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Minimax Critical Points in Ginzburg-Landau Problems with Semi-stiff Boundary Conditions: Existence and Bubbling

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Abstract

Let $\Omega \subset \mathbb{R}^2$ be smooth bounded simply connected. We consider the simplified Ginzburg-Landau energy $E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_\Omega (1 - |u|^2)^2$, where $u : \Omega \rightarrow \mathbb{C}$. We prescribe $|u| = 1$ and $\deg(u, \partial\Omega) = 1$. In this setting, there are no minimizers of E_ε . Using a mountain pass approach, we obtain existence, for large ε , of critical points of E_ε . Our analysis relies on Wente estimates and on the analysis of bubbling phenomena for Palais-Smale sequences.

1 Introduction

We consider a smooth bounded simply connected domain $\Omega \subset \mathbb{R}^2$ and set $\Gamma = \partial\Omega$. Let

$$\mathcal{E} = \{u \in H^1(\Omega; \mathbb{C}); |\operatorname{tr} u| = 1\}.$$

Here, $\operatorname{tr} u$ denotes the trace of u on Γ . If $u \in \mathcal{E}$ and we let $g = \operatorname{tr} u$, then $g \in H^{1/2}(\Gamma; \mathbb{S}^1)$, and therefore we may define the winding number (degree) of g [14, Appendix], denoted by $\deg(u, \Gamma)$ or $\deg(g, \Gamma)$. In particular, for $d \in \mathbb{N}^*$ we may define the class

$$\mathcal{E}_d = \{u \in H^1(\Omega; \mathbb{C}); |\operatorname{tr} u| = 1 \text{ on } \Gamma, \deg(u, \Gamma) = d\}.$$

For $\varepsilon \in (0, \infty]$, we consider the simplified Ginzburg-Landau energy

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_\Omega (1 - |u|^2)^2.$$

Our paper is devoted to the existence of critical points of E_ε in \mathcal{E}_d , with special focus on the case $d = 1$.

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Boundary condition $|\operatorname{tr} u| = 1$ can be regarded as a relaxation of the \mathbb{S}^1 -valued Dirichlet boundary condition. This latter condition was considered in details in the classical work [10], where the asymptotic behavior of critical points of E_ε was studied in the limit $\varepsilon \rightarrow 0$. It was shown, in particular, that zeroes (vortices) of critical points are distant from the boundary, i.e., boundary "repels" vortices. The Ginzburg-Landau equation with the Neumann boundary condition presents another extremal effect – "vortices flow through the boundary outside the domain", more precisely, there are no stable (global) minimizers with vortices [33]. The semi-stiff boundary conditions were introduced and studied in [9], [6][8]. It is intermediate between the Dirichlet and Neumann conditions in the following sense: it is the Dirichlet condition $|u| = 1$ for the modulus and the Neumann for the phase of u on the boundary.

In order to obtain nontrivial (nonconstant) critical points one can prescribe nonzero degree on the boundary. Note that the topological degree is continuous with respect to the strong convergence in $H^1(\Omega; \mathbb{C})$ and one can show that sets \mathcal{E}_d are connected components of \mathcal{E} (in the topology inherited from $H^1(\Omega; \mathbb{C})$). Thus minimizers of E_ε in \mathcal{E}_d (if they exist) are local minimizers in \mathcal{E} .

A first natural issue is existence of minimizers of E_ε in \mathcal{E}_d . When $\varepsilon = \infty$, it is easy to see that the minimizers of E_∞ in \mathcal{E}_1 are precisely the conformal representations of Ω into the unit disc \mathbb{D} ; it is also possible to characterize the minimizers E_∞ in \mathcal{E}_d for $d \geq 2$ (Corollary 3.2). When $\varepsilon < \infty$, E_ε does not attain its minimum in \mathcal{E}_d (Lemma 3.4).

Next natural question is existence of critical points. Our main result is the following.

1.1 Theorem. *There exists some $\varepsilon_0 > 0$ such that, for $\varepsilon > \varepsilon_0$, E_ε has critical points in \mathcal{E}_1 .*

The interesting features of existence/nonexistence of critical points were observed in [9]. Next a nontrivial result on existence of critical points for Ginzburg-Landau functional with semi-stiff boundary conditions was obtained in [25] for annuli by minimization of E_ε with prescribed degrees d on both connected component of the boundary. This result (in the case $d = 1$) was improved and extended in [6] to general doubly connected domains. The existence/nonexistence study for doubly connected domains was completed in [4]. Works [6] and [4] show that the existence of minimizers crucially depends on the upper energy bound obtained by minimizing the Dirichlet energy among \mathbb{S}^1 -valued maps.¹ Namely, if this bound does not exceed a certain threshold then minimizers exist for all ε , otherwise they exist for large ε and do not exist for small ε . It can be conjectured on the basis of aforementioned works that global minimizers with prescribed degrees either do not exist or, if they exist, have no zeroes (vortices). However, in [8] it was shown that critical points with zeroes (vortices) do exist (for small ε) for all prescribed degrees on components of a doubly connected domain. The method in [8] makes use of nontrivial topological structure of energy sublevel sets in the case of doubly connected domain, and critical points found in [8] are local minimizers. This approach extends to general multiply connected domains [22], but cannot be applied to find critical points in simply connected domains. An important tool in the construction of [8] is the approximate bulk degree functional, introduced in that work for doubly connected domains. The latter notion can be generalized to multiply connected domains [22]; however, it does not have an analogue for simply connected domains.

The techniques developed in the works cited above for semi-stiff boundary conditions do not lead to (locally or globally) minimizing solutions in simply connected domains. Thus a natural question arises: can one find minimax (saddle) critical points for semi-stiff boundary conditions? This question motivated the present work.

By contrast with the above references and especially [8], our approach is not based on the direct method: in Theorem 1.1, critical points are minimax type ones; their existence is obtained *via* the Mountain Pass Theorem of Ambrosetti and Rabinowitz [1] combined with an asymptotic analysis of the Palais-Smale sequences. This analysis is rather delicate, since our problem is non compact. Non

¹In turn, this bound can be explicitly expressed in terms of the H^1 -capacity of the domain.

compact problems have been broadly considered in the PDE literature for more than three decades. They include nonlinear problems with critical growth (in particular, the celebrated Yamabe problem), three body problem, Yang-Mills equations, harmonic map problem etc (see e.g. the references in [15]). In non compact problems, analysis of Palais-Smale sequences and the validity of the $(PS)_c$ (Palais-Smale condition at the energy level c) condition of Brezis, Coron and Nirenberg [17] play a crucial role. This analysis is also at the heart of the proof of Theorem 1.1.

Our paper is organized as follows. In Section 2, we recall existence and basic properties of the boundary degree for maps in \mathcal{E} , as well as some applications of the Wentz estimates, which are crucial in the analysis of the Palais-Smale sequences. In Section 3, we prove (non)existence of minimizers of E_ε . Section 4 gives the structure of maps close (in a suitable sense) to Moebius transforms, and more generally to Blaschke products. Results in Section 4 are essential for establishing the presence of a mountain pass geometry; this is achieved in Section 5. In Section 5, we rely on the Mountain Pass Theorem in order to obtain sequences of almost critical points of E_ε and an almost critical level $c > \pi$. In Section 6 we prove Theorem 1.1. The rather simple proof requires both ε large and $c < 2\pi$. Once existence of critical points is established, we determine in Section 7 their behavior as $\varepsilon \rightarrow \infty$. We next turn to the task of generalizing Theorem 1.1 to larger ε . This relies on a careful analysis of Palais-Smale sequences. Although Theorem 1.1 is about degree 1 and simply connected domains, we found interesting to present in Section 8 the analysis of Palais-Smale sequences when the boundary degree is arbitrary and the domain is allowed to be multiply connected.² We next rely on this analysis in order to obtain a slight generalization of Theorem 1.1; see Theorem 8.13.

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2 Functional setting

We start by specifying some notation used throughout the paper.

²If we specialize to simply connected domains, some of our arguments can be substantially simplified. See e.g. Lemma 2.23 and Remark 2.24.

1. \mathbb{D} , respectively \mathbb{S}^1 , denote the unit disc, respectively the unit circle. More generally, we will denote $\{z \in \mathbb{C}; |z| < r\}$, respectively $\{z \in \mathbb{C}; |z| = r\}$, by \mathbb{D}_r , respectively C_r .
2. ω and Ω will usually denote smooth open subsets of \mathbb{R}^2 .
3. If $\omega \subset \mathbb{R}^2$ is a smooth bounded open set and if $g : \partial\omega \rightarrow \mathbb{C}$, then we let $u(g)$ denote the harmonic extension of g . If in addition $g \in H^{1/2}(\partial\omega; \mathbb{C})$, then we define a semi-norm in $H^{1/2}$ via the formula

$$|g|_{H^{1/2}}^2 = \frac{1}{2} \int_{\omega} |\nabla u(g)|^2. \quad (2.1)$$

4. \wedge stands for the vector product of complex numbers: $(a_1 + ia_2) \wedge (b_1 + ib_2) = a_1 b_2 - a_2 b_1$. Similarly, the notation $u \wedge \nabla v$, with u and v complex-valued functions, denotes the vector-field $u_1 \nabla v_2 - u_2 \nabla v_1$.
5. If $(a, b) \in \mathbb{R}^2$, then $(a, b)^\perp = (-b, a)$. With complex notations, $z^\perp = iz$.
6. \cdot stands for the real scalar product. E.g., we have $(a_1 + ia_2) \cdot (b_1 + ib_2) = a_1 b_1 + a_2 b_2$, and if $u = u_1 + iu_2$ and $v = v_1 + iv_2$ are complex vectors, then $u \cdot v = u_1 \cdot v_1 + u_2 \cdot v_2$.
7. Several function spaces will appear frequently:
 - a) $\mathcal{E} = \{u \in H^1(\Omega; \mathbb{C}); |\operatorname{tr} u| = 1\}$. If, in addition, Ω is simply connected and $d \in \mathbb{Z}$, then $\mathcal{E}_d = \{u \in \mathcal{E}; \deg(u, \partial\Omega) = d\}$.
 - b) When $\Omega = \mathbb{D}$, the two above spaces are denoted \mathcal{G} , respectively \mathcal{G}_d .
 - c) We let $\mathcal{H} = H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ and $\mathcal{H}_d = \{g \in \mathcal{H}; \deg g = d\}$.

We next turn to the description of a functional setting adapted to the study of critical points.

If $g \in H^{1/2}(\mathbb{S}^1; \mathbb{C})$, then the semi-norm $|g|_{H^{1/2}}$ is easily expressed in terms of Fourier coefficients: if we write $g = \sum a_n e^{in\theta}$, then

$$|g|_{H^{1/2}}^2 = \pi \sum |n| |a_n|^2. \quad (2.2)$$

As noticed first by Boutet de Monvel and Gabber [14, Appendix], a map $g \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ has a well-defined winding number (degree), denoted $\deg(g, \mathbb{S}^1)$, or simply $\deg g$. This degree is defined as follows. On the one hand, $C^\infty(\mathbb{S}^1; \mathbb{S}^1)$ is dense in $H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ [14, Appendix].³ On the other hand, if we write in Fourier series a smooth circle-valued map g as $g = \sum a_n e^{in\theta}$, then

$$\deg g = \sum n |a_n|^2. \quad (2.3)$$

Via (2.2), we easily obtain that the degree of smooth maps is continuous with respect the $H^{1/2}$ convergence. This implies that the right side of (2.3) is an integer for each map $g \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$; this integer is the degree of g .

When \mathbb{S}^1 is replaced by a smooth simple closed planar curve Γ , the degree of a map $g \in H^{1/2}(\Gamma; \mathbb{S}^1)$ can be defined using the above procedure: we first establish density of $C^\infty(\Gamma; \mathbb{S}^1)$ into $H^{1/2}(\Gamma; \mathbb{S}^1)$, next prove continuity of the degree of smooth maps for $H^{1/2}$ convergence. An alternative equivalent approach is the following. Let Ω be the interior of Γ and fix some conformal representation $\Phi \in$

³The point here is not density of smooth maps, but density of smooth circle-valued maps.

$C^\infty(\overline{\Omega}; \overline{\mathbb{D}})$. Let $\Psi = \Phi|_\Gamma : \Gamma \rightarrow \mathbb{S}^1$. If $g \in H^{1/2}(\Gamma; \mathbb{S}^1)$, then we have $g \circ \Psi^{-1} \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ and thus we may set

$$\deg(g, \Gamma) = \deg(g \circ \Psi^{-1}, \mathbb{S}^1). \quad (2.4)$$

Formula (2.4) reduces the study of degree to the case where $\Gamma = \mathbb{S}^1$ and $\Omega = \mathbb{D}$. Existence of the degree suffices to describe the functional setting we need. However, in subsequent sections we will use further properties of the degree. For the convenience of the reader, these properties are recalled below. The results we present are well-known to the experts but difficult to find in the literature. We follow mainly unpublished lecture notes of a graduate course of H. Brezis at Paris 6. Other useful references are [14, 19, 20, 13, 18].

2.1 Lemma. 1. $C^\infty(\Gamma; \mathbb{S}^1)$ is dense in $H^{1/2}(\Gamma; \mathbb{S}^1)$.

2. The degree of $H^{1/2}(\Gamma; \mathbb{S}^1)$ maps is continuous with respect to strong $H^{1/2}$ -convergence.

3. The degree of $H^{1/2}(\Gamma; \mathbb{S}^1)$ maps is not continuous with respect to weak $H^{1/2}$ -convergence.

Proof. The first two items rephrase the beginning of this section. For the third one, we let $\Gamma = \mathbb{S}^1$ and proceed as follows: let $g_a(z) = \frac{z-a}{1-az}$, $z \in \mathbb{S}^1$, $a \in (0, 1)$. Then $g_a \rightarrow -1$ as $a \rightarrow 1$ and $\deg g_a = 1$, but $\deg(-1) = 0$. \square

Lack of continuity with respect to weak convergence makes the minimization of E_ϵ in \mathcal{E}_d non trivial.

We continue with a result relating the degree of $H^{1/2}$ maps to the more familiar degree of continuous maps.

2.2 Lemma. Let $u \in H^1(\mathbb{D}; \mathbb{C}) \cap C(\mathbb{D}; \mathbb{C})$. Assume that $1/2 \leq |u| \leq 2$ in a neighbourhood of \mathbb{S}^1 and that $|\text{tr } u| = 1$. Then, for r close to 1, we have

$$\deg(\text{tr } u, \mathbb{S}^1) = \deg(u, C_r). \quad (2.5)$$

In the remaining part of this section, unless if stated otherwise, Ω is a smooth bounded simply connected domain and $\Gamma = \partial\Omega$.

We continue with two useful formulas giving the degree. The first one is merely an interpretation of (2.3).

2.3 Lemma. Let $g \in H^{1/2}(\Gamma; \mathbb{S}^1)$ and let $u \in H^1(\Omega; \mathbb{C})$ be such that $\text{tr } u = g$. Then

$$\deg g = \frac{1}{2i\pi} \left\langle \frac{\partial u}{\partial \tau}, u \right\rangle_{H^{-1/2}, H^{1/2}} = \frac{1}{2\pi} \int_\Gamma u \wedge \frac{\partial u}{\partial \tau} \quad (2.6)$$

and

$$\deg g = \frac{1}{\pi} \int_\Omega \text{Jac } u. \quad (2.7)$$

The second equality in (2.6) is valid only when g is sufficiently smooth, say $g \in C^1$. In this paper, we always use this notation, but the integral has to be understood in the sense of the $H^{1/2} - H^{-1/2}$ duality.

2.4 Corollary. Let $u \in H^1(\Omega; \mathbb{C})$.

1. If $|u| = 1$ on $\partial\Omega$, then

$$\int_{\Omega} |\nabla u|^2 \geq 2\pi |\deg(\text{tr } u, \partial\Omega)|. \quad (2.8)$$

2. If $|u| \geq \rho > 0$ on $\partial\Omega$, then

$$\int_{\Omega} |\nabla u|^2 \geq 2\pi \rho^2 |\deg(\text{tr } u, \partial\Omega)|. \quad (2.9)$$

Proof. The first conclusion above is obtained by combining (2.7) with the pointwise inequality $2|\text{Jac } u| \geq |\nabla u|^2$. The second one is obtained by applying (2.8) to $\Phi(u)$, where $\Phi(z) = \begin{cases} z/\rho, & \text{if } |z| \leq \rho \\ z/|z|, & \text{if } |z| > \rho \end{cases}$. \square

As one may expect, the degree of $H^{1/2}$ maps inherits some well-known properties of the degree of continuous maps.

2.5 Lemma. \deg is locally constant in $H^{1/2}(\Gamma; \mathbb{S}^1)$.

2.6 Lemma. Let $g, h \in H^{1/2}(\Gamma; \mathbb{S}^1)$. Then the following hold.

1. If g is continuous, then the degree of g in the sense $H^{1/2}$ maps is the same as the degree of g in the sense of continuous maps.
2. $\deg(gh) = \deg g + \deg h$.
3. $\deg(g/h) = \deg g - \deg h$.
4. $\deg g = 0 \iff g = e^{i\psi}$ for some $\psi \in H^{1/2}(\Gamma; \mathbb{R})$.
More generally, $\deg g = d \iff g = g_0 e^{i\psi}$ for some $\psi \in H^{1/2}(\Gamma; \mathbb{R})$ and a fixed smooth reference map $g_0 \in C^\infty(\Gamma; \mathbb{S}^1)$ of degree d .

Item 4. above gives a first characterization of degree zero maps. Further characterizations are related to existence of circle-valued extensions.

2.7 Lemma. 1. Let $u \in H^1(\Omega; \mathbb{C})$ be such that $\text{essinf } |u| > 0$. Then we may write $u = \rho e^{i\varphi}$, where $\rho = |u| \in H^1(\Omega; \mathbb{R})$ and $\varphi \in H^1(\Omega; \mathbb{R})$.

2. Let $g \in H^{1/2}(\Gamma; \mathbb{S}^1)$. Then

$$\begin{aligned} \deg g = 0 &\iff g = \text{tr } u \text{ for some } u \in H^1(\Omega; \mathbb{S}^1) \\ &\iff g = \text{tr } u \text{ for some } u \in H^1(\Omega; \mathbb{C}) \text{ such that } \text{essinf } |u| > 0. \end{aligned}$$

3. Let $g \in H^{1/2}(\Gamma; \mathbb{S}^1)$ be such that $\deg g = 0$. Write $g = e^{i\psi}$, with $\psi \in H^{1/2}(\Gamma; \mathbb{R})$. Then

$$H_g^1(\Omega; \mathbb{S}^1) := \{u \in H^1(\Omega; \mathbb{C}); |u| = 1 \text{ and } \text{tr } u = g\} = \{e^{i\varphi}; \varphi \in H^1(\Omega; \mathbb{R}) \text{ and } \text{tr } \varphi = \psi\}$$

and, for $C > 0$,

$$\{u \in H^1(\Omega; \mathbb{C}); \text{essinf } |u| \geq C, \text{tr } u = g\} = \{\rho e^{i\varphi}; \rho, \varphi \in H^1(\Omega; \mathbb{R}), \text{tr } \varphi = \psi, \text{essinf } \rho \geq C\}.$$

Let us also note that, if $u = \rho e^{i\varphi}$ with $\varphi \in H^1$ and $\rho \in H^1 \cap L^\infty$ such that $\text{essinf } \rho > 0$, then we have the identities

$$|\nabla u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \varphi|^2 \quad \text{and} \quad \nabla \varphi = \left(\frac{u}{\rho} \right) \wedge \nabla \left(\frac{u}{\rho} \right). \quad (2.10)$$

As for continuous maps, in a multiply connected domain ω a circle-valued H^1 map u need not have a phase φ as in Lemma 2.7. However, as a consequence of Lemma 2.7 1, if $\text{essinf } |u| > 0$ then we may locally write

$$u = \rho e^{i\varphi}, \quad \text{with } \rho = |u| \text{ and such that (2.10) holds.} \quad (2.11)$$

In particular, $\nabla \varphi$ is globally defined. Global existence of φ itself is governed by the next result, which is the H^1 -counterpart of a well-known property of continuous circle-valued maps.

2.8 Lemma. Let $\omega \subset \mathbb{R}^2$ be a smooth bounded domain. Let $u \in H^1(\omega; \mathbb{C})$ be such that $\text{essinf } |u| > 0$. Then

$$u = |u| e^{i\varphi}, \quad \text{with } \varphi \in H^1(\omega; \mathbb{R}) \iff \deg \left(\frac{u}{|u|}, \gamma \right) = 0, \quad \forall \gamma \text{ component of } \partial\omega.$$

2.9 Corollary. Let $\omega \subset \mathbb{R}^2$ be a smooth bounded domain. Let $u_j \in H^1(\omega; \mathbb{C})$, $j \in \llbracket 1, 2 \rrbracket$, be such that $0 < \text{essinf } |u_j| \leq \text{esssup } |u_j| < \infty$, $j \in \llbracket 1, 2 \rrbracket$. Assume that $\deg \left(\frac{u_1}{|u_1|}, \gamma \right) = \deg \left(\frac{u_2}{|u_2|}, \gamma \right)$, for each component γ of $\partial\omega$.

Then we may write $u_2 = u_1 \eta e^{i\psi}$, where $\eta = \left| \frac{u_2}{u_1} \right| \in H^1(\omega; \mathbb{R})$ and $\psi \in H^1(\omega; \mathbb{R})$.

An immediate consequence of Lemma 2.1 3 is that the class \mathcal{E}_d (and in particular \mathcal{G}_d) is closed with respect to strong H^1 convergence, but it is not closed with respect to the weak H^1 convergence. However, we do have weak closedness in absence of vortices, as explained in the next couple of results.

Let $\omega \subset \mathbb{R}^2$ be smooth and bounded. Let Γ_j , $j \in \llbracket 1, k \rrbracket$, be the components of $\partial\omega$. Consider, for $\lambda > 0$, the class $\mathcal{H}_\lambda := \{u \in H^1(\omega; \mathbb{C}); |u| \geq \lambda\}$. For $u \in \mathcal{H}_\lambda$, let $\mathbf{d}(u) \in \mathbb{Z}^k$, $\mathbf{d}_j(u) := \deg(u/|u|, \Gamma_j)$.

2.10 Lemma. Let $(u_n) \subset \mathcal{H}_\lambda$ be such that $u_n \rightharpoonup u$. Then, for large n , we have $\mathbf{d}(u_n) = \mathbf{d}(u)$.

In particular, for every $\mathbf{d} \in \mathbb{Z}^k$, the class

$$\mathcal{H}_{\lambda, \mathbf{d}} := \{u \in \mathcal{H}_\lambda; \mathbf{d}(u) = \mathbf{d}\}$$

is weakly closed.

Proof. The above lemma follows essentially from the results of White on the existence of homotopical invariants [36], but we present below a simple direct proof.

It is straightforward that, if $u_n \rightharpoonup u$, then $\frac{u_n}{|u_n|} \rightharpoonup \frac{u}{|u|}$. Therefore, we may assume all the maps circle-valued.

