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Scaling law and reduced models for epitaxially strained crystalline films

M. Goldman ^{*} B. Zwicknagl [†]

Abstract

A variational model for the epitaxial deposition of a film on a rigid substrate in the presence of a crystallographic misfit is studied. The scaling behavior of the minimal energy in terms of the volume of the film and the amplitude of the misfit is considered, and reduced models in the various regimes are derived by Γ -convergence methods. Depending on the relation between the thickness of the film and the amplitude of the misfit, the surface or the elastic energy contribution dominate, and in the critical case the two contributions balance. In particular, the formation of islands is proven if the amplitude of the misfit is large compared to the volume of the film.

1 Introduction

We study a free energy functional introduced in [41] (see also [8, 19, 25]) to model the epitaxial deposition of a film on a rigid substrate when there is a crystallographic misfit between the two solids. The energy consists of the stored elastic energy in the film and the interfacial energy of its free surface. Precisely, we consider

$$F_{d,e_0}(u, h) := \int_{\Omega_h} W(\nabla u) dx dy + \int_0^1 \sqrt{1 + |h'|^2} dx, \quad (1)$$

where the Lipschitz function $h : [0, 1] \rightarrow [0, \infty)$ describes the profile of the deposited film, $\Omega_h := \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq h(x)\}$, and $u \in W^{1,p}(\Omega_h; \mathbb{R}^2)$ is the planar displacement. Note that we restrict ourselves to two-dimensional morphologies which correspond to three-dimensional configurations with planar symmetry (see Section 5 for some results in the higher-dimensional setting). We consider film profiles of fixed volume $d > 0$, i.e., $\int_0^1 h(x) dx = d$, and impose the boundary conditions $h(0) = h(1) = 0$ (see Figure 1). The parameter $e_0 \in \mathbb{R}$ stands for the amplitude of the crystallographic misfit between the film and the rigid substrate, and is introduced in this model via the boundary condition $u(x, 0) = e_0(x, 0)$. This condition forces the film to be strained. The free energy density W is nonnegative and

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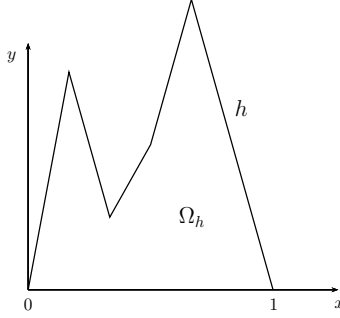


Figure 1: A height profile h

satisfies a p -growth condition (see Section 2.3).

In recent years, the mathematical analysis of this model has been devoted to the geometrically linear small strain approximation (see [19, 25, 8]), corresponding to small deformations, in which W depends only on the symmetrized gradient $E(u) := \nabla u + \nabla^T u$. Since we are mainly interested in the regime of large mismatch e_0 , the small strain hypothesis is not applicable and instead we focus on the geometrically nonlinear case. However, our analysis applies also to the geometrically linear setting with only minor modifications. We note that the functional (1) is normalized by setting a typical surface energy per unit length to one, i.e., the material parameters are all collected in W .

It has been observed experimentally and numerically that the qualitative shape of the optimal profile depends on the volume d of the film and on the misfit e_0 (see [8, 26, 30, 42]). If the volume is small then the film typically forms a flat layer. If the volume of the film is sufficiently large then the flat configuration is no longer stable and atoms often tend to rearrange in such a way that part of the bulk energy is released. The competition between elastic and surface energy then gives rise to the formation of islands on the substrate surface. Such island formation is used for many optical and optoelectronic applications, such as quantum dot lasers.

Our main result is the following scaling law for the energy (see Theorem 3.1): There are positive constants C_1 and C_2 such that for all choices of parameters d and e_0 ,

$$C_1 \max \left\{ 1, d, e_0^{p/3} d^{2/3} \right\} \leq \inf F_{d,e_0} \leq C_2 \max \left\{ 1, d, e_0^{p/3} d^{2/3} \right\}.$$

We subsequently consider the scaling regimes separately and derive reduced limiting functionals in the spirit of Γ -convergence in the three scaling regimes and for the transition regime $d \simeq e_0^p$. Our results agree qualitatively with the experimentally observed morphologies. Consider first large volumes $d \rightarrow \infty$. If the amplitudes of the misfits $e_0 \rightarrow \infty$ are such that $\frac{e_0^p}{d} \rightarrow 0$, i.e., $\inf F_{d,e_0} \simeq d$, then the surface energy dominates, and the main energy contribution comes from the non-horizontal parts of the boundary. Indeed, in this case the limiting

reduced functional does not depend on the elastic strain and charges only the vertical parts of the boundary of Ω_h (see Proposition 4.2). In this regime, the flat configuration is optimal (see Lemma 2.6). In the transition regime in which $d \simeq e_0^p \rightarrow +\infty$, surface and elastic contributions compete (see Proposition 4.3). Consequently, the minimizer of the reduced model is either the flat configuration or an island, depending on the elastic properties of the material. Finally, if the amplitudes of the misfits are large compared to the volumes, i.e., if $e_0 \rightarrow +\infty$ and $\frac{e_0^p}{d} \rightarrow +\infty$, then $\inf F_{d,e_0} \simeq e_0^{p/3} d^{2/3}$, and the main contribution to the energy comes from the elastic part. Precisely, the limiting functional is now defined on the space of (at most) countable sums of Dirac masses. Using a compactness argument (see Proposition 4.8), this proves that in this regime, low-energy sequences have to converge to sum of Dirac masses. This shows that the formation of islands is energetically favorable in this regime. To the best of our knowledge, this work is the first rigorous mathematical justification of the formation of islands in relation to the amplitude of the mismatch between crystallographic lattices in epitaxially strained films.

For small volumes $d \rightarrow 0$ and small amplitudes of the misfits $e_0 \rightarrow 0$ such that $e_0^p d^2 \rightarrow 0$, we have $\inf F_{d,e_0} \simeq 1$, and the limiting reduced functional is simply a constant (see Proposition 4.1). In particular, for low-energy sequences in this regime neither the elastic energy nor the non-horizontal parts of the boundary contribute significantly to the energy. If, however, $e_0^p d^2 \rightarrow \infty$, then $\inf F_{d,e_0} \simeq e_0^{p/3} d^{2/3}$, and, as discussed above for the case of large volumes, low-energy sequences converge to sums of Dirac masses, and the formation of islands is energetically favorable.

In the last part of the paper we adapt our method to prove the scaling law in higher dimension for the Dirichlet elastic energy (see Proposition 5.1). Even though our method seems to be quite flexible, the richness of the geometry makes the extension to more general energy functionals difficult (see [12] for some results). Let us finally notice that in higher dimension, the elastic part of the energy can be seen as a repulsive nonlocal term giving the problem a flavor of the Ohta-Kawasaki model which has recently received a lot of attention (see [33, 15, 13, 29, 2]).

We note that in the regimes in which the elastic energy is not dominated by the surface energy, the recession function of W at infinity plays an essential role in the limit. The latter is defined by

$$W^\infty(A) := \limsup_{t \rightarrow +\infty} \frac{W(tA)}{t^p} \quad \text{for } A \in \mathbb{R}^{2 \times 2},$$

and appears naturally in problems for functionals with linear growth (see [21, 22]). However, to the best of our knowledge, this is the first time that it appears in the context of $p > 1$ growth.

Previous work, starting with [11], has been devoted to proving regularity and other qualitative properties of islands, once they are formed. In [19] the authors prove that minimizing configurations are smooth outside of a finite number of cusps and cuts. Subsequently, it

was proven in [25] that for small values of the mismatch, the flat configuration is always minimizing no matter how thick is the film. Further, they proved that for larger (but fixed) values of e_0 , there are increasing thresholds for the volume d such that the flat configuration changes from being an absolute minimizer to being only a local minimizer, and is no longer a local minimizer above a certain threshold for d . Finally, they showed that even though the flat configuration is no longer minimizing, there exists another threshold below which the minimizers are smooth.

Our analysis bears similarities with dimension reduction problems (see [5, 18, 23, 24, 31, 36]). In contrast to these works, we study the behavior for small and for big volumes. Furthermore, the interaction between elastic and surface energy creates a complex behavior, i.e., the reference configuration is not given but is instead one of the unknowns. Another feature of (1) is the absence of regularizing second order terms which are often used for obtaining scaling laws in elasticity (see [34, 16, 6]). Notice also that in contrast with these works, our proof of the lower bound for the scaling law in dimension two does not rely on interpolation inequalities.

We deal only with the stationary setting, but we refer to [20, 39] for recent results on the time evolution problem. Finally, we point out that it might be interesting to investigate the model without the hypothesis that the domain of the film is a subgraph (see [10]). We expect to observe a similar behavior where typical shapes of islands are not rectangles but balls (see [42]).

The plan of the paper is as follows: In Section 2 we set the notation and collect some preliminary results concerning both the surface energy and the elastic energy. In Section 3 we establish the scaling law for F_{d,e_0} , and in Section 4 we derive and discuss reduced models in the various scaling regimes. Finally in Section 5, we show how our methods can be extended in higher dimension for quadratic energies.

2 Notation and preliminary results

Set $I := [0, 1]$, and throughout the text denote by C and c constants that may vary from line to line and from expression to expression. For $\Omega \subset \mathbb{R}^2$, we denote by $\mathcal{H}^1(\Omega)$ its one-dimensional Hausdorff measure, and by $|\Omega|$ its Lebesgue measure. Given two sets $A, B \subset \mathbb{R}^2$, we define their Hausdorff distance as $d_{\mathcal{H}}(A, B) := \inf\{r > 0 : A \subset N(B, r) \text{ and } B \subset N(A, r)\}$, where $N(A, r) := \{x \in \mathbb{R}^2 : d(x, A) < r\}$ (with $d(x, A)$ denoting the distance from x to A). For $c \in \mathbb{R}$ we let δ_c be the Dirac mass at the point c . Given a vector $(x, y) \in \mathbb{R}^2$, we denote by $|(x, y)| := (x^2 + y^2)^{1/2}$ its Euclidean norm. For a matrix $A \in \mathbb{R}^{2 \times 2}$, we consider the Frobenius norm $|A|^2 := \text{Tr}(A^T A)$.

