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| Petru Mironescu. Lifting of S^1 -valued maps in sums of Sobolev spaces. 2008. hal-00747663

HAL Id: hal-00747663

<https://hal.science/hal-00747663>

Preprint submitted on 31 Oct 2012

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LIFTING OF \mathbb{S}^1 -VALUED MAPS IN SUMS OF SOBOLEV SPACES

PETRU MIRONESCU

ABSTRACT. We describe, in terms of lifting, the closure of smooth \mathbb{S}^1 -valued maps in the space $W^{s,p}((-1, 1)^N; \mathbb{S}^1)$. (Here, $0 < s < \infty$ and $1 \leq p < \infty$.) This description follows from an estimate for the phase of smooth maps: let $0 < s < 1$, let $\varphi \in C^\infty([-1, 1]^N; \mathbb{R})$ and set $u = e^{i\varphi}$. Then we may split $\varphi = \varphi_1 + \varphi_2$, where the smooth maps φ_1 and φ_2 satisfy

$$(*) \quad |\varphi_1|_{W^{s,p}} \leq C|u|_{W^{s,p}} \quad \text{and} \quad \|\nabla\varphi_2\|_{L^{sp}}^{sp} \leq C|u|_{W^{s,p}}^p.$$

(*) was proved for $s = 1/2$, $p = 2$ and arbitrary space dimension N by Bourgain and Brezis [3] and for $N = 1$, $p > 1$ and $s = 1/p$ by Nguyen [14].

Our proof is a sort of continuous version of the Bourgain-Brezis approach (based on paraproducts). Estimate (*) answers (and generalizes) a question of Bourgain, Brezis, and the author [5].

1. INTRODUCTION

In [4], the authors addressed the problem of lifting of \mathbb{S}^1 -valued maps in Sobolev spaces:

$(L_{s,p})$ Given an arbitrary $u \in W^{s,p}(Q; \mathbb{S}^1)$, is there a $\varphi \in W^{s,p}(Q; \mathbb{R})$ such that $u = e^{i\varphi}$?

Here, $0 < s < \infty$, $1 \leq p < \infty$ and $Q = (-1, 1)^N$. The complete answer is [4]

SPACE DIMENSION N	SIZE OF s	SIZE OF sp	ANSWER TO $(L_{s,p})$
$N = 1$	ANY	ANY	YES
$N \geq 2$	$0 < s < 1$	$0 < sp < 1$	YES
$N \geq 2$	$0 < s < 1$	$1 \leq sp < N$	NO
$N \geq 2$	$0 < s < 1$	$sp \geq N$	YES
$N \geq 2$	$s \geq 1$	$1 \leq sp < 2$	NO
$N \geq 2$	$s \geq 1$	$sp \geq 2$	YES

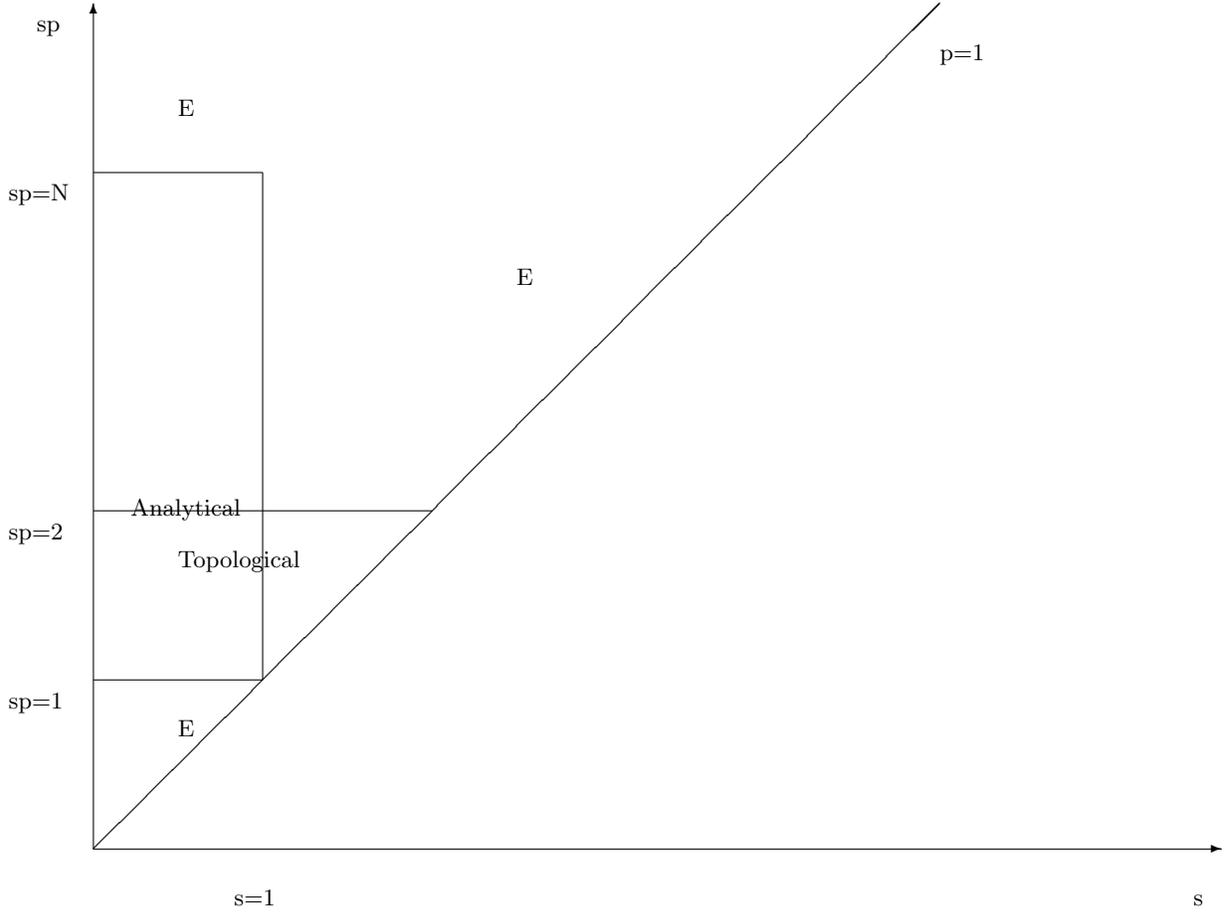
The non existence results rely on two kinds of counterexamples: **topological** and **analytical**.

Topological counterexamples. One may prove (see Proposition 1) that, if there is lifting in $W^{s,p}$, then $C^\infty(\overline{Q}; \mathbb{S}^1)$ is dense in $W^{s,p}(Q; \mathbb{S}^1)$. Thus the answer to $(L_{s,p})$ is no whenever $C^\infty(\overline{Q}; \mathbb{S}^1)$ is not dense in $W^{s,p}(Q; \mathbb{S}^1)$. When $1 \leq sp < 2$, the typical "topological counterexample" is the

Date: June 23, 2008.

The author thanks H.-M. Nguyen for sending him the paper [14] and for stimulating discussions. He warmly thanks H. Brezis for his comments on the paper.

map $Q \ni x \mapsto \frac{(x_1, x_2)}{|(x_1, x_2)|}$ which belongs to $W^{s,p}(Q; \mathbb{S}^1)$ but cannot be approximated by smooth maps in the $W^{s,p}$ -norm. (This goes back essentially to [16]; for a proof, see e. g. [9].) Such a counterexample does not exist outside the "topological region" $1 \leq sp < 2$. Indeed, when $sp < 1$ or $sp \geq 2$, $C^\infty(\overline{Q}; \mathbb{S}^1)$ is dense in $W^{s,p}(Q; \mathbb{S}^1)$ [9]. Thus, topological counterexamples are confined to the topological region.



In the "E" regions, there is lifting. The "topological" region is a trapezoid, the "analytical" one a rectangle

Analytical counterexamples. In the region $1 < sp < N$ and $0 < s < 1$, one may prove non existence of lifting as follows [4]: pick some $\psi \in W^{1,sp}(Q; \mathbb{R}) \setminus W^{s,p}(Q; \mathbb{R})$ (such a ψ exists, by the Sobolev "non embedding" $W^{1,sp}(Q) \not\subset W^{s,p}(Q)$). Let $u := e^{i\psi}$. Then $u \in W^{1,sp} \cap L^\infty$, so that $u \in W^{s,p}(Q; \mathbb{S}^1)$, by the Gagliardo-Nirenberg embedding $W^{1,sp} \cap L^\infty \subset W^{s,p}$.