Let $f \in C^\infty(\bar{\omega})$. Then we have the identity [6, (2.2)]

$$2 \int_\omega f \text{Jac } u_n = \int_{\partial\omega} f u_n \wedge \frac{\partial u_n}{\partial \tau} + \int_\omega \left(\frac{\partial f}{\partial y} u_n \wedge \frac{\partial u_n}{\partial x} - \frac{\partial f}{\partial x} u_n \wedge \frac{\partial u_n}{\partial y} \right). \quad (2.12)$$

Since u_n is circle-valued, we have $\text{Jac } u_n \equiv 0$ [18]. Combining this fact with (2.6) we find that, if f is such that $f = 1$ on Γ_j and $f = 0$ on $\partial\omega \setminus \Gamma_j$, then (2.12) becomes

$$\deg(u_n, \Gamma_j) = -\frac{1}{2\pi} \int_{\omega} \left(\frac{\partial f}{\partial y} u_n \wedge \frac{\partial u_n}{\partial x} - \frac{\partial f}{\partial x} u_n \wedge \frac{\partial u_n}{\partial y} \right). \quad (2.13)$$

We conclude *via* the fact that the right-hand side of (2.13) is continuous with respect to the weak H^1 convergence of uniformly bounded maps. \square

By combining Lemma 2.10 with Corollary 2.9 and with (2.10), we obtain the following straightforward consequence, whose proof is left to the reader.

2.11 Lemma. Let ω be as above. Let $(u_n) \subset \mathcal{H}_\lambda$ satisfy $u_n \rightharpoonup u$ and $|u_n| \leq \Lambda$.

Then, for large n , we may write $u_n = u \eta_n e^{i\psi_n}$, with $\lambda/\Lambda \leq \eta_n \leq \Lambda/\lambda$ and $\eta_n \rightarrow 1$, $\psi_n \rightarrow 0$ in $H^1(\omega; \mathbb{R})$.

The next result is essentially due to Brezis and Nirenberg [20, Theorem A3.2].

2.12 Lemma. Let $u \in H^1(\Omega; \mathbb{C})$ satisfy $\Delta u \in L^\infty$ and $|\text{tr } u| = 1$. Then

$$\lim_{z \rightarrow \partial\Omega} |u(z)| = 1. \quad (2.14)$$

Proof. When $\Delta u = 0$, (2.14) follows by combining [20, Theorem A3.2] with the embedding $H^{1/2}(\Gamma) \hookrightarrow VMO$.

In the general case, we write $u = v + w$, with v harmonic and $\text{tr } w = 0$. Then $w \in C^0(\overline{\Omega})$ and thus $\lim_{z \rightarrow \partial\Omega} w(z) = 0$. We conclude *via* the fact that $\lim_{z \rightarrow \partial\Omega} |v(z)| = 1$. \square

We will also need the following version of Lemma 2.12.

2.13 Lemma. Let $u_j, u \in H^1(\Omega; \mathbb{C})$ be such that $|\text{tr } u_j| = 1$, $u_j \rightarrow u$ in H^1 and $|\Delta u_j| \leq C$. Then

$$|u_j| \rightarrow |u| \quad \text{uniformly in } \overline{\Omega}. \quad (2.15)$$

Proof. By standard interior estimates, we have $u_j \rightarrow u$ uniformly on compacts of Ω .

In order to describe the boundary behavior of u_j , write, as in the previous lemma, $u_j = v_j + w_j$ and $u = v + w$. The proof of [20, Theorem A3.2] gives the following uniform estimate:

$$\liminf_{z \rightarrow \partial\Omega} \inf_j |v_j(z)| = 1. \quad (2.16)$$

On the other hand, $w_j \rightarrow w$ uniformly in $\overline{\Omega}$, by standard elliptic estimates. We conclude by combining this uniform convergence with (2.16). \square

We next return to our initial problem of finding critical points of E_ε in \mathcal{E}_d and transfer it from Ω to \mathbb{D} . For this purpose, we consider a fixed conformal representation $\Phi : \Omega \rightarrow \mathbb{D}$. Let $w = \text{Jac } \Phi^{-1} \in C^\infty(\overline{\mathbb{D}}; (0, +\infty))$ and set $\beta = \frac{1}{\varepsilon^2} w$. We associate to Φ and ε the energy

$$F_\beta(u) = \frac{1}{2} \int_{\mathbb{D}} |\nabla u|^2 + \frac{1}{4} \int_{\mathbb{D}} \beta(x) (1 - |u|^2)^2, \quad (2.17)$$

and to \mathcal{E} and \mathcal{E}_d the classes $\mathcal{G} = \{u \in H^1(\mathbb{D}; \mathbb{C}); |\operatorname{tr} u| = 1\}$, respectively

$$\mathcal{G}_d = \{u \in H^1(\mathbb{D}; \mathbb{C}); |\operatorname{tr} u| = 1, \deg(u, \mathbb{S}^1) = d\}. \quad (2.18)$$

In view of the conformal invariance of the Dirichlet integral, it is easy to see that finding critical points (or minimizers) of E_ε in \mathcal{E}_d is equivalent to finding critical points (or minimizers) of F_β in \mathcal{G}_d . It will be convenient to consider the energy F_β for more general weights. In what follows, we always assume that

$$\beta \text{ is non negative and essentially bounded.} \quad (2.19)$$

We continue with a brief discussion concerning the critical points of F_β .

2.14 Definition. By a critical point of F_β we mean a solution of

$$\begin{cases} -\Delta u = \beta u(1 - |u|^2) & \text{in } \mathbb{D} \\ |\operatorname{tr} u| = 1 & \text{on } \mathbb{S}^1 \\ \int_{\mathbb{D}} (u \wedge \nabla u) \cdot \nabla \zeta = 0 & \forall \zeta \in H^1(\mathbb{D}) \\ \deg(u, \mathbb{S}^1) = d \end{cases} \quad (2.20)$$

The above definition is motivated by the fact that (2.20) consists of the Euler-Lagrange equations obtained by variations of u with compact support in \mathbb{D} and by variations of the type ue^{tf} , where $f \in C^k(\bar{\mathbb{D}}; \mathbb{R})$.

2.15 Lemma. Let u belong to \mathcal{G}_d . Then u is a critical point if and only if $u \in W^{2,p}(\mathbb{D})$, $p < \infty$, and

$$\begin{cases} -\Delta u = \beta u(1 - |u|^2) & \text{in } \mathbb{D} \\ |\operatorname{tr} u| = 1 & \text{on } \mathbb{S}^1 \\ u \wedge \frac{\partial u}{\partial \nu} = 0 & \text{on } \mathbb{S}^1 \\ \deg(u, \mathbb{S}^1) = d \end{cases} \quad (2.21)$$

In addition, if β is smooth, then so is u .

Proof. Follow the argument in [5, Lemma 4.4]. □

For further use, let us mention the maximum principle, essentially established in [10].

2.16 Lemma. Let $u \in \mathcal{G}$ be a critical point of F_β with respect to its own boundary condition $g \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$. Then $|u| \leq 1$ in \mathbb{D} .

In our analysis we also rely on the following Price Lemma [6, Lemma 1].

2.17 Lemma. Let $\omega \subset \mathbb{R}^2$ be smooth and bounded. Let Γ_j , $j \in \llbracket 1, k \rrbracket$, be the components of $\partial\Omega$. Let $(u_n) \subset H^1(\omega; \mathbb{C})$ satisfy: $u_n \rightharpoonup u$ in $H^1(\omega)$, $|\operatorname{tr} u_n| = 1$, $\deg(u_n, \Gamma_j) \equiv d_j$. Let $\beta \in L^\infty(\omega)$. Then

$$\liminf F_\beta(u_n) \geq F_\beta(u) + \pi \sum_{j=1}^k |d_j - \deg(u, \Gamma_j)|. \quad (2.22)$$

We end this section by recalling one of the important tools in our proofs, the Wentzell estimates [35] in the sharp form of Bethuel and Ghidaglia [11], and some of their applications.

In the remaining part of this section, $\omega \subset \mathbb{R}^2$ is assumed to be smooth and bounded.

2.18 Lemma. Let $f \in H_0^1(\omega; \mathbb{R})$ and $g, h \in H^1(\omega; \mathbb{R})$. Let $u \in W_0^{1,1}(\omega)$ be the solution of $\Delta u = \text{Jac}(g, h)$, where

$$\text{Jac}(g, h) := \nabla g \cdot \nabla^\perp h. \quad (2.23)$$

Then:

1. We have $u \in C(\bar{\omega}) \cap H^1(\omega)$ and

$$\|u\|_{L^\infty} \leq 2\|\nabla g\|_{L^2}\|\nabla h\|_{L^2}, \quad (2.24)$$

$$\|\nabla u\|_{L^2} \leq \sqrt{2}\|\nabla g\|_{L^2}\|\nabla h\|_{L^2}. \quad (2.25)$$

In particular, the map

$$[H^1(\mathbb{D}; \mathbb{R})]^2 \ni (g, h) \mapsto u \in C(\bar{\omega}) \cap H^1(\omega)$$

is continuous.

2. We have

$$\left| \int_\omega f \text{Jac}(g, h) \right| \leq \sqrt{2}\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}\|\nabla h\|_{L^2}. \quad (2.26)$$

2.19 Lemma. Let

$$\mathbf{H} := \{h \in H^1(\omega; \mathbb{R}); \Delta h = 0\}. \quad (2.27)$$

Let $f \in H_0^1(\omega; \mathbb{R})$, $g \in H^1(\omega; \mathbb{R})$ and $h \in \mathbf{H}$. Then

$$\left| \int_\omega f \nabla g \cdot \nabla h \right| \leq C(\omega)\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}\|\nabla h\|_{L^2}. \quad (2.28)$$

Proof. We start with the simpler case where ω is simply connected. If $h \in \mathbf{H}$, let h^* denote the harmonic conjugate of h , normalized by $\int_{\partial\omega} h^* = 0$. Then from (2.23) and since $\nabla h^* = \nabla^\perp h$, we have

$$\int_\omega f \nabla g \cdot \nabla h = - \int_\omega f \text{Jac}(g, h^*). \text{ Using (2.26), we obtain (2.28) with } C(\omega) = \sqrt{2}.$$

We next turn to a multiply connected domain. Let ω_j , $j \in \llbracket 1, k \rrbracket$, be the bounded components of $\mathbb{R}^2 \setminus \bar{\omega}$. Fix $a_j \in \omega_j$ and let $X_j(x) := \frac{1}{2\pi} (\nabla \ln(x - a_j))^\perp$.⁴ Note that each X_j is closed. Let $X := (\nabla h)^\perp$. By a standard mean value argument, we may find curves $\Gamma_j \subset \omega$, $j \in \llbracket 1, k \rrbracket$, such that the following hold:

1. Γ_j is homotopic in $\bar{\omega}$ to $\partial\omega_j$.⁵

$$2. \left| \int_{\Gamma_j} X \cdot \tau \right| \leq C\|\nabla h\|_{L^2}.$$

Let $c_j := \int_{\Gamma_j} X \cdot \tau$ and set $Y := X - \sum_j c_j X_j$. By construction, the vector field Y is closed, and satisfies

$\|Y\|_{L^2} \leq C\|\nabla h\|_{L^2}$ and $\int_{\Gamma_j} Y \cdot \tau = 0$, $\forall j$.⁶ Since the Γ_j 's span the homotopy group $\pi_1(\omega)$, this implies

⁴Recall that $(a, b)^\perp = (-b, a)$.

⁵In particular, these curves span the homotopy group $\pi_1(\omega)$.

⁶The latter property follows from the identity $\int_{\Gamma_j} X_l = \begin{cases} 1, & \text{if } j = l \\ 0, & \text{if } j \neq l \end{cases}$.

that $\int_{\Gamma} Y \cdot \tau = 0$ for each smooth closed curve $\Gamma \subset \omega$. Thus we may write $Y = \nabla u$ for some u , and it is easy to see that $\|\nabla u\|_{L^2} \leq C\|\nabla h\|_{L^2}$. Using (2.26), we find that

$$\left| \int_{\omega} f \nabla g \cdot \nabla h \right| = \left| \int_{\omega} f \text{Jac}(g, u) + \sum_j c_j \int_{\omega} f \nabla g \cdot X_j^{\perp} \right| \leq C(\omega) \|\nabla f\|_{L^2} \|\nabla g\|_{L^2} \|\nabla h\|_{L^2}. \quad \square$$

Recall that given a function f on $\partial\omega$ we let $u(f)$ denote the harmonic extension of f .

2.20 Lemma. If $f, g \in H^{1/2}(\partial\omega)$, then $u := u(fg) - u(f)u(g)$ belongs to $C(\bar{\omega}) \cap H_0^1(\omega)$ and we have

$$\|u\|_{L^\infty} \leq C(\omega) |f|_{H^{1/2}} |g|_{H^{1/2}} \quad (2.29)$$

and

$$\|u\|_{H^1} \leq C(\omega) |f|_{H^{1/2}} |g|_{H^{1/2}}. \quad (2.30)$$

Proof. It suffices to establish the above estimates when f and g are smooth. In this case, u is smooth and we have⁷

$$\begin{cases} -\Delta u = 2\nabla u(f) \cdot \nabla u(g) & \text{in } \omega \\ u = 0 & \text{on } \partial\omega \end{cases}. \quad (2.31)$$

If we multiply (2.31) by u and use (2.28), we find that

$$\int_{\omega} |\nabla u|^2 \leq C(\omega) \|\nabla u\|_{L^2} \|\nabla u(f)\|_{L^2} \|\nabla u(g)\|_{L^2} \leq C(\omega) \|\nabla u\|_{L^2} |f|_{H^{1/2}} |g|_{H^{1/2}},$$

whence (2.30).

On the other hand, by the proof of Lemma 2.19, we may write $\nabla u(g) = -\nabla v - c_j X_j^{\perp}$, with $\|\nabla v\|_{L^2} \leq C|g|_{H^{1/2}}$ and $|c_j| \leq C|g|_{H^{1/2}}$. We find that

$$\Delta u = 2\text{Jac}(u(f), v) - 2\sum_j c_j (\nabla u(f)) \wedge X_j. \quad (2.32)$$

By combining (2.32) with Lemma 2.18 1 and with standard elliptic estimates, we find that

$$\|u\|_{L^\infty} \leq C(\omega) (\|\nabla u(f)\|_{L^2} \|\nabla v\|_{L^2} + \sum |c_j| \|\nabla u(f)\|_{L^2} \|X_j\|_{L^\infty}) \leq C(\omega) |f|_{H^{1/2}} |g|_{H^{1/2}}. \quad \square$$

From the above, if $f_n \rightarrow f$ and $g_n \rightarrow g$ in $H^{1/2}$, then for the corresponding u 's we have $u_n \rightarrow u$ in $H^1(\omega)$. This conclusion can be strengthened as follows.

2.21 Lemma. Let $f, g_n, g \in H^{1/2}(\partial\omega)$ be such that $g_n \rightarrow g$ in $H^{1/2}$. Then $u(fg_n) - u(f)u(g_n) \rightarrow 0$ strongly in $H^1(\omega)$.

As a consequence, if $f_n \rightarrow f$ and $g_n \rightarrow g$ in $H^{1/2}$ then $u(f_n g_n) - u(f_n)u(g_n) \rightarrow 0$ strongly in $H^1(\omega)$.

Proof. We may assume that $g = 0$. Let $u_n := u(fg_n) - u(f)u(g_n)$. By Lemma 2.20, we have $|u_n| \leq C$ and $u_n \rightarrow 0$ in $H^1(\omega)$. Thus $u_n \nabla u(f) \rightarrow 0$ in $L^2(\omega)$. Using (2.31), we find that

$$\int_{\omega} |\nabla u_n|^2 = 2 \int_{\omega} (u_n \nabla u(f)) \cdot \nabla u(g_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad \square$$

⁷For further use, let us note that a byproduct of the proof is that (2.31) still holds for arbitrary f and g .

2.22 Remark. Lemma 2.21 can be rephrased as follows: if $g_n \rightarrow g$, then $\nabla u(f) \cdot \nabla u(g_n) \rightarrow 0$ strongly in $H^{-1}(\omega)$. The above proof leads to the following more general fact: if $u \in H^1(\omega)$ and $v_n \rightarrow v$ in $H^1(\omega)$, with v_n harmonic, then $\nabla u \cdot \nabla v_n \rightarrow \nabla u \cdot \nabla v$ strongly in $H^{-1}(\omega)$.⁸

2.23 Lemma. Let $f, g_n \in H^{1/2}(\partial\omega; \mathbb{S}^1)$. Assume that $g_n \rightarrow 1$ in $H^{1/2}$. Then

$$\int_{\omega} |\nabla u(f g_n)|^2 = \int_{\omega} |\nabla u(f)|^2 + \int_{\omega} |\nabla u(g_n)|^2 + o(1) \quad \text{as } n \rightarrow \infty. \quad (2.33)$$

Equivalently, $|f g_n|_{H^{1/2}}^2 = |f|_{H^{1/2}}^2 + |g_n|_{H^{1/2}}^2 + o(1)$.

Proof. Let $u := u(f)$ and $v_n := u(g_n)$. In view of Lemma 2.21, (2.33) is equivalent to

$$\int_{\omega} |u \nabla v_n + v_n \nabla u|^2 = \int_{\omega} |\nabla u|^2 + \int_{\omega} |\nabla v_n|^2 + o(1). \quad (2.34)$$

Using the maximum principle Lemma 2.16 combined with the fact that $v_n \rightarrow 1$ pointwise and $\nabla v_n \rightarrow 0$ in $L^2(\omega)$, we find that

$$\int_{\omega} (u \nabla v_n) \cdot (v_n \nabla u) = \int_{\omega} \nabla v_n \cdot (v_n \bar{u} \nabla u) \rightarrow 0.$$

By the above, (2.34) amounts to

$$\int_{\omega} |u|^2 |\nabla v_n|^2 + \int_{\omega} |v_n|^2 |\nabla u|^2 = \int_{\omega} |\nabla v_n|^2 + \int_{\omega} |\nabla u|^2 + o(1). \quad (2.35)$$

In turn, (2.35) is easily obtained by combining the following ingredients:

1. $|u(z)| \rightarrow 1$ uniformly as $\text{dist}(z, \partial\omega) \rightarrow 0$ (cf Lemma 2.12).
2. $\int_K |\nabla v_n|^2 \rightarrow 0$ on each compact $K \subset \omega$.
3. $|v_n| \leq 1$ (cf Lemma 2.16).
4. $v_n \rightarrow 1$ uniformly on compacts of ω . □

2.24 Remark. The proof of (2.33) is much simpler when ω is simply connected. Indeed, by conformal invariance of the quantities we consider, we may assume that $\omega = \mathbb{D}$. In this case, if $f = \sum a_n e^{in\theta}$, then Parseval's identity combined with (2.2) yields

$$\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|f(x) - f(y)|^2}{|x - y|^2} dx dy = 4\pi^2 \sum |n| |a_n|^2 = 4\pi |f|_{H^{1/2}}^2.$$

Thus (2.33) amounts to

$$\begin{aligned} \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|f(x)g_n(x) - f(y)g_n(y)|^2}{|x - y|^2} dx dy &= \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|f(x) - f(y)|^2}{|x - y|^2} dx dy \\ &\quad + \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|g_n(x) - g_n(y)|^2}{|x - y|^2} dx dy + o(1). \end{aligned} \quad (2.36)$$

In turn, using the fact that $|f| = |g_n| = 1$, (2.36) is equivalent to

$$\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} F_n \cdot G_n \rightarrow 0, \quad \text{with } F_n(x, y) = \bar{f}(x)g_n(y) \frac{f(x) - f(y)}{|x - y|} \quad \text{and } G_n(x, y) = \frac{g_n(x) - g_n(y)}{|x - y|}. \quad (2.37)$$

We obtain (2.37) using the fact that $F_n \rightarrow 0$ and $G_n \rightarrow 0$ in $L^2(\mathbb{S}^1 \times \mathbb{S}^1)$.

⁸When ω is simply connected, see [23, Lemma 4.2].

3 The basic example

We discuss here the minimization of F_β in \mathcal{G}_d with $d \in \mathbb{N}^*$.⁹

Recall that a Blaschke product is a map of the form

$$B_{\alpha, a_1, \dots, a_d}(z) = \alpha \prod_{j=1}^d \frac{z - a_j}{1 - \overline{a_j}z}, \quad z \in \overline{\mathbb{D}}, d \in \mathbb{N}^*, \alpha \in \mathbb{S}^1, a_j \in \mathbb{D}, \forall j \in \llbracket 1, d \rrbracket.$$

More specifically, we will call such a product a d -Blaschke product. In the special case $d = 1$, a Blaschke products reduces to a Moebius transform

$$M_{\alpha, a}(z) = \alpha \frac{z - a}{1 - \overline{a}z}, \quad z \in \overline{\mathbb{D}}, \alpha \in \mathbb{S}^1, a \in \mathbb{D}.$$

For further use, let us also define $M_a = M_{1, a}$ and the restriction of M_a to \mathbb{S}^1 :

$$N_a : \mathbb{S}^1 \rightarrow \mathbb{S}^1, N_a(z) = M_a(z), \quad z \in \mathbb{S}^1. \quad (3.1)$$

3.1 Lemma. Assume that $\beta = 0$. Then u minimizes F_0 in \mathcal{G}_d if and only if u is a d -Blaschke product.

Proof. We argue as in [5, Section 4.1]. Since $|\nabla u|^2 \geq 2 \text{Jac } u$, we have

$$F_0(u) = \frac{1}{2} \int_{\mathbb{D}} |\nabla u|^2 \geq \int_{\mathbb{D}} \text{Jac } u = \pi \deg(u, \mathbb{S}^1) = \pi d, \quad (3.2)$$

the second equality following from (2.7). Equality in (3.2) requires $|\nabla u|^2 = 2 \text{Jac } u$ a.e., which amounts to u holomorphic. In particular, if $g = \text{tr } u$, then $u = u(g)$.

On the other hand, if $g \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ and if $u = u(g)$ is the harmonic extension of g , then Lemma 2.12 implies

$$\lim_{|z| \rightarrow 1} |u(z)| = 1. \quad (3.3)$$

In order to complete the proof of Lemma 3.1, it suffices to combine (3.3) with the holomorphy of u and with the following well-known result: let $u \in \text{Hol}(\mathbb{D})$.¹⁰ Then

$$\lim_{|z| \rightarrow 1} |u(z)| = 1 \text{ uniformly} \iff u \text{ is a Blaschke product.}$$

For the sake of completeness, we recall the proof of " \implies " (implication " \impliedby " being obvious). Let

z_1, \dots, z_d be the zeroes of u in \mathbb{D} , counted with their multiplicities. Let $v(z) = \prod_{j=1}^d \frac{z - z_j}{1 - \overline{z_j}z}$ and set

$w = \frac{u}{v} \in \text{Hol}(\mathbb{D})$. Then $w \neq 0$ in \mathbb{D} and $\lim_{|z| \rightarrow 1} |w(z)| = 1$ uniformly. Thus $w = e^f$, where $f \in \text{Hol}(\mathbb{D})$ satisfies $\lim_{|z| \rightarrow 1} \text{Re } f(z) = 0$. By the maximum principle, we have $\text{Re } f \equiv 0$, and thus $\text{Im } f \equiv \text{const}$. Therefore, $u = \alpha v$ for some $\alpha \in \mathbb{S}^1$. \square

3.2 Corollary. Let $g \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ have degree $d > 0$. Then $|g|_{H^{1/2}}^2 \geq \pi d$, with equality if and only if g is a d -Blaschke product.

