in F$ .

So far, we have a map  $H_{j+1}g$  defined on  $\mathcal{C}^{j+1}$ . We extend it homogeneously first to  $\mathcal{C}^{j+2}$ , next to  $\mathcal{C}^{j+3}$  and so on. We end up with  $u_{\varepsilon,T,j} := H_N \circ H_{N-1} \circ \dots \circ H_{j+1}g$ .

We may now state the main result in [9]:

Assume that  $u \in W^{s,p}(\mathbb{R}^N)$ , with  $0 < s < 1$  and  $sp < N$ . Set  $j = [sp]$  (the largest integer not exceeding  $sp$ ). Then there is a sequence  $\varepsilon_k \rightarrow 0$  and, for each  $k$ , there is a full measure set  $A_k \subset \mathbb{R}^N$  such that  $u_{\varepsilon_k,T_k,j} \rightarrow u$  in  $W^{s,p}$  whenever  $T_k \in A_k$ .<sup>1</sup>

We now return to the proof of b).

The set  $A_k$  being of full measure, we may assume that the  $T_k$ 's have been chosen such that:

- (i)  $u|_{\mathcal{C}^1} \in W^{s,p}$ ;
- (ii) there is a cube  $Q_k$  containing  $\bar{Q}$  such that  $\psi|_{\mathcal{C}^1 \cap Q_k} \in W^{1,1}$ . In addition, we may assume that  $Q_k$  is union of some cubes in  $\mathcal{G}$ .

Let now  $B$  be the boundary of a square  $S \in \mathcal{C}^2 \cap Q_k$ . Since  $u|_B \in W^{s,p}$ , we may lift, locally on  $B$ ,  $u = e^{i\varphi}$  with  $\varphi \in W^{s,p}$ . (Recall that in one dimension, lifting always exists.) Since  $2\pi\mathbb{Z}$ -valued maps in  $W^{s,p} + W^{1,1}$  are locally constant [4], we find that  $\psi|_B \in W^{s,p}$ . Thus, restricted to  $\mathcal{C}^1 \cap Q_k$ ,  $u$  has a lifting in  $W^{s,p}$ .

It is easy to see that, if  $f \in W^{\sigma,q}(\mathcal{C}^l)$ , where  $0 < \sigma < 1$  and  $\sigma q < l + 1$ , then  $H_{l+1}f \in W^{\sigma,q}(\mathcal{C}^{l+1})$  [9]. Applying this property with  $\sigma = s$ ,  $q = p$  and  $l = 2, \dots, N$ , we find that, in  $Q$ , we have  $u_{\varepsilon_k,T_k,1} = e^{i\psi_k}$ , where  $\psi_k \in W^{s,p}(Q; \mathbb{R})$ . Since clearly  $e^{i\psi_k} \in X^{s,p}$ , we find that  $u \in X^{s,p}$ .  $\square$

**Proposition 3.** *Let  $s \geq 1$ . If  $u \in X^{s,p}$ , then  $u$  lifts in  $W^{s,p}(Q; \mathbb{R})$ .*

*Proof.* When  $s \geq 1$ , a map  $u = u_1 + iu_2 \in W^{s,p}(Q; \mathbb{S}^1)$  has a lifting in  $W^{s,p}(Q; \mathbb{R})$  if and only if the vector field  $Y := u_1 \nabla u_2 - u_2 \nabla u_1$  is closed in the distribution sense, i. e. if (\*)  $\frac{\partial Y_j}{\partial x_k} = \frac{\partial Y_k}{\partial x_j}$  in  $\mathcal{D}'(Q)$ ,  $j, k = 1, \dots, N$  [4].

When  $u = e^{i\varphi}$  with  $\varphi \in C^2$ , (\*) becomes  $\frac{\partial^2 \varphi}{\partial x_j \partial x_k} = \frac{\partial^2 \varphi}{\partial x_k \partial x_j}$  and is clearly satisfied. Since the

<sup>1</sup>In the proof of Proposition 2, we will apply this approximation result to the  $\mathbb{R}^2$ -valued map  $u$  when  $s = 1/p$  (and thus  $j = 1$ ).

mapping  $W^{s,p}(Q; \mathbb{S}^1) \ni u \mapsto Y \in L^1(Q) \subset \mathcal{D}'(Q)$  is clearly continuous, we find that (\*) still holds for  $u \in X^{s,p}$ .  $\square$

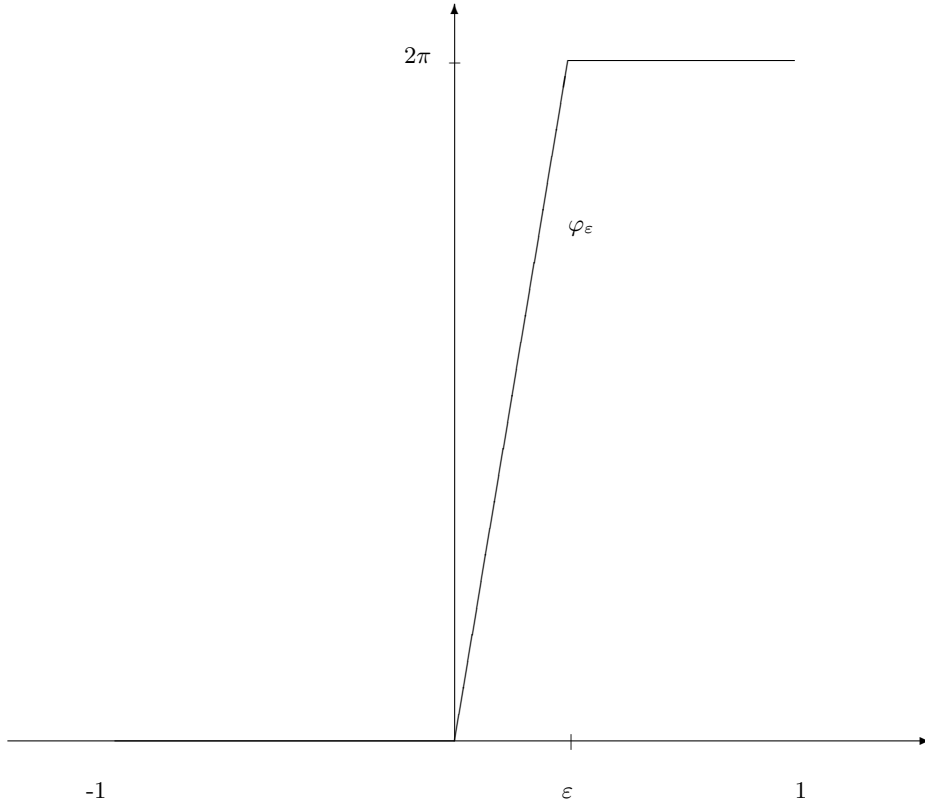
### 3. HEURISTICS OF THE PROOF OF THEOREM 2

Recall that, in one dimension, lifting always exists. One may thus hope that, for each  $s$  and  $p$  and for each  $u \in W^{s,p}((-1, 1); \mathbb{S}^1)$ , one may find a lifting  $\psi \in W^{s,p}((-1, 1); \mathbb{R})$  of  $u$  satisfying in addition an estimate of the form  $\|\psi\|_{W^{s,p}} \leq F(\|u\|_{W^{s,p}})$ . This is indeed true except when  $sp = 1$  [4].