2.1 Γ -convergence

We briefly recall the definition and main properties of Γ -convergence (see [17, 9]).

Definition 2.1. *Let X be a topological space. We say that a sequence of functions $\{F_n\}$, $F_n : X \rightarrow \mathbb{R}$, (sequentially) Γ -converges to $F : X \rightarrow \mathbb{R}$ and write $F_n \xrightarrow{\Gamma} F$ if the following two conditions are satisfied.*

- (i) *If $\{x_n\} \subset X$ satisfies $\sup F_n(x_n) < +\infty$ then there exists a subsequence (not relabeled) such that $x_n \rightarrow x \in X$ and*

$$\liminf_{n \rightarrow +\infty} F_n(x_n) \geq F(x).$$

- (ii) *For every $x \in X$ there exists a sequence $\{x_n\} \subset X$ such that $x_n \rightarrow x$ and*

$$\limsup_{n \rightarrow +\infty} F_n(x_n) \leq F(x).$$

The main property of Γ -convergence is the following:

Theorem 2.2. *Suppose a sequence $\{F_n\}$ with $F_n : X \rightarrow \mathbb{R}$ Γ -converges to F and there is some $C > 0$ such that $\inf_{x \in X} F_n(x) \leq C$ for all $n \in \mathbb{N}$. Then*

$$\inf_{x \in X} F_n(x) \rightarrow \inf_{x \in X} F(x).$$

Moreover, if x_n is a minimizer of F_n then there is a subsequence of $\{x_n\}$ that converges to a minimizer of F .

2.2 The Surface Energy

We review some results from [19, 8] adapted to our setting. If $h : [0, 1] \rightarrow [0, +\infty)$ is lower semicontinuous, then set

$$\Omega_h := \{(x, y) : 0 \leq x \leq 1, 0 \leq y < h(x)\}$$

and denote the pointwise variation of h by

$$\text{Var } h := \sup \left\{ \sum_{i=1}^n |h(x_i) - h(x_{i-1})| : P := \{x_1, \dots, x_n\} \text{ is a partition of } [0, 1] \right\}.$$

If $\text{Var } h$ is finite then h is said to be of bounded pointwise variation (see [4]). If h is a function of bounded pointwise variation then we define

$$\begin{aligned} h^-(x) &:= \min \{h(x^+), h(x^-)\} = \liminf_{z \rightarrow x} h(z), \\ h^+(x) &:= \max \{h(x^+), h(x^-)\} = \limsup_{z \rightarrow x} h(z), \end{aligned}$$

where $h(x^\pm) := \lim_{z \rightarrow x^\pm} h(z)$, and

$$\int_0^1 |h'| := \text{Var } h + h(0^+) + h(1^-) .$$

We denote by Γ_{cuts} the at most countable collection of vertical cuts,

$$\Gamma_{cuts} := \{(x, y) : x \in (0, 1) \cap S(h), h(x) \leq y \leq h^-(x)\}$$

where $S(h) := \{x \in (0, 1) : h(x) < h^-(x)\}$. The following proposition is a slight adaption of results from [19, Proposition 2.2, Lemma 2.5, and Theorem 2.8].

Proposition 2.3. *Let $\{h_n\}$ be a sequence of nonnegative Lipschitz functions such that*

$$\sup_n \int_0^1 \sqrt{1 + |h'_n|^2} dx + \int_0^1 h_n dx < +\infty .$$

Then there exists a subsequence (not relabeled) such that

(i) $\mathbb{R}^2 \setminus \Omega_{h_n}$ *converges in the Hausdorff metric to* $\mathbb{R}^2 \setminus \Omega_h$ *where*

$$h := \inf \left\{ \liminf_{n \rightarrow \infty} h_n(x_n) : x_n \rightarrow x \right\}$$

is lower semicontinuous and of bounded pointwise variation;

(ii) $h_n \rightarrow h$ *in* $L^1(0, 1)$;

(iii) *for every sequence* $\{a_n\} \subset \mathbb{R}$ *with* $a_n \rightarrow 0$

$$\liminf_{n \rightarrow +\infty} \int_0^1 \sqrt{a_n + |h'_n|^2} dx \geq \int_0^1 |h'| + 2\mathcal{H}^1(\Gamma_{cuts}) .$$

Conversely, if h is nonnegative, lower semicontinuous and of bounded pointwise variation, then there exists a sequence $\{h_n\}$ of Lipschitz functions such that $0 \leq h_n \leq h$, $\mathbb{R}^2 \setminus \Omega_{h_n}$ converges in the Hausdorff metric to $\mathbb{R}^2 \setminus \Omega_h$, and

$$\lim_{n \rightarrow +\infty} \int_0^1 \sqrt{a_n + |h'_n|^2} dx = \int_0^1 |h'| + 2\mathcal{H}^1(\Gamma_{cuts})$$

for every sequence $\{a_n\}$ *with* $a_n \rightarrow +\infty$.

By Proposition 2.3, following the argument of [19, Theorem 2.8], we obtain the next result.

Proposition 2.4. *For every nonnegative, lower semicontinuous function h of bounded pointwise variation, and for every $u \in W^{1,p}(\Omega_h; \mathbb{R}^2)$, there exists a sequence $\{h_n\}$ of nonnegative Lipschitz functions with $\int_0^1 h_n dx = \int_0^1 h dx$ such that*

$$\lim_{n \rightarrow +\infty} \int_0^1 \sqrt{a_n + |h'_n|^2} dx = \int_0^1 |h'| + 2\mathcal{H}^1(\Gamma_{cuts})$$

for every sequence $\{a_n\} \subset \mathbb{R}$ *with* $a_n \rightarrow +\infty$, *and a sequence* $\{u_n\} \subset W^{1,p}(\Omega_{h_n}; \mathbb{R}^2)$ *which converges strongly (locally) to* u *in* $W^{1,p}$.

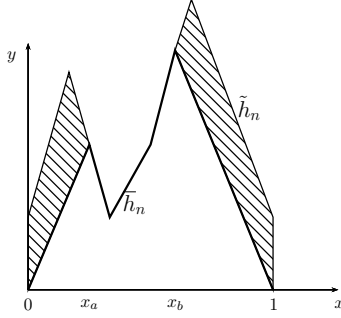


Figure 2: Construction of \bar{h}_n from \tilde{h}_n (see proof of Proposition 2.4)

Proof. By Proposition 2.3, there exists a sequence $\{h_n\}$ of Lipschitz functions such that $h_n \leq h$ and

$$\lim_{n \rightarrow +\infty} \int_0^1 \sqrt{a_n + |h'_n|^2} dx = \int_0^1 |h'| + 2\mathcal{H}^1(\Gamma_{cuts})$$

for every sequence $a_n \rightarrow +\infty$. Set

$$\varepsilon_n := \int_0^1 (h(x) - h_n(x)) dx$$

and $\tilde{h}_n := h_n + 2\varepsilon_n$. For $a > 0$ such that $\{y = \tilde{h}_n(x)\} \cap \{y = ax\} \neq \emptyset$, let $(x_a, ax_a) = (x_a, \tilde{h}_n(x_a))$ be the first point of the intersection between $\{y = \tilde{h}_n(x)\}$ and $\{y = ax\}$ (see Figure 2), i.e.

$$x_a := \inf \{x \in (0, 1) : ax = h_n(x)\}.$$

Choose $a > 0$ such that $|\Omega_{\tilde{h}_n} \cap \{(x, y) : 0 \leq x \leq x_a, y \geq ax\}| = \frac{\varepsilon_n}{2}$. Similarly, define $b > 0$ and $x_b > x_a$ such that $|\Omega_{\tilde{h}_n} \cap \{(x, y) : x_b \leq x \leq 1, y \geq b(1-x)\}| = \frac{\varepsilon_n}{2}$. Then

$$\bar{h}_n(x) := \begin{cases} ax & \text{if } 0 \leq x \leq x_a, \\ \tilde{h}_n(x) & \text{if } x_a \leq x \leq x_b, \\ b(1-x) & \text{if } x_b \leq x \leq 1, \end{cases}$$

and

$$u_n(x) := \begin{cases} e_0(x, 0) & \text{if } 0 \leq y \leq 2\varepsilon_n, \\ u(x, y - 2\varepsilon_n) & \text{if } y \geq 2\varepsilon_n, \end{cases}$$

satisfy the required properties. □

Remark 2.5. The Dirichlet boundary condition is often relaxed. Precisely, instead of restricting ourselves to Lipschitz functions h with $h(0) = h(1) = 0$, we can, equivalently,

consider arbitrary Lipschitz functions h if the surface energy term is replaced by

$$\int_0^1 \sqrt{1 + |h'|^2} dx + h(0) + h(1).$$

This kind of relaxation is very classical for functionals with linear growth (see [27]).

Finally, we state an isoperimetric inequality which will be used throughout the paper.

Lemma 2.6. *Let $h : [a, b] \rightarrow \mathbb{R}$ be a Lipschitz function on the segment $[a, b]$ with $h(a) = h(b) = 0$, and let $\int_a^b h(x) dx =: d$. Then*

$$\int_a^b \sqrt{1 + |h'|^2} dx \geq 2 \frac{d}{b-a}. \quad (2)$$

Proof. Let \bar{x} be a point where h attains its maximum on $[a, b]$ and set $\bar{h} := h(\bar{x})$. Then

$$\int_a^b \sqrt{1 + |h'|^2} dx \geq \left| \int_a^{\bar{x}} h' dx \right| + \left| \int_{\bar{x}}^b h' dx \right| = 2\bar{h} \geq 2 \frac{d}{b-a}.$$

□

2.3 The Strain Energy

In this section we collect assumptions and basic properties of the strain energy term.