This u does not lift as $u = e^{i\varphi}$ with $\varphi \in W^{s,p}(Q; \mathbb{R})$. Argue by contradiction: since $e^{2(\varphi-\psi)} = 1$, we have $\eta := \varphi - \psi \in (W^{1,sp} + W^{s,p})(Q; 2\pi\mathbb{Z})$. Thus η is constant a. e. [4] (this uses $sp \geq 1$), so

that $\psi \in W^{s,p}$, contradiction.

When $sp = 1$ and $0 < s < 1$, we still have non lifting. However, the above argument has to be slightly modified: one has to construct explicitly some $\psi \in W^{1,1}(Q; \mathbb{R}) \setminus W^{s,p}(Q; \mathbb{R})$ and $e^{i\psi} \in W^{s,p}(Q; \mathbb{S}^1)$ ($\psi(x) = |x|^{-\alpha}$ for appropriate $\alpha > 0$ will do it). Note that, when $sp = 1$, the property $\psi \in W^{1,1}$ does not imply $u \in W^{s,p}$; this follows from the Gagliardo-Nirenberg "non embedding" $W^{1,1} \cap L^\infty \not\subset W^{s,p}$. (Note the contrast with the embedding $W^{1,sp} \cap L^\infty \subset W^{s,p}$, valid when $sp > 1$.)

This argument implies non existence of lifting in the "analytical region" $1 \leq sp < N$, $0 < s < 1$.

Unlike the topological counterexamples, the "analytical" ones belong to the subspace

$$X^{s,p} := \overline{C^\infty(\overline{Q}; \mathbb{S}^1)}^{W^{s,p}}$$

(see Proposition 2).

As we have already seen, in the region "analytical\topological" there are no topological counterexamples, since $C^\infty(\overline{Q}; \mathbb{S}^1)$ is dense in $W^{s,p}(Q; \mathbb{S}^1)$ when $0 < s < 1$ and $sp \geq 2$. On the other hand, there are no analytical counterexamples (=maps in $X^{s,p}$ that do not lift) in the "topological\analytical" region. Indeed, if $s \geq 1$ and $u \in X^{s,p}$, then u has a lifting in $W^{s,p}(Q; \mathbb{R})$ (Proposition 3).

The main purpose of the present paper is to prove that the analytical counterexamples we presented above are the only ones.

Theorem 1. *Assume that $0 < s < 1$, $sp \geq 1$ and that $u \in X^{s,p}$. Then $u = e^{i\varphi}$ for some $\varphi \in W^{s,p}(Q; \mathbb{R}) + W^{1,sp}(Q; \mathbb{R})$.*

This relies essentially on the following estimate for the lifting of smooth maps.

Theorem 2. *Let $0 < s < 1$ and $1 \leq p < \infty$. Let $\varphi \in C^\infty(\overline{Q}; \mathbb{R})$ and set $u := e^{i\varphi}$. Then we may split $\varphi = \varphi_1 + \varphi_2$, where the maps $\varphi_j \in C^\infty(Q; \mathbb{R})$, $j = 1, 2$, satisfy*

$$(1) \quad |\varphi_1|_{W^{s,p}} \leq C|u|_{W^{s,p}}$$

and

$$(2) \quad \|\nabla \varphi_2\|_{L^{sp}}^{sp} \leq C|u|_{W^{s,p}}^p.$$

Here, $|\cdot|_{W^{s,p}}$ stands for the Gagliardo sem-norm

$$\begin{aligned} |u|_{W^{s,p}} &= \left(\iint_{Q \times Q} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p} \\ &\sim \left(\sum_{j=1}^N \int_{(-1,1)^{N+1}} \frac{\left| u\left(\sum_{k \neq j} x_k e_k + t e_j \right) - u\left(\sum_{k \neq j} x_k e_k + s e_j \right) \right|^p}{|t - s|^{1+sp}} \otimes_{k \neq j} dx_k dt ds \right)^{1/p}. \end{aligned}$$

Two special cases of Theorem 2 were already known. In [3], Bourgain and Brezis established Theorem 2 when $s = 1/2$, $p = 2$ and N is arbitrary. Their proof adapts to the case $1 < p \leq 2$,

$s = 1/p$ (and arbitrary N).

In [14], Nguyen proved Theorem 2 when $N = 1$, $p > 1$ and $s = 1/p$ (without the restriction $p \leq 2$).

The argument there adapts to the case where N is arbitrary, provided $sp > 1$.

Thus the really new cases are a) $sp < 1$ and b) $N \geq 2$, $sp = 1$, $p > 2$.

Theorem 1 was conjectured in [5], [12]. The results presented here were announced in [13].

The paper is organized as follows: in Section 2, we explain how lifting and density are related. In Section 3, we explain the proof of Theorem 2 and why this proof is a cousin of the Bourgain-Brezis argument. In Section 4, we establish the main estimates needed in the proof of Theorem 2. The proofs of Theorems 1 and 2 are presented in Section 5. Finally, in Section 6, we characterize $X^{s,p}$ in terms of lifting.

2. DENSITY VS LIFTING

In this section, we discuss the connection between density of $C^\infty(\overline{Q}; \mathbb{S}^1)$ in $W^{s,p}(Q; \mathbb{S}^1)$ and existence of lifting.

Proposition 1. *Assume that the answer to $(L_{s,p})$ is yes. Then $C^\infty(\overline{Q}; \mathbb{S}^1)$ is dense in $W^{s,p}(Q; \mathbb{S}^1)$.*

Proof. Let first $s \leq 1$. Write an arbitrary $u \in W^{s,p}(Q; \mathbb{S}^1)$ as $u = e^{i\varphi}$ with $\varphi \in W^{s,p}(Q; \mathbb{R})$. Since the map $W^{s,p}(Q; \mathbb{R}) \ni \varphi \mapsto e^{i\varphi} \in W^{s,p}(Q; \mathbb{S}^1)$ is clearly continuous, the conclusion follows by approximating φ with smooth maps.

Let now $s > 1$. If $u \in W^{s,p}(Q; \mathbb{S}^1)$ and if $\varphi \in W^{s,p}(Q; \mathbb{R})$ is a lifting of u , then actually φ belongs also to $W^{1,sp}(Q; \mathbb{R})$ [4]. We conclude by using the fact that the map $W^{1,sp}(Q; \mathbb{R}) \cap W^{s,p}(Q; \mathbb{R}) \ni \varphi \mapsto e^{i\varphi} \in W^{s,p}(Q; \mathbb{S}^1)$ is continuous [7, 11]. \square

As a byproduct of the proof, we obtain the following

Corollary 1. *Assume that $u \in W^{s,p}(Q; \mathbb{S}^1)$ has a lifting in $W^{s,p}(Q; \mathbb{R})$. Then u is in $X^{s,p}$.*

Proposition 2. *Assume that $0 < s < 1$ and $sp \geq 1$. Let $\psi \in W^{1,sp}(Q; \mathbb{R})$ and set $u = e^{i\psi}$. Then*

a) *When $sp > 1$, we have $u \in X^{s,p}$.*

b) *When $sp = 1$ and $u \in W^{s,p}(Q; \mathbb{S}^1)$, we have $u \in X^{s,p}$.*

Proof. a) The mapping $W^{1,sp}(Q; \mathbb{R}) \ni \psi \mapsto e^{i\psi} \in W^{s,p}(Q; \mathbb{S}^1)$ being continuous (this is an easy consequence of the Gagliardo-Nirenberg inclusion $W^{1,sp} \cap L^\infty \subset W^{s,p}$), the conclusion is immediate.

b) We may assume that $N \geq 2$, for otherwise $X^{s,p} = W^{s,p}(Q; \mathbb{S}^1)$. The main ingredient we use in the proof is the approximation technique for $W^{s,p}$ -maps (when $0 < s < 1$ and $sp \leq N$) in [9], which is recalled below.

We first extend u by reflections across ∂Q . Since $0 < s < 1$, this procedure will yield a map in $W^{s,p}((-2, 2)^N; \mathbb{S}^1)$. We next extend this map to a map in $W^{s,p}(\mathbb{R}^N; \mathbb{R}^2)$. We finally obtain a map, still denoted u , which is in $W^{s,p}(\mathbb{R}^N)$ and is, in addition, \mathbb{S}^1 -valued in a neighborhood of \overline{Q} . Moreover, u has, in a neighborhood of \overline{Q} , a $W^{1,1}$ -lifting (still denoted ψ).