[There is a parallel between this situation and the case of lifting in classes of smooth maps. It is easy to see that, when  $u \in C^k([-1, 1]; \mathbb{S}^1)$  for some  $k \geq 1$ , the derivatives of the smooth lifting  $\varphi$  of  $u$  are controlled by those of  $u$ . However, when  $k = 0$ , the uniform norm of  $u$  is always 1, while the one of  $\varphi$  is arbitrary, so that there is no control of  $\varphi$  in terms of  $u$ .

It is thus not a surprise that there is no control in  $W^{1/p,p}$ , which is the space that "almost" embeds in  $C^0$ .]

Indeed, let  $\varphi_\varepsilon$  be as in the picture below:



Unlike its phase  $\varphi_\varepsilon$ ,  $e^{i\varphi_\varepsilon}$  remains bounded in  $W^{1/p,p}((-1, 1))$  ( $p > 1$ )

Since  $\varphi_\varepsilon \rightarrow \chi_{(0,1]}$  and non constant step functions are not in  $W^{1/p,p}$ , it follows that  $\|\varphi_\varepsilon\|_{W^{1/p,p}} \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ .

Consider now  $u_\varepsilon := e^{i\varphi_\varepsilon}$ . It is easy to see that  $u_1 \in W^{1/p,p}(\mathbb{R})$  (while  $\varphi_1$  belongs only to  $W_{loc}^{1/p,p}(\mathbb{R})$ , but not to  $W^{1/p,p}(\mathbb{R})$ ). Since  $u_\varepsilon(\cdot) = u_1(\cdot/\varepsilon)$  and the  $W^{1/p,p}$ -semi-norm is scale invariant in  $\mathbb{R}$ , it follows that  $u_\varepsilon$  remains bounded in  $W^{1/p,p}$  as  $\varepsilon \rightarrow 0$ .

[This does not contradict Theorem 2, since  $\varphi_\varepsilon$  is bounded in  $W^{1,1}$ .]

This is a typical situation where one needs the  $W^{1,sp}$ -part,  $\varphi_2$ , of  $\varphi$ : the case where  $u$  is strongly oscillating. In contrast, the  $W^{s,p}$ -part,  $\varphi_1$ , is controlled by  $u$  provided  $u$  has small oscillations.

In practice, we obtain the decomposition  $\varphi = \varphi_1 + \varphi_2$  as follows: we will derive a formula for the lifting  $\varphi$  of maps with small amplitude oscillations. When  $u$  is arbitrary, this formula will be used to define  $\varphi_1$ . We next simply set  $\varphi_2 := \varphi - \varphi_1$ .

In order to derive the formula of  $\varphi_1$ , assume that  $u \in W^{s,p}(Q; \mathbb{S}^1)$  is close to the constant 1. Assume for simplicity that  $u$  has been extended to  $\mathbb{R}^N$  as an  $\mathbb{S}^1$ -valued map that equals 1 at infinity; we still denote by  $u$  this extension. Let  $v = v(x, \varepsilon)$  be an extension by averages of  $u$  to  $\mathbb{R}^N \times \mathbb{R}_+$  (thus  $v(x, \varepsilon) = u * \rho_\varepsilon(x)$ , where  $\rho$  is a suitable mollifier). Since  $u$  is close to 1,  $v$  is also close to 1. In particular,  $v$  is far away from 0, so that  $w := v/|v|$  is as smooth as  $v$ .

We may write (at least when  $\varepsilon > 0$ )  $w = e^{i\psi}$  for some smooth  $\psi$ . Since  $u$  is 1 at infinity, we have  $\lim_{\varepsilon \rightarrow \infty} w(x, \varepsilon) = 1$ , which suggests that we may pick  $\psi$  such that  $\lim_{\varepsilon \rightarrow \infty} \psi(x, \varepsilon) = 0$ . This allows to write formally  $u(x) = e^{i\varphi(x)}$ , where  $\varphi(x) := \psi(x, 0)$  and

$$\varphi(x) = -\psi(x, \varepsilon) \Big|_{\varepsilon=0}^{\varepsilon=\infty} = -\int_0^\infty \frac{\partial \psi}{\partial \varepsilon}(x, \varepsilon) d\varepsilon = -\int_0^\infty w(x, \varepsilon) \wedge \frac{\partial w}{\partial \varepsilon}(x, \varepsilon) d\varepsilon.$$

It turns out that this is the right formula  $\varphi_1$ , provided we choose a more convenient  $w$ .

We may now give the explicit splitting  $\varphi = \varphi_1 + \varphi_2$  in the proof of Theorem 2. Let  $\varphi \in C^\infty(\overline{Q}; \mathbb{R})$  and set  $u = e^{i\varphi}$ . Then we may extend  $u$  to some compactly supported map in  $W^{s,p}(\mathbb{R}^N)$ , still denoted  $u$ , such that  $|u| \leq 3$  and  $|u|_{W^{s,p}(\mathbb{R}^N)} \leq C|u|_{W^{s,p}(\mathbb{R}^N)}$  (Lemma 8). Let  $\rho \in C_0^\infty$  be a mollifier (whose precise properties will be specified in Section 4). Let  $v(x, \varepsilon) = u * \rho_\varepsilon(x)$ .