⁹The reader may wonder what happens when $d \leq 0$. When $d = 0$, it is clear that minimizers of F_α are precisely the constants of modulus 1. The case $d < 0$ is obtained from the case $d > 0$ by complex conjugation.

¹⁰We denote by $\text{Hol}(\Omega)$ the class of holomorphic functions in Ω .

Fix now a smooth simply connected domain Ω and a conformal representation $\Phi : \Omega \rightarrow \mathbb{D}$. Then we define a (generalized) d -Blaschke product¹¹ by the formula

$$B_{\alpha, a_1, \dots, a_d, \Phi}(z) = \alpha \prod_{j=1}^d \frac{\Phi(z) - a_j}{1 - \bar{a}_j \Phi(z)}, \quad z \in \bar{\Omega}, d \in \mathbb{N}^*, \alpha \in \mathbb{S}^1, a_j \in \mathbb{D}, \forall j \in \llbracket 1, d \rrbracket,$$

and a (generalized) Moebius transform *via*

$$M_{\alpha, a, \Phi}(z) = \alpha \frac{\Phi(z) - a}{1 - \bar{a} \Phi(z)}, \quad z \in \bar{\Omega}, \alpha \in \mathbb{S}^1, a \in \mathbb{D}.$$

Using Lemma 3.1 and the invariance of the Dirichlet integral under conformal representations, we obtain

3.3 Corollary. The minimizers of E_∞ in \mathcal{E}_d are precisely the d -Blaschke products.

The next results implies, in particular, that the infimum of E_ε in \mathcal{E}_d is not attained when $\varepsilon < \infty$.

3.4 Lemma. Let $\beta \neq 0$. If $d \neq 0$ then F_β does not attain its minimum in \mathcal{G}_d .

Proof. Let $u \in \mathcal{G}_d$. Then $F_\beta(u) \geq F_0(u) = \pi d$. We find that $\inf_{\mathcal{G}_d} F_\beta \geq \pi d$. On the other hand, consider the d -Blaschke product $u = \prod_{j=1}^d \frac{\Phi - a_j}{1 - \bar{a}_j \Phi}$ and let $a_j \rightarrow -1, \forall j$. Then $u \rightarrow 1$ in $H^1(\Omega)$, and in particular $F_\beta(u) \rightarrow \pi d$. We find that $\inf_{\mathcal{G}_d} F_\beta = \pi d$. In order to prove that the minimum is not attained, argue by contradiction: assume that u minimizes F_β in \mathcal{F} . Then $F_\beta(u) = F_0(u) = \pi d$. Thus u is a d -Blaschke product and $\int_\Omega \beta (1 - |u|^2)^2 = 0$. This is impossible, since $|u| < 1$ in \mathbb{D} and thus $\beta (1 - |u|^2)^2 \neq 0$. \square

We end this section with a strong improvement of Lemma 3.1.

3.5 Lemma. The critical points of E_∞ in \mathcal{E}_d are precisely:

- a) the d -Blaschke products if $d > 0$.
- b) the conjugates of $(-d)$ -Blaschke products if $d < 0$.
- c) constants of modulus 1 if $d = 0$.

After our work was completed, we learned that the above result has also been obtained independently by V. Millot and Y. Sire [30].

Proof. Since the equation (2.21) of critical points is invariant by conformal representations, we may assume that $\Omega = \mathbb{D}$.

We rely on the properties of the Hopf differential, for which we send the reader to [26, Chapter 4]. If $u : U \rightarrow \mathbb{C}$ is a harmonic function, where U is a domain in \mathbb{C} , then the function

$$\omega : U \rightarrow \mathbb{C}, \quad \omega(z) = 4(\partial_z u)(\partial_z \bar{u}) = (\partial_x u - i\partial_y u)(\partial_x \bar{u} - i\partial_y \bar{u}),$$

is holomorphic, and $\omega = 0$ is equivalent to either u being holomorphic or u being anti-holomorphic.

¹¹In Ω and with respect to Φ , but this will be tacitly understood in what follows.

Let u satisfy (2.21) with $\beta = 0$. Then $u \in C^\infty(\overline{\mathbb{D}})$, by Lemma 2.15. If we write, locally near some point $z_0 \in \mathbb{S}^1$, $u = \rho e^{i\varphi}$, with ρ and φ smooth, then the boundary conditions in (2.21) give

$$\frac{\partial \rho}{\partial \tau} = 0, \quad \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \mathbb{S}^1. \quad (3.4)$$

Using (3.4), we easily find that

$$z^2 \omega(z) = \left(\frac{\partial \rho}{\partial \nu} \right)^2 - \left(\frac{\partial \varphi}{\partial \tau} \right)^2, \quad \forall z \in \mathbb{S}^1. \quad (3.5)$$

We find that the holomorphic function $z \mapsto z^2 \omega(z)$ is real, thus constant, in $\overline{\mathbb{D}}$. On the other hand, this function vanishes at the origin, so that $\omega = 0$, and therefore u is holomorphic or anti-holomorphic. Since $|u| = 1$ on \mathbb{S}^1 , we obtain the desired conclusion as in the proof of Lemma 3.1. \square

4 Moebius and almost Moebius transforms

In this section, we describe harmonic maps u which are "close" to Moebius maps. Recall that we defined the class

$$\mathcal{H}_d = \{g \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1); \deg g = d\}.$$

Recall that we denoted by N_α the restriction to \mathbb{S}^1 of the Moebius transform $M_{1,\alpha}$. Recall also Corollary 3.2: for $g \in \mathcal{H}_1$ we have $|g|_{H^{1/2}}^2 \geq \pi$, with equality if and only if $g = \alpha N_\alpha$ for some $\alpha \in \mathbb{D}$ and $\alpha \in \mathbb{S}^1$.

4.1 Theorem. *There exists some $\delta_0 > 0$ and a function $f : (0, \delta_0) \rightarrow (0, \infty)$ such that $\lim_{\delta \rightarrow 0} f(\delta) = 0$ with the following property: if $g \in \mathcal{H}_1$ satisfies $|g|_{H^{1/2}}^2 < \pi + \delta$ for some $\delta < \delta_0$, then:*

1. *The harmonic extension $u = u(g)$ of g has exactly one zero, $a = a(u) = a(g)$.*
2. *If we write $g = N_\alpha e^{i\psi}$ with $\psi \in H^{1/2}(\mathbb{S}^1; \mathbb{R})$, then $|\psi|_{H^{1/2}} \leq f(\delta)$.*
3. *The map $g \mapsto a$ is continuous.*
4. *In addition, given $r \in (0, 1)$ and $\mu > 0$, we may pick δ_0 such that the above hold and $\| \alpha u \circ M_{-\alpha} - \text{Id} \|_{C^2(\mathbb{D}_r)} < \mu$ for some appropriate $\alpha \in \mathbb{S}^1$.*

Before proceeding to the proof, let us make two comments. First, using repeatedly Lemma 2.6, we find that $\frac{g}{N_\alpha}$ has degree zero, and thus we may write $g = N_\alpha e^{i\psi}$ for some $\psi \in H^{1/2}(\mathbb{S}^1; \mathbb{R})$. The point in item 2. is that $|\psi|_{H^{1/2}}$ is small when $|g|_{H^{1/2}}^2$ is close to π . Second, an equivalent and possibly more illuminating formulation of item 4. is the following: for sufficiently small δ_0 and for g as above, we have $\| u \circ M_{\alpha,a}^{-1} - \text{Id} \|_{C^2(\mathbb{D}_r)} < \mu$ for some $\alpha \in \mathbb{S}^1$; that is, u is close, in an appropriate sense, to a Moebius transform.

Proof. Step 1. u has to vanish somewhere.

Indeed, otherwise we have $u \in C(\mathbb{D}; \mathbb{C} \setminus \{0\})$ and $\lim_{|z| \rightarrow 1} |u(z)| = 1$. Thus $|u| \geq \alpha > 0$ for some α . By Lemma

2.7, we may write $u = |u| e^{i\varphi}$, with $\varphi \in H^1(\Omega; \mathbb{R})$. The fact that $\text{tr } e^{i\varphi} = g$ combined with the degree formula (2.7) leads to the contradiction

$$1 = \deg(g, \mathbb{S}^1) = \frac{1}{\pi} \int_{\mathbb{D}} \text{Jac}(|u| e^{i\varphi}) = 0.$$

Let now $a \in \mathbb{D}$ be one of the (possibly several) zeroes of u . Set $v = u \circ M_{-a} = u \circ M_a^{-1}$. Then v is harmonic, $v(0) = 0$, and $v|_{\mathbb{S}^1} = h$, with $h = g \circ N_{-a}$. In addition, we have $|h|_{H^{1/2}} = |g|_{H^{1/2}}$, by conformal invariance of the $H^{1/2}$ -semi-norm.

Step 2. Proof of 4.

Argue by contradiction: there are some $\mu > 0$, $r \in (0, 1)$ such that

$$\inf_{\alpha \in \mathbb{S}^1} \|\alpha v_n - \text{Id}\|_{C^2(\mathbb{D}_r)} \geq \mu$$

for some sequence (v_n) of harmonic maps such that

$$\begin{cases} v_n(0) = 0 \\ h_n := v_n|_{\mathbb{S}^1} \text{ has modulus 1 and degree 1.} \\ |h_n|_{H^{1/2}}^2 \leq \pi + \frac{1}{n} \end{cases}$$

We find that, possibly up to a subsequence, $v_n \rightarrow v$ in $C_{loc}^\infty(\mathbb{D})$, and $h_n \rightarrow h \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$. In addition, we have $v = u(h)$. The limit v satisfies

$$\begin{cases} \|\alpha v - \text{Id}\|_{C^2(\mathbb{D}_r)} \geq \mu, \forall \alpha \in \mathbb{S}^1 \\ v(0) = 0 \\ |h|_{H^{1/2}}^2 \leq \pi \end{cases}.$$

We consider first the case where $\deg(h, \mathbb{S}^1) = 1$. Since $|h|_{H^{1/2}}^2 \leq \pi$, Corollary 3.2 combined with $v(0) = 0$ leads to $v = \gamma \text{Id}$ for some $\gamma \in \mathbb{S}^1$ and this is impossible.

Thus $\deg(h, \mathbb{S}^1) \neq 1$. Then the Price Lemma 2.17 gives:

$$\pi = \lim \frac{1}{2} \int_{\mathbb{D}} |\nabla v_n|^2 \geq \frac{1}{2} \int_{\mathbb{D}} |\nabla v|^2 + \pi |\deg(h_n, \mathbb{S}^1) - \deg(h, \mathbb{S}^1)| = \frac{1}{2} \int_{\mathbb{D}} |\nabla v|^2 + \pi |1 - \deg(h, \mathbb{S}^1)|. \quad (4.1)$$

Therefore, v is a constant of modulus 1. This contradicts $v(0) = 0$.

In the remaining part of the proof, we assume that $|g|_{H^{1/2}}^2 \leq \pi + \delta$, where $\delta < \delta_0$ and δ_0 is to be fixed later.

Step 3. For $0 < s < r < 1$ and for sufficiently small δ_0 (depending on s and r), we have $|v| \geq s$ on $\mathbb{D} \setminus \overline{\mathbb{D}}_r$.

Indeed, let $\mu > 0$. By Step 2 we have

$$\frac{1}{2} \int_{\mathbb{D}_r} |\nabla v|^2 \geq \pi r^2 - \mu \text{ and } |v| \geq r - \mu \text{ on } C_r,$$

provided δ_0 is sufficiently small. In particular, we have

$$\frac{1}{2} \int_{\mathbb{D} \setminus \overline{\mathbb{D}}_r} |\nabla v|^2 \leq \pi - \pi r^2 + \mu + \delta_0 \text{ and } |v| \geq r - \mu \text{ on } \partial(\mathbb{D} \setminus \overline{\mathbb{D}}_r). \quad (4.2)$$

Estimate (4.2) combined with [23, Example 3.5 c)] and [23, Theorem 3.6] imply that $|v| \geq r - 2\mu$ in $\mathbb{D} \setminus \overline{\mathbb{D}}_r$, provided δ_0 is small. We conclude by taking $\mu = \frac{r-s}{2}$.

Step 4. The map u has exactly one zero.

Indeed, by combining Step 3 with Step 2, we find that v has exactly one zero for small δ_0 (and that this zero is located near the origin). Whence the conclusion for u .

Step 5. The continuity of $g \mapsto a$.

This follows essentially from uniqueness, but some care is needed, since in principle zeroes could escape to the boundary. Let $g_n \rightarrow g$ in $H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$. Then there is some $r > 0$ independent of n such

$$|u(g_n)| \geq \frac{1}{2} \quad \text{in } \mathbb{D} \setminus \bar{\mathbb{D}}_r, \forall n. \quad (4.3)$$

This uniform version of (2.14) can be easily obtained by following the proof of [20, Theorem A3.2]. The continuity of $g \mapsto a$ is a straightforward consequence of (4.3).

Step 6. We prove 2.

Since $|\cdot|_{H^{1/2}}$ is invariant by composition with restrictions to \mathbb{S}^1 of Moebius transforms, we may assume that $u(0) = 0$. We fix some $r \in (0, 1)$ close to 1. For small δ , we may write, in $\mathbb{D} \setminus \bar{\mathbb{D}}_r$, $u = |u|e^{i(\theta+\varphi)}$, with $1/2 \leq |u| \leq 1$ and $\varphi \in H^1(\mathbb{D} \setminus \bar{\mathbb{D}}_r; \mathbb{R})$. Since $\deg(u, \mathbb{S}^1) = 1$ and $u \in C^\infty$, we have

$$\nabla\varphi = \left(\frac{ue^{-i\theta}}{|u|} \right) \wedge \nabla \left(\frac{ue^{-i\theta}}{|u|} \right) \quad \text{in } \mathbb{D} \setminus \bar{\mathbb{D}}_r.$$

We find that $\|\nabla\varphi\|_{L^2(\mathbb{D} \setminus \bar{\mathbb{D}}_r)} \leq C$, and 2. follows by taking the semi-norm of the trace of φ on \mathbb{S}^1 . \square

We continue with a similar result, where control of the phase ψ at higher energy levels of g is compensated by loss of information on the zero a .

4.2 Theorem. *There exists a function $h : (0, 2\pi) \rightarrow (0, \infty)$ such that: if $g \in \mathcal{H}_1$ satisfies $|g|_{H^{1/2}}^2 \leq 2\pi - \delta$ for some $\delta \in (0, 2\pi)$, then:*

1. *We may write $g = N_a e^{i\psi}$ for some $\psi \in H^{1/2}(\mathbb{S}^1; \mathbb{R})$ such that $|\psi|_{H^{1/2}} \leq h(\delta)$.*
2. *In addition, we may take the point a to be a zero of $u(g)$, and the zeroes of $u(g)$ are mutually close, in the following sense: there exist some $R = R(\delta) \in (0, 1)$ and $\mu = \mu(\delta) \in (0, 1)$ such that, if a is a zero of $u(g)$, then $|u(g)| \geq \mu$ in $\mathbb{D} \setminus M_a^{-1}(\bar{\mathbb{D}}_R)$.*

Before proceeding to the proof, let us note the following more informative form of item 2:

$$\text{if } u(g)(0) = 0, \text{ then } |u(g)| \geq \mu \text{ outside } \bar{\mathbb{D}}_R. \quad (4.4)$$

Proof. As in the proof of Theorem 4.1, Step 6, it suffices to establish item 2 in the special case where $u(g)(0) = 0$. We argue by contradiction: we assume that there exist a sequence $(g_n) \subset \mathcal{H}_1$ and a sequence $(z_n) \subset \mathbb{D}$ such that:

$$|g_n|_{H^{1/2}}^2 \leq 2\pi - \delta, \quad |z_n| \rightarrow 1, \quad u_n := u(g_n) \text{ satisfies } u_n(0) = 0 \text{ and } |u_n(z_n)| \rightarrow 0. \quad (4.5)$$

Possibly after passing to a subsequence, we may also assume that, for some $D \in \mathbb{Z}$, $g \in \mathcal{H}_D$ and $u = u(g)$, we have

$$u_n \rightarrow u \text{ in } C_{loc}^\infty(\mathbb{D}), \quad u(0) = 0. \quad (4.6)$$

We first prove that $D = 1$. Indeed, by the argument leading to (4.1) we have

$$2\pi > 2\pi - \delta \geq \frac{1}{2} \int_{\mathbb{D}} |\nabla u|^2 + \pi|1 - D|. \quad (4.7)$$

By combining (4.7) with Corollary 3.2, we find that $D = 0$ or $D = 1$. If $D = 0$, then $\int_{\mathbb{D}} |\nabla u|^2 \leq \pi - \delta$. This inequality combined with [23, Theorem 3.6], implies the existence of some $\lambda > 0$ such that $|u| \geq \lambda$ in \mathbb{D} ; this contradicts (4.6). Thus $D = 1$.

Let $s \in (0, 1)$ to be fixed later. By Lemma 2.12, there is some $R \in (0, 1)$ such that $|u(z)| > s$ on $C(0, R)$, and the same holds for u_j provided j is large. On the other hand, we have $\deg(u, C_R) = 1$ provided R is chosen sufficiently close to 1; this follows by combining Lemma 2.2 with Lemma 2.12. By (4.6), for large n we have $\deg(u_n, C_R) = 1$. Using (2.9), we find that

$$\frac{1}{2} \int_{\mathbb{D}_R} |\nabla u_n|^2 \geq \pi s^2. \quad (4.8)$$

On the other hand, since $|u_n(z_n)| \rightarrow 0$ and $|u_n| \geq s$ on $\partial(\mathbb{D} \setminus \overline{D}_R)$, we obtain, by applying [23, Theorem 3.6], that

$$\liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{D} \setminus \overline{D}_R} |\nabla u_n|^2 \geq \pi s^2. \quad (4.9)$$

For s sufficiently close to 1, we obtain a contradiction by combining (4.8) with (4.9).

Once existence of R and μ as in item 2 is established, the proof of item 1 is identical to the one of Theorem 4.1, item 2: we repeat Step 6 in the proof of Theorem 4.1, and rely on (4.1). \square

4.3 Remark. It is not clear whether the restriction $|g|_{H^{1/2}}^2 < 2\pi$ is optimal in Theorem 4.2. However, *some* bound is required. Here is an example with a sketch of proof. Let $g = g_{b,c,d} = \frac{N_b N_c}{N_d}$. Then $|g|_{H^{1/2}}^2 \leq 9\pi$. If we let, say $b \rightarrow 1$, $c \rightarrow -1$ and $d \rightarrow \iota$, we claim that there is no $a = a(b, c, d)$ such that Theorem 4.2 1. holds. Argue by contradiction: otherwise, after passing to a subsequence, $a \rightarrow \tilde{a} \in \overline{\mathbb{D}}$. The limit \tilde{a} cannot be close to 1, -1 and ι at the same time; say $\tilde{a} \neq 1$. Then, near $z = 1$, $\frac{g}{N_a}$ is close to αN_b for some $\alpha \in \mathbb{S}^1$. Using the fact that the $H^{1/2}$ -semi-norm of the phase of N_b (computed near $z = 1$) tends to ∞ as $b \rightarrow 1$, and the fact that the phase is unique modulo 2π [13], we find that Theorem 4.2 does not hold for g as above.

The next result explains that, at low energy, the lack of weak compactness of the class \mathcal{H}_1 is due solely to Moebius transforms.

4.4 Corollary. Let $t < 2\pi$ and let $\mathcal{H}_1^t = \{g \in \mathcal{H}_1; |g|_{H^{1/2}}^2 \leq t\}$.

Then \mathcal{H}_1^t is weakly closed modulo Moebius transforms: If $(g_n) \subset \mathcal{H}_1^t$, then there exists N_{a_n} such that the sequence $(g_n \circ N_{a_n}^{-1})$ is weakly compact in \mathcal{H}_1^t .

In addition, we may take a_n to be any zero of $u(g_n)$.

In particular, for each $t < 2\pi$ and $a_0 \in \mathbb{D}$, the class

$$\{g \in \mathcal{H}_1; |g|_{H^{1/2}}^2 \leq t, u(g) \text{ vanishes at } a_0\}$$

is weakly compact.

We continue by presenting another consequence of Theorem 4.2.

4.5 Lemma. Let

$$\beta \in L^\infty(\mathbb{D}), \beta \geq 0, \beta \neq 0. \quad (4.10)$$

Consider a Lebesgue point $a_0 \in \mathbb{D}$ of β at which the approximate limit of β is $b > 0$. Then

$$c(\beta, a_0) := \inf\{F_\beta(v); g = \text{tr } v \in \mathcal{H}_1, u(g) \text{ vanishes at } a_0\} > \pi. \quad (4.11)$$

Proof. If $c(\beta, a) \geq 2\pi$, there is nothing to prove. Otherwise, we apply Corollary 4.4 and obtain that the minimum is attained in (4.11). Argue by contradiction and assume that $c(\beta, a_0) = \pi$. If v attains the minimum in (4.11) then, by Corollary 3.2, $v = M_{\alpha, a}$ and

$$\int_{\mathbb{D}} \beta(1 - |v|^2)^2 = 0. \quad (4.12)$$

Our choice of a implies that

$$\lim_{r \rightarrow 0} \int_{B(a, r)} \beta(1 - |v|^2)^2 = b(1 - |v|^2)^2(a). \quad (4.13)$$

We obtain a contradiction by combining (4.12) and (4.13). \square

Though this will not be used in the subsequent analysis, we found useful to mention that part of Theorem 4.1 still holds for higher degrees.¹²

4.6 Theorem. *Let $d \in \mathbb{N}$. Then there exists some $\delta_0 > 0$ and a function $f : (0, \delta_0) \rightarrow (0, \infty)$ such that $\lim_{\delta \rightarrow 0} f(\delta) = 0$ with the following property: if $g \in \mathcal{H}_d$ satisfies $|g|_{H^{1/2}}^2 < \pi d + \delta$ for some $\delta < \delta_0$, then we may write $g = N_{a_1} \dots N_{a_d} e^{i\psi}$ for some $a_1, \dots, a_d \in \mathbb{D}$ and for some $\psi \in H^{1/2}(\mathbb{S}^1; \mathbb{R})$ such that $|\psi|_{H^{1/2}} \leq f(\delta)$.*

Consequently, if $u \in \mathcal{G}_d$ is a harmonic function and if $\frac{1}{2} \int_{\mathbb{D}} |\nabla u|^2 < \pi d + \delta$, then $u = B_{1, a_1, \dots, a_d} u(e^{i\psi}) + w$, where $\psi \in H^{1/2}(\mathbb{S}^1; \mathbb{R})$, $w \in H_0^1(\mathbb{D}; \mathbb{C})$, $|\psi|_{H^{1/2}} \leq f(\delta)$ and $\|w\|_{H^1} \leq f(\delta)$.