Definition 2.7. *A Borel measurable function $W : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ is quasiconvex if for every $A \in \mathbb{R}^{2 \times 2}$ and every bounded domain $\Omega \subset \mathbb{R}^2$*

$$W(A) \leq \frac{1}{|\Omega|} \int_{\Omega} W(A + \nabla \phi) dx dy \quad \text{for all } \phi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2). \quad (3)$$

Remark 2.8. *It is well known that in the definition of quasiconvexity we could have equivalently asked that (3) only holds for the unit cube $(0, 1)^2$ (see [28]).*

We consider energy densities $W : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ satisfying the following hypotheses:

(H1) $W \geq 0$;

(H2) W is quasiconvex;

(H3) there exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ and $1 < p < \infty$ such that

$$\alpha(|A|^p + 1) \geq W(A) \geq \frac{1}{\alpha}(|A|^p - 1) \quad \text{for all } A \in \mathbb{R}^{2 \times 2}.$$

We remark that we assume (H2) only for simplicity of notation. Indeed, the analysis goes through exactly in the same way provided that in the results the function W is replaced by its quasiconvex envelope, i.e. the largest quasiconvex function which is below W . Hypothesis (H3) does not allow for potentials that blow-up when $\det y \rightarrow 0^+$, which corresponds to the non-interpenetration condition (see [7] for a discussion of this issue). By (H3) and a classical result of Marcellini (see [28, Lemma 5.2]), W is locally p -Lipschitz continuous. Moreover, by (H3) again and by density, the class of test functions in (3) can be enlarged to be $W_0^{1,p}(\Omega; \mathbb{R}^2)$.

We consider the recession function of W at infinity defined by

$$W^\infty(A) := \limsup_{t \rightarrow +\infty} \frac{W(tA)}{t^p} \quad \text{for } A \in \mathbb{R}^{2 \times 2}.$$

By definition, W^∞ is p -homogeneous and satisfies the growth condition $\alpha |A|^p \geq W^\infty(A) \geq \frac{1}{\alpha} |A|^p$. Using the quasiconvexity of W and the Fatou Lemma, we see that W^∞ is also quasiconvex. Therefore, W^∞ is also locally p -Lipschitz continuous. The recession function arises naturally in relaxation and Γ -convergence problems for functional with linear growth (see [21, 22]). However, to the best of our knowledge, this is the first time that it appears in the context of $p > 1$ growth. We will need the following additional assumption on the rate of convergence of $\frac{W(tA)}{t^p}$ to $W^\infty(A)$ (see [22]):

(H4) There exist $0 < m < p$, $\gamma > 0$ and $L > 0$ such that for $t|A| \geq L$,

$$\left| W^\infty(A) - \frac{W(tA)}{t^p} \right| \leq \gamma \frac{|A|^{p-m}}{t^m}.$$

Remark 2.9. For $p > 1$ the functions $W(\cdot) := \text{dist}^p(\cdot, SO_2)$ satisfy (H4).

The main geometrical objects that will appear in the analysis are rectangles $[0, \ell] \times [0, L]$ for some $\ell, L > 0$. We will use the following rescaling estimate.

Lemma 2.10. *If $u \in W^{1,p}([0, \ell] \times [0, L]; \mathbb{R}^2)$, with $u(x, 0) = e_0(x, 0)$, then*

$$\int_{[0, \ell] \times [0, L]} W(\nabla u) \, dx dy \geq e_0^p \ell^2 \min_{v(x, 0) = (x, 0)} \int_{[0, 1] \times [0, L/\ell]} \frac{W(e_0 \nabla v)}{e_0^p} \, dx dy, \quad (4)$$

where here, and in the sequel, the minimum is taken among all $v \in W^{1,p}([0, 1] \times [0, L/\ell]; \mathbb{R}^2)$ with boundary condition $v(x, 0) = (x, 0)$.

Proof. For $u \in W^{1,p}([0, 1] \times [0, L/\ell]; \mathbb{R}^2)$ set $\tilde{u}(x, y) := \frac{1}{\ell} u(\ell x, \ell y)$. Then $\tilde{u} \in W^{1,p}([0, 1] \times [0, L/\ell]; \mathbb{R}^2)$ with $\nabla \tilde{u}(x, y) = \nabla u(\ell x, \ell y)$, and thus

$$\begin{aligned} \min_{u(x, 0) = e_0(x, 0)} \int_{[0, \ell] \times [0, L]} W(\nabla u) \, dx dy &= \ell^2 \min_{\tilde{u}(x, 0) = e_0(x, 0)} \int_{[0, 1] \times [0, L/\ell]} W(\nabla \tilde{u}) \, dx dy \\ &= \ell^2 \min_{v(x, 0) = (x, 0)} \int_{[0, 1] \times [0, L/\ell]} W(e_0 \nabla v) \, dx dy. \end{aligned}$$

□

Notice that for every e_0 ,

$$\phi(t, e_0) := \min_{v(x,0)=(x,0)} \int_{[0,1] \times [0,t]} \frac{W(e_0 \nabla v)}{e_0^p} dx dy$$

is a non-decreasing function of t , which is bounded above by some constant independent of e_0 (take any smooth function with bounded support and use (H3)).

Remark 2.11. We recall that Korn-type inequalities or related rigidity results in the geometrically non-linear setting, degenerate if the Lipschitz constants of the domain blow-up, or if $L \rightarrow +\infty$ in infinite strips of the form $[0, \ell] \times [0, L[$ (see [35, 23]).

The next lemma describes the behavior of the elastic energy for small thickness. It can be seen as a dimension reduction argument (see [36, 5]). The proof extends almost verbatim to the geometrically linear setting, i.e. when W depends only on the symmetric part of the gradient.

Lemma 2.12. *There holds*

$$\lim_{\varepsilon \rightarrow 0^+} \min_{v(x,0)=(x,0)} \frac{1}{\varepsilon} \int_{[0,1] \times [0,\varepsilon]} |\nabla v|^p dx dy \geq 1. \quad (5)$$

Proof. In order to work on a ε -independent domain, we make a classical change of variables (see [14, Ch.1 Sect. 1.3]), i.e. for $v = (v_1, v_2) \in W^{1,p}([0,1] \times [0,\varepsilon]; \mathbb{R}^2)$ we set $w(x, y) := (v_1(x, \varepsilon y), \varepsilon v_2(x, \varepsilon y)) \in W^{1,p}([0,1]^2; \mathbb{R}^2)$. Notice that if $v(x, 0) = (x, 0)$, then also $w(x, 0) = (x, 0)$. The rescaled functional to be considered then reads

$$J_\varepsilon(w) := \int_{[0,1]^2} \left| \begin{pmatrix} \frac{\partial w_1}{\partial x} & \frac{1}{\varepsilon} \frac{\partial w_1}{\partial y} \\ \frac{1}{\varepsilon} \frac{\partial w_2}{\partial x} & \frac{1}{\varepsilon^2} \frac{\partial w_2}{\partial y} \end{pmatrix} \right|^p dx dy.$$

If w^ε is a minimizer of J_ε subject to the condition $w^\varepsilon(x, 0) = (x, 0)$ then $\|w^\varepsilon\|_{W^{1,p}}$ is bounded above uniformly in ε . In particular, up to a subsequence, $\{w^\varepsilon\}$ converges weakly in $W^{1,p}$ to a function w with $\frac{\partial w_1}{\partial y} = \frac{\partial w_2}{\partial x} = \frac{\partial w_2}{\partial y} = 0$. Furthermore, $w(x, 0) = (x, 0)$ since for $p > 1$ the space $W^{1,p}([0,1]^2)$ is compactly embedded in some $L^q([0,1])$ (see [1, Theorem 6.3]). Thus

$$\liminf_{\varepsilon \rightarrow 0^+} J_\varepsilon(w^\varepsilon) \geq \int_{[0,1]^2} \left| \frac{\partial w_1}{\partial x} \right|^p dx dy = 1.$$

□

We next examine the scaling of the elastic energy when the thickness of the film tends to infinity. The proof can be extended along the same lines to the geometrically linear setting.

Lemma 2.13. *Let W satisfy (H2), (H3) and (H4). If $f : \mathbb{R} \rightarrow \mathbb{R}$ is such that $f(t) \rightarrow +\infty$ for $t \rightarrow +\infty$, then*

$$\lim_{e_0 \rightarrow +\infty} \min_{v(x,0)=(x,0)} \int_{[0,1] \times [0,f(e_0)]} \frac{W(e_0 \nabla v)}{e_0^p} dx dy = \min_{v(x,0)=(x,0)} \int_{[0,1] \times [0,+\infty)} W^\infty(\nabla v) dx dy. \quad (6)$$

Proof. Set

$$G_{e_0}(v) := \int_{[0,1] \times [0, f(e_0)]} \frac{W(e_0 \nabla v)}{e_0^p} dx dy \quad \text{and} \quad G(v) := \int_{[0,1] \times [0, +\infty)} W^\infty(\nabla v) dx dy .$$

By Theorem 2.2, it suffices to show that $G_{e_0} \xrightarrow{\Gamma} G$ with respect to the (sequential) weak topology of $W_{\text{loc}}^{1,p}([0,1] \times [0, +\infty))$ as $e_0 \rightarrow \infty$. Consider first $\{v_{e_0}\}$ such that $v_{e_0}(x, 0) = (x, 0)$ and $\sup G_{e_0}(v_{e_0}) < +\infty$. Then by (H3), for every $M > 0$ the functions v_{e_0} are uniformly bounded in $W^{1,p}([0,1] \times [0, M])$, and thus there exists a subsequence converging weakly in $W_{\text{loc}}^{1,p}$ to a function v .