Assume first that  $W^{s,p} \cap L^\infty$  is contained in  $W^{1-1/2sp, 2sp}$ , i. e. that  $s \geq 1 - \frac{1}{2sp}$  and  $sp \leq 1$ .<sup>2</sup> Then

Theorem 2 works with

$$(3) \quad \varphi_1(x) = -\int_0^\infty v(x, \varepsilon) \wedge \frac{\partial v}{\partial \varepsilon}(x, \varepsilon) d\varepsilon, \quad \varphi_2 = \varphi - \varphi_1.$$

In general (i. e., when we do not assume that  $W^{s,p} \cap L^\infty$  is contained in  $W^{1-1/2sp, 2sp}$ ), one has to project  $v$  on  $\mathbb{S}^1$ . More specifically, let  $\Pi \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$  be such that  $\Pi(z) = z/|z|$  near  $\mathbb{S}^1$  and set  $w := \Pi(v)$ . Then we may choose, in the proof of Theorem 2,

$$(4) \quad \varphi_1(x) = -\int_0^\infty w(x, \varepsilon) \wedge \frac{\partial w}{\partial \varepsilon}(x, \varepsilon) d\varepsilon, \quad \varphi_2 = \varphi - \varphi_1.$$

<sup>2</sup>This covers the case  $s = 1/2, p = 2$  treated in [3].



We end this section by comparing our approach to the Bourgain-Brezis one [3]. The fact that  $\varphi_1$  given by (3) belongs to  $W^{s,p}$  is reminiscent from standard estimates on para-products (for a quick introduction, see, e. g., [10]). Recall that (for a suitable mollifier  $\rho$ ) the Littlewood-Paley decomposition of a function  $f$  is  $f = \sum_{j \geq 0} LP_j(f)$ , where  $LP_0(f) = f * \rho$  and, for  $j \geq 1$ ,  $LP_j(f) = f * \rho_{2^{-j}} - f * \rho_{2^{-j+1}}$ . Recall also that  $W^{s,p} \cap L^\infty$  is an algebra. Thus, if  $f, g \in W^{s,p} \cap L^\infty$ , then  $\sum_{j,k} LP_j(f)LP_k(g) \in W^{s,p}$ . The paraproducts technique yields slightly more:

$$(5) \quad \text{each of the sums } \sum_{j \leq k} LP_j(f)LP_k(g) \text{ and } \sum_{j > k} LP_j(f)LP_k(g) \text{ is in } W^{s,p}.$$

It is well-known to the experts (though difficult to find in the literature) that each "Littlewood-Paley like" (i. e., via sequences) property of some functions space has continuous analogues.<sup>3</sup> These analogues are obtained by replacing  $LP_j(f)$  by an integral, e. g.

$$f = LP_0(f) + \sum_{j=-\infty}^{\infty} (f * \rho_{2^{-j}} - f * \rho_{2^{-j+1}}) = LP_0(f) - \int_0^1 f * \frac{\partial}{\partial \varepsilon}(\rho_\varepsilon) d\varepsilon;$$

another used decomposition is  $f = - \int_0^\infty f * \frac{\partial}{\partial \varepsilon}(\rho_\varepsilon) d\varepsilon$ . Equivalently, if  $F(x, \varepsilon) = f * \rho_\varepsilon(x)$  is the extension of  $f$  to  $\mathbb{R}^N \times \mathbb{R}_+$ , then  $f(x) = - \int_0^\infty \frac{\partial F}{\partial \varepsilon}(x, \varepsilon) d\varepsilon$ .

With  $F, G$  the extensions of  $f, g \in W^{s,p} \cap L^\infty$ , an analogue of (5) is

$$(6) \quad x \mapsto \int_0^\infty F(x, \varepsilon) \frac{\partial G}{\partial \varepsilon}(x, \varepsilon) d\varepsilon \in W^{s,p}, \quad \forall f, g \in W^{s,p} \cap L^\infty.$$

Property (6) is true<sup>4</sup> and implies that the function  $\varphi_1$  given by (3) is in  $W^{s,p}$ . The same conclusion holds for the  $\varphi_1$  given by (4), but the argument is more involved.

We may now compare our construction to the Bourgain-Brezis one: their splitting is  $\varphi_1 = \sum_{j \leq k} LP_j(u) \wedge LP_k(u)$ ,  $\varphi_2 = \varphi - \varphi_1$ . This is nothing else than a discrete analogue of (3). However, it seems difficult to cover the case  $sp \neq 1$  using this decomposition.

<sup>3</sup>Example #1: one may define a square function of an  $L^p$ -function  $f$  defined in  $\mathbb{R}^N$  either through the formula  $S_1 f(x) := \left( \sum |LP_j f(x)|^2 \right)^{1/2}$ , or through  $S_2 f(x) := \left( \int_0^\infty (f * \rho_\varepsilon(x))^2 \frac{d\varepsilon}{\varepsilon} \right)^{1/2}$ , for appropriate  $\rho$ . In both cases, we have the "square function theorem" of Littlewood and Paley  $S_j(f) \sim \|f\|_{L^p}$ ,  $1 < p < \infty$  [17] II. 6. Example #2: functions  $f$  in Triebel-Lizorkin spaces can be characterized both in terms of the Littlewood-Paley decomposition of  $f$  and in terms of the behavior of the solution of the heat equation with initial condition  $f$  [18] 2.12.

<sup>4</sup>This is presumably well-known. For another  $\rho$ , (6) is nothing else but the conclusion of Lemma 19.

## 4. ESTIMATES FOR EXTENSIONS BY AVERAGES

Throughout this section,  $u \in \text{Lip}(\mathbb{R}^N)$  is such that  $|u| \leq 3$  and  $u$  is constant outside some compact<sup>5</sup>. We set, for  $\varepsilon > 0$ ,  $v(x, \varepsilon) = u * \rho_\varepsilon(x)$ .

We assume that the mollifier  $\rho$  satisfies<sup>6</sup>:

$$\rho \in C_0^\infty(\mathbb{R}^N), \quad \rho \geq 0, \quad \text{supp } \rho \subset B(0, 2), \quad \rho = 0 \text{ in } B(0, 1).$$

$C$  will denote a constant depending on  $s, p, N$ , but not on  $u$  (provided  $|u| \leq 3$ ).