Proof. The last part of the theorem follows by combining the first part of the theorem with Lemma 2.20.

In order to prove the first part, it suffices to establish the following fact: if $(g_n) \subset \mathcal{H}_d$ is such that $|g_n|_{H^{1/2}}^2 \rightarrow \pi d$, then, possibly up to a subsequence and for large n , we may write $g_n = N_{a_1(n)} \dots N_{a_d(n)} e^{i\psi_n}$, with $|\psi_n|_{H^{1/2}} \rightarrow 0$ as $n \rightarrow \infty$. The proof of this fact is by induction on d .

Step 1. The case where $d = 0$.

By Lemma 2.13, we have $|u(g_n)| \rightarrow 1$ uniformly in \mathbb{D} . By Lemma 2.7 and (2.10), for large n we may write $u(g_n) = \rho_n e^{i\varphi_n}$, with $\nabla \varphi_n \rightarrow 0$ in $L^2(\mathbb{D})$. We conclude by letting $\psi_n = \text{tr } \varphi_n$.

Step 2. The case where $d \geq 1$.

By Step 1 in the proof of Theorem 4.1, the map $u(g_n)$ has to vanish somewhere. Since our hypotheses and conclusions are invariant by conformal transforms, we may assume that $u(g_n)(0) = 0$. Up to a subsequence, we have $g_n \rightarrow g$, with $g \in \mathcal{H}_D$ for some $D \in \mathbb{Z}$. By combining (2.8) with the Price Lemma 2.17, we find that $D \in \llbracket 0, d \rrbracket$ and that $|g|_{H^{1/2}}^2 = \pi D$. By Corollary 3.2, $u(g)$ is a D -Blaschke product. Since in addition we have $u(g)(0) = 0$, we find that $D \in \llbracket 1, d \rrbracket$. If $D = d$, then we actually have $g_n \rightarrow g$ strongly in $H^{1/2}$, and thus $g_n/g \rightarrow 1$. In this case, Step 1 implies that $g_n = g e^{i\psi_n}$, with $|\psi_n|_{H^{1/2}} \rightarrow 0$, and we are done.

We next turn to the more delicate case where $D < d$. Let $h_n := g_n/g$, so that $h_n \in \mathcal{H}_{d-D}$ and $h_n \rightarrow 1$. By Lemma 2.23 combined with the fact that $|g|_{H^{1/2}}^2 = \pi D$, we have $|h_n|_{H^{1/2}}^2 \rightarrow \pi(d-D)$. By the induction hypothesis,¹³ we may write, for large n , $h_n = N_{a_1(n)} \dots N_{a_{d-D}(n)} e^{i\psi_n}$, with $|\psi_n|_{H^{1/2}} \rightarrow 0$. We obtain the desired conclusion by noting that $g_n = g h_n$ and that g is the trace of a D -Blaschke product. \square

¹²Or for negative degrees, but this is simply obtained by complex conjugation.

¹³Recall that $D \geq 1$ and thus $d - D < d$.

5 Mountain pass approach

Let us start by recalling the standard mountain pass situation; see e.g. [29, Chapter 4]. Let $K_0 \subset K$ be compact metric spaces, X a metric space and $J : X \rightarrow \mathbb{R}$. Consider a fixed map $\chi \in C(K_0; X)$ and the minimization problem

$$c := \inf \left\{ \max_K J \circ F; F \in C(K; X), F = \chi \text{ on } K_0 \right\}. \quad (5.1)$$

Mountain pass geometry is characterized by the inequality

$$c > c_1 := \max_{K_0} J \circ \chi. \quad (5.2)$$

Under additional assumptions on X and J , (5.2) leads to existence of a critical point x of J such that $J(x) = c$.

In our case, we let $K = \overline{\mathbb{D}}_r$, where $0 < r < 1$, and $K_0 = C_r$. Then the most straightforward approach would be to let $J = F_\beta$, $X = \mathcal{G}$ be the set of complex-valued H^1 maps with modulus 1 on the boundary, and $\chi(a) = M_a$. Since \mathcal{G} is not a linear space, however, we need to parametrize it in an appropriate way. We present a first way of doing this in this section, which allows to prove the main result of this paper but is not defined for small values of ε . A second approach will be presented in the next section.

When β is small, it is possible to modify the above construction as follows.¹⁴ Assume that

$$\lambda_1(-\Delta - \beta) > 0, \quad (5.3)$$

that is that there is some $\delta > 0$ such that

$$(1 - \delta) \int_{\mathbb{D}} |\nabla v|^2 \geq \int_{\mathbb{D}} \beta v^2, \quad \forall \beta \in H_0^1(\mathbb{D}; \mathbb{R}).$$

For such β and fixed boundary condition $g \in H^{1/2}(\mathbb{S}^1; \mathbb{C})$, the energy functional F_β is strictly convex in the affine space

$$H_g^1(\mathbb{D}; \mathbb{C}) = \{u \in H^1(\mathbb{D}; \mathbb{C}); \text{tr } u = g\}.$$

Thus, the problem

$$\begin{cases} -\Delta u = \beta u(1 - |u|^2) & \text{in } \mathbb{D} \\ u = g & \text{on } \mathbb{S}^1, \end{cases} \quad (5.4)$$

has a unique solution, which we denote $T(g)$. Our function space is now $X^* = H^{1/2}(\mathbb{S}^1; \mathbb{R})$ and we let, for any $\psi \in X^*$,

$$J^*(\psi) = F_\beta(T(e^{i\psi})).$$

For $a \in C_r$, the restriction of M_a to \mathbb{S}^1 can be written $e^{i(\theta + \psi_a)}$ for some $\psi_a \in H^{1/2}(\mathbb{S}^1; \mathbb{R})$.¹⁵ We let, for any $a \in C_r$, $\chi^*(a) = \psi_a$. This defines a map from C_r into X^* .

¹⁴Of particular interest for us is the fact that the assumption (5.3) is satisfied when in the original problem of finding critical points for E_ε we take a large ε .

¹⁵ ψ_a is actually smooth.

5.1 Lemma. Assume (4.10) and (5.3). Then, for r sufficiently close to 1, we have

$$c^* := \inf \left\{ \max_K J^* \circ F; F \in C(K; X^*), F = \chi^* \text{ on } K_0 \right\} > c_1^* := \max_{K_0} J^* \circ \chi^*. \quad (5.5)$$

Proof. It is easy to see that $|M_a| \rightarrow 1$ in $L^4(\mathbb{D})$ as $|a| \rightarrow 1$, and thus $c_1^* \rightarrow \pi$ as $r \rightarrow 1$. Fix a Lebesgue point a_0 of β . Then, if $r > |a_0|$, for any $F \in C(K; \mathcal{G})$ such that $F = \chi$ on C_r there exists some $z_0 \in \mathbb{D}_r$ such that the harmonic extension of $\text{tr } F(z_0)$ has a zero at a_0 . Indeed from Theorem 4.1 and using the notations there, the map

$$G : \bar{\mathbb{D}}_r \rightarrow \mathbb{D}, \quad \bar{\mathbb{D}}_r \ni z \xrightarrow{G} a(\text{tr } F(z)) \in \mathbb{D}$$

is continuous, and then since it is the identity on C_r and using the Brouwer fixed point theorem, it must take the value a_0 .

Now, from Lemma 4.5 we find that $J^*(F(z_0)) \geq c(\beta, a_0) > \pi$, therefore $c^* > c(\beta, a_0) > \pi$. Choosing r close enough to 1 so that $c_1^* < c(\beta, a_0)$ proves the result. \square

We now prove that J^* is C^1 (Lemma 5.6). As an immediate consequence, we will characterize the Palais-Smale sequences associated to (5.5) (Corollary 5.11). Before proceeding, let us recall a useful elementary fact.

5.2 Lemma. Let \mathcal{X}, \mathcal{Y} be normed spaces. Let \mathcal{Z} be a dense subspace of \mathcal{X} . Let $\mathcal{F} \in C(\mathcal{X}; \mathcal{Y})$ and $\mathcal{T} \in C(\mathcal{X}; B(\mathcal{X}; \mathcal{Y}))$ be such that

$$\frac{\partial \mathcal{F}}{\partial z}(x) = \mathcal{T}(x)(z), \quad \forall x, z \in \mathcal{Z}. \quad (5.6)$$

Then $\mathcal{F} \in C^1$ and (5.6) holds for every $x, z \in \mathcal{X}$.

Here, $B(\mathcal{X}; \mathcal{Y})$ denotes the space of bounded linear operators from \mathcal{X} into \mathcal{Y} . In practice, we will apply the above with $\mathcal{Z} = C^\infty$, and the point will be to check the continuity of \mathcal{F} and \mathcal{T} .

We start with a straightforward consequence of the embedding $H^1(\mathbb{D}) \hookrightarrow L^4(\mathbb{D})$.

5.3 Lemma. The map F_β is C^1 in $H^1(\mathbb{D}; \mathbb{C})$ and

$$F'_\beta(u)(v) = \int_{\mathbb{D}} \nabla u \cdot \nabla v - \int_{\mathbb{D}} \beta u \cdot v(1 - |u|^2), \quad \forall u, v \in H^1(\mathbb{D}; \mathbb{C}). \quad (5.7)$$

Similarly, the map

$$G : H^1(\mathbb{D}; \mathbb{C}) \rightarrow \mathbb{R}, \quad H^1(\mathbb{D}; \mathbb{C}) \ni u \mapsto \frac{1}{4} \int_{\mathbb{D}} \beta(1 - |u|^2)^2,$$

is C^1 , and $G'(u)(v) = - \int_{\mathbb{D}} \beta u \cdot v(1 - |u|^2)$.

Recall that \cdot denotes the real scalar product.

5.4 Lemma. Assume that (5.3) holds. Then the map $g \rightarrow T(g)$, where $T(g)$ denotes the unique solution to (5.4), is C^1 in a neighborhood of \mathcal{H} .

Proof. Consider the maps

$$V : H^{1/2}(\mathbb{S}^1; \mathbb{C}) \rightarrow H^1(\mathbb{D}; \mathbb{C}), \quad g \xrightarrow{V} u(g)$$

and

$$U : H_0^1(\mathbb{D}; \mathbb{C}) \times H^{1/2}(\mathbb{S}^1; \mathbb{C}) \rightarrow H^{-1}(\mathbb{D}; \mathbb{C}), \quad (v, g) \xrightarrow{U} -\Delta v - \beta(v + u(g))(1 - |v + u(g)|^2).$$

Thanks to the embedding $H^1(\mathbb{D}) \hookrightarrow L^4(\mathbb{D})$ and to the continuity of V , it is easy to see that $U \in C^1$ and that the partial differential of U in the v variable is given by

$$\frac{\partial U}{\partial v}(v, g)(w) = -\Delta w - \beta(1 - |v + u(g)|^2)w + 2\beta(w \cdot (v + u(g)))(v + u(g)), \quad \forall w \in H_0^1(\mathbb{D}; \mathbb{C}).$$

The conclusion of the lemma follows *via* the implicit function theorem if we prove that the operator $W = \frac{\partial U}{\partial v}(v, g)$ is invertible at each couple $(T(g) - u(g), g)$ with $g \in \mathcal{H}$. Since W is symmetric in $H_0^1(\mathbb{D}; \mathbb{C})$,¹⁶ it suffices to prove that the quadratic form Q associated to W is definite positive. We note that

$$Q(w) = \int_{\mathbb{D}} |\nabla w|^2 - \int_{\mathbb{D}} \beta(1 - |T(g)|^2)|w|^2 + 2 \int_{\mathbb{D}} \beta(w \cdot T(g))^2 \geq \int_{\mathbb{D}} |\nabla w|^2 - \int_{\mathbb{D}} \beta(1 - |T(g)|^2)|w|^2,$$

and positivity follows from (5.3) combined with the maximum principle Lemma 2.16. \square

The next result is reminiscent of the fact that H^1 harmonic functions have traces of the normal derivative on the boundary.

5.5 Lemma. Let $g \in \mathcal{H}$ and let u satisfy (5.4). Then the vector field $u \wedge \nabla u$ has a trace $\text{tr}(u \wedge \nabla u)$ on \mathbb{S}^1 , and this trace belongs to $H^{-1/2}(\mathbb{S}^1)$.

If, in addition, (5.3) holds, then the map

$$Y : \mathcal{H} \rightarrow H^{-1/2}(\mathbb{S}^1), \quad \mathcal{H} \ni g \xrightarrow{Y} \text{tr}(T(g) \wedge \nabla T(g)) \in H^{-1/2}(\mathbb{S}^1),$$

is continuous.

Proof. The vector field $u \wedge \nabla u$ belongs to L^2 and (by (5.4)) is divergence free. Existence of the trace is then standard; let us briefly recall the argument. By the (L^2 -version of the) Poincaré lemma, we may write $u \wedge \nabla u = -\nabla^\perp h$ for some $h \in H^1(\mathbb{D})$. We then set

$$\text{tr} \left(u \wedge \frac{\partial u}{\partial v} \right) = \frac{\partial}{\partial \tau} \text{tr} h \in H^{-1/2}(\mathbb{S}^1),$$

where τ stands for the tangential derivative. When u is smooth, say $u \in C^1$, $\frac{\partial}{\partial \tau} \text{tr} h$ is nothing else than $u \wedge \frac{\partial u}{\partial v}$; with an abuse of notation, we keep the notation $u \wedge \frac{\partial u}{\partial v}$ even if u is merely H^1 .

We note the integration by parts formula

$$\int_{\mathbb{D}} (u \wedge \nabla u) \cdot \nabla \zeta = \left\langle u \wedge \frac{\partial u}{\partial v}, \text{tr} \zeta \right\rangle_{H^{-1/2}, H^{1/2}} = \int_{\mathbb{S}^1} \left(u \wedge \frac{\partial u}{\partial v} \right) \zeta, \quad \forall \zeta \in H^1(\mathbb{D}; \mathbb{R}). \quad (5.8)$$

¹⁶Here, we endow \mathbb{C} with the scalar product of \mathbb{R}^2 and we identify $H_0^1(\mathbb{D}; \mathbb{C})$ with a space of \mathbb{R}^2 -valued functions.

Although the last integral in (5.8) is defined only when u is sufficiently smooth, we will use the integral notation for a general u .

The second part of the lemma is obtained as follows: let $g_n \rightarrow g$ in \mathcal{H} . Set $u_n = T(g_n)$ and $u = T(g)$. Then $u_n \rightarrow u$ in H^1 since T is continuous. Since $|u_n| \leq 1$ (Lemma 2.16), we find that $u_n \wedge \nabla u_n \rightarrow u \wedge \nabla u$ in L^2 . If we normalize the corresponding potentials h_n such that $\int_{\mathbb{D}} h_n = 0$, then $h_n \rightarrow h$ in H^1 . This implies convergence of $\text{tr}(u_n \wedge \nabla u_n)$ to $\text{tr}(u \wedge \nabla u)$ in $H^{-1/2}$. \square

5.6 Lemma. Assume (5.3). Then we have $J^* \in C^1$ and, if $u := T(N_0 e^{i\psi})$,

$$J^{*'}(\psi)(\eta) = \int_{\mathbb{S}^1} \left(u \wedge \frac{\partial u}{\partial v} \right) \eta = \int_{\mathbb{D}} (u \wedge \nabla u) \cdot \nabla \zeta, \quad \forall \psi, \eta \in H^{1/2}(\mathbb{S}^1; \mathbb{R}), \forall \zeta \in H^1(\mathbb{D}; \mathbb{R}) \text{ such that } \text{tr } \zeta = \eta. \quad (5.9)$$

More generally, the same holds if we consider the map

$$H^{1/2}(\mathbb{S}^1; \mathbb{R}) \ni \psi \xrightarrow{J_d^*} F_\beta(T(N_0^d e^{i\psi})), \quad \text{where } d \in \mathbb{Z}.$$

Proof. Note that the equality of the two integrals in (5.9) is a consequence of (5.8). We rely on Lemma 5.2. Clearly, J^* is continuous. On the other hand, if we let $\zeta = u(\eta)$ in (5.9) and we use the continuity of $H^{1/2} \ni \psi \mapsto u \wedge \nabla u \in L^2$, then we see that the second integral in (5.9) defines a map

$$\mathcal{F} \in C(H^{1/2}(\mathbb{S}^1; \mathbb{R}), H^{-1/2}(\mathbb{S}^1; \mathbb{R})), \quad \mathcal{F}(\psi)(\eta) := \int_{\mathbb{D}} (u \wedge \nabla u) \cdot \nabla u(\eta).$$

Therefore, it suffices to prove that (5.9) holds when ψ and η are smooth and when $\zeta = u(\eta)$. For such ψ and η , let $g_t = N_0 e^{i(\psi + t\eta)}$ and set $u_t = T(g_t)$, $u = u_0$. It is easy to see that $t \mapsto g_t \in \mathcal{H}$ is smooth. By Lemma 5.3 and Lemma 5.4, the maps $t \mapsto u_t$ and $t \mapsto J^*(\psi + t\eta)$ are C^1 , and

$$\frac{d}{dt}[J^*(\psi + t\eta)] = \int_{\mathbb{D}} \nabla u_t \cdot \nabla \left(\frac{d}{dt} u_t \right) - \int_{\mathbb{D}} \beta u_t \cdot \left(\frac{d}{dt} u_t \right) (1 - |u_t|^2). \quad (5.10)$$

On the other hand, if $v \in H^1(\Omega; \mathbb{C})$ is such that $v = iu\zeta$ on \mathbb{S}^1 , where $\zeta \in C^1(\mathbb{S}^1; \mathbb{R})$, and if u solves (5.4), then

$$\int_{\mathbb{D}} \nabla u \cdot \nabla v - \int_{\mathbb{D}} \beta(u \cdot v)(1 - |u|^2) = \int_{\mathbb{S}^1} \left(u \wedge \frac{\partial u}{\partial v} \right) \zeta. \quad (5.11)$$

Indeed, when u is smooth, this is a consequence of (5.4) and of the identity

$$\frac{\partial u}{\partial v} \cdot (iu\zeta) = \left(u \wedge \frac{\partial u}{\partial v} \right) \zeta \quad \text{on } \mathbb{S}^1.$$

The general case follows by approximation, using Lemma 2.1 1., and the fact that given β satisfying (5.3) we may find a uniformly bounded sequence (β_n) of smooth function satisfying (5.3) and such that $\beta_n \rightarrow \beta$.¹⁷

We next note that $u_t = N_0 e^{i(\psi + t\eta)}$ on \mathbb{S}^1 , and therefore $\frac{\partial}{\partial t} u_t = iu_t \eta$ on \mathbb{S}^1 . Thus, in (5.11), we may take $u = u_t$ and $v = \frac{\partial}{\partial t} u_t$. We obtained the first identity in (5.9) by combining this remark with (5.10). \square

¹⁷For an alternative argument not relying on (5.3), see Remark 5.7 below.

5.7 Remark. Here is an alternative proof of (5.11), valid without the assumption (5.3). In view of (5.4), (5.11) holds when $v \in H_0^1$, so that it suffices to prove (5.11) for a special v such that $v = \iota u \zeta$ on \mathbb{S}^1 . Consider a C^1 extension of ζ , still denoted ζ , and let $v = \iota u \zeta$. For this v , (5.11) is nothing else but (5.8).

5.8 Remark. Lemma 5.6 hides a small miracle. Recall that J^* is constructed as follows:

$$J^* = F_\beta \circ T \circ S, \quad H^{1/2}(\mathbb{S}^1; \mathbb{R}) \ni \psi \xrightarrow{S} g = e^{\iota \psi} \xrightarrow{T} u = T(g) \xrightarrow{F_\beta} F_\beta(u).$$

By Lemmas 5.4 and 5.3, we know that T and F_β are C^1 . We also know that J^* is C^1 (Lemma 5.6). However, S is *not* C^1 ; see Lemma 5.9 below. Since J^* is the composition of two operators, one, $F_\beta \circ T$, smooth, the other one, S , non smooth, the conclusion of Lemma 5.6 is that the smoothing effect prevails. Smoothness comes from the main ingredient of the proof of Lemma 5.6, which is existence of the boundary trace of the vector field $u \wedge \nabla u$ for a solution u of (5.4). In turn, this relies basically on the maximum principle. All in all, this turns $F_\beta \circ T$ (and finally J^*) into a smooth operator. On the other hand, we note that S is almost smooth: it is clearly Lipschitz.

5.9 Lemma. S is not differentiable. More generally, if $g \in \mathcal{H}$, then the map

$$S_g : H^{1/2}(\mathbb{S}^1; \mathbb{R}) \rightarrow \mathcal{H}, \quad H^{1/2}(\mathbb{S}^1; \mathbb{R}) \ni \psi \xrightarrow{S_g} g e^{\iota \psi} \in \mathcal{H}$$

is not differentiable.

Proof. We start by reducing the general case to the special case $g = 1$. Let $d = \deg g$ and write $g = N_0^d e^{\iota \varphi}$, with $\varphi \in H^{1/2}(\mathbb{S}^1; \mathbb{R})$ (cf Lemma 2.6 4). Multiplication with N_0^d being clearly a linear continuous bijective operator in $H^{1/2}$, we see that S_g is differentiable at ψ if and only if S is differentiable at $\psi + \varphi$.

We continue by finding functions ψ such that S is not differentiable at ψ . It is easy to see that, for $\eta \in C^\infty(\mathbb{S}^1; \mathbb{R})$, we have

$$\lim_{t \rightarrow 0} \frac{S(\psi + t\eta) - S(\psi)}{t} = A(\psi, \eta) := \iota S(\psi)\eta, \quad (5.12)$$

the limit being considered in $H^{1/2}$. If S is differentiable at ψ , then (5.12) holds for every $\eta \in H^{1/2}(\mathbb{S}^1; \mathbb{R})$, and the map $A(\psi, \eta)$ belongs to $H^{1/2}$. The heuristics for concluding is the following: $H^{1/2} \cap L^\infty$ is an algebra, but $H^{1/2}$ is not. Thus, if we pick a function $\eta \in H^{1/2} \cap L^\infty$, then $A(\psi, \eta) \in H^{1/2}$, but this need not hold when η is merely $H^{1/2}$. Formally, we continue as follows: if S is differentiable at ψ , then $B(\psi) := \iota e^{\iota \psi} \psi \in H^{1/2}$, and in particular $C(\psi) := \psi \cos \psi$ belongs to $H^{1/2}$. We recall the following result of Bourdaud and Kateb [12]: a superposition operator $\psi \mapsto G \circ \psi$ acts on $H^{1/2}$ if and only if G is Lipschitz. Though the result in [12] is stated in $H^{1/2}(\mathbb{R}^n)$, the construction of counterexamples yields, for a non Lipschitz G , compactly supported maps ψ such that $G \circ \psi \notin H^{1/2}$; see [31, proof of Theorem 1, Section 5.3.1]. Thus, even on the circle, G has to be Lipschitz. We conclude by noting that $t \mapsto t \cos t$ is not Lipschitz. \square

We are now in position to construct sequences of almost critical points as a consequence of the Mountain Pass Theorem of Ambrosetti and Rabinowitz [1], which will be used in the following form [29, Theorem 4.3, p. 77].