We next explain how to approximate maps as above (which are $W^{s,p}$, \mathbb{S}^1 -valued and with a $W^{1,1}$ -lifting near \overline{Q}) by maps with a simple structure. To each $\varepsilon > 0$ and $T \in \mathbb{R}^N$ we may associate

a unique grid $\mathcal{G} = \mathcal{G}_{T,\varepsilon}$ of size 2ε passing through T , namely $\mathcal{G} = \bigcup_{m \in \mathbb{Z}^N} (T + (0, 2\varepsilon)^N)$. For $j = 0, \dots, N$, one may define the j^{th} -dimensional skeleton $\mathcal{C}^j = \mathcal{C}_{T,\varepsilon}^j$ of \mathcal{G} as follows: \mathcal{C}^N is \mathbb{R}^N , i. e., the union of the cubes in \mathcal{G} . \mathcal{C}^{N-1} is the union of the faces of the cubes in \mathcal{G} . By backward induction, \mathcal{C}^j is the union of the (geometrical) boundaries of the $(j+1)$ -dimensional faces that form \mathcal{C}^{j+1} .

Let $u : \mathbb{R}^N \rightarrow \mathbb{R}$. To each ε , T and j we may associate a map $u_{\varepsilon,T,j} : \mathbb{R}^N \rightarrow \mathbb{R}^M$ as follows: let $g = g_{T,\varepsilon,j}$ be the restriction of u to $\mathcal{C}^j = \mathcal{C}_{T,\varepsilon}^j$. We extend g to \mathcal{C}^{j+1} "homogenously". More specifically, let C be the center of some face F of \mathcal{C}^{j+1} . Then the homogeneous extension of g from ∂F to F is the map which is constant on segments joining C to points of ∂F . Equivalently, the desired extension is $H_{j+1}g$, where $H_{j+1}g(x) = g\left(\frac{\varepsilon}{\|x - C\|_\infty}(x - C)\right)$ if $x \in F$.

So far, we have a map $H_{j+1}g$ defined on \mathcal{C}^{j+1} . We extend it homogeneously first to \mathcal{C}^{j+2} , next to \mathcal{C}^{j+3} and so on. We end up with $u_{\varepsilon,T,j} := H_N \circ H_{N-1} \circ \dots \circ H_{j+1}g$.

We may now state the main result in [9]:

Assume that $u \in W^{s,p}(\mathbb{R}^N)$, with $0 < s < 1$ and $sp < N$. Set $j = [sp]$ (the largest integer not exceeding sp). Then there is a sequence $\varepsilon_k \rightarrow 0$ and, for each k , there is a full measure set $A_k \subset \mathbb{R}^N$ such that $u_{\varepsilon_k,T_k,j} \rightarrow u$ in $W^{s,p}$ whenever $T_k \in A_k$.¹

We now return to the proof of b).

The set A_k being of full measure, we may assume that the T_k 's have been chosen such that:

(i) $u|_{\mathcal{C}^1} \in W^{s,p}$;

(ii) there is a cube Q_k containing \bar{Q} such that $\psi|_{\mathcal{C}^1 \cap Q_k} \in W^{1,1}$. In addition, we may assume that Q_k is union of some cubes in \mathcal{G} .

Let now B be the boundary of a square $S \in \mathcal{C}^2 \cap Q_k$. Since $u|_B \in W^{s,p}$, we may lift, locally on B , $u = e^{i\varphi}$ with $\varphi \in W^{s,p}$. (Recall that in one dimension, lifting always exists.) Since $2\pi\mathbb{Z}$ -valued maps in $W^{s,p} + W^{1,1}$ are locally constant [4], we find that $\psi|_B \in W^{s,p}$. Thus, restricted to $\mathcal{C}^1 \cap Q_k$, u has a lifting in $W^{s,p}$.

It is easy to see that, if $f \in W^{\sigma,q}(\mathcal{C}^l)$, where $0 < \sigma < 1$ and $\sigma q < l + 1$, then $H_{l+1}f \in W^{\sigma,q}(\mathcal{C}^{l+1})$ [9]. Applying this property with $\sigma = s$, $q = p$ and $l = 2, \dots, N$, we find that, in Q , we have $u_{\varepsilon_k,T_k,1} = e^{i\psi_k}$, where $\psi_k \in W^{s,p}(Q; \mathbb{R})$. Since clearly $e^{i\psi_k} \in X^{s,p}$, we find that $u \in X^{s,p}$. \square

Proposition 3. *Let $s \geq 1$. If $u \in X^{s,p}$, then u lifts in $W^{s,p}(Q; \mathbb{R})$.*

Proof. When $s \geq 1$, a map $u = u_1 + iu_2 \in W^{s,p}(Q; \mathbb{S}^1)$ has a lifting in $W^{s,p}(Q; \mathbb{R})$ if and only if the vector field $Y := u_1 \nabla u_2 - u_2 \nabla u_1$ is closed in the distribution sense, i. e. if (*) $\frac{\partial Y_j}{\partial x_k} = \frac{\partial Y_k}{\partial x_j}$ in $\mathcal{D}'(Q)$, $j, k = 1, \dots, N$ [4].

When $u = e^{i\varphi}$ with $\varphi \in C^2$, (*) becomes $\frac{\partial^2 \varphi}{\partial x_j \partial x_k} = \frac{\partial^2 \varphi}{\partial x_k \partial x_j}$ and is clearly satisfied. Since the

¹In the proof of Proposition 2, we will apply this approximation result to the \mathbb{R}^2 -valued map u when $s = 1/p$ (and thus $j = 1$).

mapping $W^{s,p}(Q; \mathbb{S}^1) \ni u \mapsto Y \in L^1(Q) \subset \mathcal{D}'(Q)$ is clearly continuous, we find that (*) still holds for $u \in X^{s,p}$. \square

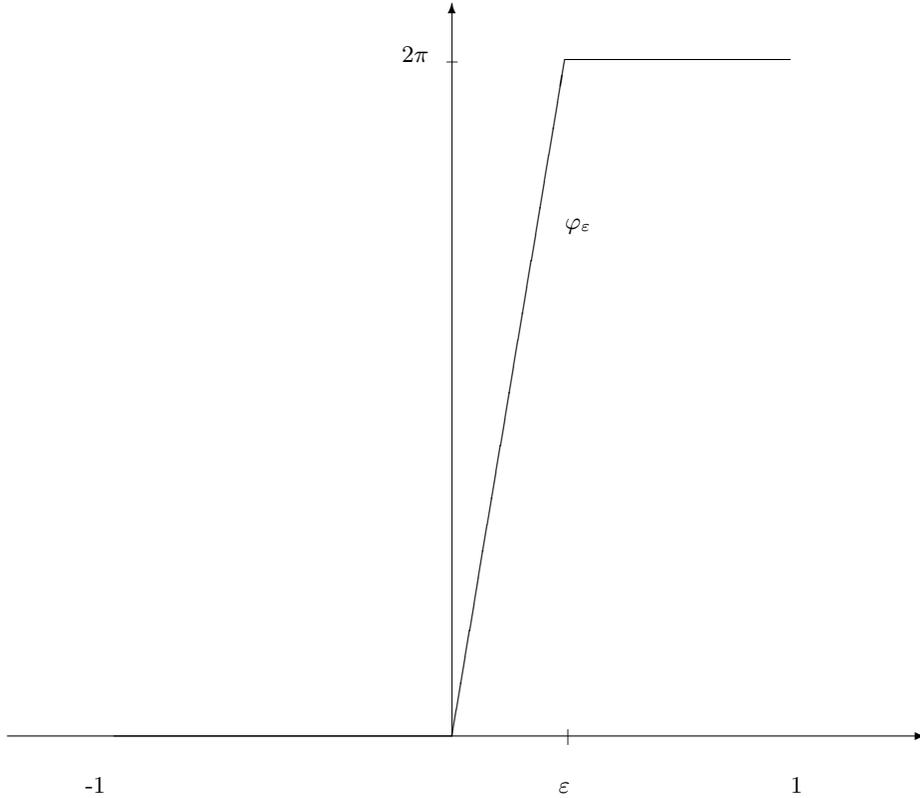
3. HEURISTICS OF THE PROOF OF THEOREM 2

Recall that, in one dimension, lifting always exists. One may thus hope that, for each s and p and for each $u \in W^{s,p}((-1, 1); \mathbb{S}^1)$, one may find a lifting $\psi \in W^{s,p}((-1, 1); \mathbb{R})$ of u satisfying in addition an estimate of the form $\|\psi\|_{W^{s,p}} \leq F(\|u\|_{W^{s,p}})$. This is indeed true except when $sp = 1$ [4].