**Lemma 1.** *Assume that  $0 < s < 1$ ,  $1 \leq p < \infty$ . Then*

$$(7) \quad \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{p-sp-1} |Dv(x, \varepsilon)|^p d\varepsilon dx \leq C |u|_{W^{s,p}}^p.$$

*Proof.* Set  $\zeta_j := \partial_j \rho$ ,  $j = 1, \dots, N$ , and  $\zeta_0 := -\sum_{j=1}^N \partial_j(x_j \rho)$ . It is easy to see that

$$(8) \quad \frac{\partial}{\partial x_j} v(x, \varepsilon) = \frac{1}{\varepsilon} u * (\zeta_j)_\varepsilon(x) \quad \text{and} \quad \frac{\partial}{\partial \varepsilon} v(x, \varepsilon) = \frac{1}{\varepsilon} u * (\zeta_0)_\varepsilon(x).$$

Noting that each  $\zeta_j$  is supported in  $B(0, 2)$  and has zero integral, we find that (7) is a consequence of

$$(9) \quad \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{-sp-1} |u * \zeta_\varepsilon(x)|^p d\varepsilon dx \leq C(\zeta) |u|_{W^{s,p}}^p, \quad \forall \zeta \in C_0^\infty(B(0, 2)) \text{ such that } \int \zeta = 0.$$

In order to prove (9), we note that

$$|u * \zeta_\varepsilon(x)|^p = \left| \frac{1}{\varepsilon^N} \int_{B(0, 2\varepsilon)} (u(x-y) - u(x)) \zeta\left(\frac{y}{\varepsilon}\right) dy \right|^p \leq \frac{C}{\varepsilon^N} \int_{B(0, 2\varepsilon)} |u(x-y) - u(x)|^p dy.$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{-sp-1} |u * \zeta_\varepsilon(x)|^p d\varepsilon dx &\leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{|y|/2}^\infty \frac{1}{\varepsilon^{N+sp+1}} d\varepsilon |u(x-y) - u(x)|^p dy dx \\ &= C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dx dy = C |u|_{W^{s,p}}^p. \end{aligned}$$

□

**Lemma 2.** *Assume that  $0 < s < 1$ ,  $1 \leq p < \infty$ . Then*

$$(10) \quad I := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} \left( \int_0^{|x-y|} |Dv(x, \varepsilon)| d\varepsilon \right)^p dx dy \leq C |u|_{W^{s,p}}^p.$$

<sup>5</sup>The second assumption is used only in Lemmas 5-7.

<sup>6</sup>Assumptions on  $\rho$  are not crucial. The results in this section are true for any reasonable mollifier. However, our assumptions make the proofs simpler.

*Proof.* Let  $\alpha$  be such that  $\frac{1}{p} - 1 < \alpha < \frac{1}{p} - 1 + s$ . We first note that, for  $r > 0$  and  $p > 1$ , we have

$$(11) \quad \left( \int_0^r |Dv(x, \varepsilon)| d\varepsilon \right)^p = \left( \int_0^r \frac{|Dv(x, \varepsilon)|}{\varepsilon^\alpha} \varepsilon^\alpha d\varepsilon \right)^p \leq \int_0^r \frac{|Dv(x, \varepsilon)|^p}{\varepsilon^{\alpha p}} d\varepsilon \left( \int_0^r \varepsilon^{\alpha p/(p-1)} d\varepsilon \right)^{p-1} \\ = Cr^{\alpha p + p - 1} \int_0^r \frac{|Dv(x, \varepsilon)|^p}{\varepsilon^{\alpha p}} d\varepsilon$$

(here we use the fact that  $\frac{\alpha p}{p-1} > -1$ , which is equivalent to  $\alpha > \frac{1}{p} - 1$ ). On the other hand, it is immediate that the conclusion of (11) still holds when  $p = 1$ .

If we write  $y = x + r\omega$ , with  $r = |x - y|$  and  $\omega \in \mathbb{S}^{N-1}$  and use (11) with  $r = |x - y|$ , we find that

$$I = C \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^\infty \frac{1}{r^{sp+1}} \left( \int_0^r |Dv(x, \varepsilon)| d\varepsilon \right)^p dr d\omega dx \\ \leq C \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^\infty r^{\alpha p + p - sp - 2} \int_0^r \frac{|Dv(x, \varepsilon)|^p}{\varepsilon^{\alpha p}} d\varepsilon dr d\omega dx \\ = C \int_{\mathbb{R}^N} \int_0^\infty \frac{|Dv(x, \varepsilon)|^p}{\varepsilon^{\alpha p}} \int_\varepsilon^\infty r^{\alpha p + p - sp - 2} dr d\varepsilon dx \\ = C \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{p-sp-1} |Dv(x, \varepsilon)|^p d\varepsilon dx \leq C |u|_{W^{s,p}}^p.$$

Here, we rely on the inequality  $\alpha p + p - sp - 2 < -1$  (which amounts to  $\alpha < \frac{1}{p} - 1 + s$ ) and on Lemma 1.  $\square$

**Lemma 3.** *Assume that  $0 < s < 1$ ,  $1 \leq p < \infty$ . Then*

$$(12) \quad J := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{1}{|x - y|^{N+sp}} \left( \int_{|x-y|}^\infty |Dv(x, \varepsilon) - Dv(y, \varepsilon)| d\varepsilon \right)^p dx dy \leq C |u|_{W^{s,p}}^p.$$

*Proof.* In view of (8), it suffices to prove that

$$(13) \quad \tilde{J} := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{1}{|x - y|^{N+sp}} \left( \int_{|x-y|}^\infty \frac{|u * \zeta_\varepsilon(x) - u * \zeta_\varepsilon(y)|}{\varepsilon} d\varepsilon \right)^p dx dy \leq C |u|_{W^{s,p}}^p$$

under the assumptions  $\zeta \in C_0^\infty(B(0, 2))$ ,  $\int \zeta = 0$  and  $\zeta = 0$  in  $B(0, 1)$ .

If we set  $\Phi := D\zeta$ , then  $D_x(u * \zeta_\varepsilon) = \frac{1}{\varepsilon} u * \Phi_\varepsilon$ ,  $\Phi \in C_0^\infty(B(0, 2))$ ,  $\int \Phi = 0$  and  $\Phi = 0$  in  $B(0, 1)$ .