We now show that for $M > 0$ fixed

$$\liminf_{e_0 \rightarrow +\infty} \int_{[0,1] \times [0, M]} \frac{W(e_0 \nabla v_{e_0})}{e_0^p} dx dy \geq \int_{[0,1] \times [0, M]} W^\infty(\nabla v) dx dy. \quad (7)$$

The proof of (7) is an adaptation of the semicontinuity proof for quasiconvex functionals (see [28, Thm. 5.4]), and we sketch it only for the reader's convenience. Let $R := [0,1] \times [0, M]$. *Step 1.* If v is affine and $v_{e_0} = v$ on ∂R , then $v_{e_0} = v + \phi_{e_0}$ for some $\phi_{e_0} \in W_0^{1,p}(R)$, and by (3) and (H4), for $e_0 |A| \geq L$ we have

$$\int_R \frac{W(e_0 \nabla v_{e_0})}{e_0^p} dx dy \geq M \frac{W(e_0 A)}{e_0^p} \geq M \left(W^\infty(A) - \gamma \frac{|A|^{p-m}}{e_0^m} \right), \quad (8)$$

and (7) follows as $e_0 \rightarrow +\infty$.

Step 2. If v is affine but $v_{e_0} \neq v$ on ∂R , then we modify v_{e_0} near $\partial\Omega$. Precisely, for $R_0 \Subset R$, $K \in \mathbb{N}$ and $i = 1, \dots, K$, consider the sets

$$R_i := \left\{ x \in R : \text{dist}(x, R_0) < \frac{i}{K} \rho \right\}$$

where $\rho := \frac{1}{2} \text{dist}(R_0, \partial R)$, and introduce cut-off functions $\psi_i \in C_0^1(R_i)$ with

$$0 \leq \psi_i \leq 1, \quad \psi_i = 1 \text{ in } R_{i-1}, \quad \text{and} \quad |\nabla \psi_i| \leq \frac{2(K+1)}{\rho}.$$

Set $w_{e_0}^i := v + \psi_i(v_{e_0} - v)$. Since $w_{e_0}^i = v$ on ∂R , by (8)

$$\int_R \frac{W(e_0 \nabla w_{e_0}^i)}{e_0^p} dx dy \geq \int_R W^\infty(\nabla v) dx dy - M \gamma \frac{|A|^{p-m}}{e_0^m}.$$

Thus, since $v_{e_0} \rightharpoonup v$ weakly in $W^{1,p}(R)$, by (H3), summing the inequalities over i , dividing by K and letting $e_0 \rightarrow +\infty$, we find that for some $C > 0$

$$\int_{R_0} W^\infty(\nabla v) dx dy \leq \liminf_{e_0 \rightarrow +\infty} \int_R \frac{W(e_0 \nabla u_{e_0})}{e_0^p} dx dy + \frac{C}{K}$$

which for $K \rightarrow +\infty$ and $R_0 \rightarrow R$ implies (7).

Step 3. Consider a countable family of squares Q_i with pairwise disjoint interiors and covering R . We apply Step 2 in each square to the functions $\langle z_i, x \rangle$, where

$$z_i := \frac{1}{|Q_i|} \int_{Q_i} \nabla v \, dxdy,$$

and to the sequence $w_{e_0}^i(x) := v_{e_0}(x) - u(x) + \langle z_i, x \rangle$, which weakly converges to $\langle z_i, x \rangle$ in $W^{1,p}(R)$. If the diameter of the squares tends to zero then (7) follows, and so for every $M > 0$,

$$\liminf_{e_0 \rightarrow +\infty} G_{e_0}(v_{e_0}) \geq \liminf_{e_0 \rightarrow +\infty} \int_{[0,1] \times [0,M]} \frac{W(e_0 \nabla v_{e_0})}{e_0^p} \, dxdy \geq \int_{[0,1] \times [0,M]} W^\infty(\nabla v) \, dxdy.$$

We let $M \rightarrow +\infty$ to get $\liminf_{e_0 \rightarrow +\infty} G_{e_0}(v_{e_0}) \geq G(v)$.

On the other hand, if $v \in W^{1,p}([0,1] \times [0, +\infty))$, then v can be approximated in energy by functions with compact support and for these functions, using the growth conditions (H3) and the reverse Fatou Lemma, we conclude that

$$\limsup_{e_0 \rightarrow +\infty} G_{e_0}(v) \leq G(v).$$

□

Define

$$C_W := \min_{v(x,0)=(x,0)} \int_{[0,1] \times [0,+\infty)} W^\infty(\nabla v) \, dxdy. \quad (9)$$

Remark 2.14. If $W(\cdot) = |\cdot|^2$, one obtains by Fourier methods that (see [16, Appendix])

$$C_W = \frac{1}{\pi^3} \sum_{k>0} \frac{1}{k^3} (1 + (-1)^{k+1})^2.$$

3 Scaling law

In this section we consider the qualitative behavior of the minimal energy in terms of its scaling regimes.

Theorem 3.1. *Suppose that W satisfies (H1) and (H2). Then there are constants $C_1, C_2 > 0$ such that, for all $d, e_0 > 0$,*

$$C_1 \max \left\{ 1, d, e_0^{p/3} d^{2/3} \right\} \leq \inf F_{d,e_0} \leq C_2 \max \left\{ 1, d, e_0^{p/3} d^{2/3} \right\}.$$

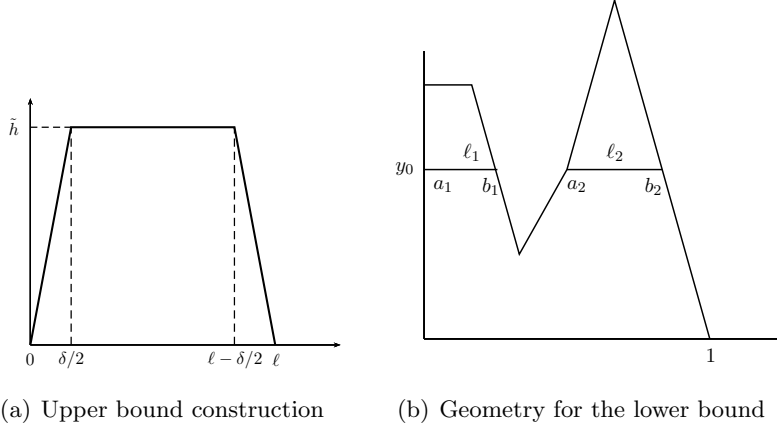


Figure 3: Constructions in the proof of Theorem 3.1

Proof. Upper bound.

For $\ell \leq 1$ and $0 < \delta \ll \frac{\ell}{2}$ set $\tilde{h} := \frac{2d}{2\ell - \delta}$, and consider (see Figure 3(a))

$$h(x) := \begin{cases} \frac{2\tilde{h}}{\delta}x & \text{if } 0 \leq x \leq \frac{\delta}{2}, \\ \tilde{h} & \text{if } \frac{\delta}{2} \leq x \leq \ell - \frac{\delta}{2}, \\ -\frac{2\tilde{h}}{\delta}x + \frac{2\tilde{h}}{\delta}\ell & \text{if } \ell - \frac{\delta}{2} \leq x \leq \ell, \\ 0 & \text{if } \ell \leq x \leq 1. \end{cases}$$

Let u be the restriction to Ω_h of $\tilde{u} : I \times [0, \infty) \rightarrow \mathbb{R}$ given by

$$\tilde{u}(x, y) = \begin{cases} (e_0 x (1 - \frac{1}{\ell}y), 0) & \text{if } 0 \leq y \leq \ell, \\ 0 & \text{else.} \end{cases}$$

Then

$$\begin{aligned} F_{d, e_0}(u, h) &= \int_{\Omega_h} W(\nabla u) dy dx + 1 + \frac{4d}{2\ell - \delta} \\ &\leq C e_0^p \int_0^\ell \int_0^\ell \left(\left(1 - \frac{1}{\ell}y\right)^p + \left(\frac{1}{\ell}x\right)^p \right) dy dx + \alpha d + 1 + \frac{4d}{2\ell - \delta} \\ &\leq C \left(e_0^p \ell^2 + 1 + d + \frac{d}{\ell} \right) \end{aligned}$$

If $e_0^p \leq d$, then choose $\ell := 1$, and if $e_0^p \geq d$ choose $\ell := d^{1/3} e_0^{-p/3} \leq 1$. It follows that

$$e_0^p \ell^2 + 1 + d + \frac{d}{\ell} \leq 2 \max \left\{ 1, d, e_0^{p/3} d^{2/3} \right\},$$

and this completes the proof of the upper bound.

Lower bound: By definition of the surface energy, $F_{d,e_0}(u, h) \geq 1$, and by Lemma 2.6 $F_{d,e_0}(u, h) \geq 2d$. It remains to show that $F_{d,e_0}(u, h) \geq e_0^{p/3} d^{2/3}$ if $e_0^{p/3} d^{2/3} \geq \max\{1, d\}$.

Let $y_0 := \frac{d}{2\sqrt{e_0^{p/3} d^{2/3}}}$, $\ell := \mathcal{H}^1(\Omega_h \cap (I \times \{y_0\}))$ and $I_\ell := \Omega_h \cap (I \times \{y_0\})$. Denote by $[a_i, b_i]$ the finitely many (pairwise disjoint) maximal connected components of I_ℓ (see Figure 3(b)), i.e., $I_\ell = \cup_{i=1}^n [a_i, b_i]$, and for $i = 1, \dots, n$, set $\ell_i := b_i - a_i$ and $d_i := |\Omega_h \cap ([a_i, b_i] \times [y_0, +\infty))|$. Then $\sum_{i=1}^n \ell_i = \ell$ and $\sum_{i=1}^n d_i \geq d - y_0 \geq Cd$.