[There is a parallel between this situation and the case of lifting in classes of smooth maps. It is easy to see that, when $u \in C^k([-1, 1]; \mathbb{S}^1)$ for some $k \geq 1$, the derivatives of the smooth lifting φ of u are controlled by those of u . However, when $k = 0$, the uniform norm of u is always 1, while the one of φ is arbitrary, so that there is no control of φ in terms of u .

It is thus not a surprise that there is no control in $W^{1/p,p}$, which is the space that "almost" embeds in C^0 .]

Indeed, let φ_ε be as in the picture below:



Unlike its phase φ_ε , $e^{i\varphi_\varepsilon}$ remains bounded in $W^{1/p,p}((-1, 1))$ ($p > 1$)

Since $\varphi_\varepsilon \rightarrow \chi_{(0,1]}$ and non constant step functions are not in $W^{1/p,p}$, it follows that $\|\varphi_\varepsilon\|_{W^{1/p,p}} \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Consider now $u_\varepsilon := e^{i\varphi_\varepsilon}$. It is easy to see that $u_1 \in W^{1/p,p}(\mathbb{R})$ (while φ_1 belongs only to $W_{loc}^{1/p,p}(\mathbb{R})$, but not to $W^{1/p,p}(\mathbb{R})$). Since $u_\varepsilon(\cdot) = u_1(\cdot/\varepsilon)$ and the $W^{1/p,p}$ -semi-norm is scale invariant in \mathbb{R} , it follows that u_ε remains bounded in $W^{1/p,p}$ as $\varepsilon \rightarrow 0$.

[This does not contradict Theorem 2, since φ_ε is bounded in $W^{1,1}$.]

This is a typical situation where one needs the $W^{1,sp}$ -part, φ_2 , of φ : the case where u is strongly oscillating. In contrast, the $W^{s,p}$ -part, φ_1 , is controlled by u provided u has small oscillations.

In practice, we obtain the decomposition $\varphi = \varphi_1 + \varphi_2$ as follows: we will derive a formula for the lifting φ of maps with small amplitude oscillations. When u is arbitrary, this formula will be used to define φ_1 . We next simply set $\varphi_2 := \varphi - \varphi_1$.

In order to derive the formula of φ_1 , assume that $u \in W^{s,p}(Q; \mathbb{S}^1)$ is close to the constant 1. Assume for simplicity that u has been extended to \mathbb{R}^N as an \mathbb{S}^1 -valued map that equals 1 at infinity; we still denote by u this extension. Let $v = v(x, \varepsilon)$ be an extension by averages of u to $\mathbb{R}^N \times \mathbb{R}_+$ (thus $v(x, \varepsilon) = u * \rho_\varepsilon(x)$, where ρ is a suitable mollifier). Since u is close to 1, v is also close to 1. In particular, v is far away from 0, so that $w := v/|v|$ is as smooth as v .

We may write (at least when $\varepsilon > 0$) $w = e^{i\psi}$ for some smooth ψ . Since u is 1 at infinity, we have $\lim_{\varepsilon \rightarrow \infty} w(x, \varepsilon) = 1$, which suggests that we may pick ψ such that $\lim_{\varepsilon \rightarrow \infty} \psi(x, \varepsilon) = 0$. This allows to write formally $u(x) = e^{i\varphi(x)}$, where $\varphi(x) := \psi(x, 0)$ and

$$\varphi(x) = -\psi(x, \varepsilon) \Big|_{\varepsilon=0}^{\varepsilon=\infty} = -\int_0^\infty \frac{\partial \psi}{\partial \varepsilon}(x, \varepsilon) d\varepsilon = -\int_0^\infty w(x, \varepsilon) \wedge \frac{\partial w}{\partial \varepsilon}(x, \varepsilon) d\varepsilon.$$

It turns out that this is the right formula φ_1 , provided we choose a more convenient w .

We may now give the explicit splitting $\varphi = \varphi_1 + \varphi_2$ in the proof of Theorem 2. Let $\varphi \in C^\infty(\overline{Q}; \mathbb{R})$ and set $u = e^{i\varphi}$. Then we may extend u to some compactly supported map in $W^{s,p}(\mathbb{R}^N)$, still denoted u , such that $|u| \leq 3$ and $|u|_{W^{s,p}(\mathbb{R}^N)} \leq C|u|_{W^{s,p}(\mathbb{R}^N)}$ (Lemma 8). Let $\rho \in C_0^\infty$ be a mollifier (whose precise properties will be specified in Section 4). Let $v(x, \varepsilon) = u * \rho_\varepsilon(x)$.

Assume first that $W^{s,p} \cap L^\infty$ is contained in $W^{1-1/2sp, 2sp}$, i. e. that $s \geq 1 - \frac{1}{2sp}$ and $sp \leq 1$.² Then

Theorem 2 works with

$$(3) \quad \varphi_1(x) = -\int_0^\infty v(x, \varepsilon) \wedge \frac{\partial v}{\partial \varepsilon}(x, \varepsilon) d\varepsilon, \quad \varphi_2 = \varphi - \varphi_1.$$

In general (i. e., when we do not assume that $W^{s,p} \cap L^\infty$ is contained in $W^{1-1/2sp, 2sp}$), one has to project v on \mathbb{S}^1 . More specifically, let $\Pi \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ be such that $\Pi(z) = z/|z|$ near \mathbb{S}^1 and set $w := \Pi(v)$. Then we may choose, in the proof of Theorem 2,

$$(4) \quad \varphi_1(x) = -\int_0^\infty w(x, \varepsilon) \wedge \frac{\partial w}{\partial \varepsilon}(x, \varepsilon) d\varepsilon, \quad \varphi_2 = \varphi - \varphi_1.$$

²This covers the case $s = 1/2, p = 2$ treated in [3].

We end this section by comparing our approach to the Bourgain-Brezis one [3]. The fact that φ_1 given by (3) belongs to $W^{s,p}$ is reminiscent from standard estimates on para-products (for a quick introduction, see, e. g., [10]). Recall that (for a suitable mollifier ρ) the Littlewood-Paley decomposition of a function f is $f = \sum_{j \geq 0} LP_j(f)$, where $LP_0(f) = f * \rho$ and, for $j \geq 1$, $LP_j(f) = f * \rho_{2^{-j}} - f * \rho_{2^{-j+1}}$. Recall also that $W^{s,p} \cap L^\infty$ is an algebra. Thus, if $f, g \in W^{s,p} \cap L^\infty$, then $\sum_{j,k} LP_j(f)LP_k(g) \in W^{s,p}$. The paraproducts technique yields slightly more:

$$(5) \quad \text{each of the sums } \sum_{j \leq k} LP_j(f)LP_k(g) \text{ and } \sum_{j > k} LP_j(f)LP_k(g) \text{ is in } W^{s,p}.$$

It is well-known to the experts (though difficult to find in the literature) that each "Littlewood-Paley like" (i. e., via sequences) property of some functions space has continuous analogues.³ These analogues are obtained by replacing $LP_j(f)$ by an integral, e. g.

$$f = LP_0(f) + \sum_{j=-\infty}^{\infty} (f * \rho_{2^{-j}} - f * \rho_{2^{-j+1}}) = LP_0(f) - \int_0^1 f * \frac{\partial}{\partial \varepsilon}(\rho_\varepsilon) d\varepsilon;$$

another used decomposition is $f = - \int_0^\infty f * \frac{\partial}{\partial \varepsilon}(\rho_\varepsilon) d\varepsilon$. Equivalently, if $F(x, \varepsilon) = f * \rho_\varepsilon(x)$ is the extension of f to $\mathbb{R}^N \times \mathbb{R}_+$, then $f(x) = - \int_0^\infty \frac{\partial F}{\partial \varepsilon}(x, \varepsilon) d\varepsilon$.