5.10 Theorem. Let $K_0 \subset K$ be compact metric spaces. Let X be a Banach space and $J \in C^1(X; \mathbb{R})$. Fix $\chi \in C(K_0; X)$ and define

$$M = \{F \in C(K; X); F = \chi \text{ on } K_0\}.$$

Assume that

$$c := \inf_{F \in M} \max_{t \in K} J(F(t)) > c_1 := \max_{t \in K_0} J(\chi(t)). \quad (5.13)$$

Then for every $\delta > 0$ there exists $x_\delta \in X$ such that

$$c - \delta \leq J(x_\delta) \leq c + \delta \quad (5.14)$$

and

$$\|J'(x_\delta)\| \leq \sqrt{\delta}. \quad (5.15)$$

By combining this theorem with Lemmas 5.1 and 5.6, we obtain the following

5.11 Corollary. Assume that (5.3) holds. Let c be defined by (5.5). Then, for each $\delta > 0$, there is some $\psi_\delta \in H^{1/2}(\mathbb{S}^1; \mathbb{R})$ such that $u_\delta = T(N_0 e^{i\psi_\delta})$ satisfies

$$c^* - \delta \leq J^*(\psi_\delta) = F_\beta(u_\delta) \leq c^* + \delta \quad (5.16)$$

and

$$\left| \int_{\mathbb{S}^1} \left(u_\delta \wedge \frac{\partial u_\delta}{\partial v} \right) \eta \right| \leq \sqrt{\delta} |\zeta|_{H^{1/2}}, \quad \forall \eta \in H^{1/2}(\mathbb{S}^1; \mathbb{R}). \quad (5.17)$$

In particular, the corollary applies if we take a large ε in the original energy E_ε . Note that, by (5.8), (5.17) is equivalent to

$$\left| \int_{\mathbb{D}} (u_\delta \wedge \nabla u_\delta) \cdot \nabla \zeta \right| \leq \sqrt{\delta} \|\nabla \zeta\|_{L^2}, \quad \forall \zeta \in H^1(\mathbb{D}; \mathbb{R}). \quad (5.18)$$

5.12 Corollary. Assume that (5.3) holds. Then every weak limit of u_δ as $\delta \rightarrow 0$ is a critical point of F_β in \mathcal{G} , i.e., a solution of (2.21).

Proof. Assume that (possibly along a subsequence) $u_\delta \rightharpoonup u$ in H^1 . By the maximum principle Lemma 2.16, on the one hand u satisfies $-\Delta u = \beta u(1 - |u|^2)$. On the other hand, $u_\delta \wedge \nabla u_\delta \rightharpoonup u \wedge \nabla u$ in L^2 , and thus (using (5.18)) u satisfies (2.20), which is equivalent to (2.21), by Lemma 2.15. \square

6 Proof of Theorem 1.1

We establish here the following generalization of Theorem 1.1.

6.1 Theorem. Assume that (5.3) holds and that the value c^* in (5.5) satisfies

$$c^* < 2\pi. \quad (6.1)$$

Then c^* is a critical value of F_β in \mathcal{G}_1 , i.e., there exists a solution u of (2.21) with $d = 1$ such that $F_\beta(u) = c^*$.

Before proceeding to the proof, let us note that, with the choice $F(a) = N_a$, $a \in \overline{\mathbb{D}}_r$, in (5.5), and for $\beta = \frac{1}{\varepsilon^2} \text{Jac } \Phi^{-1}$ in (2.17), where $\Phi : \Omega \rightarrow \mathbb{D}$ is a conformal representation, we find that

$$c^* \leq \max_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} |\nabla M_a|^2 + \frac{1}{4\varepsilon^2} \int_{\mathbb{D}} (1 - |M_a|^2)^2 \text{Jac } \Phi^{-1} \right) \leq \pi + \frac{|\Omega|}{4\varepsilon^2}.$$

The latter quantity is $< 2\pi$ for large ε . Thus Theorem 6.1 generalizes indeed Theorem 1.1.

Proof. Let u_δ be as in Corollary 5.11. By combining Corollary 5.11 with Corollary 5.12 and with Lemma 2.10, we see that the conclusion of the theorem holds provided we find some $r \in (0, 1)$ and some $\lambda > 0$ such that (possibly along a subsequence $\delta_n \rightarrow 0$)

$$|u_{\delta_n}(z)| \geq \lambda \quad \text{when } |z| \geq r. \quad (6.2)$$

Validity of (6.2) is established by contradiction: assuming that there exist sequences $\delta_n \rightarrow 0$ and a_n such that

$$u_{\delta_n}(a_n) \rightarrow 0 \quad \text{and} \quad |a_n| \rightarrow 1, \quad (6.3)$$

we will prove that

$$F_\beta(u_{\delta_n}) \rightarrow \pi, \quad (6.4)$$

which contradicts Lemma 5.1.

Before proceeding to the proof, we introduce lighter notation: we write u_n instead of u_{δ_n} and denote by (u_n) any subsequence extracted from (u_n) , and the same for (a_n) .

We start by rescaling the sequence (u_n) : we let

$$\Phi_n = M_{-a_n}, \quad v_n = u \circ \Phi_n, \quad \beta_n = \beta \circ \Phi_n \text{Jac } \Phi_n, \quad F_n(w) = \frac{1}{2} \int_{\mathbb{D}} |\nabla w|^2 + \frac{1}{4} \int_{\mathbb{D}} \beta_n (1 - |w|^2)^2.$$

In particular, using Corollary 5.11 and the conformal invariance of the Dirichlet integral, we find that the rescaled sequence $(v_n) \subset \mathcal{G}_1$ satisfies, with $c_n \rightarrow 0$,

$$\begin{cases} -\Delta v_n = \beta_n v_n (1 - |v_n|^2) & \text{in } \mathbb{D} \\ |\text{tr } v_n| = 1 \\ \left| \int_{\mathbb{D}} (v_n \wedge \nabla v_n) \cdot \nabla \zeta \right| \leq c_n \|\nabla \zeta\|_{L^2}, \quad \forall \zeta \in H^1(\mathbb{D}). \\ v_n(0) \rightarrow 0 \\ F_n(v_n) = F_\beta(u_n) \rightarrow c^* \end{cases} \quad (6.5)$$

We start by collecting some straightforward properties of β_n and v_n .

6.2 Lemma. 1. $\beta_n \rightarrow 0$ uniformly on compacts. In particular, (v_n) converges, up to a subsequence and in $C_{loc}^{1,\alpha}(\mathbb{D})$, $0 < \alpha < 1$, to some solution $v \in \mathcal{G}$ of (2.21) with $\beta = 0$ and such that $v(0) = 0$. Moreover, (by Lemma 3.5) $v \in \mathcal{G}_d$ for some $d \neq 0$, and v is either a d -Blaschke product, or the conjugate of such a product.

2. $\lambda_1(-\Delta - \beta_n) > 0$. In particular, v_n is a minimizer of F_{β_n} with respect with its own boundary condition.

Proof. The first item follows from (6.5) and Lemma 2.16 combined with standard elliptic estimates, Corollary 5.12 and Lemma 3.5. The second one follows from (5.3). \square

Step 1 (in the proof of Theorem 6.1). Let v be as in Lemma 6.2 Then $v = M_{\alpha,0}$ for some $\alpha \in \mathbb{S}^1$. Indeed, we adapt the argument leading to (4.1): we have, by the Price Lemma 2.17 and Corollary 3.2,

$$2\pi > c^* \geq \liminf \frac{1}{2} \int_{\mathbb{D}} |\nabla v_n|^2 \geq \frac{1}{2} \int_{\mathbb{D}} |\nabla v|^2 + \pi|1-d| \geq \pi|d| + \pi|1-d|,$$

where $d = \deg(v, \mathbb{S}^1)$. We find that $d = 0$ or $d = 1$. Since $d \neq 0$ (by Lemma 6.2 1), we obtain the desired conclusion by combining Lemma 6.2 1 with the fact that $v(0) = 0$.

Step 2. There exist $r \in (0, 1)$ and $\lambda > 0$ such that $|v_n(z)| \geq \lambda$ when $|z| \geq r$.

In order to prove this, we argue by contradiction and assume that up to a subsequence we have $v_n(z_n) \rightarrow 0$, where the points $z_n \in \mathbb{D}$ are such that $|z_n| \rightarrow 1$. By repeating the arguments leading to (4.8) and (4.9), we obtain that, for each $s \in (0, 1)$, there is some $R \in (0, 1)$ such that

$$\liminf \frac{1}{2} \int_{\mathbb{D}_R} |\nabla v_n|^2 \geq \pi s^2 \tag{6.6}$$

and

$$\liminf \frac{1}{2} \int_{\mathbb{D} \setminus \bar{\mathbb{D}}_R} |\nabla v_n|^2 \geq \pi s^2. \tag{6.7}$$

Indeed, (6.6) is obtained exactly as (4.8). On the other hand, (4.9) relies on [23, Theorem 3.6]. In order to apply, this result requires that v_n minimizes F_n with respect to its own boundary conditions. This holds indeed in our case, by Lemma 6.2 3.

By combining (6.6) with (6.7), we find that $\liminf \frac{1}{2} \int_{\mathbb{D}} |\nabla v_n|^2 \geq 2\pi$. But this contradicts the assumption $c^* < 2\pi$.

Step 3. We have (the contradiction) $F_n(v_n) \rightarrow \pi$.

This step is a consequence of Step 2, of (6.5) and of Lemma 6.3 below.

We start by stating the technical assumptions required in Lemma 6.3. For simplicity, we stated Lemma 6.3 when the underlying domain is \mathbb{D} , but this is not relevant for our analysis.

We consider two sequences (β_n) and (v_n) and two maps γ and v such that:

$$\beta_n, \gamma \in L^\infty(\mathbb{D}), \beta_n, \gamma \geq 0, \tag{6.8}$$

$$\beta_n \rightarrow \gamma \text{ uniformly on compacts of } \mathbb{D}, \tag{6.9}$$

$$v_n, v \in H^1(\mathbb{D}; \mathbb{C}), |\operatorname{tr} v_n| = 1, v_n \rightharpoonup v \text{ in } H^1, \tag{6.10}$$

$$-\Delta v_n = \beta_n v_n (1 - |v_n|^2) \text{ in } \mathbb{D}, \tag{6.11}$$

$$\left| \int_{\mathbb{D}} (v_n \wedge \nabla v_n) \cdot \nabla \zeta \right| \leq c_n \|\nabla \zeta\|_{L^2}, \forall \zeta \in H^1(\mathbb{D}), \text{ with } c_n \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{6.12}$$

Unlike our next hypothesis (6.13), the above assumptions are naturally satisfied by sequences of almost critical points obtained *via* the functional J^* (as in Corollary 5.11).

6.3 Lemma. Assume that (6.8)-(6.12) hold. Assume in addition that there exists some $\lambda > 0$ such that

$$|v_n(z)| \geq \lambda \quad \forall z \in \mathbb{D} \text{ such that } |z| \geq 1 - \lambda. \tag{6.13}$$

Then we have

$$\int_{\mathbb{D}} |\nabla v_n|^2 \rightarrow \int_{\mathbb{D}} |\nabla v|^2 \quad (6.14)$$

and

$$\int_{\mathbb{D}} \beta_n (1 - |v_n|^2)^2 \rightarrow \int_{\mathbb{D}} \gamma (1 - |v|^2)^2. \quad (6.15)$$

Equivalently, we have $v_n \rightarrow v$ strongly in H^1 and

$$F_n(v_n) \rightarrow F_\gamma(v). \quad (6.16)$$

We complete Step 3 by taking, in the above lemma, $\gamma = 0$ and $v = M_{\alpha,0}$ as in Step 1. \square

Proof of Lemma 6.3. It suffices to establish (6.14)-(6.15) along a subsequence.

By the maximum principle Lemma 2.16, we have $|v_n|, |v| \leq 1$ in \mathbb{D} . By standard elliptic estimates,

$$v_n \rightarrow v \quad \text{in } W_{loc}^{2,p}(\mathbb{D}), \quad \forall p < \infty. \quad (6.17)$$

By (6.9) and (6.17), we find that (6.14) and (6.15) hold if we replace \mathbb{D} by $\mathbb{D}_{1-\varepsilon}$ for each $\varepsilon > 0$. On the other hand, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{D} \setminus \bar{\mathbb{D}}_{1-\varepsilon}} (|\nabla v|^2 + \gamma(1 - |v|^2)^2) = 0.$$

Therefore, it suffices to prove that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\mathbb{D} \setminus \bar{\mathbb{D}}_{1-\varepsilon}} (|\nabla v_n|^2 + \beta_n(1 - |v_n|^2)^2) = 0. \quad (6.18)$$

By (6.17), $v_n \rightarrow v$ uniformly on compacts of \mathbb{D} . Combining this fact with (6.13), we find that for large n we have

$$d := \deg \left(\frac{v}{|v|}, C_r \right) = \deg \left(\frac{v_n}{|v_n|}, C_r \right), \quad \forall r \in [1 - \delta, 1].$$

By Corollary 2.9 applied with $u^1 = \left(\frac{z}{|z|} \right)^d$, $u^2 = v_n$, $\omega = \mathbb{D} \setminus \bar{\mathbb{D}}_{1-\lambda}$, we may write, in ω , $v_n = \rho_n e^{i(d\theta + \varphi_n)}$, and similarly $v = \rho e^{i(d\theta + \varphi)}$, with $\lambda \leq \rho_n, \rho \leq 1$, and $\varphi_n, \varphi \in H^1(\omega)$. By (2.10), we find that, possibly after extracting suitable multiples of 2π , we have $\varphi_n \rightarrow \varphi$ and $\rho_n \rightarrow \rho$ in $H^1(\omega)$. On the other hand, by (6.17), we have $\varphi_n \rightarrow \varphi$ and $\rho_n \rightarrow \rho$ in $C_{loc}^1(\omega)$. We also note the fact that $v \in C^1(\bar{\mathbb{D}})$, by Lemma 2.15.

We next translate the properties of v_n in terms of ρ_n and φ_n : (6.11) and (6.12) imply that

$$\begin{cases} \operatorname{div}(\rho_n^2 \nabla(d\theta + \varphi_n)) = 0 & \text{in } \omega \\ -\Delta \rho_n = \beta_n \rho_n (1 - \rho_n)^2 - \rho_n |\nabla(d\theta + \varphi_n)|^2 & \text{in } \omega \\ \operatorname{tr} \rho_n = 1 & \text{on } \mathbb{S}^1 \\ v_n \wedge \nabla v_n = \rho_n^2 \nabla(d\theta + \varphi_n) & \text{in } \omega \\ \left| \int_{\mathbb{D}} (v_n \wedge \nabla v_n) \cdot \nabla \zeta \right| \leq c_n \|\nabla \zeta\|_{L^2}, & \forall \zeta \in H^1(\mathbb{D}) \end{cases} \quad (6.19)$$

Let $0 < \varepsilon < \lambda$. Since $\varphi_n \rightarrow \varphi$ in $C^1(C_{1-\varepsilon})$, we find that the function $\varphi_n - \varphi$, defined in $\mathbb{D} \setminus \overline{\mathbb{D}}_{1-\varepsilon}$, has an extension $\zeta_n \in H^1(\mathbb{D})$ such that $\|\nabla \zeta_n\|_{L^2(\mathbb{D}_{1-\varepsilon})} \rightarrow 0$. Using the fact that

$$\rho_n^2 \nabla(d\theta) \rightarrow \rho^2 \nabla(d\theta) \text{ and } \rho_n^2 \nabla \varphi_n \rightarrow \rho^2 \nabla \varphi \text{ in } L^2(\omega),$$

we find that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_{\mathbb{D}} (v_n \wedge \nabla v_n) \cdot \nabla \zeta_n = \lim_{n \rightarrow \infty} \int_{\mathbb{D} \setminus \overline{\mathbb{D}}_{1-\varepsilon}} (v_n \wedge \nabla v_n) \cdot \nabla \zeta_n \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{D} \setminus \overline{\mathbb{D}}_{1-\varepsilon}} [\rho_n^2 \nabla(d\theta + \varphi_n)] \cdot \nabla(\varphi_n - \varphi) = \lim_{n \rightarrow \infty} \int_{\mathbb{D} \setminus \overline{\mathbb{D}}_{1-\varepsilon}} \rho_n^2 |\nabla \varphi_n|^2 - \int_{\mathbb{D} \setminus \overline{\mathbb{D}}_{1-\varepsilon}} \rho^2 |\nabla \varphi|^2, \end{aligned}$$

which implies easily that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{D} \setminus \overline{\mathbb{D}}_{1-\varepsilon}} \rho_n^2 |\nabla(d\theta + \varphi_n)|^2 = \int_{\mathbb{D} \setminus \overline{\mathbb{D}}_{1-\varepsilon}} \rho^2 |\nabla(d\theta + \varphi)|^2,$$

and in particular

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{D} \setminus \overline{\mathbb{D}}_{1-\varepsilon}} \rho_n^2 |\nabla(d\theta + \varphi_n)|^2 = 0. \quad (6.20)$$

We next multiply by $\eta_n = 1 - \rho_n$ the equation satisfied by ρ_n and find that

$$\int_{\mathbb{D} \setminus \overline{\mathbb{D}}_{1-\varepsilon}} \left(|\nabla \rho_n|^2 + \frac{\rho_n}{1 + \rho_n} \beta_n (1 - |v_n|^2)^2 \right) = \int_{\mathbb{D} \setminus \overline{\mathbb{D}}_{1-\varepsilon}} \rho_n \eta_n |\nabla(d\theta + \varphi_n)|^2 + \int_{C_{1-\varepsilon}} \eta_n \frac{\partial \rho_n}{\partial \nu}, \quad (6.21)$$

ν being the normal exterior to $\mathbb{D}_{1-\varepsilon}$.

We next note that, since $v \in C^1(\overline{\mathbb{D}})$ and $|v| = 1$ on \mathbb{S}^1 , we have

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{C_{1-\varepsilon}} \eta_n \frac{\partial \rho_n}{\partial \nu} = \lim_{\varepsilon \rightarrow 0} \int_{C_{1-\varepsilon}} (1 - \rho) \frac{\partial \rho}{\partial \nu} = 0. \quad (6.22)$$

By combining (6.21) with (6.20), (6.22) and with the assumption (6.13), we find that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\mathbb{D} \setminus \overline{\mathbb{D}}_{1-\varepsilon}} (|\nabla \rho_n|^2 + \beta_n (1 - |v_n|^2)^2) = 0. \quad (6.23)$$

We obtain (6.18) (and thus complete the proof of Lemma 6.3) by combining (6.20) with (6.23) and with the identity (2.10). \square

By combining (6.2) (whose validity has been established in the course of the proof of Theorem 6.1) with Lemma 6.3 applied to the original sequence (u_n) and to the weights $\beta_n = \beta$ and $\gamma = \beta$, we obtain the following improvement of Theorem 6.1.

6.4 Theorem. *Assume that (5.3) holds and that $c^* < 2\pi$. Then J^* satisfies the Palais-Smale condition at level c^* . In particular, the infimum is attained in (5.5).*

7 Asymptotic behavior of critical points as $\varepsilon \rightarrow \infty$

We describe here the asymptotic behavior, when $\varepsilon \rightarrow \infty$, of critical points of E_ε in \mathcal{E}_1 obtained by the mountain pass approach described in Section 5. For simplicity, we transfer the problem on \mathbb{D} (as explained at the end of Section 2) and consider the energy F_β , where

$$\beta = \beta_\varepsilon = \frac{1}{\varepsilon^2} \text{Jac } \Phi^{-1} = \frac{1}{\varepsilon^2} w. \quad (7.1)$$

In order to emphasize dependence on ε , we denote the number c^* associated to F_β as c_ε^* .

Before stating our main result in this section, let us translate the mountain pass approach associated to J^* into a more tractable one involving critical points of the Ginzburg-Landau equations with respect to their own boundary conditions. Under assumption (5.3), we may identify a map $g \in H^{1/2}(\mathbb{S}^1; \mathbb{C})$ with the solution $u = T(g)$ of (5.4). Let us denote by Z_ε the set of solutions of (5.4) (with arbitrary g):

$$Z_\varepsilon = \left\{ u \in \mathcal{G}; -\Delta u = \frac{1}{\varepsilon^2} w u (1 - |u|^2) \right\}. \quad (7.2)$$

Let us also denote, for each $u \in H^1(\mathbb{D}; \mathbb{C})$, by \tilde{u} the harmonic extension of $\text{tr } u$, i.e., $\tilde{u} = u(\text{tr } u)$.

7.1 Lemma. Assume that (5.3) holds. Then we have

$$c_\varepsilon^* = \min \left\{ \max F_\beta(F(a)); F \in C(\overline{\mathbb{D}}_r; Z_\varepsilon), \widetilde{F(a)} = M_a \text{ on } C_r \right\}. \quad (7.3)$$

Proof. Indeed, “ \leq ” is clear. For the opposite inequality, let $G \in C(\overline{\mathbb{D}}_r; X^*)$ be such that $G = \chi^*$ on C_r . If we define $F(a) := \widetilde{G(a)}$, then F is continuous (by Lemma 5.4), and clearly $\widetilde{F(a)} = M_a$ on C_r . On the other hand, assumption (5.3) implies that maps in Z_ε are minimizers of F_β with respect to their own boundary datum, and thus we have $F_\beta(F(a)) \leq F_\beta(G(a))$. \square

We next turn to the description of the asymptotic behavior of u_ε as $\varepsilon \rightarrow \infty$. Clearly, the Moebius transforms $M_{\alpha,a}$ satisfy

$$\int_{\mathbb{D}} w(1 - |M_{\alpha,a}|^2)^2 > 0, \quad \forall \alpha \in \mathbb{S}^1, \forall a \in \mathbb{D} \quad \text{and} \quad \lim_{|a| \rightarrow 1} \int_{\mathbb{D}} w(1 - |M_{\alpha,a}|^2)^2 = 0.$$

Therefore, the maximization problem

$$M := \max_{\substack{\alpha \in \mathbb{D} \\ \alpha \in \mathbb{S}^1}} \int_{\mathbb{D}} w(1 - |M_{\alpha,a}|^2)^2 \quad (7.4)$$

has a solution and, if $M_{\alpha,a}$ is a maximizer, then so is $M_{\gamma,a}$ for every $\gamma \in \mathbb{S}^1$. In addition, there is some $r_0 < 1$ such that every maximizer in (7.4) satisfies $|a| \leq r_0$. In what follows, we always assume that $r_0 < r < 1$.