If $\ell_i \leq \left(\frac{d}{e_0^p}\right)^{1/3}$ for all $i = 1, \dots, n$, then by (2) applied to $h - y_0$ we obtain

$$\begin{aligned} F_{d,e_0}(u, h) &\geq \sum_{i=1}^n \int_{a_i}^{b_i} \sqrt{1 + |h'|^2} dx \\ &\geq \sum_{i=1}^n \frac{d_i}{\ell_i} \geq \left(\frac{e_0^p}{d}\right)^{1/3} \sum_{i=1}^n d_i \\ &\geq C e_0^{p/3} d^{2/3}. \end{aligned}$$

Otherwise, the largest connected component of I_ℓ , say I_{ℓ_1} , satisfies $\ell_1 \geq \left(\frac{d}{e_0^p}\right)^{1/3}$.

By Lemma 2.12 there exists a small constant $c < 1$ such that for $\varepsilon \leq c$,

$$\min_{u(x,0)=(x,0)} \frac{1}{\varepsilon} \int_{[0,1] \times [0,\varepsilon]} |\nabla u|^p dx dy \geq \frac{1}{2}.$$

Since $c > 0$ is a universal constant, and since $e_0^{p/3} d^{2/3} \geq \max\{1, d\}$ implies in particular that $e_0 \geq 1$, we may, without loss of generality, assume that $e_0^{p/2} \geq \frac{1}{4c}$. If $\left(\frac{d}{e_0^p}\right)^{1/3} \leq \ell_1 \leq \frac{y_0}{c}$, by (4) and (H3),

$$\begin{aligned} F_{d,e_0}(u, h) &\geq \int_{[a_1, b_1] \times [0, y_0]} W(\nabla u) dx dy \\ &\geq \frac{1}{\alpha} \left(\frac{d}{e_0^p}\right)^{2/3} e_0^p \min_{v(x,0)=(x,0)} \int_{[0,1] \times [0,c]} |\nabla v|^p dx dy - \frac{1}{\alpha} \ell_1 y_0 \\ &\geq \frac{c}{2\alpha} e_0^{p/3} d^{2/3} - \frac{1}{\alpha} y_0 \\ &\geq \frac{c}{2\alpha} e_0^{p/3} d^{2/3} - \frac{1}{\alpha} \frac{d}{\sqrt{e_0^{p/3} d^{2/3}}} \\ &\geq \frac{c}{4\alpha} e_0^{p/3} d^{2/3}, \end{aligned}$$

where we used the fact that $\frac{d}{\sqrt{e_0^{p/3} d^{2/3}}} \leq \frac{c}{4} e_0^{p/3} d^{2/3}$, since $e_0^{p/2} \geq \frac{1}{4c}$.

Otherwise, if $\ell_1 \geq y_0/c$, then, by (4), (H3), and (5),

$$\begin{aligned} F_{d,e_0}(u, h) &\geq \frac{1}{\alpha} \ell_1^2 e_0^p \min_{v(x,0)=(x,0)} \int_{[0,1] \times [0,y_0/\ell_1]} |\nabla v|^p \, dxdy - \frac{1}{\alpha} \ell_1 y_0 \\ &\geq \frac{1}{2\alpha} \ell_1^2 e_0^p \frac{y_0}{\ell_1} - \frac{1}{\alpha} e_0^{p/3} d^{2/3} \\ &\geq \frac{1}{2\alpha} (e_0^{p/3} d^{2/3})^2 - \frac{1}{\alpha} e_0^{p/3} d^{2/3} \geq C e_0^{p/3} d^{2/3}. \end{aligned}$$

□

4 Reduced models

To study the asymptotic behavior of F_{d,e_0} in the various regimes determined in Theorem 3.1, we set $\tilde{h} := h/d$, $\Omega_{\tilde{h}} := \{(x, y) : (x, dy) \in \Omega_h\}$, and $\tilde{u}(x, y) = u(x, dy)$. Dropping the tildes, the energy now reads

$$F_{d,e_0}(u, h) = d \left[\int_{\Omega_h} W \left(\frac{\partial u}{\partial x}, \frac{1}{d} \frac{\partial u}{\partial y} \right) dxdy + \int_0^1 \sqrt{\frac{1}{d^2} + |h'|^2} dx \right]$$

for (u, h) such that $\int_0^1 h dx = 1$, and $u \in W^{1,p}(\Omega_h)$ with $u(x, 0) = e_0(x, 0)$.

In the regime in which $F_{d,e_0} \simeq 1$, the limiting functional is constant (see Proposition 4.1), while for $F_{d,e_0} \simeq d$, the main contribution to the reduced energy comes from the surface part (see Proposition 4.2), and the flat configuration is optimal. In the transition regime, in which $d \simeq e_0^p \rightarrow +\infty$, elastic and surface energy compete (see Proposition 4.3), and the minimizer of the reduced model is either the flat configuration or an island, depending on the elastic properties of the material, specifically on the constant C_W in (9). Finally, in the regime in which $F_{d,e_0} \simeq e_0^{p/3} d^{2/3}$, the elastic energy dominates. By a corresponding compactness result (see Proposition 4.8), we conclude that in this regime the formation of islands is favored.

4.1 The trivial regime $F_{d,e_0} \simeq 1$

If $F_{d,e_0} \simeq 1$, $d \rightarrow 0$ and $e_0^p = o(\frac{1}{d^2})$, then the limit functional is a constant. In this regime, the surface and the elastic part of the energy do not contribute significantly. Given a probability measure $\mu := h \, dx$, let

$$G_{d,e_0}(\mu) := \min_{u(x,0)=e_0(x,0)} F_{d,e_0}(u, h).$$

Proposition 4.1. *Suppose that W satisfies (H1) and (H2), $d \rightarrow 0$ and $e_0^p = o(\frac{1}{d^2})$, then $\{G_{d,e_0}\}$ Γ -converges with respect to the weak-* convergence of measures to the constant functional equal to 1 on probability measures.*

Proof. The compactness follows from the weak-* compactness of sequences of probability measures on compact sets. Since the energy is always bounded from below by 1, the lower bound is immediate. For the upper bound, let ρ_d be a smooth convolution kernel such that $|\rho'_d| \leq \frac{1}{\sqrt{d}}$, and set $h_d(x) := \int_0^1 \rho_d(x-t) d\mu(t)$. Then $\int_0^1 h_d(x) dx = 1$, $\{h_d dx\}$ weak-* converges to μ , and

$$\limsup_{d \rightarrow 0} d \left[\int_0^1 \sqrt{\frac{1}{d^2} + |h'_d|^2} dx \right] \leq 1.$$

The functions h_d can be modified in such a way that $h_d(0) = h_d(1) = 0$ (see Proposition 2.4). Consider the function $u(x, y) := e_0(x, 0)$. By the p growth condition (H3), we then have

$$d \int_{\Omega_h} W \left(\frac{\partial u}{\partial x}, \frac{1}{d} \frac{\partial u}{\partial y} \right) dx dy \leq \alpha (e_0^p d^2 + d^2),$$

so $\lim_{d \rightarrow 0} d \int_{\Omega_h} W \left(\frac{\partial u}{\partial x}, \frac{1}{d} \frac{\partial u}{\partial y} \right) dx dy = 0$ as $d \rightarrow +\infty$. □

4.2 The surface dominant regime $F_{d,e_0} \simeq d$

If $F_{d,e_0} \simeq d$, $d \rightarrow +\infty$ and $e_0^p = o(d)$, dividing the energy by d , we obtain the rescaled energy

$$F_d(u, h) := \int_{\Omega_h} W \left(\frac{\partial u}{\partial x}, \frac{1}{d} \frac{\partial u}{\partial y} \right) dx dy + \int_0^1 \sqrt{\frac{1}{d^2} + |h'|^2} dx.$$

In this regime, the surface energy is the dominating term, and the limit functional is given by

$$\bar{F}(h) := \int_0^1 |h'| + 2\mathcal{H}^1(\Gamma_{cuts}).$$

Proposition 4.2. *Suppose that W satisfies (H1) and (H2), and let $\{(u_d, h_d)\}$ be such that $\sup_d F_d(u_d, h_d) < +\infty$. Then the sets $\mathbb{R}^2 \setminus \Omega_{h_d}$ converge $\mathbb{R}^2 \setminus \Omega_h$ in the Hausdorff metric, where $h(x) := \inf \{\liminf h_d(x_d) : x_d \rightarrow x\}$ and*

$$\liminf_{d \rightarrow +\infty} F_d(u_d, h_d) \geq \bar{F}(h).$$

Moreover, for every nonnegative lower semicontinuous function h with bounded pointwise variation and $\int_0^1 h(x) dx = 1$, there exists a sequence $\{(u_d, h_d)\}$ where $h_d : I \rightarrow \mathbb{R}$ are non-negative Lipschitz functions with $\int_0^1 h_d(x) dx = 1$, $h_d(0) = h_d(1) = 0$, and $u_d \in W^{1,p}(\Omega_{h_d}; \mathbb{R}^2)$, such that $\mathbb{R}^2 \setminus \Omega_{h_d}$ converge to $\mathbb{R}^2 \setminus \Omega_h$ and

$$\limsup_{d \rightarrow +\infty} F_d(u_d, h_d) \leq \bar{F}(h).$$

Proof. The compactness and liminf inequality follow from Proposition 2.3. For the limsup inequality, without loss of generality we may assume that h is Lipschitz continuous (see Proposition 2.4). Choose $h_d := h$ and

$$u_d(x, y) := \begin{cases} e_0(x(1 - dy), 0) & \text{if } y \leq \frac{1}{d}, \\ 0 & \text{if } y \geq \frac{1}{d}. \end{cases}$$