With F, G the extensions of $f, g \in W^{s,p} \cap L^\infty$, an analogue of (5) is

$$(6) \quad x \mapsto \int_0^\infty F(x, \varepsilon) \frac{\partial G}{\partial \varepsilon}(x, \varepsilon) d\varepsilon \in W^{s,p}, \quad \forall f, g \in W^{s,p} \cap L^\infty.$$

Property (6) is true⁴ and implies that the function φ_1 given by (3) is in $W^{s,p}$. The same conclusion holds for the φ_1 given by (4), but the argument is more involved.

We may now compare our construction to the Bourgain-Brezis one: their splitting is $\varphi_1 = \sum_{j \leq k} LP_j(u) \wedge LP_k(u)$, $\varphi_2 = \varphi - \varphi_1$. This is nothing else than a discrete analogue of (3). However, it seems difficult to cover the case $sp \neq 1$ using this decomposition.

³Example #1: one may define a square function of an L^p -function f defined in \mathbb{R}^N either through the formula $S_1 f(x) := \left(\sum |LP_j f(x)|^2 \right)^{1/2}$, or through $S_2 f(x) := \left(\int_0^\infty (f * \rho_\varepsilon(x))^2 \frac{d\varepsilon}{\varepsilon} \right)^{1/2}$, for appropriate ρ . In both cases, we have the "square function theorem" of Littlewood and Paley $S_j(f) \sim \|f\|_{L^p}$, $1 < p < \infty$ [17] II. 6. Example #2: functions f in Triebel-Lizorkin spaces can be characterized both in terms of the Littlewood-Paley decomposition of f and in terms of the behavior of the solution of the heat equation with initial condition f [18] 2.12.

⁴This is presumably well-known. For another ρ , (6) is nothing else but the conclusion of Lemma 19.

4. ESTIMATES FOR EXTENSIONS BY AVERAGES

Throughout this section, $u \in \text{Lip}(\mathbb{R}^N)$ is such that $|u| \leq 3$ and u is constant outside some compact⁵. We set, for $\varepsilon > 0$, $v(x, \varepsilon) = u * \rho_\varepsilon(x)$.

We assume that the mollifier ρ satisfies⁶:

$$\rho \in C_0^\infty(\mathbb{R}^N), \quad \rho \geq 0, \quad \text{supp } \rho \subset B(0, 2), \quad \rho = 0 \text{ in } B(0, 1).$$

C will denote a constant depending on s, p, N , but not on u (provided $|u| \leq 3$).

Lemma 1. *Assume that $0 < s < 1$, $1 \leq p < \infty$. Then*

$$(7) \quad \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{p-sp-1} |Dv(x, \varepsilon)|^p d\varepsilon dx \leq C |u|_{W^{s,p}}^p.$$

Proof. Set $\zeta_j := \partial_j \rho$, $j = 1, \dots, N$, and $\zeta_0 := -\sum_{j=1}^N \partial_j(x_j \rho)$. It is easy to see that

$$(8) \quad \frac{\partial}{\partial x_j} v(x, \varepsilon) = \frac{1}{\varepsilon} u * (\zeta_j)_\varepsilon(x) \quad \text{and} \quad \frac{\partial}{\partial \varepsilon} v(x, \varepsilon) = \frac{1}{\varepsilon} u * (\zeta_0)_\varepsilon(x).$$

Noting that each ζ_j is supported in $B(0, 2)$ and has zero integral, we find that (7) is a consequence of

$$(9) \quad \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{-sp-1} |u * \zeta_\varepsilon(x)|^p d\varepsilon dx \leq C(\zeta) |u|_{W^{s,p}}^p, \quad \forall \zeta \in C_0^\infty(B(0, 2)) \text{ such that } \int \zeta = 0.$$

In order to prove (9), we note that

$$|u * \zeta_\varepsilon(x)|^p = \left| \frac{1}{\varepsilon^N} \int_{B(0, 2\varepsilon)} (u(x-y) - u(x)) \zeta\left(\frac{y}{\varepsilon}\right) dy \right|^p \leq \frac{C}{\varepsilon^N} \int_{B(0, 2\varepsilon)} |u(x-y) - u(x)|^p dy.$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{-sp-1} |u * \zeta_\varepsilon(x)|^p d\varepsilon dx &\leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{|y|/2}^\infty \frac{1}{\varepsilon^{N+sp+1}} d\varepsilon |u(x-y) - u(x)|^p dy dx \\ &= C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dx dy = C |u|_{W^{s,p}}^p. \end{aligned}$$

□

Lemma 2. *Assume that $0 < s < 1$, $1 \leq p < \infty$. Then*

$$(10) \quad I := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} \left(\int_0^{|x-y|} |Dv(x, \varepsilon)| d\varepsilon \right)^p dx dy \leq C |u|_{W^{s,p}}^p.$$

⁵The second assumption is used only in Lemmas 5-7.

⁶Assumptions on ρ are not crucial. The results in this section are true for any reasonable mollifier. However, our assumptions make the proofs simpler.

Proof. Let α be such that $\frac{1}{p} - 1 < \alpha < \frac{1}{p} - 1 + s$. We first note that, for $r > 0$ and $p > 1$, we have

$$(11) \quad \left(\int_0^r |Dv(x, \varepsilon)| d\varepsilon \right)^p = \left(\int_0^r \frac{|Dv(x, \varepsilon)|}{\varepsilon^\alpha} \varepsilon^\alpha d\varepsilon \right)^p \leq \int_0^r \frac{|Dv(x, \varepsilon)|^p}{\varepsilon^{\alpha p}} d\varepsilon \left(\int_0^r \varepsilon^{\alpha p/(p-1)} d\varepsilon \right)^{p-1} \\ = Cr^{\alpha p + p - 1} \int_0^r \frac{|Dv(x, \varepsilon)|^p}{\varepsilon^{\alpha p}} d\varepsilon$$

(here we use the fact that $\frac{\alpha p}{p-1} > -1$, which is equivalent to $\alpha > \frac{1}{p} - 1$). On the other hand, it is immediate that the conclusion of (11) still holds when $p = 1$.

If we write $y = x + r\omega$, with $r = |x - y|$ and $\omega \in \mathbb{S}^{N-1}$ and use (11) with $r = |x - y|$, we find that

$$I = C \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^\infty \frac{1}{r^{sp+1}} \left(\int_0^r |Dv(x, \varepsilon)| d\varepsilon \right)^p dr d\omega dx \\ \leq C \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^\infty r^{\alpha p + p - sp - 2} \int_0^r \frac{|Dv(x, \varepsilon)|^p}{\varepsilon^{\alpha p}} d\varepsilon dr d\omega dx \\ = C \int_{\mathbb{R}^N} \int_0^\infty \frac{|Dv(x, \varepsilon)|^p}{\varepsilon^{\alpha p}} \int_\varepsilon^\infty r^{\alpha p + p - sp - 2} dr d\varepsilon dx \\ = C \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{p-sp-1} |Dv(x, \varepsilon)|^p d\varepsilon dx \leq C |u|_{W^{s,p}}^p.$$

Here, we rely on the inequality $\alpha p + p - sp - 2 < -1$ (which amounts to $\alpha < \frac{1}{p} - 1 + s$) and on Lemma 1. \square

Lemma 3. *Assume that $0 < s < 1$, $1 \leq p < \infty$. Then*

$$(12) \quad J := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{1}{|x - y|^{N+sp}} \left(\int_{|x-y|}^\infty |Dv(x, \varepsilon) - Dv(y, \varepsilon)| d\varepsilon \right)^p dx dy \leq C |u|_{W^{s,p}}^p.$$

Proof. In view of (8), it suffices to prove that

$$(13) \quad \tilde{J} := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{1}{|x - y|^{N+sp}} \left(\int_{|x-y|}^\infty \frac{|u * \zeta_\varepsilon(x) - u * \zeta_\varepsilon(y)|}{\varepsilon} d\varepsilon \right)^p dx dy \leq C |u|_{W^{s,p}}^p$$

under the assumptions $\zeta \in C_0^\infty(B(0, 2))$, $\int \zeta = 0$ and $\zeta = 0$ in $B(0, 1)$.

If we set $\Phi := D\zeta$, then $D_x(u * \zeta_\varepsilon) = \frac{1}{\varepsilon} u * \Phi_\varepsilon$, $\Phi \in C_0^\infty(B(0, 2))$, $\int \Phi = 0$ and $\Phi = 0$ in $B(0, 1)$.