7.2 Theorem. Let u_ε be obtained via (7.3). Then, possibly up to a subsequence, $u_\varepsilon \rightarrow M_{\alpha,a}$ strongly in $H^1(\mathbb{D})$, where $M_{\alpha,a}$ is a solution of (7.4).

7.3 Remark. Equivalently, Theorem 7.2 states that, if v_ε is a critical point in \mathcal{E}_1 obtained by the mountain pass approach in Ω and for the original energy E_ε , then the family (v_ε) converges, possibly up to a subsequence, strongly in $H^1(\Omega)$ to a maximizer of

$$M := \max_{\substack{a \in \Omega \\ \alpha \in \mathbb{S}^1}} \int_{\Omega} (1 - |M_{\alpha, a, \Phi}|^2)^2.$$

Proof. Step 1. We have

$$c_\varepsilon^* = \pi + \frac{1}{4\varepsilon^2}M + o\left(\frac{1}{\varepsilon^2}\right) \quad \text{as } \varepsilon \rightarrow \infty. \quad (7.5)$$

Indeed, on the one hand the upper bound $c_\varepsilon^* \leq \pi + \frac{1}{4\varepsilon^2}M$ is trivial: consider, as a competitor in (7.3), the map F given by $\widetilde{F}(a) = M_a$ for every $a \in \overline{\mathbb{D}}_r$.

For the lower bound, fix some $\xi \in \overline{\mathbb{D}}_r$ such that M_ξ is a maximizer in (7.4). Let F be a competitor in (7.3) such that $F_\beta(F(a)) < c_\varepsilon^* + \frac{1}{\varepsilon^4}$ for each $a \in \overline{\mathbb{D}}_r$. By Theorem 4.1 3, for large ε the map

$$G : \overline{\mathbb{D}}_r \rightarrow \mathbb{D}, \quad G(a) = \text{the zero of } \widetilde{F}(a)$$

is continuous and satisfies $G(a) = a$ if $|a| = r$. By the Brouwer fixed point theorem, we may find some $a = a_\varepsilon$ such that $\widetilde{F}(a)$ vanishes at $z = \xi$. Since $\frac{1}{2} \int_{\mathbb{D}} |\nabla \widetilde{F}(a)|^2 \leq c_\varepsilon^*$, we find, *via* Corollary 4.4 and Corollary 3.2, that $F(a)$ converges, up to a subsequence, strongly in $H^1(\mathbb{D})$ to $M_{\alpha, \xi}$ for some $\alpha \in \mathbb{S}^1$. We find that

$$\frac{1}{2} \int_{\mathbb{D}} |\nabla \widetilde{F}(a)|^2 + \frac{1}{4\varepsilon^2} \int_{\mathbb{D}} w(1 - |\widetilde{F}(a)|^2)^2 \geq \pi + \frac{1}{4\varepsilon^2}M + o\left(\frac{1}{\varepsilon^2}\right) \quad \text{as } \varepsilon \rightarrow \infty. \quad (7.6)$$

The lower bound in (7.5) is obtained by combining (7.6) with the following straightforward consequence of the equation (5.4) satisfied by every $u \in Z_\varepsilon$ and of the maximum principle Lemma 2.16.

7.4 Lemma. Let $u \in Z_\varepsilon$. Then

$$\|\nabla u - \nabla \tilde{u}\|_{L^2(\mathbb{D})} + \|u - \tilde{u}\|_{L^\infty(\mathbb{D})} = O\left(\frac{1}{\varepsilon^2}\right), \quad (7.7)$$

$$\int_{\mathbb{D}} |\nabla u|^2 = \int_{\mathbb{D}} |\nabla \tilde{u}|^2 + O\left(\frac{1}{\varepsilon^4}\right) \quad (7.8)$$

and

$$\int_{\mathbb{D}} w(1 - |u|^2)^2 = \int_{\mathbb{D}} w(1 - |\tilde{u}|^2)^2 + O\left(\frac{1}{\varepsilon^2}\right). \quad (7.9)$$

To summarize: we have the expansion (7.5) of the energy. Using the identity $|\nabla u|^2 = \text{Jac } u + 4|\partial_{\bar{z}} u|^2$ and formula (2.7), we find that

$$2 \int_{\mathbb{D}} |\partial_{\bar{z}} u_\varepsilon|^2 + \frac{1}{4\varepsilon^2} \int_{\mathbb{D}} w(|u_\varepsilon|^2 - 1)^2 = \frac{1}{4\varepsilon^2}M + o\left(\frac{1}{\varepsilon^2}\right). \quad (7.10)$$

In order to complete the proof of Theorem 7.2, it suffices to show that

$$\int_{\mathbb{D}} |\partial_{\bar{z}} u_\varepsilon|^2 = o\left(\frac{1}{\varepsilon^2}\right). \quad (7.11)$$

Indeed, assume for the moment that (7.11) holds. As in the proof of Lemma 6.2, Step 1, we see that any possible weak limit of the u_ε 's is either a constant of modulus 1, or a Moebius map $M_{\alpha,a}$. By (7.10) and (7.11), the former case cannot hold. By (7.5), (7.10), (7.11) and Corollary 3.2, we find that $u_\varepsilon \rightarrow M_{\alpha,a}$ in $H^1(\mathbb{D})$, where $M_{\alpha,a}$ is a maximizer in (7.4).

Step 2. Proof of (7.11).

We follow the main lines of [7]. Let h_ε solve

$$\begin{cases} \nabla^\perp h_\varepsilon = u_\varepsilon \wedge \nabla u_\varepsilon & \text{in } \mathbb{D} \\ h_\varepsilon = 0 & \text{on } \mathbb{S}^1 \end{cases}$$

(existence of h_ε follows from the fact that $\operatorname{div}(u_\varepsilon \wedge \nabla u_\varepsilon) = 0$ in \mathbb{D} and $u_\varepsilon \wedge \frac{\partial u_\varepsilon}{\partial \nu} = 0$ on \mathbb{S}^1). Define

$$v_\varepsilon = \frac{1 - |u_\varepsilon|^2}{2} + h_\varepsilon. \text{ The following is straightforward.}$$

7.5 Lemma. We have

$$\begin{cases} \Delta h_\varepsilon = 2 \operatorname{Jac} u_\varepsilon & \text{in } \mathbb{D} \\ h_\varepsilon = 0 & \text{on } \mathbb{S}^1, \end{cases} \quad (7.12)$$

$$\begin{cases} \Delta v_\varepsilon = \frac{w}{\varepsilon^2} |u_\varepsilon|^2 (1 - |u_\varepsilon|^2) - 4 |\partial_{\bar{z}} u_\varepsilon|^2 & \text{in } \mathbb{D} \\ v_\varepsilon = 0 & \text{on } \mathbb{S}^1 \end{cases} \quad (7.13)$$

and

$$|\nabla v_\varepsilon|^2 = 4 |u_\varepsilon|^2 |\partial_{\bar{z}} u_\varepsilon|^2. \quad (7.14)$$

The key ingredient of the proof of Step 2 is the following

7.6 Lemma. We have

$$\|v_\varepsilon\|_{L^\infty(\mathbb{D})} \rightarrow 0. \quad (7.15)$$

Proof. For large ε , let $a = a_\varepsilon$ be the unique zero of $\widetilde{u}_\varepsilon$ (cf Theorem 4.1 1). Set $U_\varepsilon = u_\varepsilon \circ M_{-a}$ and $H_\varepsilon = h_\varepsilon \circ M_{-a}$. As in Step 1, Corollaries 4.4 and 3.2 combined with (7.5) imply that, up to a subsequence,

$$\widetilde{U}_\varepsilon \rightarrow M_{\gamma,0} = \gamma \operatorname{Id} \quad \text{strongly in } H^1(\mathbb{D}).$$

By (7.7), we also have

$$U_\varepsilon \rightarrow M_{\gamma,0} = \gamma \operatorname{Id} \quad \text{strongly in } H^1(\mathbb{D}). \quad (7.16)$$

Using Lemma 2.18 1. combined with (7.16) and with the fact that (by (7.12)) H_ε satisfies

$$\begin{cases} \Delta H_\varepsilon = 2 \operatorname{Jac} U_\varepsilon & \text{in } \mathbb{D} \\ H_\varepsilon = 0 & \text{on } \mathbb{S}^1, \end{cases}$$

we find that

$$H_\varepsilon(z) \rightarrow \frac{1}{2}(|z|^2 - 1) \quad \text{uniformly in } \overline{\mathbb{D}} \text{ as } \varepsilon \rightarrow 0. \quad (7.17)$$

On the other hand, by combining Lemma 2.13 with (7.16), we find that

$$|\widetilde{U}_\varepsilon(z)| \rightarrow |z| \quad \text{uniformly in } \overline{\mathbb{D}}.$$

The above convergence combined with (7.7) implies that

$$|U_\varepsilon(z)| \rightarrow |z| \quad \text{uniformly in } \overline{\mathbb{D}}. \quad (7.18)$$

(7.15) is obtained by combining (7.18) with (7.17). \square

Step 2 continued. If we multiply (7.13) by v_ε and take (7.10), (7.14) and Lemma 7.6 into account, we find that

$$\int_{\mathbb{D}} |u_\varepsilon|^2 |\partial_{\bar{z}} u_\varepsilon|^2 = o\left(\frac{1}{\varepsilon^2}\right). \quad (7.19)$$

In view of (7.18) and of the conformal invariance of the integral in (7.19), estimate (7.19) implies in particular that

$$\int_{\mathbb{D} \setminus \mathbb{D}_{1/2}} |\partial_{\bar{z}} U_\varepsilon|^2 = o\left(\frac{1}{\varepsilon^2}\right). \quad (7.20)$$

Therefore, in order to complete the proof of Step 2 it suffices to prove that

$$\int_{\mathbb{D}_{1/2}} |\partial_{\bar{z}} U_\varepsilon|^2 = o\left(\frac{1}{\varepsilon^2}\right). \quad (7.21)$$

Estimate (7.21) is obtained *via* the equation satisfied by $\partial_{\bar{z}} U_\varepsilon$. In view of (5.4), we have

$$\Delta(\partial_{\bar{z}} U_\varepsilon) = \frac{1}{\varepsilon^2} \partial_{\bar{z}} \underbrace{(\beta \circ M_{-a_\varepsilon} \text{Jac } M_{-a_\varepsilon} U_\varepsilon (1 - |U_\varepsilon|^2))}_{\gamma_\varepsilon} \quad \text{in } \mathbb{D}. \quad (7.22)$$

Noting that γ_ε is uniformly bounded on compacts of \mathbb{D} , we find, by standard estimates and (7.20), that

$$\|\partial_{\bar{z}} U_\varepsilon\|_{L^2(\mathbb{D}_{1/2})} \leq C \|\Delta(\partial_{\bar{z}} U_\varepsilon)\|_{H^{-1}(\mathbb{D}_{3/4})} + C \|\partial_{\bar{z}} U_\varepsilon\|_{L^2(\mathbb{D}_{3/4} \setminus \mathbb{D}_{1/2})} \leq C \frac{1}{\varepsilon^2} + o\left(\frac{1}{\varepsilon}\right).$$

This implies (7.21). The proof of Theorem 7.2 is complete. \square

7.7 Remark. The construction of critical points of E_ε in \mathcal{E}_d and Theorem 7.2 can be generalized to the case of magnetic Ginzburg-Landau functional, whose minimizers with prescribed degrees were studied in [7] for $\varepsilon \geq \sqrt{2}$. This shows that critical points with prescribed degree one still exist when $\varepsilon < \sqrt{2}$. However, their type changes when passing the critical value $\sqrt{2}$, namely when $\varepsilon > \sqrt{2}$ we have minimizers while for $\varepsilon < \sqrt{2}$ they become minimax critical points.

8 Bubbling analysis for small ε

When ε is small instead of large, our proof of Theorem 1.1 breaks down, although we expect its conclusion to remain true. In fact the very definition of J^* is valid only for large ε since it requires the uniqueness of solutions to (5.4).

Nevertheless we present in this section an alternative min-max setting which is valid for any $\varepsilon > 0$, and for which the proof of Theorem 1.1 can be carried out up to and including the analysis of Palais-Smale sequences, i.e., the analog of Corollary 4.4. We believe this result is interesting in its own right.

Let $X^\sharp := H_0^1(\mathbb{D}; \mathbb{C}) \times H^{1/2}(\mathbb{S}^1; \mathbb{R})$ and

$$U : X^\sharp \rightarrow H^1(\mathbb{D}; \mathbb{C}), \quad X^\sharp \ni (v, \psi) \mapsto U(v, \psi) := v + u(N_0 e^{i\psi}).$$

Recall that N_0 is the identity of \mathbb{S}^1 and that $u(w)$ is the harmonic extension of w . Set $J^\sharp(v, \psi) := F_\beta(U(v, \psi))$.

As in the previous construction, let $K = \overline{\mathbb{D}}_r$, $K_0 = C_r$ and write $N_a = N_0 e^{i\psi_a}$. Define

$$\chi^\sharp : C_r \rightarrow X^\sharp, \quad C_r \ni a \mapsto \chi^\sharp(a) = (0, \psi_a).$$

We have $U \circ \chi^\sharp(a) = M_a$.

We also define

$$c^\sharp := \inf \left\{ \max_K J^\sharp \circ F; F \in C(K; X^\sharp), F = \chi^\sharp \text{ on } K_0 \right\}, \quad c_1^\sharp := \max_{K_0} J^\sharp \circ \chi^\sharp.$$

By repeating the proof of Lemma 5.1, we find the following

8.1 Lemma. Assume (4.10). Then, for r sufficiently close to 1, we have $c^\sharp > c_1^\sharp$.

We next establish the analog of Lemma 5.6 and Corollary 5.11 for J^\sharp ; this will require more involved arguments. We start with some straightforward consequences of Lemma 5.2.

8.2 Lemma. Let $1 \leq p < \infty$. Let $u_0 \in H^{1/2} \cap L^\infty(\mathbb{S}^1; \mathbb{C})$. Then the map

$$\mathcal{F} : H^{1/2}(\mathbb{S}^1; \mathbb{R}) \rightarrow L^p(\mathbb{S}^1; \mathbb{C}), \quad H^{1/2}(\mathbb{S}^1; \mathbb{R}) \ni \psi \xrightarrow{\mathcal{F}} u_0 e^{i\psi},$$

is C^1 , and $\mathcal{F}'(\psi)(\eta) = i u_0 e^{i\psi} \eta$.

Proof. The point to be checked is that $\mathcal{F}(\psi)(\eta) := i u_0 e^{i\psi} \eta$ defines a map $\mathcal{F} \in C(H^{1/2}; B(H^{1/2}; L^p))$. This follows from

$$\|\mathcal{F}(\psi_1) - \mathcal{F}(\psi_2)\| \leq \sup_{\|\eta\|_{H^{1/2}} \leq 1} \|(\psi_1 - \psi_2)\eta\|_{L^p} \leq \sup_{\|\eta\|_{H^{1/2}} \leq 1} \|\psi_1 - \psi_2\|_{L^{2p}} \|\eta\|_{L^{2p}} \leq C \|\psi_1 - \psi_2\|_{H^{1/2}},$$

by the Sobolev embedding $H^{1/2}(\mathbb{S}^1) \hookrightarrow L^{2p}(\mathbb{S}^1)$. □

8.3 Lemma. Let $1 \leq p < \infty$. Let $u_0 \in H^{1/2} \cap L^\infty(\mathbb{S}^1; \mathbb{C})$. Then the map

$$\mathcal{F} : H_0^1(\mathbb{D}; \mathbb{C}) \times H^{1/2}(\mathbb{S}^1; \mathbb{R}) \rightarrow L^p(\mathbb{D}), \quad H_0^1(\mathbb{D}; \mathbb{C}) \times H^{1/2}(\mathbb{S}^1; \mathbb{R}) \ni (v, \psi) \xrightarrow{\mathcal{F}} v + u(u_0 e^{i\psi}),$$

is C^1 , and $\mathcal{F}'(v, \psi)(w, \eta) = w + u(i u_0 e^{i\psi} \eta)$.

Proof. It suffices to combine the previous lemma with the embedding $H^1(\mathbb{D}) \hookrightarrow L^p(\mathbb{D})$ and with the continuity of the map $L^p(\mathbb{S}^1) \ni f \mapsto u(f) \in L^p(\mathbb{D})$. \square

By combining the above result with Lemma 5.3, we obtain

8.4 Lemma. With u_0 as above and $\beta \in L^\infty(\mathbb{D})$, the map

$$\mathcal{F} : H_0^1(\mathbb{D}; \mathbb{C}) \times H^{1/2}(\mathbb{S}^1; \mathbb{R}) \rightarrow L^p(\mathbb{D}; \mathbb{C}), \quad H_0^1(\mathbb{D}; \mathbb{C}) \times H^{1/2}(\mathbb{S}^1; \mathbb{R}) \ni (v, \psi) \xrightarrow{\mathcal{F}} \frac{1}{4} \int_{\mathbb{D}} \beta \left(1 - |v + u(u_0 e^{i\psi})|^2\right)^2,$$

is C^1 , and, if we set $u := v + u(u_0 e^{i\psi})$, then

$$\mathcal{F}'(v, \psi)(w, \eta) = - \int_{\mathbb{D}} \beta u \cdot (w + u(iu_0 e^{i\psi} \eta)) (1 - |u|^2).$$

8.5 Lemma. The map

$$\mathcal{F} : H^{1/2}(\mathbb{S}^1; \mathbb{R}) \rightarrow \mathbb{R}, \quad H^{1/2}(\mathbb{S}^1; \mathbb{R}) \ni \psi \xrightarrow{\mathcal{F}} \frac{1}{2} \int_{\mathbb{D}} |\nabla u(u_0 e^{i\psi})|^2,$$

is C^1 , and, if $u = u(u_0 e^{i\psi})$, then

$$\mathcal{F}'(\psi)(\eta) = \int_{\mathbb{S}^1} \left(u \wedge \frac{\partial u}{\partial v} \right) \eta = \int_{\mathbb{D}} (u \wedge \nabla u) \cdot \nabla \zeta, \quad \forall \psi, \eta \in H^{1/2}(\mathbb{S}^1; \mathbb{R}), \quad \forall \zeta \in H^1(\mathbb{D}; \mathbb{R}) \text{ such that } \text{tr } \zeta = \eta. \quad (8.1)$$

Proof. The equality of the two integrals in (8.1) is justified as in the proof of (5.9).

In order to prove that $\mathcal{F}'(\psi)(\eta) = \mathcal{T}(\psi)(\eta)$, where $\mathcal{T}(\psi)(\eta) := \int_{\mathbb{D}} (u \wedge \nabla u) \cdot \nabla u(\eta)$, we rely on Lemma 5.2. The map $\psi \mapsto u \wedge \nabla u \in L^2(\mathbb{D})$ is continuous,¹⁸ and thus $\mathcal{T} \in C(H^{1/2}; H^{-1/2})$. On the other hand, it is clear that \mathcal{F} is continuous. Therefore, it remains to prove that (8.1) holds when ψ and η are smooth. By replacing u_0 with $u_0 e^{i\psi}$, we may assume that $\psi = 0$. We have

$$u_0 e^{it\eta} = u_0(1 + it\eta) + R(t), \quad \text{with } \|R(t)\|_{H^{1/2}} \leq Ct^2 \text{ as } t \rightarrow 0,$$

and thus

$$\mathcal{F}(t\eta) = \frac{1}{2} \int_{\mathbb{D}} |\nabla u|^2 + t \int_{\mathbb{D}} \nabla u \cdot \nabla u(iu_0 \eta) + S(t), \quad \text{with } \|S(t)\|_{H^{1/2}} \leq Ct^2 \text{ as } t \rightarrow 0. \quad (8.2)$$

Using (8.2) and (5.11) with $\beta = 0$, we find

$$\frac{\partial \mathcal{F}}{\partial \eta}(0) = \int_{\mathbb{D}} \nabla u \cdot \nabla(iu_0 \eta) = \int_{\mathbb{S}^1} \left(u \wedge \frac{\partial u}{\partial v} \right) \eta = \mathcal{T}(0)(\eta). \quad \square$$

An immediate consequence of Lemma 8.5 is the analog of Lemma 5.6.

8.6 Lemma. J^\sharp is C^1 . With $u = v + u(N_0 e^{i\psi})$ and $U = u(N_0 e^{i\psi})$ we have

$$J^\sharp(v, \psi)(w, \eta) = \int_{\mathbb{D}} \nabla v \cdot \nabla w + \int_{\mathbb{D}} (U \wedge \nabla U) \cdot \nabla \zeta - \int_{\mathbb{D}} \beta u \cdot (w + u(iN_0 e^{i\psi} \eta)) (1 - |u|^2) \quad (8.3)$$

for every $\zeta \in H^1(\mathbb{D})$ such that $\text{tr } \zeta = \eta$.

¹⁸This argument was already used in the proof of Lemma 5.5.

Proof. It suffices to note that $J^\sharp(v, \psi) = \mathcal{F}(\psi) + \frac{1}{2} \int_{\mathbb{D}} |\nabla v|^2$, where \mathcal{F} is the map from Lemma 8.5.¹⁹ \square

We next turn to the properties of the Palais-Smale sequences associated to J^\sharp . Note that an application of Theorem 5.10 in conjunction with Lemma 8.1 leads to a sequence (v_n, ψ_n) such that, with $u_n^* := v_n + u(N_0 e^{i\psi_n}) \in \mathcal{G}$, we have

$$J^{\sharp'}(v_n, \psi_n) \rightarrow 0, \quad J^\sharp(v_n, \psi_n) \rightarrow c^\sharp, \quad u_n^* \rightharpoonup u \text{ in } H^1. \quad (8.4)$$

In the asymptotic analysis of the Palais-Smale sequences, the smoothness of u_n^* is not sufficient. In the remaining part of this section, we find a better Palais-Smale sequence and establish its main properties. More specifically, we let $u_n := v + u(N_0 e^{i\psi_n})$, where $v \in H_0^1(\mathbb{D})$ is such that $v_n \rightharpoonup v$. In the original variables of J^\sharp , this amounts to replacing v_n by v . The next result is the analog of Corollary 5.11.

8.7 Lemma. Assume that (v_n, ψ_n) satisfy (8.4), with $u_n^* := v_n + u(N_0 e^{i\psi_n})$, and that $v_n \rightharpoonup v$. Then, letting

$$u_n := v + u(N_0 e^{i\psi_n}),$$

the sequence (u_n) has the following properties.