Then by the p growth condition (H3) we get

$$\begin{aligned} \int_{\Omega_h} W\left(\frac{\partial u}{\partial x}, \frac{1}{d} \frac{\partial u}{\partial y}\right) dx dy &\leq \int_0^1 \int_0^{1/d} \alpha(e_0^p |(1 - dy, -x)|^p + 1) dx dy \\ &\leq \alpha\left(\frac{e_0^p}{d} \int_0^1 \int_0^1 |(1 - y, x)|^p dy dx + \frac{1}{d}\right), \end{aligned}$$

and thus $\limsup \int_{\Omega_h} W\left(\frac{\partial u}{\partial x}, \frac{1}{d} \frac{\partial u}{\partial y}\right) dx dy = 0$ as $d \rightarrow +\infty$. \square

4.3 The limit case $d = e_0^p$

We now focus on the limit case

$$d = e_0^p \rightarrow +\infty.$$

We expect that, in this regime, elastic and surface energy will compete and interplay in the limit. Dividing the energy by d , we obtain the rescaled energy

$$F_d(u, h) := \int_{\Omega_h} W\left(\frac{\partial u}{\partial x}, \frac{1}{d} \frac{\partial u}{\partial y}\right) dx dy + \int_0^1 \sqrt{\frac{1}{d^2} + |h'|^2} dx.$$

We first give a compactness and lower-bound result.

Proposition 4.3. *Suppose that W satisfies (H1)–(H4). If $\sup F_d(u_d, h_d) < +\infty$ then $\{\mathbb{R}^2 \setminus \Omega_{h_d}\}$ converges in the Hausdorff topology to $\mathbb{R}^2 \setminus \Omega_h$, where $h(x) := \inf \{\liminf h_d(x_d) : x_d \rightarrow x\}$. Denote the boundary layers by*

$$BL_h := \{(x, y) : (x, y) \in [0, 1] \times \mathbb{R}^+, 0 < h(x)\}$$

and

$$BL_h^d := \{(x, y) : (x, y) \in BL_h, y \leq \log(d) < d h_d(x)\}.$$

Then $\{BL_h^d\}$ converges in the local Hausdorff topology to BL_h . Moreover, if $v_d : BL_h^d \rightarrow \mathbb{R}^2$ is defined by $v_d(x, y) := \frac{1}{e_0} u_d(x, \frac{y}{d})$, then there exists $v \in W^{1,p}(BL_h; \mathbb{R}^2)$ with $v(x, 0) = (x, 0)$, $v_d \chi_{BL_h^d} \rightharpoonup v$ locally weakly in $W_{loc}^{1,p}(BL_h; \mathbb{R}^2)$, and

$$\liminf_{d \rightarrow +\infty} F_d(u_d, h_d) \geq \int_{BL_h} W^\infty(\nabla v) dx dy + \int_0^1 |h'| + 2\mathcal{H}^1(\Gamma_{cuts}).$$

Proof. Since $|\Omega_{h_d}| = 1$ and $\left\{ \int_0^1 \sqrt{\frac{1}{d^2} + |h'|^2} dx \right\}$ is uniformly bounded, the compactness and lower-semicontinuity for the surface part follow from Proposition 2.3, and the convergence of $\{BL_h^d\}$ follows from the Hausdorff convergence of $\mathbb{R}^2 \setminus \Omega_{h_d}$.

We now focus on the elastic energy. Changing variables $z = dy$ and dividing by e_0 , we get

$$\int_{BL_h^d} \frac{W(e_0 \nabla v_d)}{e_0^p} dx dy \leq C$$

which, by the p -growth condition (H3), implies that $\|\nabla v_d\|_{L^p(BL_h^d \cap \{y \leq M\})}$ is bounded for every M . Hence, as in Lemma 2.12, $\|v_d\|_{L^p(BL_h^d \cap \{y \leq M\})}$ is also bounded, and so there exists a subsequence that converges locally to $v \in W^{1,p}(BL_h)$ with $v(x, 0) = (x, 0)$. The liminf inequality follows from the weak convergences, just as in Lemma 2.13. \square

Remark 4.4. A more precise description of the limit energy takes into account also the elastic energy outside the boundary layer. Let $I_y := \{t \in I : (t, y) \in \Omega_{h_d}\}$, and set

$$\psi_d(x, y) := u(t(x), y),$$

where $t(x)$ is the right endpoint of the connected component of I_y containing x . Then, by Poincaré's inequality,

$$\int_{I_y} |u(x, y) - \psi_d(x, y)|^p dx \leq \int_{I_y} \left| \frac{\partial u}{\partial y}(x, y) \right|^p dx.$$

Integrating this inequality we get

$$\int_{\Omega_{h_d} \cap \{y \geq \log(d)/d\}} |u - \psi_d|^p dx dy \leq \int_{\Omega_{h_d} \cap \{y \geq \log(d)/d\}} \left| \frac{\partial u_d}{\partial x} \right|^p dx dy.$$

Define $\tilde{u}_d := u_d - \psi_d$. Since $\frac{\partial \psi_d}{\partial x} = 0$, the sequences $\{\tilde{u}_d\}$ and $\left\{ \frac{\partial \tilde{u}_d}{\partial x} \right\}$ are bounded in L^p , and this implies that there exists a subsequence (not relabeled) such that $\tilde{u}_d \rightharpoonup u \in L^p(\Omega_h)$ and $\frac{\partial \tilde{u}_d}{\partial x} \rightharpoonup \frac{\partial u}{\partial x} \in L^p(\Omega_h)$. Now, since $\left| \frac{1}{d} \frac{\partial u_d}{\partial y} \right|_{L^p}$ is bounded, we have that $\frac{1}{d} \frac{\partial u_d}{\partial y} \rightharpoonup w \in L^p(\Omega_h)$, and in view of the fact that for every smooth φ with compact support in Ω_h

$$\begin{aligned} \int_{\Omega_h} w \frac{\partial \varphi}{\partial x} dx dy &= \lim_{d \rightarrow +\infty} \int_{\Omega_{h_d} \cap \{y \geq \log(d)/d\}} \frac{1}{d} \frac{\partial u_d}{\partial y} \frac{\partial \varphi}{\partial x} dx dy \\ &= \lim_{d \rightarrow +\infty} \int_{\Omega_{h_d} \cap \{y \geq \log(d)/d\}} \frac{1}{d} \frac{\partial u_d}{\partial x} \frac{\partial \varphi}{\partial y} dx dy = 0 \end{aligned}$$

we find that $\frac{\partial w}{\partial x} = 0$. Note that we could have add to the limiting energy an elastic part depending on $\frac{\partial u}{\partial x}$ and w . We decided not to include this in the limit since it is expected that for minimizing sequences, the elastic energy is only contained in the boundary layer.

We are now in position to prove the limsup inequality:

Proposition 4.5. *Suppose W satisfies (H1)–(H4). For every pair (v, h) with $v \in W^{1,p}(BL_h; \mathbb{R}^2)$, $v(x, 0) = (x, 0)$, and $h : I \rightarrow \mathbb{R}$ a nonnegative lower semicontinuous function of bounded pointwise variation, $\int_0^1 h(x) dx = 1$, there exists a sequence $\{u_d\} \subset W^{1,p}(\Omega_h; \mathbb{R}^2)$ with $u_d(x, 0) = e_0(x, 0)$, such that $\frac{1}{e_0} u_d(x, \frac{y}{d}) \rightharpoonup v$ locally weakly in $W^{1,p}(BL_h; \mathbb{R}^2)$ and*

$$\lim_{d \rightarrow +\infty} F_d(u_d, h) = \int_{BL_h} W^\infty(\nabla v) dx dy + \int_0^1 |h'| + 2\mathcal{H}^1(\Gamma_{cuts}).$$

Proof. We may assume that h is Lipschitz continuous (see Proposition 2.4), and that v has bounded support. Define the recovery sequence by

$$u_d(x, y) := \begin{cases} e_0 v(x, dy) & \text{if } y \leq \frac{\log(d)}{d}, \\ e_0 v(x, \log(d)) \left(2 - \frac{d}{\log(d)} y\right) & \text{if } \frac{\log(d)}{d} \leq y \leq \frac{2\log(d)}{d}, \\ 0 & \text{if } y \geq \frac{2\log(d)}{d}. \end{cases}$$

□

We now characterize the minimizers of the limiting functional.

Proposition 4.6. *Assume that W satisfies (H1)–(H4), and let C_W be defined as in (9). If $C_W \geq 1$, then the minimum of*

$$\int_{BL_h} W^\infty(\nabla v) dx dy + \int_0^1 |h'| + 2\mathcal{H}^1(\Gamma_{cuts})$$

is attained for $h = \frac{1}{\ell_{\min}} \chi_{[0, \ell_{\min}]}$ with $\ell_{\min} = \left(\frac{1}{C_W}\right)^{1/3}$, and v the minimizer of the elastic energy in the corresponding boundary layer BL_h . If $C_W < 1$, then the flat configuration is minimizing.