We find that

$$|D_x(u * \zeta_\varepsilon)(x)| = \frac{1}{\varepsilon^{N+1}} \left| \int_{B(0, 2\varepsilon) \setminus B(0, \varepsilon)} (u(x+z) - u(x)) \Phi \left(\frac{z}{\varepsilon} \right) dz \right| \leq \frac{C}{\varepsilon^{N+1}} \int_{\varepsilon \leq |z| \leq 2\varepsilon} |u(x+z) - u(x)| dz.$$

Therefore, with $r := |y - x|$ and $\omega := \frac{y - x}{r}$, we have

$$\begin{aligned} |u * \zeta_\varepsilon(x) - u * \zeta_\varepsilon(y)| &= r \left| \int_0^1 D_x(u * \zeta_\varepsilon)(x + rt\omega) \cdot \omega dt \right| \\ &\leq \frac{Cr}{\varepsilon^{N+1}} \int_0^1 \int_{\varepsilon \leq |z| \leq 2\varepsilon} |u(x + rt\omega + z) - u(x + rt\omega)| dz dt, \end{aligned}$$

which implies that

$$\begin{aligned} (14) \quad \int_{|x-y|}^\infty \frac{|u * \zeta_\varepsilon(x) - u * \zeta_\varepsilon(y)|}{\varepsilon} d\varepsilon &\leq Cr \int_{|z| \geq r} \int_0^1 \int_{|z|/2}^{|z|} \frac{|u(x + rt\omega + z) - u(x + rt\omega)|}{\varepsilon^{N+2}} d\varepsilon dt dz \\ &= Cr \int_0^1 \int_{|z| \geq r} \frac{|u(x + rt\omega + z) - u(x + rt\omega)|}{|z|^{N+1}} dz dt. \end{aligned}$$

Let now α be such that $\frac{N}{p} - N + s - 1 < \alpha < \frac{N}{p} - N$. We perform the following calculation only for $p > 1$, but the reader may easily see that its conclusion still holds for $p = 1$. Using (14), we find that

$$\begin{aligned} (15) \quad K &:= \left(\int_{|x-y|}^\infty \frac{|u * \zeta_\varepsilon(x) - u * \zeta_\varepsilon(y)|}{\varepsilon} d\varepsilon \right)^p \\ &\leq Cr^p \left(\int_0^1 \int_{|z| \geq r} \frac{|u(x + rt\omega + z) - u(x + rt\omega)|}{|z|^{N+1+\alpha}} |z|^\alpha dz dt \right)^p \\ &\leq Cr^p \int_0^1 \int_{|z| \geq r} \frac{|u(x + rt\omega + z) - u(x + rt\omega)|^p}{|z|^{Np+p+\alpha p}} dz dt \left(\int_{|z| \geq r} |z|^{\alpha p/(p-1)} dz \right)^{p-1} \\ &= Cr^{p+\alpha p+Np-N} \int_0^1 \int_{|z| \geq r} \frac{|u(x + rt\omega + z) - u(x + rt\omega)|^p}{|z|^{Np+p+\alpha p}} dz dt \end{aligned}$$

since $\frac{\alpha p}{p-1} < -N$ (equivalently, since $\alpha < \frac{N}{p} - N$). Inserting (15) into the definition of \tilde{J} and computing the integral in y in spherical coordinates, we obtain

$$\begin{aligned} \tilde{J} &\leq C \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^\infty \int_0^1 \int_{|z| \geq r} r^{p+\alpha p+Np-N-sp-1} \frac{|u(x + rt\omega + z) - u(x + rt\omega)|^p}{|z|^{Np+p+\alpha p}} dz dt dr d\omega dx \\ &= C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x+z) - u(x)|^p}{|z|^{Np+p+\alpha p}} \int_0^{|z|} r^{p+\alpha p+Np-N-sp-1} dr dx dz \\ &= C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x+z) - u(x)|^p}{|z|^{N+sp}} dx dz = C |u|_{W^{s,p}}^p. \end{aligned}$$

Here, we used the inequality $p+\alpha p+Np-N-sp-1 > -1$, which amounts to $\alpha > \frac{N}{p} - N + s - 1$. \square

Lemma 4. *Assume $0 < s < 1$, $1 \leq p < \infty$. Then*

$$(16) \quad L := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} \left(\int_{|x-y|}^{\infty} \frac{|x-y|}{\varepsilon} |Dv(x, \varepsilon)| d\varepsilon \right)^p dx dy \leq C|u|_{W^{s,p}}^p.$$

Proof. In spherical coordinates $y = x + r\omega$ we have

$$\begin{aligned} L &= C \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^{\infty} r^{p-sp-1} \left(\int_r^{\infty} \frac{|Dv(x, \varepsilon)|}{\varepsilon} d\varepsilon \right)^p dr d\omega dx \\ &= C \int_{\mathbb{R}^N} \int_0^{\infty} r^{p-sp-1} \left(\int_r^{\infty} \frac{|Dv(x, \varepsilon)|}{\varepsilon} d\varepsilon \right)^p dr dx, \end{aligned}$$

so that (16) amounts to

$$\tilde{L} := \int_{\mathbb{R}^N} \int_0^{\infty} r^{p-sp-1} \left(\int_r^{\infty} \frac{|Dv(x, \varepsilon)|}{\varepsilon} d\varepsilon \right)^p dr dx \leq C|u|_{W^{s,p}}^p.$$

Let α be such that $\frac{1}{p} + s - 2 < \alpha < \frac{1}{p} - 1$. We perform the calculation below for $p > 1$; clearly, its conclusion still holds for $p = 1$. We have

$$\begin{aligned} \left(\int_r^{\infty} \frac{|Dv(x, \varepsilon)|}{\varepsilon} d\varepsilon \right)^p &= \left(\int_r^{\infty} \frac{|Dv(x, \varepsilon)|}{\varepsilon^{\alpha+1}} \varepsilon^{\alpha} d\varepsilon \right)^p \\ (17) \quad &\leq \int_r^{\infty} \frac{|Dv(x, \varepsilon)|^p}{\varepsilon^{p+\alpha p}} d\varepsilon \left(\int_r^{\infty} \varepsilon^{\alpha p/(p-1)} d\varepsilon \right)^{p-1} \\ &\leq Cr^{\alpha p+p-1} \int_r^{\infty} \frac{|Dv(x, \varepsilon)|^p}{\varepsilon^{p+\alpha p}} d\varepsilon, \end{aligned}$$

since $\frac{\alpha p}{p-1} < -1$ (i. e., $\alpha < \frac{1}{p} - 1$). Combining (17) with the definition of \tilde{L} , we obtain

$$\begin{aligned} \tilde{L} &\leq C \int_{\mathbb{R}^N} \int_0^{\infty} \int_r^{\infty} |Dv(x, \varepsilon)|^p \varepsilon^{-p-\alpha p} r^{2p-sp+\alpha p-2} d\varepsilon dr dx \\ &= C \int_{\mathbb{R}^N} \int_0^{\infty} \int_0^{\varepsilon} |Dv(x, \varepsilon)|^p \varepsilon^{-p-\alpha p} r^{2p-sp+\alpha p-2} dr d\varepsilon dx \\ &= C \int_{\mathbb{R}^N} \int_0^{\infty} \varepsilon^{p-sp-1} |Dv(x, \varepsilon)|^p d\varepsilon dx \leq C|u|_{W^{s,p}}^p. \end{aligned}$$

Here, we used the fact that $2p - sp + \alpha p - 2 > -1$ (which is equivalent to $\alpha > \frac{1}{p} + s - 2$) and Lemma 1. □

Lemma 5. *For each $x \in \mathbb{R}^N$, the integral $\int_0^{\infty} |Dv(x, \varepsilon)| d\varepsilon$ is convergent.*

Proof. Since v is Lipschitz, it suffices to prove that $\int_1^\infty |Dv(x, \varepsilon)| d\varepsilon$ converges. With a the value of u at infinity, this follows from

$$|Dv(x, \varepsilon)| = |D[(u - a) * \rho_\varepsilon(x)]| = |(u - a) * D[\rho_\varepsilon(x)]| \leq \|u - a\|_{L^1} \|D[\rho_\varepsilon(x)]\|_{L^\infty} \leq \frac{C}{\varepsilon^{N+1}}.$$

□

In the same spirit, we have the following result, whose straightforward proof will be omitted

Lemma 6. *Assume u complex-valued, Lipschitz in \mathbb{R}^N and smooth in Q . Set*

$$(18) \quad w(x) = w_\Pi(x) := \int_0^\infty \Pi \circ v(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v(x, \varepsilon)) d\varepsilon, \quad \text{where } \Pi \in C^\infty(\mathbb{R}^2; \mathbb{R}^2).$$

Then w is smooth in Q and $\partial^\alpha w(x) = \int_0^\infty \partial^\alpha \left(\Pi \circ v(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v(x, \varepsilon)) \right) d\varepsilon$.