1. u is a solution of (2.21) and $-\Delta v = \beta u(1 - |u|^2)$.
2. $|u_n| \leq C$.
3. $-\Delta u_n - \beta u_n(1 - |u_n|^2) \rightarrow 0$ as $n \rightarrow \infty$ in every L^p , $p < \infty$.
4. $u_n - u_n^* \rightarrow 0$ in $H^1(\mathbb{D})$. In particular, $u_n \rightharpoonup u$ and $F_\beta(u_n) \rightarrow c^\sharp$.
5. (u_n) is a Palais-Smale sequence.
6. There exists a sequence $c_n \rightarrow 0$ such that

$$\left| \int_{\mathbb{D}} (u_n \wedge \nabla u_n) \cdot \nabla \zeta \right| \leq c_n \|\nabla \zeta\|_{L^2}, \quad \forall \zeta \in H^1(\mathbb{D}; \mathbb{R}). \quad (8.5)$$

Proof. All points except 6 are straightforward. Indeed, let $U_n := u(N_0 e^{i\psi_n})$. Since $u_n^* = v_n + U_n \rightharpoonup u$, we find that $U_n \rightharpoonup U$ for some harmonic $U \in \mathcal{G}$, and thus $v_n \rightharpoonup v := u - U$. Using the fact that (u_n^*) is a PS-sequence, we find, by passing to the limits in (8.3),²⁰ that $-\Delta v = \beta u(1 - |u|^2)$, whence the last assertion in item 1. Item 2 follows by combining the inequality $|U_n| \leq 1$ with the fact that v is smooth.²¹ Item 3 follows from the fact that $u_n \rightarrow u$ in L^p for every $p < \infty$. This item implies that $u_n \rightarrow u$ in $W^{2,p}$, $p < \infty$, and thus item 4. Item 5 is an easy consequence of the fact that $v_n \rightarrow v$ in H^1 (which follows from item 4) combined with the fact that $u_n^* - u_n \rightarrow 0$ in every L^p .

We now turn to property 6, which is at the heart of the lemma. Since $\Delta U_n = 0$ in \mathbb{D} , we have

$$\int_{\mathbb{D}} (U_n \wedge \nabla U_n) \cdot \nabla \zeta = \int_{\mathbb{D}} \operatorname{div}((U_n \wedge \nabla U_n)\zeta) = \int_{\mathbb{S}^1} \frac{\partial U_n}{\partial v} \cdot (iU_n \eta) = \int_{\mathbb{D}} \nabla U_n \cdot \nabla u(iN_0 e^{i\psi_n} \eta).$$

¹⁹With $u_0 = N_0$.

²⁰(8.3) is applied to the couple (v_n, ψ_n) , with $\eta = 0$ and for fixed w .

²¹Here, we use Lemmas 2.16 and 2.15.

We plug this into (8.3) and integrate by parts to find, with $\zeta \in H^1(\mathbb{D})$ such that $\text{tr } \zeta = \eta$, that

$$\begin{aligned}
\mathcal{J}^{\sharp'}(v, \psi_n)(0, \eta) &= \int_{\mathbb{D}} \nabla U_n \cdot \nabla u(\iota N_0 e^{i\psi_n} \eta) - \int_{\mathbb{D}} \beta u_n \cdot u(\iota N_0 e^{i\psi_n} \eta)(1 - |u_n|^2) \\
&= \int_{\mathbb{D}} \nabla u_n \cdot \nabla u(\iota N_0 e^{i\psi_n} \eta) - \int_{\mathbb{D}} \beta u_n \cdot u(\iota N_0 e^{i\psi_n} \eta)(1 - |u_n|^2) \\
&= \int_{\mathbb{S}^1} u_n \wedge \frac{\partial u_n}{\partial v} \eta - \int_{\mathbb{D}} (\Delta u_n + \beta u_n(1 - |u_n|^2)) \cdot u(\iota N_0 e^{i\psi_n} \eta) \\
&= \int_{\mathbb{D}} (u_n \wedge \nabla u_n) \cdot \nabla \zeta + \int_{\mathbb{D}} u_n \wedge \Delta u_n \zeta - \int_{\mathbb{D}} (\Delta u_n + \beta u_n(1 - |u_n|^2)) \cdot u(\iota N_0 e^{i\psi_n} \eta).
\end{aligned}$$

The above holds calculation is valid for smooth η and ζ . By density, it still holds for every $\eta \in H^{1/2}(\mathbb{S}^1; \mathbb{R})$ and every $\eta \in H^1(\mathbb{D}; \mathbb{R})$ such that $\text{tr } \zeta = \eta$. In particular, we have

$$\left| \int_{\mathbb{D}} (u_n \wedge \nabla u_n) \cdot \nabla \zeta \right| \leq \left| \mathcal{J}^{\sharp'}(v, \psi_n)(0, \eta) \right| + \int_{\mathbb{D}} |u_n \wedge \Delta u_n| |\zeta| + \int_{\mathbb{D}} |\Delta u_n + \beta u_n(1 - |u_n|^2)| |u(\iota N_0 e^{i\psi_n} \eta)|.$$

If we take, in the above inequality, ζ with zero mean and $\eta = \text{tr } \zeta$, then we obtain the conclusion of item 6, using the Poincaré inequality, the bounds

$$|\eta|_{H^{1/2}} \leq C \|\nabla \zeta\|_{L^2} \text{ and } \|u(\iota N_0 e^{i\psi_n} \eta)\|_{L^p} \leq C |\eta|_{H^{1/2}}, \quad \forall p < \infty,$$

and items 3 and 5.

Finally, the first assertion in item 1 follows by combining 6 with the second part of 1. \square

8.8 Definition. In the following we will denote by the term *bubble* either a Moebius transform or the conjugate of a Moebius transform. We will denote such a bubble by \mathcal{B}_a , with $a \in \mathbb{D}$. Thus either $\mathcal{B}_a = M_a$ or $\mathcal{B}_a = \overline{M_a}$.

A *multi-bubble* is either a (non trivial) Blaschke product, or the conjugate of such a product.

The key result is the following.

8.9 Lemma. Let $(v_n) \subset \mathcal{G}$ be a sequence of harmonic functions with the following properties:

$$v_n \rightarrow 1 \quad \text{in } H^1(\mathbb{D}); \tag{8.6}$$

$$\frac{1}{2} \int_{\mathbb{D}} |\nabla v_n|^2 \rightarrow K\pi; \tag{8.7}$$

$$\left| \int_{\mathbb{D}} (v_n \wedge \nabla v_n) \cdot \nabla \zeta \right| \leq c_n \|\nabla \zeta\|_{L^2}, \quad \forall \zeta \in H^1(\mathbb{D}; \mathbb{R}), \text{ with } c_n \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{8.8}$$

Then K is an integer and, up to a subsequence, there exist points $a_1(n), \dots, a_K(n)$ and corresponding bubbles $\mathcal{B}_{a_j(n)}$, $j \in \llbracket 1, K \rrbracket$, such that

$$|a_j(n)| \rightarrow 1 \text{ as } n \rightarrow \infty, \quad \forall j \in \llbracket 1, K \rrbracket; \tag{8.9}$$

$$v_n - \prod_{j=1}^K \mathcal{B}_{a_j(n)} \rightarrow 0 \text{ strongly in } H^1(\mathbb{D}) \text{ as } n \rightarrow \infty. \tag{8.10}$$

Proof. The proof is by induction on the integer part $[K]$ of K .

Step 1. Case where $K < 1$.

In this case, we will prove that $v_n \rightarrow 1$ strongly in $H^1(\mathbb{D})$. By [23, Corollary 3.7], if $K < 1$ then $|v_n| \geq a > 0$ for large n and some constant a . Thus we may write, globally in \mathbb{D} , $v_n = \rho_n e^{i\varphi_n}$ (Lemma 2.7 3). If we take $\zeta = \varphi_n$ in (8.8), then we find that $\nabla\varphi_n \rightarrow 0$ in L^2 . Since v_n is harmonic, we have

$$\begin{cases} \Delta\rho_n = \rho_n |\nabla\varphi_n|^2 & \text{in } \mathbb{D} \\ \rho_n = 1 & \text{on } \mathbb{S}^1, \end{cases} \text{ and therefore } \rho_n \rightarrow 1 \text{ strongly in } H^1(\mathbb{D}).^{22}$$

We conclude *via* (2.10) that $v_n \rightarrow 1$ strongly in H^1 , i.e., that (8.10) holds with $K = 0$.

Step 2. Induction step.

Assume that $[K] \geq 1$. Then the convergence of (v_n) , or any subsequence, is not strong hence there is no $a > 0$ such that $|v_n| \geq a$ holds along a subsequence. Thus we may pick $z_n \in \mathbb{D}$ such that $v_n(z_n) \rightarrow 0$. In view of (8.6), we have $|z_n| \rightarrow 1$. Let $w_n = v_n \circ M_{-z_n}$. The new sequence (w_n) satisfies (8.7), (8.8) and $w_n(0) \rightarrow 0$.

Let w be the weak limit of (w_n) . Then w is harmonic, $w(0) = 0$ and, by (8.8), w satisfies (2.21) with $\beta = 0$. In view of Lemma 3.5, there exist $\alpha_1, \dots, \alpha_L$ in \mathbb{D} such that

$$w = M_{\alpha_1} \dots M_{\alpha_L}, \quad \text{or} \quad \bar{w} = M_{\alpha_1} \dots M_{\alpha_L}. \quad (8.11)$$

In particular, $\frac{1}{2} \int_{\mathbb{D}} |\nabla w|^2 = L\pi$, with L a positive integer. Let $f := \text{tr } w$ and $g_n := \text{tr } (w_n \bar{w})$. Then $g_n \rightarrow 1$ in $H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ and we have $w_n = u(f g_n)$ and $h_n := u(g_n) \rightarrow 1$ in $H^1(\mathbb{D})$ (and $h_n \rightarrow 1$ in $C_{loc}^1(\mathbb{D})$.) By Lemmas 2.21 and 2.23, we have $w_n - w h_n \rightarrow 0$ in $H^1(\mathbb{D})$ and $\frac{1}{2} \int_{\mathbb{D}} |\nabla h_n|^2 \rightarrow (K - L)\pi$. We now turn back to the original sequence (v_n) . Set $t_n := w \circ M_{z_n}$, $y_n := h_n \circ M_{z_n}$. Then we have:

1. $v_n - t_n y_n \rightarrow 0$ in $H^1(\mathbb{D})$.
2. Letting $\alpha_j(n) = M_{-z_n}(\alpha_j)$, we have $|\alpha_j(n)| \rightarrow 1$ and either $t_n = \prod_j M_{\alpha_j(n)}$ or $\bar{t}_n = \prod_j M_{\alpha_j(n)}$.
3. $\frac{1}{2} \int_{\mathbb{D}} |\nabla y_n|^2 \rightarrow (K - L)\pi$.
4. $y_n \rightarrow 1$ in $H^1(\mathbb{D})$ (since $v_n \rightarrow 1$ and $w \circ M_{z_n} \rightarrow 1$).

We are in position to complete the proof of Lemma 8.9 by applying the induction hypothesis to the sequence (y_n) provided we establish the validity of (8.8) for the sequence (y_n) . By conformal invariance, (8.8) holds for (w_n) . On the other hand, (8.11) implies that

$$\int_{\mathbb{D}} (w \wedge \nabla w) \cdot \nabla \zeta = 0.$$

We recall that $w_n = w h_n + r_n$, where $r_n \rightarrow 0$ in $H^1(\mathbb{D})$. Since $\Delta r_n = -2\nabla w \cdot \nabla h_n \rightarrow 0$ weakly in $L^2(\Omega)$, we also have $r_n \rightarrow 0$ in $L^\infty(\mathbb{D})$. Therefore

$$w_n \wedge \nabla w_n = |h_n|^2 (w \wedge \nabla w) + |w|^2 (h_n \wedge \nabla h_n) + F_n,$$

²²This is obtained by multiplying by $\rho_n - 1$ the equation of ρ_n .

with $F_n \in L^2(\mathbb{D}; \mathbb{R}^2)$ satisfying $\|F_n\|_{L^2} \rightarrow 0$. From the above, we find that

$$\begin{aligned}
\int_{\mathbb{D}} (h_n \wedge \nabla h_n) \cdot \nabla \zeta &= \int_{\mathbb{D}} (w_n \wedge \nabla w_n) \cdot \nabla \zeta + \int_{\mathbb{D}} (1 - |w|^2)(h_n \wedge \nabla h_n) \cdot \nabla \zeta \\
&\quad + \int_{\mathbb{D}} (1 - |h_n|^2)(w \wedge \nabla w) \cdot \nabla \zeta - \int_{\mathbb{D}} (w \wedge \nabla w) \cdot \nabla \zeta - \int_{\mathbb{D}} F_n \cdot \nabla \zeta \\
&= \int_{\mathbb{D}} (w_n \wedge \nabla w_n) \cdot \nabla \zeta + \int_{\mathbb{D}} (1 - |w|^2)(h_n \wedge \nabla h_n) \cdot \nabla \zeta \\
&\quad + \int_{\mathbb{D}} (1 - |h_n|^2)(w \wedge \nabla w) \cdot \nabla \zeta - \int_{\mathbb{D}} F_n \cdot \nabla \zeta.
\end{aligned} \tag{8.12}$$

The fact that $h_n \rightarrow 1$ in $C_{loc}^1(\mathbb{D})$ and $|w(z)| \rightarrow 1$ as $|z| \rightarrow 1$ easily implies that

$$\left| \int_{\mathbb{D}} (1 - |w|^2)(h_n \wedge \nabla h_n) \cdot \nabla \zeta \right| + \left| \int_{\mathbb{D}} (1 - |h_n|^2)(w \wedge \nabla w) \cdot \nabla \zeta \right| \leq c_n \|\nabla \zeta\|_{L^2}, \quad \text{with } c_n \rightarrow 0. \tag{8.13}$$

(8.12) and (8.13) imply that (8.8) is satisfied by (h_n) . By conformal invariance, the same holds for (y_n) . \square

8.10 Remark. Lemma 8.9 is about bubbling of harmonic functions in the unit disc \mathbb{D} , and clearly this analysis extends to the case of simply connected domains. It is still possible to study the case of multiply connected domains. Such analysis, which is not relevant for the subsequent results in this section, is postponed to Section 9.

8.11 Lemma. Let (v_n, ψ_n) be a Palais-Smale sequence associated to \mathcal{J}^\sharp , and set $u_n^* := v_n + u(N_0 e^{i\psi_n})$. Assume that $u_n \rightarrow u$ in $H^1(\mathbb{D})$. Set $g_n = u_n^* \bar{u}$ and $w_n = u(g_n)$. Then

1. $u_n^* - u w_n \rightarrow 0$ strongly in $H^1(\mathbb{D})$ as $n \rightarrow \infty$.
2. $F_\beta(u_n^*) = F_\beta(u) + \frac{1}{2} \int_{\mathbb{D}} |\nabla w_n|^2 + c_n$, with $c_n \rightarrow 0$ as $n \rightarrow \infty$.
3. The sequence (w_n) satisfies the assumptions of Lemma 8.9.

Proof. Let $u_n = v + u(N_0 e^{i\psi_n})$, where $v = \lim_n v_n$ as in Lemma 8.7. By Lemma 8.7, item 1 amounts to proving that $z_n := u w_n - u_n$ converges strongly to 0 in $H^1(\mathbb{D})$. This convergence is obtained via Remark 2.22. Indeed, since u solves (2.20), since from Lemma 8.7 we have $-\Delta u_n = -\Delta v = \beta u(1 - |u|^2)$, and since w_n is harmonic, the map z_n solves

$$\begin{cases} -\Delta z_n = \beta u(w_n - 1)(1 - |u|^2) - 2\nabla u \cdot \nabla w_n & \text{in } \mathbb{D} \\ z_n = 0 & \text{on } \mathbb{S}^1. \end{cases}$$

We turn to item 2. By combining item 1 with the fact that $w_n \rightarrow 1$ weakly in H^1 and strongly in L^p , we note that item 2 amounts to proving that, as $n \rightarrow +\infty$,

$$\int_{\mathbb{D}} |\nabla(u w_n)|^2 = \int_{\mathbb{D}} |\nabla u|^2 + \int_{\mathbb{D}} |\nabla w_n|^2 + o(1). \tag{8.14}$$

The starting point in the proof of (8.14) is the identity

$$|\nabla(u w_n)|^2 = |u|^2 |\nabla w_n|^2 + |\nabla u|^2 |w_n|^2 + 2(u \nabla w_n \cdot w_n \nabla u).$$

Since w_n is harmonic and $w_n \rightarrow 1$ in H^1 we have $w_n \rightarrow 1$ in C_{loc}^1 , while $|u(x)| \rightarrow 1$ uniformly as $x \rightarrow \partial\mathbb{D}$. Therefore

$$\int |u|^2 |\nabla w_n|^2 = \int |\nabla w_n|^2 + o(1) \quad \text{and} \quad \int |\nabla u|^2 |w_n|^2 = \int |\nabla u|^2 + o(1).$$

Finally, using Remark 2.22 we find that

$$\int (u \nabla w_n) \cdot (w_n \nabla u) = o(1) \quad \text{as } n \rightarrow \infty.$$

This proves (8.14) and item 2.

As for item 3, we start from the identity

$$\int_{\mathbb{D}} ((u w_n) \wedge \nabla(u w_n)) \cdot \nabla \zeta = \int_{\mathbb{D}} (|u|^2 - 1)(w_n \wedge \nabla w_n) \cdot \nabla \zeta + \int_{\mathbb{D}} (w_n \wedge \nabla w_n) \cdot \nabla \zeta + \int_{\mathbb{D}} |w_n|^2 (u \wedge \nabla u) \cdot \nabla \zeta$$

and argue as in the proof of Lemma 8.9. \square

A straightforward combination of the two preceding lemmas implies the main result of this section.

8.12 Theorem. *Let (v_n, ψ_n) be a Palais-Smale sequence associated to J^\sharp and set $u_n^* := v_n + u(N_0 e^{i\psi_n})$. Then, up to a subsequence, there exist: a critical point u of F_β in \mathcal{G} , an integer K , points $a_1(n), \dots, a_K(n)$ and corresponding bubbles $\mathcal{B}_{a_j(n)}$, $j \in \llbracket 1, K \rrbracket$, such that*

$$|a_j(n)| \rightarrow 1 \text{ as } n \rightarrow \infty, \quad \forall j \in \llbracket 1, K \rrbracket; \tag{8.15}$$

$$u_n^* - u \prod_{j=1}^K \mathcal{B}_{a_j(n)} \rightarrow 0 \text{ strongly in } H^1(\mathbb{D}) \text{ as } n \rightarrow \infty; \tag{8.16}$$

$$F_\beta(u_n^*) = F_\beta(u) + K\pi + c_n, \quad \text{with } c_n \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{8.17}$$

In particular, we have (with c given by (8.4))

$$c = F_\beta(u) + K\pi. \tag{8.18}$$

Proof. From Lemma 8.11, we have $\|u_n^* - u w_n\|_{H^1} \rightarrow 0$, where (w_n) satisfies the hypothesis of Lemma 8.9. In order to conclude, it suffices to apply Lemma 8.9 to (w_n) . \square

This type of result has proven useful in many variational settings, especially in geometry. Let us simply mention the pioneering work of Sacks and Uhlenbeck [32] about minimal 2-spheres, the analysis of Brezis and Coron [16] of constant mean curvature surfaces, or the one of Struwe [34] of equations involving the critical Sobolev exponent. There are also abstract approaches to bubbling as in the work of Lions [28] about concentration-compactness or the characterization of noncompactness of critical embeddings in Gérard [24], Jaffard [27] or Bahouri, Cohen and Koch [2].

We end with the following obvious consequence of our analysis of the Palais-Smale sequences.

8.13 Theorem. *In the setting of Section 5, let $c^* > \pi$ be defined by (5.5). Assume that (6.1) holds, i.e., $c^* < 2\pi$. Assume in addition that*

$$\text{there is no solution } u \in \mathcal{G} \text{ of (2.21) such that } F_\beta(u) = c^* - \pi. \tag{8.19}$$

Then F_β has a critical point $u \in \mathcal{G}_1$.

In addition, F_β satisfies the $(PS)_c$ condition at the level $c = c^*$.

Theorem 8.13 combined with our next result implies Theorem 6.1 (and thus Theorem 1.1).

8.14 Lemma. Assume that (5.3) and (5.5) hold. Let u be a critical point of F_β in \mathcal{G} such that $F_\beta(u) < \pi$. Then u is a constant (and thus $F_\beta(u) = 0$).

Proof. Assumption (5.3) implies that critical points of F_β are actually minimizers of F_β with respect to their own boundary conditions. We next argue as in Step 1 in the proof of Lemma 8.9. If u is such a minimizer and $F_\beta(u) < \pi$, then there exists some $a > 0$ such that $|u| \geq a$ [23, Corollary 3.7]. Then we may write, globally in \mathbb{D} , $u = \rho e^{i\varphi}$ (Lemma 2.7 3). Assume, in addition, that u is a critical point of F_β . If we take $\zeta = \varphi$ in (2.20), then we find that φ is constant, say $\varphi = 0$. Then ρ satisfies

$$\begin{cases} -\Delta\rho = \beta\rho(1 - \rho^2) & \text{in } \mathbb{D} \\ \rho = 1 & \text{on } \mathbb{S}^1. \end{cases}$$

By multiplying this equation with $\rho - 1$, we find that $\rho \equiv 1$. \square

Open Problem 1. Let $u \in \mathcal{G}$ be a critical point of F_β . Assume that $F_\beta(u) < \pi$.²³ Is it true that u is a constant? More generally, does the same hold if we replace the smallness assumption $F_\beta(u) < \pi$ by the weaker assumption $\frac{1}{2} \int_{\mathbb{D}} |\nabla u|^2 < \pi$?

9 Bubbling analysis in multiply connected domains

In this section, we establish the analog of Theorem 8.12 in multiply connected domains Ω . To start with, this requires defining Palais-Smale sequences and bubbles. In defining Palais-Smale sequences, we can take as a starting point either Lemma 8.6 (and define a sequence (u_n^*)) or Lemma 8.7 (and define a sequence (u_n)). We adopt here the latter point of view.²⁴

9.1 Definition. Let $\beta \in L^\infty(\Omega)$. A sequence $(u_n) \subset \mathcal{E}$ is a Palais-Smale sequence for F_β if there exists a sequence $c_n \rightarrow 0$ such that

$$\left| \int_{\Omega} \nabla u_n \cdot \nabla w - \int_{\Omega} \beta u_n \cdot w(1 - |u_n|^2) \right| \leq c_n \|\nabla w\|_{L^2}, \quad \forall w \in H_0^1(\Omega; \mathbb{C}); \quad (9.1)$$

$$\left| \int_{\Omega} (u_n \wedge \nabla u_n) \cdot \nabla \zeta \right| \leq c_n \|\nabla \zeta\|_{L^2}, \quad \forall \zeta \in H^1(\Omega; \mathbb{R}). \quad (9.2)$$

We next define bubbles.

9.2 Definition. Let Γ_ℓ , $\ell \in \llbracket 1, L \rrbracket$, be the components of $\partial\Omega$. We assume that Γ_1 encloses Ω . Let ω_1 be the interior of Γ_1 , and for $\ell \geq 2$ let ω_ℓ be the exterior of Γ_ℓ in the extended plane $\mathbb{C} \cup \{\infty\}$. For each ℓ , fix a conformal representation $\Phi_\ell : \omega_\ell \rightarrow \mathbb{D}$. For $a \in \mathbb{D}$ and $\ell \in \llbracket 1, L \rrbracket$, the corresponding bubble \mathcal{B}_a^ℓ is defined as either $\mathcal{B}_a^\ell = M_{1,a,\Phi_\ell}$, or $\mathcal{B}_a^\ell = \overline{M_{1,a,\Phi_\ell}}$.