We find that

$$|D_x(u * \zeta_\varepsilon)(x)| = \frac{1}{\varepsilon^{N+1}} \left| \int_{B(0, 2\varepsilon) \setminus B(0, \varepsilon)} (u(x+z) - u(x)) \Phi \left( \frac{z}{\varepsilon} \right) dz \right| \leq \frac{C}{\varepsilon^{N+1}} \int_{\varepsilon \leq |z| \leq 2\varepsilon} |u(x+z) - u(x)| dz.$$

Therefore, with  $r := |y - x|$  and  $\omega := \frac{y - x}{r}$ , we have

$$\begin{aligned} |u * \zeta_\varepsilon(x) - u * \zeta_\varepsilon(y)| &= r \left| \int_0^1 D_x(u * \zeta_\varepsilon)(x + rt\omega) \cdot \omega dt \right| \\ &\leq \frac{Cr}{\varepsilon^{N+1}} \int_0^1 \int_{\varepsilon \leq |z| \leq 2\varepsilon} |u(x + rt\omega + z) - u(x + rt\omega)| dz dt, \end{aligned}$$

which implies that

$$\begin{aligned} (14) \quad \int_{|x-y|}^\infty \frac{|u * \zeta_\varepsilon(x) - u * \zeta_\varepsilon(y)|}{\varepsilon} d\varepsilon &\leq Cr \int_{|z| \geq r} \int_0^1 \int_{|z|/2}^{|z|} \frac{|u(x + rt\omega + z) - u(x + rt\omega)|}{\varepsilon^{N+2}} d\varepsilon dt dz \\ &= Cr \int_0^1 \int_{|z| \geq r} \frac{|u(x + rt\omega + z) - u(x + rt\omega)|}{|z|^{N+1}} dz dt. \end{aligned}$$

Let now  $\alpha$  be such that  $\frac{N}{p} - N + s - 1 < \alpha < \frac{N}{p} - N$ . We perform the following calculation only for  $p > 1$ , but the reader may easily see that its conclusion still holds for  $p = 1$ . Using (14), we find that

$$\begin{aligned} (15) \quad K &:= \left( \int_{|x-y|}^\infty \frac{|u * \zeta_\varepsilon(x) - u * \zeta_\varepsilon(y)|}{\varepsilon} d\varepsilon \right)^p \\ &\leq Cr^p \left( \int_0^1 \int_{|z| \geq r} \frac{|u(x + rt\omega + z) - u(x + rt\omega)|}{|z|^{N+1+\alpha}} |z|^\alpha dz dt \right)^p \\ &\leq Cr^p \int_0^1 \int_{|z| \geq r} \frac{|u(x + rt\omega + z) - u(x + rt\omega)|^p}{|z|^{Np+p+\alpha p}} dz dt \left( \int_{|z| \geq r} |z|^{\alpha p/(p-1)} dz \right)^{p-1} \\ &= Cr^{p+\alpha p+Np-N} \int_0^1 \int_{|z| \geq r} \frac{|u(x + rt\omega + z) - u(x + rt\omega)|^p}{|z|^{Np+p+\alpha p}} dz dt \end{aligned}$$

since  $\frac{\alpha p}{p-1} < -N$  (equivalently, since  $\alpha < \frac{N}{p} - N$ ). Inserting (15) into the definition of  $\tilde{J}$  and computing the integral in  $y$  in spherical coordinates, we obtain

$$\begin{aligned} \tilde{J} &\leq C \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^\infty \int_0^1 \int_{|z| \geq r} r^{p+\alpha p+Np-N-sp-1} \frac{|u(x + rt\omega + z) - u(x + rt\omega)|^p}{|z|^{Np+p+\alpha p}} dz dt dr d\omega dx \\ &= C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x+z) - u(x)|^p}{|z|^{Np+p+\alpha p}} \int_0^{|z|} r^{p+\alpha p+Np-N-sp-1} dr dx dz \\ &= C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x+z) - u(x)|^p}{|z|^{N+sp}} dx dz = C |u|_{W^{s,p}}^p. \end{aligned}$$

Here, we used the inequality  $p+\alpha p+Np-N-sp-1 > -1$ , which amounts to  $\alpha > \frac{N}{p} - N + s - 1$ .  $\square$

**Lemma 4.** *Assume  $0 < s < 1$ ,  $1 \leq p < \infty$ . Then*

$$(16) \quad L := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} \left( \int_{|x-y|}^{\infty} \frac{|x-y|}{\varepsilon} |Dv(x, \varepsilon)| d\varepsilon \right)^p dx dy \leq C|u|_{W^{s,p}}^p.$$

*Proof.* In spherical coordinates  $y = x + r\omega$  we have

$$\begin{aligned} L &= C \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^{\infty} r^{p-sp-1} \left( \int_r^{\infty} \frac{|Dv(x, \varepsilon)|}{\varepsilon} d\varepsilon \right)^p dr d\omega dx \\ &= C \int_{\mathbb{R}^N} \int_0^{\infty} r^{p-sp-1} \left( \int_r^{\infty} \frac{|Dv(x, \varepsilon)|}{\varepsilon} d\varepsilon \right)^p dr dx, \end{aligned}$$

so that (16) amounts to

$$\tilde{L} := \int_{\mathbb{R}^N} \int_0^{\infty} r^{p-sp-1} \left( \int_r^{\infty} \frac{|Dv(x, \varepsilon)|}{\varepsilon} d\varepsilon \right)^p dr dx \leq C|u|_{W^{s,p}}^p.$$

Let  $\alpha$  be such that  $\frac{1}{p} + s - 2 < \alpha < \frac{1}{p} - 1$ . We perform the calculation below for  $p > 1$ ; clearly, its conclusion still holds for  $p = 1$ . We have

$$(17) \quad \begin{aligned} \left( \int_r^{\infty} \frac{|Dv(x, \varepsilon)|}{\varepsilon} d\varepsilon \right)^p &= \left( \int_r^{\infty} \frac{|Dv(x, \varepsilon)|}{\varepsilon^{\alpha+1}} \varepsilon^{\alpha} d\varepsilon \right)^p \\ &\leq \int_r^{\infty} \frac{|Dv(x, \varepsilon)|^p}{\varepsilon^{p+\alpha p}} d\varepsilon \left( \int_r^{\infty} \varepsilon^{\alpha p/(p-1)} d\varepsilon \right)^{p-1} \\ &\leq C r^{\alpha p + p - 1} \int_r^{\infty} \frac{|Dv(x, \varepsilon)|^p}{\varepsilon^{p+\alpha p}} d\varepsilon, \end{aligned}$$

since  $\frac{\alpha p}{p-1} < -1$  (i. e.,  $\alpha < \frac{1}{p} - 1$ ). Combining (17) with the definition of  $\tilde{L}$ , we obtain

$$\begin{aligned} \tilde{L} &\leq C \int_{\mathbb{R}^N} \int_0^{\infty} \int_r^{\infty} |Dv(x, \varepsilon)|^p \varepsilon^{-p-\alpha p} r^{2p-sp+\alpha p-2} d\varepsilon dr dx \\ &= C \int_{\mathbb{R}^N} \int_0^{\infty} \int_0^{\varepsilon} |Dv(x, \varepsilon)|^p \varepsilon^{-p-\alpha p} r^{2p-sp+\alpha p-2} dr d\varepsilon dx \\ &= C \int_{\mathbb{R}^N} \int_0^{\infty} \varepsilon^{p-sp-1} |Dv(x, \varepsilon)|^p d\varepsilon dx \leq C|u|_{W^{s,p}}^p. \end{aligned}$$