Proof. We note that h has to be constant on each connected component of $\{h \neq 0\}$. Indeed, given an interval $[a, b]$ and a volume d , the rectangle is the shape with least vertical perimeter (i.e. minimal $\int_a^b |h'|$) for a given volume d . In the boundary layer, v has to be chosen to be the minimizer of the elastic energy. If $h = \sum d_i \chi_{[a_i, b_i]}$, then set $\ell_i := b_i - a_i$ and $h_i := \frac{d_i}{\ell_i}$. The minimal energy is given by

$$\min_{\sum_i h_i \ell_i = 1} \frac{C_W}{2} \sum_i \ell_i^2 + \sum_i h_i.$$

Assume, for the sake of contradiction, that two of the ℓ_i are non-zero, say $\ell_1 \geq \ell_2 > 0$. For $\eta \in \left[-h_1, \frac{\ell_2}{\ell_1} h_2\right]$ consider $h_1 + \eta$ and $h_2 - \eta \frac{\ell_1}{\ell_2}$. Since (ℓ_i, h_i) is minimizing and $(h_1 + \eta) \ell_1 + \left(h_2 - \eta \frac{\ell_1}{\ell_2}\right) \ell_2 = h_1 \ell_1 + h_2 \ell_2$, we find that

$$\eta - \eta \frac{\ell_1}{\ell_2} \geq 0 \quad \text{for all } \eta \in \left[-h_1, \frac{\ell_2}{\ell_1} h_2\right],$$

and hence $\ell_1 = \ell_2$, from which we deduce that $\ell_i = \ell$ for every i . The minimization problem then reduces to

$$\min_{\ell \leq 1} \frac{C_W}{2} N \ell^2 + \frac{1}{\ell},$$

where N is the number of intervals where $h \neq 0$. It is optimal to take $N = 1$ and $\ell_{\min} = \min \left\{ 1, \left(\frac{1}{C_W} \right)^{1/3} \right\}$.

□

Remark 4.7. Notice that when $W^\infty(A) = |A|^2$, then

$$C_W \leq \min_{u(x,0)=(x,0)} \int_0^1 \int_0^{+\infty} |\nabla u|^2 dy dx \leq \int_0^1 \int_0^1 (1-y)^2 + x^2 dx dy = \frac{2}{3} < 1,$$

where we used as a test function for the minimizing problem, $u(x, y) := (x(1-y) \chi_{y \leq 1}, 0)$. In particular $\ell_{\min} = 1$, and the flat configuration is optimal.

4.4 The elastic dominant regime $F_{d,e_0} \simeq e_0^{p/3} d^{2/3}$

We now turn to the case in which $e_0 \rightarrow +\infty$, $\frac{e_0^p}{d} \rightarrow +\infty$ and $e_0^p d^2 \rightarrow +\infty$, where no uniform control is available for the surface energy. In this regime, the main contribution to the energy comes from the elastic part. The relevant parameter is $\eta := \left(\frac{d}{e_0^p} \right)^{1/3} \rightarrow 0$, so that the energy scales like $\frac{d}{\eta}$. Notice that this regime also allows $d \rightarrow 0$. We thus rescale the energy $F_{d,e_0}(u, h)$ and consider

$$F_\eta(u, h) := \eta \left[\int_{\Omega_h} W \left(\frac{\partial u}{\partial x}, \frac{1}{d} \frac{\partial u}{\partial y} \right) dx dy + \int_0^1 \sqrt{\frac{1}{d^2} + |h'|^2} dx \right].$$

We first give a compactness and a liminf inequality result.

Proposition 4.8. Let $\{(u_\eta, h_\eta)\}$ be such that $\sup F_\eta(u_\eta, h_\eta) \leq C$, and set $\mu_\eta := h_\eta dx$. Then there exists a subsequence (not relabeled) such that $\{\mu_\eta\}$ weak-* converges to $\mu := \sum_{i=1}^{+\infty} d_i \delta_{c_i}$ where $d_i > 0$ satisfy $\sum_{i=1}^{+\infty} d_i = 1$. Moreover, there holds

$$\liminf_{\eta \rightarrow 0} F_\eta(u_\eta, h_\eta) \geq 3C_W^{1/3} \sum_{i=1}^{+\infty} d_i^{2/3}.$$

Proof. Since μ_η are probability measures, there exist a subsequence and a probability measure μ such that μ_η weakly-* converges to μ . We use the notation from the proof of Theorem 3.1 (see Figure 3(b)), i.e., $y_0 := \frac{1}{2\sqrt{e_0^{p/3} d^{2/3}}}$, $\ell := \mathcal{H}^1(\Omega_{h_\eta} \cap (I \times \{y_0\}))$. Let $I_\ell^\eta := \Omega_{h_\eta} \cap (I \times \{y_0\})$, $I_\ell^\eta = \cup_{i=1}^{+\infty} [a_i^\eta, b_i^\eta]$, be the decomposition into connected components, $\ell_i^\eta := b_i^\eta - a_i^\eta$, $d_i^\eta := |\Omega_{h_\eta} \cap ([a_i^\eta, b_i^\eta] \times [y_0, +\infty))|$, and assume that the d_i^η are ordered in a decreasing way.

Notice that $1 - \sum_i d_i^\eta \leq y_0$, and thus $\lim_{\eta \rightarrow 0} \sum_i d_i^\eta = 1$. Let finally $d_i := \lim_{\eta \rightarrow 0} d_i^\eta$ (which we can assume exists for every $i \in \mathbb{N}$ up to further extraction of a subsequence). Since $F_{d, e_0} \simeq e_0^{p/3} d^{2/3}$, by the lower bound for the energy in Theorem 3.1, we deduce that $\max_i \ell_i^\eta \leq C\eta$ and thus $(b_i^\eta - a_i^\eta) \rightarrow 0$ for all i . Hence, we may assume that for some $c_i \in [0, 1]$, $a_i^\eta \rightarrow c_i$ and $b_i^\eta \rightarrow c_i$. Let $\Omega_i^\eta := \Omega_{h_\eta} \cap ([a_i^\eta, b_i^\eta] \times [0, +\infty))$ and

$$F_\eta(u_\eta, h_\eta, \Omega_i^\eta) := \eta \int_{a_i^\eta}^{b_i^\eta} \sqrt{\frac{1}{d^2} dx + |h'|^2} dx + \eta \int_{\Omega_i^\eta} W\left(\frac{\partial u}{\partial x}, \frac{1}{d} \frac{\partial u}{\partial y}\right) dx dy.$$

Then for every i (see (2))

$$\begin{aligned} F_\eta(u_\eta, h_\eta, \Omega_i^\eta) &\geq \eta \int_{a_i^\eta}^{b_i^\eta} |h'_\eta| dx + \eta \int_{[a_i^\eta, b_i^\eta] \times [0, y_0]} W\left(\frac{\partial u}{\partial x}, \frac{1}{d} \frac{\partial u}{\partial y}\right) dx dy \\ &\geq 2\eta \frac{d_i^\eta}{\ell_i^\eta} + \frac{\eta}{d} \min_{u(x,0)=e_0(x,0)} \int_{[a_i^\eta, b_i^\eta] \times [0, dy_0]} W(\nabla u) dx dy \\ &\geq 2\eta \frac{d_i^\eta}{\ell_i^\eta} + \frac{\eta}{d} e_0^p (\ell_i^\eta)^2 \min_{v(x,0)=(x,0)} \int_{[0,1] \times [0, \frac{dy_0}{\ell_i^\eta}]} \frac{W(e_0 \nabla v)}{e_0^p} dx dy \\ &\geq 2\eta \frac{d_i^\eta}{\ell_i^\eta} + \frac{\eta}{d} e_0^p (\ell_i^\eta)^2 \min_{v(x,0)=(x,0)} \int_{[0,1] \times [0, \sqrt{e_0^{p/3} d^{2/3}}]} \frac{W(e_0 \nabla v)}{e_0^p} dx dy \\ &= 2\eta \frac{d_i^\eta}{\ell_i^\eta} + \frac{\eta}{d} e_0^p (\ell_i^\eta)^2 C_W (1 - \psi(\eta)) \\ &\geq 3C_W^{1/3} (1 - \psi(\eta))^{1/3} (d_i^\eta)^{2/3}, \end{aligned}$$

where $\psi(\eta) \rightarrow 0$ as $\eta \rightarrow 0$. Summing over i and letting $\eta \rightarrow 0$, we obtain the liminf inequality.

Now, for every $\varepsilon > 0$, let $V^\varepsilon := \{i \in \mathbb{N} / d_i^\eta < \varepsilon\}$. Then

$$\begin{aligned} \sum_{i \in V^\varepsilon} d_i^\eta &= \sum_{i \in V^\varepsilon} (d_i^\eta)^{1/3} (d_i^\eta)^{2/3} \leq \varepsilon^{1/3} \sum_{i \in V^\varepsilon} (d_i^\eta)^{2/3} \\ &\leq C\varepsilon^{1/3} \sum_{i \in V^\varepsilon} F_\eta(u_\eta, h_\eta, \Omega_i^\eta) \\ &\leq C\varepsilon^{1/3} F_\eta(u_\eta, h_\eta) \leq C\varepsilon^{1/3}. \end{aligned}$$

The number of islands such that $d_i^\eta > \varepsilon$ is uniformly bounded by some constant $N_\varepsilon \leq \frac{1}{\varepsilon}$. For fixed $\varepsilon > 0$ let $I^\varepsilon := (\cup_{i \in V^\varepsilon} [a_i^\eta, b_i^\eta])^c$ and $\mu_\eta^\varepsilon := h_\eta \chi_{I^\varepsilon} dx$. Then $\{\mu_\eta^\varepsilon\}$ converges weakly-* to $\mu^\varepsilon := \sum_{i=1}^{N_\varepsilon} d_i \delta_{c_i}$. Finally $\mu_\varepsilon \rightarrow \mu$ since for every $\phi \in C([0, 1])$

$$|(\mu^\varepsilon - \mu)(\phi)| = \lim_{\eta \rightarrow 0} \int_{I^\varepsilon} h_\eta \phi dx \leq C |\phi|_\infty \varepsilon^{1/3}.$$

Since μ_ε weakly-* converges $\sum_{i \in \mathbb{N}} d_i \delta_{c_i}$, this ends the proof. \square

Remark 4.9. In contrast to [13], the structure of the limiting measure seems not to follow directly from the Second Concentration-Compactness Lemma by P.L. Lions (see [38]), but rather from the scaling of the energy.