Note that, unlike the case of simply connected domains, bubbles do not belong to \mathcal{E} . However, as $a \rightarrow \Gamma_\ell$, the trace of \mathcal{B}_a^ℓ almost fulfills the condition $|\text{tr } \mathcal{B}_a^\ell| = 1$.

The analog of Theorem 8.12 is

²³But we do not make any smallness assumption on β . In particular, we do not assume (5.3).

²⁴An inspection of the proof of Lemma 8.7 shows that passing from (u_n^*) to (u_n) relies on Lemma 2.20, which is valid in arbitrary domains.

9.3 Theorem. *Let (u_n) be a Palais-Smale sequence. Then, up to a subsequence, there exist: a critical point u of F_β in \mathcal{E} , an integer K , indices $\ell_1, \dots, \ell_K \in \llbracket 1, L \rrbracket$, points $a_1(n), \dots, a_K(n) \in \Omega$ and corresponding bubbles $\mathcal{B}_{a_j(n)}^{\ell_j}$, $j \in \llbracket 1, K \rrbracket$, such that*

$$\text{dist}(a_j(n), \Gamma_{\ell_j}) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad \forall j \in \llbracket 1, K \rrbracket; \quad (9.3)$$

$$u_n - u \prod_{j=1}^K \mathcal{B}_{a_j(n)}^{\ell_j} \rightarrow 0 \text{ strongly in } H^1(\Omega) \text{ as } n \rightarrow \infty; \quad (9.4)$$

$$F_\beta(u_n) = F_\beta(u) + K\pi + c_n, \quad \text{with } c_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (9.5)$$

Proof. Up to a subsequence, we have $u_n \rightharpoonup u \in \mathcal{E}$. We define $g_n^\ell : \partial\Omega \rightarrow \mathbb{S}^1$ by

$$g_n^\ell = \begin{cases} \text{tr}(u_n/u) & \text{on } \Gamma_\ell \\ 1 & \text{on } \partial\Omega \setminus \Gamma_\ell \end{cases},$$

so that $g_n := \text{tr}(u_n/u)$ satisfies $g_n = \prod_{\ell=1}^L g_n^\ell$. Let $v_n := u(g_n)$ and $v_n^\ell := u(g_n^\ell)$. An inspection of the proof of Lemma 8.11 shows that the conclusions of the lemma hold in our case.²⁵ We complete the proof of Theorem 9.3 by combining the conclusions of Lemma 8.11 with the next couple of lemmas and with Lemma 8.9. \square

We first introduce two useful objects. Let w_n^ℓ denote the harmonic extension to ω_ℓ of the trace of g_n on Γ_ℓ . We also set $W_n^\ell : \mathbb{D} \rightarrow \mathbb{C}$, $W_n^\ell := w_n^\ell \circ (\Phi_\ell)^{-1}$.

9.4 Lemma. We have

$$\int_{\Omega} |\nabla v_n|^2 = \sum_{\ell=1}^L \int_{\Omega} |\nabla w_n^\ell|^2 + c_n, \quad \text{with } c_n \rightarrow 0 \text{ as } n \rightarrow \infty \quad (9.6)$$

and

$$v_n - \prod_{\ell=1}^L w_n^\ell \rightarrow 0 \quad \text{strongly in } H^1(\Omega) \text{ as } n \rightarrow \infty. \quad (9.7)$$

Proof. By standard estimates, the fact that $g_n^\ell \rightarrow 1$ in $H^{1/2}(\partial\Omega)$ and $g_n^\ell = 1$ on $\partial\Omega \setminus \Gamma_\ell$ implies that

$$v_n^\ell \rightarrow 1 \quad \text{in } C_{loc}^k(\overline{\Omega} \setminus \Gamma_\ell). \quad (9.8)$$

The same holds for w_n^ℓ . It is then clear that $v_n - v_n^\ell \rightarrow 0$ in $C_{loc}^k(\overline{\Omega} \setminus \Gamma_\ell)$, and this leads to the expansion

$$\int_{\Omega} |\nabla v_n|^2 = \sum_{\ell=1}^L \int_{\Omega} |\nabla v_n^\ell|^2 + c_n, \quad \text{with } c_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

and to the strong convergence

$$v_n - \prod_{\ell=1}^L v_n^\ell \rightarrow 0 \quad \text{in } C^\infty(\overline{\Omega}). \quad (9.9)$$

By the above, in order to conclude it suffices to invoke the fact that $|w_n^\ell| \leq 1$ (Lemma 2.16) and to prove that

$$w_n^\ell - v_n^\ell \rightarrow 0 \quad \text{strongly in } H^1(\Omega). \quad (9.10)$$

In turn, (9.10) is obtained by noting that $y_n^\ell := w_n^\ell - v_n^\ell$ is harmonic and that $\text{tr } y_n^\ell \rightarrow 0$ in $H^{1/2}(\partial\Omega)$. \square

²⁵With u_n instead of u_n^* .

9.5 Lemma. Up to a subsequence, the sequence $(W_n^\ell)_n$ satisfies the assumptions of Lemma 8.9.

As a consequence, up to a subsequence there are integers $K(\ell)$ and points $a_1(n), \dots, a_{K(\ell)}(n) \in \Omega$ such that:

1. $\text{dist}(a_j(n), \Gamma_\ell) \rightarrow 0$ as $n \rightarrow \infty$, $\forall j \in \llbracket 1, K(\ell) \rrbracket$.
2. $\frac{1}{2} \int_{\Omega} |\nabla w_n^\ell|^2 \rightarrow K(\ell)\pi$.
3. $w_n^\ell - \prod_{j=1}^{K(\ell)} \mathcal{B}_{a_j(n)}^\ell \rightarrow 0$ strongly in $H^1(\Omega)$.

Proof. We have to prove that the sequence $(W_n^\ell)_n$ satisfies (8.8). By conformal invariance, it suffices to check the same property for $(w_n^\ell)_n$. By (9.10), we are reduced to checking the same for $(v_n^\ell)_n$. Let $\Omega_\ell \subset \mathbb{C}$ be an open set such that $\Omega_\ell \cap \partial\Omega = \Gamma_\ell$. Let $\eta_\ell \in C_c^\infty(\Omega_\ell; \mathbb{R})$ be such that $\eta_\ell = 1$ in a neighborhood of Γ_ℓ . If $\zeta \in H^1(\Omega; \mathbb{R})$, set $\xi_\ell := \eta_\ell \zeta$, $\lambda_\ell := \zeta - \psi_\ell$. Assuming in addition that $\int_{\Omega} \zeta = 0$, we have the Poincaré type inequality

$$\|\nabla \xi_\ell\|_{L^2} + \|\nabla \lambda_\ell\|_{L^2} \leq C \|\nabla \zeta\|_{L^2}. \quad (9.11)$$

Using Lemma 8.11, (9.8) and (9.9), we find that

$$\begin{aligned} \left| \int_{\Omega} (v_n^\ell \wedge \nabla v_n^\ell) \cdot \nabla \xi_\ell \right| &\leq \left| \int_{\Omega_\ell} (v_n^\ell \wedge \nabla v_n^\ell) \cdot \nabla \xi_\ell \right| + \left| \int_{\Omega \setminus \Omega_\ell} (v_n^\ell \wedge \nabla v_n^\ell) \cdot \nabla \xi_\ell \right| \\ &\leq \left| \int_{\Omega_\ell} (v_n \wedge \nabla v_n) \cdot \nabla \xi_\ell \right| + c_n \int_{\Omega} |\nabla \xi_\ell| \leq c_n \|\nabla \xi_\ell\|_{L^2}, \quad \text{with } c_n \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (9.12)$$

Similarly, we have

$$\left| \int_{\Omega} (v_n^\ell \wedge \nabla v_n^\ell) \cdot \nabla \lambda_\ell \right| \leq c_n \|\nabla \lambda_\ell\|_{L^2}, \quad \text{with } c_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (9.13)$$

We obtain (8.8) for the sequence (v_n^ℓ) by combining (9.11)-(9.13).

Items 1 and 3 are obtained from the analog results for W_n^ℓ via composition with Φ_ℓ . As for item 2, it follows from (8.7) once we note that

$$\int_{\Omega} |\nabla w_n^\ell|^2 = \int_{\mathbb{D}} |\nabla W_n^\ell|^2 - \int_{\mathbb{D} \setminus (\Phi_\ell)^{-1}(\Omega)} |\nabla W_n^\ell|^2 = \int_{\mathbb{D}} |\nabla W_n^\ell|^2 + c_n, \quad \text{with } c_n \rightarrow 0 \text{ as } n \rightarrow \infty;$$

here, we use the fact that $W_n^\ell \rightarrow 1$ in $C_{loc}^k(\mathbb{D})$. □

We end with an application of the above analysis. We let $\Omega = \mathbb{D} \setminus \overline{\mathbb{D}}_R$, with $0 < R < 1$.²⁶ With $\mathbf{d} = (d_1, d_2) \in \mathbb{Z}^2$, we let

$$\mathcal{E}_{\mathbf{d}} := \{u \in \mathcal{E}; \deg(u, \mathbb{S}^1) = d_1, \deg(u, C_R) = d_2\}.$$

Our result is the following.

9.6 Theorem. Assume that R is sufficiently small. Then there exists some ε_0 such that, for $\varepsilon \in (\varepsilon_0, \infty]$, E_ε has critical points in $\mathcal{E}_{(1,0)}$.

²⁶Theorem 9.6 below extends to arbitrary doubly connected domains.

The above result is reminiscent of Coron's result on the existence of non trivial solution of the equation $-\Delta u = u^{(n+2)/(n-2)}$ in domains $\Omega \subset \mathbb{R}^n$ with small holes [21]. Since later Bahri and Coron [3] proved that the size of the hole is irrelevant for existence of such solutions, we address the following

Open Problem 2. Let $R \in (0, 1)$. Does there exist some $\varepsilon_0 > 0$ such that E_ε has critical points in $\mathcal{E}_{(1,0)}$ for $\varepsilon \in (\varepsilon_0, \infty]$?

Proof of Theorem 9.6. We start by describing the functional setting. We let, with $0 < r < 1$,

$$K = \overline{\mathbb{D}}_r, \quad K_0 = C_r, \quad X^\sharp = H_0^1(\Omega; \mathbb{C}) \times H^{1/2}(\partial\Omega; \mathbb{R}).$$

Let $N : \partial\Omega \rightarrow \mathbb{S}^1$, $N(z) = \begin{cases} z, & \text{if } |z| = 1 \\ 1, & \text{if } |z| = R \end{cases}$. Then we define

$$J^\sharp : X^\sharp \rightarrow \mathbb{R}, \quad H_0^1(\Omega; \mathbb{C}) \times H^{1/2}(\partial\Omega; \mathbb{R}) \ni (v, \psi) \xrightarrow{J^\sharp} E_\varepsilon(v + u(Ne^{i\psi})).$$

Note that $v + u(Ne^{i\psi}) \in \mathcal{E}_{(1,0)}$. For $a \in \mathbb{D}$, we write, as in Section 5, $N_a = N_0 e^{i\psi_a}$, and let $\eta_a = \begin{cases} \psi_a, & \text{on } \mathbb{S}^1 \\ 0, & \text{on } C_R \end{cases}$. We then set

$$\chi^\sharp \in C(K_0; X^\sharp), \quad K_0 \ni a \xrightarrow{\chi^\sharp} (0, \eta_a)$$

and define

$$c = c_{R,r,\varepsilon} = \inf \left\{ \max_K J^\sharp \circ F; F \in C(K; X^\sharp), F = \chi^\sharp \text{ on } K_0 \right\}.$$

The plan is to prove that, for small R and large ε , the compactness condition $(PS)_c$ is satisfied. This will imply existence of critical points of E_ε in $\mathcal{E}_{(1,0)}$.

To start with, note that, as in the proof of Lemma 3.1, we have

$$E_\varepsilon(u) \geq \frac{1}{2} \int_\Omega |\nabla u|^2 \geq \int_\Omega |\text{Jac } u| \geq \left| \int_\Omega \text{Jac } u \right| = \pi, \quad \forall u \in \mathcal{E}_{(1,0)}. \quad (9.14)$$

Before proceeding further, we establish few auxiliary results.

9.7 Lemma. Let $\delta > 0$. Then there exist $R_0 < 1$, $r_0 < 1$, and $\varepsilon_0 > 0$ such that $c < \pi + \delta$ provided $R < R_0$, $r > r_0$ and $\varepsilon > \varepsilon_0$.

We also have

$$\lim_{r \rightarrow 1} \max_{K_0} J^\sharp \circ \chi^\sharp = \pi. \quad (9.15)$$

Proof. Let $\delta' > 0$ to be fixed later. Pick R_0 such that for $R < R_0$ there exists some $\zeta_R \in C^\infty(\overline{\Omega}; [0, 1])$ such that $\zeta_R = 1$ near \mathbb{S}^1 , $\zeta_R = 0$ near C_R and $\int_\Omega |\nabla \zeta_R|^2 < \delta'$. This is always possible, since the H^1 -capacity of a point is zero. Define

$$P : \mathbb{D} \rightarrow \mathcal{E}_{(1,0)}, \quad D \ni a \xrightarrow{P} \zeta_R M_a + 1 - \zeta_R.$$

Let also $F(a) = (0, \eta_a)$, $a \in K$. Then $\text{tr } P(a) = \text{tr } u(Ne^{i\eta_a})$, and a straightforward calculation leads to

$$\begin{aligned} c &\leq \max_K \mathcal{J}^\sharp \circ F = \max_{a \in K} E_\varepsilon(u(Ne^{i\eta_a})) \leq \max_{a \in K} E_\varepsilon(P(a)) \\ &\leq \max_{a \in K} \left(\frac{1}{2} \int_\Omega (1 - \zeta_R)^2 |\nabla M_a|^2 + C(\delta' + \sqrt{\delta'} + \varepsilon^{-2}) \right) \leq \pi + C(\delta' + \sqrt{\delta'} + \varepsilon^{-2}). \end{aligned}$$

This implies the first part of the lemma. For the second part, let us note that, when $|a| \rightarrow 1$, we have $\bar{a}M_a \rightarrow -1$ in $C_{loc}^k(\mathbb{D})$, and this leads to

$$\limsup_{|a| \rightarrow 1} E_\varepsilon(P(a)) \leq \pi. \quad (9.16)$$

We conclude by combining (9.16) with (9.14). \square

The next result is the content of [5, Lemma D.3].

9.8 Lemma. We have

$$\inf\{E_\varepsilon(u); u \in \mathcal{E}_{(1,1)}\} \geq \min\{E_\infty(u); u \in \mathcal{E}_{(1,1)}\} = 2\pi \frac{1-R}{1+R}. \quad (9.17)$$

9.9 Lemma. Consider the minimization problem

$$m_\varepsilon = \inf\{E_\varepsilon(u); u \in \mathcal{E}_{(1,0)}\}. \quad (9.18)$$

Then

1. $m_\varepsilon = \pi$ and m_ε is not attained.
2. If (u_n) is a minimizing sequence in (9.18), then, up to a subsequence, $u_n \rightarrow \alpha \in \mathbb{S}^1$ in $H^1(\Omega)$.

Proof. Let $\Phi_t : \mathbb{C} \rightarrow \mathbb{C}$, $\Phi_t(z) = \begin{cases} z/t, & \text{if } |z| \leq t \\ z/|z| & \text{if } |z| \geq t \end{cases}$. It is easy to see that, when $a \in \mathbb{D}$ and $|a| \rightarrow 1$, we may

pick some $t = t(a) \rightarrow 1$ such that we have $\Phi_t \circ M_a \in \mathcal{E}_{(1,0)}$ and $E_\varepsilon(\Phi_t \circ M_a) \rightarrow \pi$. This together with (9.14) implies that $m_\varepsilon = \pi$. As in the proof of Lemma 3.1, if $E_\varepsilon(u) = \pi$, then u is holomorphic. However, we claim that there is no holomorphic map in $\mathcal{E}_{(1,0)}$. Indeed, otherwise we have:

- a) $u(\Omega) \subset \mathbb{D}$ (by the maximum principle).
- b) $u(\Omega) \supset \mathbb{D}$ (since the total degree of u on $\partial\Omega$ is one).

$$\text{c) } \int_\Omega |\text{Jac } u| = \pi.$$

By the above and the area formula, for a.e. $a \in \mathbb{D}$ the set $u^{-1}(a)$ is a singleton. This implies that u is one-to-one. In conclusion, $u : \Omega \rightarrow \mathbb{D}$ is a homeomorphism, which is the desired contradiction.

Consider now a minimizing sequence (u_n) and assume that $u_n \rightarrow u$. Since $u \notin \mathcal{E}_{(1,0)}$, we find that u is a constant, by the Price Lemma 2.17. This constant has to be of modulus one. \square

The next result is the analog of Theorem 4.1 in doubly connected domains. If $u \in H^1(\Omega)$, we define \tilde{u} as the harmonic extension to \mathbb{D} of the trace of u on \mathbb{S}^1 . In particular, if $u \in \mathcal{E}_{(1,0)}$, then $\tilde{u} \in \mathcal{E}_1$.

9.10 Lemma. There exists some $\delta_0 = \delta_0(R) > 0$ and a function $f : (0, \delta_0) \rightarrow (0, \infty)$ such that $\lim_{\delta \rightarrow 0} f(\delta) = 0$ with the following property: if $u \in \mathcal{E}_{(1,0)}$ satisfies $\frac{1}{2} \int_{\Omega} |\nabla u|^2 < \pi + \delta$, with $\delta < \delta_0$, then $\frac{1}{2} \int_{\mathbb{D}} |\nabla \tilde{u}|^2 < \pi + f(\delta)$.

In particular, if δ_0 is sufficiently small then there exists a continuous map

$$H : \left\{ u \in \mathcal{E}_{(1,0)}; \frac{1}{2} \int_{\Omega} |\nabla u|^2 < \pi + \delta_0 \right\} \rightarrow \mathbb{D}, \quad u \xrightarrow{H} a(\tilde{u}), \quad (9.19)$$

where $a(\tilde{u})$ is the unique zero of \tilde{u} .

Moreover, we have

$$|H(u)| \rightarrow 1 \quad \text{when} \quad \frac{1}{2} \int_{\mathbb{D}} |\nabla u|^2 \rightarrow \pi. \quad (9.20)$$

Proof. For the first part of the lemma, it suffices to prove that, if $(u_n) \subset \mathcal{E}_{(1,0)}$ and $\frac{1}{2} \int_{\Omega} |\nabla u_n|^2 \rightarrow \pi$, then (possibly along a subsequence) $\frac{1}{2} \int_{\mathbb{D}} |\nabla \tilde{u}_n|^2 \rightarrow \pi$. By (9.14), we may assume that u_n is harmonic. By Lemma 9.9 2, we have, up to a subsequence, $u_n \rightarrow \alpha$ in $C_{loc}^k(\Omega)$. Thus, for $R < r < 1$, the restriction of u_n to $\mathbb{D} \setminus \bar{\mathbb{D}}_r$ has an extension v_n to \mathbb{D} such that

$$\frac{1}{2} \int_{\mathbb{D}} |\nabla \tilde{u}_n|^2 \leq \frac{1}{2} \int_{\mathbb{D}} |\nabla v_n|^2 \leq \frac{1}{2} \int_{\Omega} |\nabla u_n|^2 + c_n \quad \text{with } c_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The second part of the theorem follows from Theorem 4.1. As for the last part, consider a sequence $(u_n) \subset \mathcal{E}_{(1,0)}$ such that $\frac{1}{2} \int_{\Omega} |\nabla u_n|^2 \rightarrow \pi$. By Lemma 9.9, we have, up to a subsequence, $u_n \rightarrow \alpha \in \mathbb{S}^1$, and thus $\tilde{u}_n \rightarrow \alpha$, whence the conclusion. \square

Proof of Theorem 9.6 completed. We start by proving that $c > \pi$. Indeed, let $\delta_0 = \delta_0(R)$ be such that H given by (9.19) is well-defined and satisfies $|H(u)| \geq 1/2$. We claim that $c \geq \pi + \delta_0$. Indeed, otherwise let $F \in C(K; X^\sharp)$ be such that $F = \chi^\sharp$ on K_0 and $J^\sharp \circ F < \pi + \delta_0$. Let

$$G : \bar{\mathbb{D}}_r \rightarrow \mathbb{D} \setminus \mathbb{D}_{1/2}, \quad G := H \circ U^\sharp \circ F.$$

Here,

$$U^\sharp : X^\sharp \rightarrow \mathcal{E}_{(1,0)}, \quad X^\sharp \ni (v, \psi) \xrightarrow{U^\sharp} v + u(Ne^{i\psi}).$$

By construction, G is continuous and we have $G = \text{Id}$ on C_r . This contradicts Brouwer's fixed point theorem.

The fact that $c > \pi$ combined with (9.15) implies that for r close to 1 we have

$$c > \max_{K_0} J^\sharp \circ \chi^\sharp, \quad (9.21)$$

and that c does not depend on r close to 1.

(9.21), combined with the proofs of Lemmas 8.6 and 8.7 and with Lemma 9.7 and Theorem 9.3 leads to the following: if $R < R_0$, $r > r_0$ and $\varepsilon > \varepsilon_0$, then:

1. $\pi < c < \min \left\{ 2\pi, \pi + 2\pi \frac{1-R}{1+R} \right\}$.

2. There exists a sequence $(u_n) \subset \mathcal{E}_{(1,0)}$ as in Theorem 9.3.²⁷

We complete the proof of Theorem 9.6 if we prove that the integer K in Theorem 9.3 is zero. Let u be as in Theorem 9.3, and let $\mathbf{D} \in \mathbb{Z}^2$ be such that $u \in \mathcal{E}_{\mathbf{D}}$. By Theorem 9.3 and the fact that $c < 2\pi$, one of the following cases occurs:

1. $K = 0$ and $\mathbf{D} = (1, 0)$, which is the desired conclusion.

2. $K = 1$, $\mathbf{D} = (1, 1)$ and $E_\varepsilon(u) = c - \pi$.

3. $K = 1$, $\mathbf{D} = (0, 0)$ and $E_\varepsilon(u) = c - \pi$.

The second case is ruled out thanks to Lemma 9.8 and to the fact that $c - \pi < 2\pi \frac{1-R}{1+R}$.

We next turn to the third case. Since $\frac{1}{2} \int_{\Omega} |\nabla u|^2 < \pi$,²⁸ [23, Corollary 3.7] implies that $|u| \geq a > 0$ in Ω . By Lemma 2.8, in Ω we may globally write $u = \rho e^{i\varphi}$. As in the proof of Lemma 8.14, this implies that u is constant. This contradicts the fact that $c > \pi$.

The proof of Theorem 9.6 is complete. □

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²⁷With E_ε instead of F_β .

²⁸Thanks to the condition $c < 2\pi$.

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