Here, we used the fact that  $2p - sp + \alpha p - 2 > -1$  (which is equivalent to  $\alpha > \frac{1}{p} + s - 2$ ) and Lemma 1. □

**Lemma 5.** *For each  $x \in \mathbb{R}^N$ , the integral  $\int_0^{\infty} |Dv(x, \varepsilon)| d\varepsilon$  is convergent.*

*Proof.* Since  $v$  is Lipschitz, it suffices to prove that  $\int_1^\infty |Dv(x, \varepsilon)| d\varepsilon$  converges. With  $a$  the value of  $u$  at infinity, this follows from

$$|Dv(x, \varepsilon)| = |D[(u - a) * \rho_\varepsilon(x)]| = |(u - a) * D[\rho_\varepsilon(x)]| \leq \|u - a\|_{L^1} \|D[\rho_\varepsilon(x)]\|_{L^\infty} \leq \frac{C}{\varepsilon^{N+1}}.$$

□

In the same spirit, we have the following result, whose straightforward proof will be omitted

**Lemma 6.** *Assume  $u$  complex-valued, Lipschitz in  $\mathbb{R}^N$  and smooth in  $Q$ . Set*

$$(18) \quad w(x) = w_\Pi(x) := \int_0^\infty \Pi \circ v(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v(x, \varepsilon)) d\varepsilon, \quad \text{where } \Pi \in C^\infty(\mathbb{R}^2; \mathbb{R}^2).$$

*Then  $w$  is smooth in  $Q$  and  $\partial^\alpha w(x) = \int_0^\infty \partial^\alpha \left( \Pi \circ v(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v(x, \varepsilon)) \right) d\varepsilon$ .*

We may now prove that the map  $\varphi_1$  defined in (4) is in  $W^{s,p}$  (plus norm control).

**Lemma 7.** *Assume that  $u \in Lip(\mathbb{R}^N; \mathbb{C})$  satisfies  $|u| \leq 3$ . Let  $w$  be defined by (18). Then*

$$(19) \quad |w|_{W^{s,p}} \leq C|u|_{W^{s,p}}.$$

*Proof.* Set  $a(x, y) := \int_0^{|x-y|} \Pi \circ v(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) d\varepsilon$  and  $b(x, y) := \int_{|x-y|}^\infty \Pi \circ v(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) d\varepsilon$ , so that  $w(x) = a(x, y) + b(x, y)$ . On the one hand, we have

$$(20) \quad \begin{aligned} |w(x) - w(y)| &\leq |a(x, y)| + |a(y, x)| + |b(x, y) - b(y, x)| \\ &\leq |b(x, y) - b(y, x)| + C \left( \int_0^{|x-y|} |Dv(x, \varepsilon)| d\varepsilon + \int_0^{|x-y|} |Dv(y, \varepsilon)| d\varepsilon \right). \end{aligned}$$

On the other hand, we have

$$(21) \quad \begin{aligned} b(x, y) - b(y, x) &= \int_{|x-y|}^\infty \left( \Pi \circ v(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) - \Pi \circ v(y, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(y, \varepsilon) \right) d\varepsilon \\ &= \int_{|x-y|}^\infty (\Pi \circ v(x, \varepsilon) - \Pi \circ v(y, \varepsilon)) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) d\varepsilon \\ &\quad + \int_{|x-y|}^\infty \Pi \circ v(y, \varepsilon) \left( \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) - \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(y, \varepsilon) \right) d\varepsilon. \end{aligned}$$

Since (with  $\zeta_j$  as in (8))

$$(22) \quad |D(\Pi \circ v)(x, \varepsilon)| \leq \frac{C}{\varepsilon} \sum_{j=0}^N \|u\|_{L^\infty} \|(\zeta_j)_\varepsilon\|_{L^1} \leq \frac{C}{\varepsilon},$$

we find that  $|\Pi \circ v(x, \varepsilon) - \Pi \circ v(y, \varepsilon)| \leq \frac{C|x-y|}{\varepsilon}$ , which yields

$$\begin{aligned}
(23) \quad |b(x, y) - b(y, x)| &\leq C \int_{|x-y|}^{\infty} \left| \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) - \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(y, \varepsilon) \right| d\varepsilon \\
&\quad + C \int_{|x-y|}^{\infty} \frac{|x-y|}{\varepsilon} \left| \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) \right| d\varepsilon \\
&\leq C \left( \int_{|x-y|}^{\infty} |Dv(x, \varepsilon) - Dv(y, \varepsilon)| d\varepsilon + \int_{|x-y|}^{\infty} \frac{|x-y|}{\varepsilon} |Dv(x, \varepsilon)| d\varepsilon \right).
\end{aligned}$$

By combining (20)-(23) to Lemmas 2-4, we find that

$$|w|_{W^{s,p}}^p \leq C(I + J + L) \leq C|u|_{W^{s,p}}^p.$$

□

## 5. PROOF OF THEOREMS 1 AND 2

*Proof of Theorem 2.* Let  $\varphi \in C^\infty(\overline{Q}; \mathbb{S}^1)$  and set  $u := e^{i\varphi}$ .

**Lemma 8.** *The map  $u$  has a  $C^1$ -extension to  $\mathbb{R}^N$ , still denoted  $u$ , such that  $|u| \leq 3$ ,  $u$  is constant outside  $(-2, 2)^N$  and  $|u|_{W^{s,p}(\mathbb{R}^N)} \leq C|u|_{W^{s,p}(Q)}$ .*

*Proof.* Let  $P : W^{s,p}(Q) \rightarrow W^{s,p}(\mathbb{R}^N)$  be a linear continuous extension operator such that:  $P$  extends Lipschitz maps to Lipschitz maps,  $P$  does not increase the  $L^\infty$ -norm and  $\text{supp } Pv \subset (-2, 2)^N$ ,  $\forall v \in W^{s,p}$ . Let  $a$  be the average of  $u$  on  $Q$ . It is easy to see that  $\tilde{u} := a + P(u - a)$  is Lipschitz and satisfies  $|\tilde{u}| \leq 3$ . In addition, we have

$$|\tilde{u}|_{W^{s,p}(\mathbb{R}^N)} = |P(u - a)|_{W^{s,p}(\mathbb{R}^N)} \leq C\|u - a\|_{W^{s,p}(Q)} \leq C|u - a|_{W^{s,p}(Q)} = C|u|_{W^{s,p}(Q)};$$

we have used the Poincaré inequality  $\|u - a\|_{W^{s,p}(Q)} \leq C|u - a|_{W^{s,p}(Q)}$ , valid since  $u - a$  has zero average. □

Let then  $\Pi \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$  to be chosen later and set  $\varphi_1(x) := - \int_0^\infty \Pi \circ v(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) d\varepsilon$ .