We may now prove that the map φ_1 defined in (4) is in $W^{s,p}$ (plus norm control).

Lemma 7. *Assume that $u \in Lip(\mathbb{R}^N; \mathbb{C})$ satisfies $|u| \leq 3$. Let w be defined by (18). Then*

$$(19) \quad |w|_{W^{s,p}} \leq C|u|_{W^{s,p}}.$$

Proof. Set $a(x, y) := \int_0^{|x-y|} \Pi \circ v(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) d\varepsilon$ and $b(x, y) := \int_{|x-y|}^\infty \Pi \circ v(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) d\varepsilon$, so that $w(x) = a(x, y) + b(x, y)$. On the one hand, we have

$$(20) \quad \begin{aligned} |w(x) - w(y)| &\leq |a(x, y)| + |a(y, x)| + |b(x, y) - b(y, x)| \\ &\leq |b(x, y) - b(y, x)| + C \left(\int_0^{|x-y|} |Dv(x, \varepsilon)| d\varepsilon + \int_0^{|x-y|} |Dv(y, \varepsilon)| d\varepsilon \right). \end{aligned}$$

On the other hand, we have

$$(21) \quad \begin{aligned} b(x, y) - b(y, x) &= \int_{|x-y|}^\infty \left(\Pi \circ v(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) - \Pi \circ v(y, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(y, \varepsilon) \right) d\varepsilon \\ &= \int_{|x-y|}^\infty (\Pi \circ v(x, \varepsilon) - \Pi \circ v(y, \varepsilon)) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) d\varepsilon \\ &\quad + \int_{|x-y|}^\infty \Pi \circ v(y, \varepsilon) \left(\frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) - \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(y, \varepsilon) \right) d\varepsilon. \end{aligned}$$

Since (with ζ_j as in (8))

$$(22) \quad |D(\Pi \circ v)(x, \varepsilon)| \leq \frac{C}{\varepsilon} \sum_{j=0}^N \|u\|_{L^\infty} \|(\zeta_j)_\varepsilon\|_{L^1} \leq \frac{C}{\varepsilon},$$

we find that $|\Pi \circ v(x, \varepsilon) - \Pi \circ v(y, \varepsilon)| \leq \frac{C|x-y|}{\varepsilon}$, which yields

$$\begin{aligned}
(23) \quad |b(x, y) - b(y, x)| &\leq C \int_{|x-y|}^{\infty} \left| \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) - \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(y, \varepsilon) \right| d\varepsilon \\
&\quad + C \int_{|x-y|}^{\infty} \frac{|x-y|}{\varepsilon} \left| \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) \right| d\varepsilon \\
&\leq C \left(\int_{|x-y|}^{\infty} |Dv(x, \varepsilon) - Dv(y, \varepsilon)| d\varepsilon + \int_{|x-y|}^{\infty} \frac{|x-y|}{\varepsilon} |Dv(x, \varepsilon)| d\varepsilon \right).
\end{aligned}$$

By combining (20)-(23) to Lemmas 2-4, we find that

$$|w|_{W^{s,p}}^p \leq C(I + J + L) \leq C|u|_{W^{s,p}}^p.$$

□

5. PROOF OF THEOREMS 1 AND 2

Proof of Theorem 2. Let $\varphi \in C^\infty(\overline{Q}; \mathbb{S}^1)$ and set $u := e^{i\varphi}$.

Lemma 8. *The map u has a C^1 -extension to \mathbb{R}^N , still denoted u , such that $|u| \leq 3$, u is constant outside $(-2, 2)^N$ and $|u|_{W^{s,p}(\mathbb{R}^N)} \leq C|u|_{W^{s,p}(Q)}$.*

Proof. Let $P : W^{s,p}(Q) \rightarrow W^{s,p}(\mathbb{R}^N)$ be a linear continuous extension operator such that: P extends Lipschitz maps to Lipschitz maps, P does not increase the L^∞ -norm and $\text{supp } Pv \subset (-2, 2)^N$, $\forall v \in W^{s,p}$. Let a be the average of u on Q . It is easy to see that $\tilde{u} := a + P(u - a)$ is Lipschitz and satisfies $|\tilde{u}| \leq 3$. In addition, we have

$$|\tilde{u}|_{W^{s,p}(\mathbb{R}^N)} = |P(u - a)|_{W^{s,p}(\mathbb{R}^N)} \leq C\|u - a\|_{W^{s,p}(Q)} \leq C|u - a|_{W^{s,p}(Q)} = C|u|_{W^{s,p}(Q)};$$

we have used the Poincaré inequality $\|u - a\|_{W^{s,p}(Q)} \leq C|u - a|_{W^{s,p}(Q)}$, valid since $u - a$ has zero average. □

Let then $\Pi \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ to be chosen later and set $\varphi_1(x) := - \int_0^\infty \Pi \circ v(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) d\varepsilon$.

By Lemma 19, φ_1 belongs to $W^{s,p}$ and satisfies $|\varphi_1|_{W^{s,p}} \leq C|u|_{W^{s,p}}$.

We set, for $x \in \overline{Q}$, $\varphi_2(x) := \varphi(x) - \varphi_1(x)$.

Lemma 9. *Assume that $\Pi(z) = z$, $\forall z \in \mathbb{S}^1$. Then*

$$(24) \quad D\varphi_2(x) = -2 \int_0^\infty \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) \wedge D_x(\Pi \circ v)(x, \varepsilon) d\varepsilon.$$

Proof. Using Lemma 6 and the identity $D\varphi = u \wedge Du$ we have, with $w := \Pi \circ v$,

$$\begin{aligned} D\varphi_2(x) &= D\varphi(x) + \int_0^\infty D_x w(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} w(x, \varepsilon) d\varepsilon + \int_0^\infty w(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} D_x w(x, \varepsilon) d\varepsilon \\ &= w(x, \varepsilon) \wedge D_x w(x, \varepsilon)|_{\varepsilon=0} + \int_0^\infty D_x w(x, \varepsilon) \wedge \frac{\partial}{\partial \varepsilon} w(x, \varepsilon) d\varepsilon + w(x, \varepsilon) \wedge D_x w(x, \varepsilon) \Big|_{\varepsilon=0}^{\varepsilon=\infty} \\ &\quad - \int_0^\infty \frac{\partial}{\partial \varepsilon} w(x, \varepsilon) \wedge D_x w(x, \varepsilon) d\varepsilon = -2 \int_0^\infty \frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) \wedge D_x (\Pi \circ v)(x, \varepsilon) d\varepsilon. \end{aligned}$$

□

The remaining part of the proof of Theorem is essentially a variant of the proof of Theorem 0.1 in [5]. Up to now, the proof requires only $\Pi(u) = u$.⁷ From now on, we will require that Π is an approximate projection on \mathbb{S}^1 , e.g. we assume $\Pi(z) = \frac{z}{|z|}$ when $|z| \geq \frac{1}{2}$.

Lemma 10. *Let, for $x \in \overline{Q}$, $d(x) := \inf\{\varepsilon > 0 ; |v(x, \varepsilon)| = 1/2\}$. Then*

$$(25) \quad \int_Q \frac{1}{d(x)^{sp}} dx \leq C |u|_{W^{s,p}(Q)}^p.$$

Proof. Let x be such that $d(x)$ is finite. Since $|u(x) - v(x, d(x))| \geq 1/2$, we have

$$1/2 \leq |u(x) - v(x, d(x))| \leq |v(x, \cdot)|_{C^s(\mathbb{R}_+)} d(x)^s \leq C |v(x, \cdot)|_{W^{s+1/p,p}(\mathbb{R}_+)} d(x)^s,$$

so that

$$\int_Q \frac{1}{d(x)^{sp}} dx \leq C \int_Q |v(x, \cdot)|_{W^{s+1/p,p}(\mathbb{R}_+)}^p \leq C |v|_{W^{s+1/p,p}}^p \leq C |u|_{W^{s,p}(\mathbb{R}^N)}^p \leq C |u|_{W^{s,p}(Q)}^p,$$

by the Besov lemma [2].