We finally prove the limsup inequality result in this regime.

Proposition 4.10. *If $\mu := \sum_{i=1}^{+\infty} d_i \delta_{c_i}$ then there exist a sequence $\{(u_\eta, h_\eta)\}$ of functions $u_\eta \in W^{1,p}(\Omega_{h_\eta}; \mathbb{R}^2)$, and nonnegative Lipschitz functions h_η such that $\mu_\eta := h_\eta dx$ are probability measures that weakly-* converge to μ , and*

$$\limsup_{\eta \rightarrow 0} F_\eta(u_\eta, h_\eta) \leq 3C_W^{1/3} \sum_{i=1}^{+\infty} d_i^{2/3}.$$

Proof. Every measure $\mu = \sum_{i=1}^{+\infty} d_i \delta_{c_i}$ can be approximated in energy by the measures $\mu_N := \sum_{i=1}^N d_i \delta_{c_i}$, and by slightly moving the points c_i , we may assume, without loss of generality, that none of them is 0 or 1. For these measures, we construct a recovery sequence as follows: Set $\ell_i := \left(\frac{d_i}{C_W}\right)^{1/3} \eta$, $h_i := \frac{d_i}{\ell_i}$, and let h_η a Lipschitz function very close to $\sum_{i=1}^N h_i \chi_{(c_i - \ell_i/2, c_i + \ell_i/2)}$, and let u_η be the minimizer of the elastic energy in Ω_{h_η} . \square

Notice that the minimizer of the limit functional, i.e.,

$$\min \left\{ \sum_{i=1}^{+\infty} d_i^{2/3} : \sum_{i=1}^{+\infty} d_i = 1 \right\}$$

is given by a single Dirac mass, i.e. $d_1 = 1$ and $d_i = 0$ for $i > 1$.

Remark 4.11. (i) As already noted above, the fact that the limiting energy charges only Dirac masses can be interpreted as a confirmation of the formation of islands when the mismatch is large compared to the thickness of the film.

(ii) The idea of looking to the limit in the space of measure instead of in some functional space has already been extensively used in the study of the Ohta-Kawasaki model (see [13, 29]), and in the study of vortices in Ginzburg-Landau model (see [32, 40]), but is also behind the idea of the blow-up method of Fonseca and Müller (see [22]).

5 Scaling law in dimension three for the Dirichlet energy

In this section, we show that in higher dimension, a scaling law similar to the one of Theorem 3.1 holds. Let $D = D(0, 1) \subset \mathbb{R}^2$ be the unit disk (but it could be replaced by any smooth bounded domain) and consider

$$F_{d,e_0}(u, h) := \int_{\Omega_h} |\nabla u|^2 dx dy + \int_D \sqrt{1 + |\nabla h|^2} dx ,$$

where $\Omega_h := \{(x, y) : x \in D, 0 \leq y \leq h(x)\}$, $u(x, 0) = e_0(x, 0)$ for all $x \in D$, and $\int_D h \, dx = d$. The following scaling law holds.

Proposition 5.1. *Let $d \geq 2$ then there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \max \left\{ 1, d, e_0^{1/2} d^{3/4} \right\} \leq \inf F_{d, e_0} \leq C_2 \max \left\{ 1, d, e_0^{1/2} d^{3/4} \right\}.$$

Proof. The upper bound is very similar to the two dimensional case, replacing rectangles by cylinders.

For the lower bound, we proceed along the lines of the proof of Theorem 3.1. We assume that $e_0^{1/2} d^{3/4} \geq \max(1, d)$. Let $\Omega := \{x \in D : h(x) > 1\}$. Then $|\Omega_h \cap \{y \geq 1\}| \geq 1$ and thus

$$F(u, h) \geq \int_{\Omega \times [0, 1]} |\nabla u|^2 \, dxdy + \int_{\Omega} |\nabla h| \, dx \geq \int_{\Omega \times [0, 1]} |\nabla u|^2 \, dxdy + d\lambda_1(\Omega), \quad (10)$$

where $\lambda_1(\Omega)$ is the Cheeger constant of Ω defined as (see for instance [3])

$$\lambda_1(\Omega) := \min_{\int_{\Omega} h \, dx = 1} \int_{\Omega} |\nabla h| \, dx + \int_{\partial\Omega} |h| \, dx = \min_{E \subset \Omega} \frac{P(E)}{|E|},$$

where $P(E)$ denotes the perimeter of the set E . For a fixed domain $\Omega \subset D$, we want to estimate from below

$$\min_{u(x, 0) = (x, 0)} \int_{\Omega \times [0, 1]} |\nabla u|^2 \, dxdy.$$

Let us notice that by reducing Ω we can assume that Ω is calibrable meaning that $\lambda_1(\Omega) = \frac{P(\Omega)}{|\Omega|}$. Let us then consider the minimizer u of this problem which satisfies,

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \times (0, 1) \\ u(x, 0) = (x, 0) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega \times (0, 1) \cup \Omega \times \{1\}. \end{cases}$$

Let μ_k^2 be the increasing eigenvalues of the Laplacian in Ω with Neumann boundary conditions (so that $\mu_1 = 0$) and let ϕ_k be the associated eigenfunctions (normalized so that $\int_{\Omega} \phi_k^2 = 1$). We can then decompose u on the basis ϕ_k and get

$$u(x, y) = \sum_{k \geq 1} u_k(y) \phi_k(x)$$

where u_k is a vector valued function. From the equation we get

$$u_k'' - \mu_k^2 u_k = 0 \quad u_k(0) = u_k \quad u_k'(1) = 0$$

where $u_k = \int_{\Omega} u(x, 0) \phi_k \, dx$ and thus $u_k(y) = \alpha \exp(\mu_k y) + \beta \exp(-\mu_k y)$ so that we find

$$u_k(y) = \frac{\exp(-2\mu_k)}{1 + \exp(-2\mu_k)} u_k \exp(\mu_k y) + \frac{1}{1 + \exp(-2\mu_k)} u_k \exp(-\mu_k y)$$

We then get that (using that $\int_{\Omega} |\nabla \phi_k|^2 dx = \mu_k^2$)

$$\int_{\Omega \times [0,1]} |\nabla u|^2 dx dy = \sum_{k \geq 1} \int_0^1 |u_k(y)|^2 + \mu_k^2 |u_k(y)|^2 dy = \sum_{k \geq 2} (1 - \exp(-2\mu_k)) \mu_k |u_k|^2. \quad (11)$$

By Cheeger's inequality (see [37]), for $k \geq 2$

$$\mu_k \geq \mu_2 \geq \frac{1}{4} \lambda_1(\Omega)$$

and since $\Omega \subset D$, we also have $\lambda_1(\Omega) \geq \lambda_1(D)$ and thus

$$\int_{\Omega \times [0,1]} |\nabla u|^2 dx dy \geq \left(1 - \exp\left(-\frac{\lambda_1(D)}{2}\right)\right) \frac{\lambda_1(\Omega)}{4} \sum_{k \geq 2} |u_k|^2.$$

By Plancherel formula, there holds

$$\sum_{k \geq 2} |u_k|^2 = \int_{\Omega} |x|^2 dx - \frac{1}{|\Omega|} \sum_{i=1}^n \left(\int_{\Omega} x_i dx \right)^2.$$

Since

$$\int_{\Omega} |x|^2 dx - \frac{1}{|\Omega|} \sum_{i=1}^n \left(\int_{\Omega} x_i dx \right)^2 = \frac{1}{2|\Omega|} \int_{\Omega \times \Omega} |x - z|^2 dx dz,$$

equation (11) transforms into

$$\int_{\Omega \times [0,1]} |\nabla u|^2 dx dy \geq C \lambda_1(\Omega) \frac{1}{|\Omega|} \int_{\Omega \times \Omega} |x - z|^2 dx dz,$$

where $C := \frac{1}{8}(1 - \exp(-\frac{\lambda_1(D)}{2}))$. It thus suffices to show the lower bound

$$e_0^2 \lambda_1(\Omega) \frac{1}{|\Omega|} \int_{\Omega \times \Omega} |x - z|^2 dx dz + d \lambda_1(\Omega) \geq C e_0^{1/2} d^{3/4}.$$

For that, we employ the interpolation inequality (recall that $\lambda_1(\Omega) = \frac{P(\Omega)}{|\Omega|}$)

$$P(\Omega)^{4/5} \left(\int_{\Omega \times \Omega} |x - z|^2 dx dz \right)^{1/5} \geq C |\Omega|, \quad (12)$$

which corresponds to [33, (5.3)] with $\alpha = -2$ (their proof carries over verbatim to this choice). Finally, dividing (12) by $|\Omega|$, raising it to the power $5/4$, and multiplying it by $e_0^{1/2} d^{3/4}$, yields the desired lower bound. \square

Remark 5.2. (i) Notice that

$$\min_{u(x,0)=(x,0)} \int_{\Omega \times [0,1]} |\nabla u|^2 \, dx dy \simeq |(x,0)|_{H^{1/2}(\Omega)}^2 = \int_{\Omega \times \Omega} \frac{1}{|x-z|} \, dx dz$$

thus (at least formally), the right handside of (10) looks very similar to the Ohta-Kawasaki functional which has received a lot of attention in the last few years (see [33, 15, 13, 29, 2]).

(ii) The proof of Theorem 5.1 can be extended similarly to any space dimension.

(iii) In order to remove the condition $d \geq 2$, we would need the reverse of Cheeger's inequality. Unfortunately, it is known that Buser's inequality does not hold for problems with Neumann boundary conditions (see [37]).

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