By Lemma 19,  $\varphi_1$  belongs to  $W^{s,p}$  and satisfies  $|\varphi_1|_{W^{s,p}} \leq C|u|_{W^{s,p}}$ .

We set, for  $x \in \overline{Q}$ ,  $\varphi_2(x) := \varphi(x) - \varphi_1(x)$ .

**Lemma 9.** *Assume that  $\Pi(z) = z$ ,  $\forall z \in \mathbb{S}^1$ . Then*

$$(24) \quad D\varphi_2(x) = -2 \int_0^\infty \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) \wedge D_x(\Pi \circ v)(x, \varepsilon) d\varepsilon.$$

*Proof.* Using Lemma 6 and the identity  $D\varphi = u \wedge Du$  we have, with  $w := \Pi \circ v$ ,

$$\begin{aligned} D\varphi_2(x) &= D\varphi(x) + \int_0^\infty D_x w(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} w(x, \varepsilon) d\varepsilon + \int_0^\infty w(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} D_x w(x, \varepsilon) d\varepsilon \\ &= w(x, \varepsilon) \wedge D_x w(x, \varepsilon)|_{\varepsilon=0} + \int_0^\infty D_x w(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} w(x, \varepsilon) d\varepsilon + w(x, \varepsilon) \wedge D_x w(x, \varepsilon) \Big|_{\varepsilon=0}^{\varepsilon=\infty} \\ &\quad - \int_0^\infty \frac{\partial}{\partial \varepsilon} w(x, \varepsilon) \wedge D_x w(x, \varepsilon) d\varepsilon = -2 \int_0^\infty \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) \wedge D_x (\Pi \circ v)(x, \varepsilon) d\varepsilon. \end{aligned}$$

□

The remaining part of the proof of Theorem is essentially a variant of the proof of Theorem 0.1 in [5]. Up to now, the proof requires only  $\Pi(u) = u$ .<sup>7</sup> From now on, we will require that  $\Pi$  is an approximate projection on  $\mathbb{S}^1$ , e.g. we assume  $\Pi(z) = \frac{z}{|z|}$  when  $|z| \geq \frac{1}{2}$ .

**Lemma 10.** *Let, for  $x \in \overline{Q}$ ,  $d(x) := \inf\{\varepsilon > 0 ; |v(x, \varepsilon)| = 1/2\}$ . Then*

$$(25) \quad \int_Q \frac{1}{d(x)^{sp}} dx \leq C |u|_{W^{s,p}(Q)}^p.$$

*Proof.* Let  $x$  be such that  $d(x)$  is finite. Since  $|u(x) - v(x, d(x))| \geq 1/2$ , we have

$$1/2 \leq |u(x) - v(x, d(x))| \leq |v(x, \cdot)|_{C^s(\mathbb{R}_+)} d(x)^s \leq C |v(x, \cdot)|_{W^{s+1/p,p}(\mathbb{R}_+)} d(x)^s,$$

so that

$$\int_Q \frac{1}{d(x)^{sp}} dx \leq C \int_Q |v(x, \cdot)|_{W^{s+1/p,p}(\mathbb{R}_+)}^p \leq C |v|_{W^{s+1/p,p}}^p \leq C |u|_{W^{s,p}(\mathbb{R}^N)}^p \leq C |u|_{W^{s,p}(Q)}^p,$$

by the Besov lemma [2].

□

**Lemma 11.** *We have*

$$(26) \quad \int_Q |D\varphi_2|^{sp} dx \leq C |u|_{W^{s,p}(Q)}^p.$$

*Proof.* Set  $\Omega := \{(x, \varepsilon) ; x \in Q, 0 < \varepsilon < d(x)\}$ . In  $\Omega$ , we have  $|\Pi \circ v| \equiv 1$ , so that  $\frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) \wedge D_x (\Pi \circ v)(x, \varepsilon) \equiv 0$  in  $\Omega$ . In view of Lemma 9 and (22), we find that

$$\begin{aligned} \int_Q |D\varphi_2|^{sp} dx &\leq \int_Q \left( \int_{d(x)}^\infty |D(\Pi \circ v)(x, \varepsilon)|^2 d\varepsilon \right)^{sp} dx \\ &\leq C \int_Q \left( \int_{d(x)}^\infty \frac{1}{\varepsilon^2} d\varepsilon \right)^{sp} dx \leq C \int_Q \frac{1}{d(x)^{sp}} dx \leq C |u|_{W^{s,p}(Q)}^p. \end{aligned}$$

□

<sup>7</sup>When  $W^{s,p} \cap L^\infty$  is contained in  $W^{1-1/2sp, 2sp}$ , Lemma 11 below is valid without any other restriction.



The proof of Theorem 2 is complete.  $\square$

*Proof of Theorem 1.* When  $p > 1$  and  $sp > 1$ , Theorem 1 is an immediate consequence of Theorem 2 (one passes to the weak limits in (1) and (2)). Some care is needed when  $p = 1$  or  $sp = 1$ .