□

Lemma 11. *We have*

$$(26) \quad \int_Q |D\varphi_2|^{sp} dx \leq C |u|_{W^{s,p}(Q)}^p.$$

Proof. Set $\Omega := \{(x, \varepsilon) ; x \in Q, 0 < \varepsilon < d(x)\}$. In Ω , we have $|\Pi \circ v| \equiv 1$, so that $\frac{\partial}{\partial \varepsilon} (\Pi \circ v)(x, \varepsilon) \wedge D_x (\Pi \circ v)(x, \varepsilon) \equiv 0$ in Ω . In view of Lemma 9 and (22), we find that

$$\begin{aligned} \int_Q |D\varphi_2|^{sp} dx &\leq \int_Q \left(\int_{d(x)}^\infty |D(\Pi \circ v)(x, \varepsilon)|^2 d\varepsilon \right)^{sp} dx \\ &\leq C \int_Q \left(\int_{d(x)}^\infty \frac{1}{\varepsilon^2} d\varepsilon \right)^{sp} dx \leq C \int_Q \frac{1}{d(x)^{sp}} dx \leq C |u|_{W^{s,p}(Q)}^p. \end{aligned}$$

□

⁷When $W^{s,p} \cap L^\infty$ is contained in $W^{1-1/2sp, 2sp}$, Lemma 11 below is valid without any other restriction.

The proof of Theorem 2 is complete. \square

Proof of Theorem 1. When $p > 1$ and $sp > 1$, Theorem 1 is an immediate consequence of Theorem 2 (one passes to the weak limits in (1) and (2)). Some care is needed when $p = 1$ or $sp = 1$.

Lemma 12. *Let $u \in X^{s,p}$. Then there is a sequence $\{u_k\}_{k \geq 0} \subset C^\infty(\overline{Q}; \mathbb{S}^1)$ such that $u = \prod_{k=0}^{\infty} u_k$*

and $|u_k|_{W^{s,p}} \leq 2^{-k+1}|u|_{W^{s,p}}, \forall k$.

Proof. Recall that

$$(27) \quad \text{if } f_j \rightarrow f \text{ in } W^{s,p}, g_j \rightarrow g \text{ in } W^{s,p}, \|f_j\|_{L^\infty} \leq C, \|g_j\|_{L^\infty} \leq C, \text{ then } f_j g_l \xrightarrow{j,l \rightarrow \infty} fg \text{ in } W^{s,p}.$$

Consider a sequence $\{h_j\} \subset C^\infty(\overline{Q}; \mathbb{S}^1)$ such that $h_j \rightarrow u$. We may assume that $|h_j|_{W^{s,p}} \leq 2|u|_{W^{s,p}}, \forall j$. Using (27) with $f_j = h_j$ and $g_j = \overline{h_j}$, we find easily a sequence $j_k \rightarrow \infty$ such that $\|h_{j_{k+1}} \overline{h_{j_k}} - 1\|_{W^{s,p}} \leq 2^{-k}|u|_{W^{s,p}}, k \geq 0$. If we set $u_0 = h_{j_0}$ and $u_k = h_{j_k} \overline{h_{j_{k-1}}}, k \geq 1$, then the sequence $\{u_k\}$ has the required properties. \square

In view of Theorem 2, we may write each u_k as $e^{i(\varphi_1^k + \varphi_2^k)}$, where $|\varphi_1^k|_{W^{s,p}} \leq C2^{-k}|u|_{W^{s,p}}$ and $\|D\varphi_2^k\|_{L^{sp}} \leq C2^{-k/s}|u|_{W^{s,p}}^{1/s}$.

Let $\varphi_1 := \sum_k \varphi_1^k$. Clearly, $\varphi_1 \in W^{s,p}$ and $|\varphi_1|_{W^{s,p}} \leq C|u|_{W^{s,p}}$. On the other hand, set $\psi := \sum_k (\varphi_2^k - \int_Q \varphi_2^k)$, which satisfies $\psi \in W^{1,sp}$ and $\|D\psi\|_{L^{sp}}^{sp} \leq C|u|_{W^{s,p}}^p$. In addition, the map $ue^{-i(\varphi_1 + \psi)}$ is constant. It follows that, for an appropriate $\alpha \in \mathbb{R}$, φ_1 and $\varphi_2 := \alpha + \psi$ satisfy $u = e^{i(\varphi_1 + \varphi_2)}$ and the estimates (1)-(2).

6. CHARACTERIZATION OF $X^{s,p}$ IN TERMS OF LIFTING

The results in [4], [8] and [9] give the following information about $X^{s,p}$

SPACE DIMENSION N	SIZE OF s	SIZE OF sp	DESCRIPTION OF $X^{s,p}$
ANY	$0 < s < 1$	$0 < sp < 1$	$X^{s,p} = W^{s,p}(Q; \mathbb{S}^1) = \{e^{i\varphi} ; \varphi \in W^{s,p}(Q; \mathbb{R})\}$
$N \geq 2$	$0 < s < 1$	$1 \leq sp < 2$	$X^{s,p} \neq W^{s,p}(Q; \mathbb{S}^1), X^{s,p} \neq \{e^{i\varphi} ; \varphi \in W^{s,p}(Q; \mathbb{R})\}$
$N \geq 3$	$0 < s < 1$	$2 \leq sp < N$	$X^{s,p} = W^{s,p}(Q; \mathbb{S}^1), X^{s,p} \neq \{e^{i\varphi} ; \varphi \in W^{s,p}(Q; \mathbb{R})\}$
ANY	$0 < s < 1$	$sp \geq N$	$X^{s,p} = W^{s,p}(Q; \mathbb{S}^1) = \{e^{i\varphi} ; \varphi \in W^{s,p}(Q; \mathbb{R})\}$
$N = 1$	$s \geq 1$	ANY	$X^{s,p} = W^{s,p}(Q; \mathbb{S}^1) = \{e^{i\varphi} ; \varphi \in W^{s,p} \cap W^{1,sp}\}$
$N \geq 2$	$s \geq 1$	$1 \leq sp < 2$	$X^{s,p} \neq W^{s,p}(Q; \mathbb{S}^1), X^{s,p} = \{e^{i\varphi} ; \varphi \in W^{s,p} \cap W^{1,sp}\}$
$N \geq 2$	$s \geq 1$	$sp \geq 2$	$X^{s,p} = W^{s,p}(Q; \mathbb{S}^1) = \{e^{i\varphi} ; \varphi \in W^{s,p} \cap W^{1,sp}\}$

The following result completes the description of $X^{s,p}$ in terms of lifting⁸

Theorem 3. *Assume that $0 < s < 1$ and $1 \leq sp < N$.*

a) *If $sp > 1$, then $X^{s,p} = \{e^{i\varphi} ; \varphi \in W^{s,p} + W^{1,sp}\}$.*

b) *If $sp = 1$, then $X^{s,p} = W^{s,p}(Q; \mathbb{S}^1) \cap \{e^{i\varphi} ; \varphi \in W^{s,p} + W^{1,1}\}$.*

Proof. "⊂" follows from Theorem 1.

"⊃" Let $u = e^{i(\varphi_1 + \varphi_2)} \in W^{s,p}(Q; \mathbb{S}^1)$, with $\varphi_1 \in W^{s,p}$ and $\varphi_2 \in W^{1,sp}$. Set $u_j := e^{i\varphi_j}$. By Corollary 1, $u_1 \in X^{s,p}$. On the other hand, $W^{s,p}(Q; \mathbb{S}^1)$ is a group, so that $u_2 \in W^{s,p}(Q; \mathbb{S}^1)$. By Proposition 2, $u_2 \in X^{s,p}$. By (27), $X^{s,p}$ is a group, so that $u = u_1 u_2 \in X^{s,p}$. \square

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⁸There is another description available, in terms of distributional jacobian $T(u)$ (for its definition, see [1],[5],[6]). For $1 \leq sp < 2$, we have $X^{s,p} = \{u \in W^{s,p}(Q; \mathbb{S}^1) ; T(u) = 0\}$, result due to Bousquet [6] when $s \geq 1$ and Ponce [15] when $s < 1$.