**Lemma 12.** *Let  $u \in X^{s,p}$ . Then there is a sequence  $\{u_k\}_{k \geq 0} \subset C^\infty(\overline{Q}; \mathbb{S}^1)$  such that  $u = \prod_{k=0}^{\infty} u_k$*

*and  $|u_k|_{W^{s,p}} \leq 2^{-k+1}|u|_{W^{s,p}}, \forall k$ .*

*Proof.* Recall that

$$(27) \quad \text{if } f_j \rightarrow f \text{ in } W^{s,p}, g_j \rightarrow g \text{ in } W^{s,p}, \|f_j\|_{L^\infty} \leq C, \|g_j\|_{L^\infty} \leq C, \text{ then } f_j g_l \xrightarrow{j,l \rightarrow \infty} fg \text{ in } W^{s,p}.$$

Consider a sequence  $\{h_j\} \subset C^\infty(\overline{Q}; \mathbb{S}^1)$  such that  $h_j \rightarrow u$ . We may assume that  $|h_j|_{W^{s,p}} \leq 2|u|_{W^{s,p}}, \forall j$ . Using (27) with  $f_j = h_j$  and  $g_j = \overline{h_j}$ , we find easily a sequence  $j_k \rightarrow \infty$  such that  $\|h_{j_{k+1}} \overline{h_{j_k}} - 1\|_{W^{s,p}} \leq 2^{-k}|u|_{W^{s,p}}, k \geq 0$ . If we set  $u_0 = h_{j_0}$  and  $u_k = h_{j_k} \overline{h_{j_{k-1}}}, k \geq 1$ , then the sequence  $\{u_k\}$  has the required properties.  $\square$

In view of Theorem 2, we may write each  $u_k$  as  $e^{i(\varphi_1^k + \varphi_2^k)}$ , where  $|\varphi_1^k|_{W^{s,p}} \leq C2^{-k}|u|_{W^{s,p}}$  and  $\|D\varphi_2^k\|_{L^{sp}} \leq C2^{-k/s}|u|_{W^{s,p}}^{1/s}$ .

Let  $\varphi_1 := \sum_k \varphi_1^k$ . Clearly,  $\varphi_1 \in W^{s,p}$  and  $|\varphi_1|_{W^{s,p}} \leq C|u|_{W^{s,p}}$ . On the other hand, set  $\psi := \sum_k (\varphi_2^k - \int_Q \varphi_2^k)$ , which satisfies  $\psi \in W^{1,sp}$  and  $\|D\psi\|_{L^{sp}}^{sp} \leq C|u|_{W^{s,p}}^p$ . In addition, the map  $ue^{-i(\varphi_1 + \psi)}$  is constant. It follows that, for an appropriate  $\alpha \in \mathbb{R}$ ,  $\varphi_1$  and  $\varphi_2 := \alpha + \psi$  satisfy  $u = e^{i(\varphi_1 + \varphi_2)}$  and the estimates (1)-(2).

## 6. CHARACTERIZATION OF $X^{s,p}$ IN TERMS OF LIFTING

The results in [4], [8] and [9] give the following information about  $X^{s,p}$

SPACE DIMENSION $N$	SIZE OF $s$	SIZE OF $sp$	DESCRIPTION OF $X^{s,p}$
ANY	$0 < s < 1$	$0 < sp < 1$	$X^{s,p} = W^{s,p}(Q; \mathbb{S}^1) = \{e^{i\varphi} ; \varphi \in W^{s,p}(Q; \mathbb{R})\}$
$N \geq 2$	$0 < s < 1$	$1 \leq sp < 2$	$X^{s,p} \neq W^{s,p}(Q; \mathbb{S}^1), X^{s,p} \neq \{e^{i\varphi} ; \varphi \in W^{s,p}(Q; \mathbb{R})\}$
$N \geq 3$	$0 < s < 1$	$2 \leq sp < N$	$X^{s,p} = W^{s,p}(Q; \mathbb{S}^1), X^{s,p} \neq \{e^{i\varphi} ; \varphi \in W^{s,p}(Q; \mathbb{R})\}$
ANY	$0 < s < 1$	$sp \geq N$	$X^{s,p} = W^{s,p}(Q; \mathbb{S}^1) = \{e^{i\varphi} ; \varphi \in W^{s,p}(Q; \mathbb{R})\}$
$N = 1$	$s \geq 1$	ANY	$X^{s,p} = W^{s,p}(Q; \mathbb{S}^1) = \{e^{i\varphi} ; \varphi \in W^{s,p} \cap W^{1,sp}\}$
$N \geq 2$	$s \geq 1$	$1 \leq sp < 2$	$X^{s,p} \neq W^{s,p}(Q; \mathbb{S}^1), X^{s,p} = \{e^{i\varphi} ; \varphi \in W^{s,p} \cap W^{1,sp}\}$
$N \geq 2$	$s \geq 1$	$sp \geq 2$	$X^{s,p} = W^{s,p}(Q; \mathbb{S}^1) = \{e^{i\varphi} ; \varphi \in W^{s,p} \cap W^{1,sp}\}$

The following result completes the description of  $X^{s,p}$  in terms of lifting<sup>8</sup>

**Theorem 3.** *Assume that  $0 < s < 1$  and  $1 \leq sp < N$ .*

a) *If  $sp > 1$ , then  $X^{s,p} = \{e^{i\varphi} ; \varphi \in W^{s,p} + W^{1,sp}\}$ .*

b) *If  $sp = 1$ , then  $X^{s,p} = W^{s,p}(Q; \mathbb{S}^1) \cap \{e^{i\varphi} ; \varphi \in W^{s,p} + W^{1,1}\}$ .*

*Proof.* "⊂" follows from Theorem 1.

"⊃" Let  $u = e^{i(\varphi_1 + \varphi_2)} \in W^{s,p}(Q; \mathbb{S}^1)$ , with  $\varphi_1 \in W^{s,p}$  and  $\varphi_2 \in W^{1,sp}$ . Set  $u_j := e^{i\varphi_j}$ . By Corollary 1,  $u_1 \in X^{s,p}$ . On the other hand,  $W^{s,p}(Q; \mathbb{S}^1)$  is a group, so that  $u_2 \in W^{s,p}(Q; \mathbb{S}^1)$ . By Proposition 2,  $u_2 \in X^{s,p}$ . By (27),  $X^{s,p}$  is a group, so that  $u = u_1 u_2 \in X^{s,p}$ .  $\square$

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UNIVERSITÉ DE LYON, UNIVERSITÉ LYON 1, CNRS, UMR 5208 INSTITUT CAMILLE JORDAN, BÂTIMENT DU DOYEN JEAN BRACONNIER, 43, BLVD DU 11 NOVEMBRE 1918, F - 69200 VILLEURBANNE CEDEX, FRANCE  
*E-mail address:* mironescu@math.univ-lyon1.fr

<sup>8</sup>There is another description available, in terms of distributional jacobian  $T(u)$  (for its definition, see [1],[5], [6]). For  $1 \leq sp < 2$ , we have  $X^{s,p} = \{u \in W^{s,p}(Q; \mathbb{S}^1) ; T(u) = 0\}$ , result due to Bousquet [6] when  $s \geq 1$  and Ponce [15] when  $s < 1$ .