Markovian loop clusters on graphs

Yves Le Jan, Sophie Lemaire

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YVES LE JAN AND SOPHIE LEMAIRE

Abstract. The loop clusters of Poissonian ensembles of Markov loops on a finite or countable graph induce a coalescent process on the vertices of the graph. After a description of some general properties of the coalescent process, we study the loop clusters defined by a simple random walk killed at a constant rate on three different graphs: the integer number line $\mathbb{Z}$, the d-dimensional integer lattice $\mathbb{Z}^d$ with $d \geq 2$ and the complete graph.

Introduction

The notion of Poissonian ensembles of Markov loops (loop soups) was introduced by Lawler and Werner in [8] in the context of two dimensional Brownian motion (it already appeared informally in [13]): the loops of a Brownian loop soup on a domain $D \in \mathbb{C}$ are the points of a Poisson point process with intensity $\alpha\mu$ where α is a positive real and μ is the Brownian loop measure on D. Loop clusters induced by a Brownian loop soup were used to give a construction of the conformal loop ensembles (CLE) in [14] and [12]. They are defined as follows: *two loops ℓ and ℓ' are said to be in the same cluster if one can find a finite chain of loops $\ell_0, \ldots, \ell_n$ such that $\ell_0 = \ell$, $\ell_n = \ell'$ and for all $i \in \{1, \ldots, n\}$ $\ell_{i-1} \cap \ell_i \neq \emptyset$*. Poissonian ensembles of Markov loops can also be defined on graphs. They were studied in details in [9] and [6]. In particular Poissonian ensembles of loops on the integer lattice $\mathbb{Z}^2$ induced by simple (nearest neighbor) random walks give a discrete version of Brownian loop soup (see [7]).

The aim of this paper is to study loop clusters induced by Markov loop ensembles on graphs. The facts presented here, which are built on the results presented in [9] are more elementary than in the Brownian loop soup theory but point out that Poissonian ensembles of Markov loops and related partitions are of more general interest. The examples we develop show relations with several theories: coalescence, percolation and random graphs.

In section 1, we recall the different objects needed to define the Markovian loop clusters on a graph and the coalescent process induced by the partition of vertices they define. In section 2, we state general properties of the clusters and establish some formulae useful to study the semigroup of the coalescent process. In the last three sections we address several aspects of the loop clusters induced by a simple random walk killed at a constant rate κ on different graphs:

- On the integer number line $\mathbb{Z}$, the loop cluster model reduces to a renewal process. We establish a convergence result by rescaling space by $\sqrt{\varepsilon}$, killing rate by ε and let ε converge to 0.

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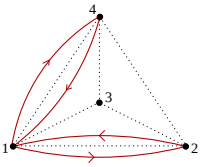
- The loop cluster model can be seen as a percolation model with two parameters α and κ. Bernoulli percolation appears as a limit if α and κ tend to $+\infty$ so that $\frac{\alpha}{\kappa^2}$ converges to a positive real. We show that a non-trivial percolation threshold occurs for the loop clusters model on the integer lattice $\mathbb{Z}^d$ with $d \geq 2$.
- As a last example, we consider the complete graph. We give a simple construction of the coalescent process on the complete graph and we deduce a construction of a coalescent process on the interval $[0, 1]$. Letting the size of the graph increase to $+\infty$, we determine the asymptotic distribution of the coalescence time: it appears to be essentially the same as in the Erdös-Rényi random graph model (see [4]) though the lack of independence makes the proof significantly more difficult.

1. Setting

We consider a finite or countable simple graph $\mathcal{G} = (X, \mathcal{E})$ endowed with positive conductances C_e, $e \in \mathcal{E}$ and a positive measure $\kappa = \{\kappa_x,\ x \in X\}$ (called the *killing measure*). We denote by $C_{x,y}$ the conductance of the edge between vertices x and y and set $\lambda_x = \sum_{y \in X, \{x,y\} \in \mathcal{E}} C_{x,y} + \kappa_x$ for every $x \in X$. The conductances and the killing measure κ induce a sub-stochastic matrix P: $P_{x,y} = \frac{C_{x,y}}{\lambda_x} \mathbb{1}_{\{\{x,y\} \in \mathcal{E}\}}\ \forall x, y \in X$. We assume that P is irreducible.

1.1. **Loop measure.** A discrete based loop ℓ of length $n \in \mathbb{N}^*$ on $\mathcal{G}$ is defined as an element of X^n which can be extended to an infinite periodic sequence. Two based loops of length n are said equivalent if they only differ by a shift of their coefficients. A discrete loop is an equivalent class of based loops for this equivalent relation (an example is drawn in Figure 1). Let $\mathcal{DL}(X)$ (resp. $\mathcal{D\dot{L}}(X)$) denote the set of discrete loops (resp. discrete based loops) of length at least 2 on X. We associate to each based loop $\ell = (x_1, \ldots, x_n)$ of length $n \geq 2$ the weight $\dot{\mu}(\ell) = \frac{1}{n} P_{x_1,x_2} P_{x_2,x_3} \ldots P_{x_n,x_1}$. This defines a measure $\dot{\mu}$ on $\mathcal{DL}(X)$ which is invariant by the shift and therefore induces a measure μ on $\mathcal{DL}(X)$.



Representation of a loop ℓ of length 4 on the complete graph K_4 corresponding to the equivalence class of the based loop $\dot{\ell} = (4, 1, 2, 1)$; the measure of this loop is

$$\mu(\ell) = P_{4,1} P_{1,2} P_{2,1} P_{1,4}.$$

Figure 1.

Remark 1.1 (*Doob's h-transform*)**.** If h is a positive function on X such that $(P - I)h \leq 0$, a new set of conductances $C^{\{h\}}$ and a new killing measure $\kappa^{\{h\}}$ can be defined as follows: $C^{\{h\}}_{x,y} = h(x)h(y)C_{x,y}$ for every $\{x, y\} \in \mathcal{E}$ and $\kappa^{\{h\}}_x = h(x)[(I - P)h](x)\lambda_x$ for every $x \in X$. This modification corresponds to the Doob's h-transform: the associated transition matrix $P^{\{h\}}$ verifies $P^{\{h\}}_{x,y} = \frac{h(y)}{h(x)} P_{x,y}\ \forall x, y \in X$. It is a self-adjoint operator on $\mathrm{L}^2(h^2\lambda)$ since $P^{\{h\}} = T_h^{-1} P T_h$

with $T_h : \begin{array}{ccc} \mathrm{L}^2(h^2\lambda) & \to & \mathrm{L}^2(\lambda) \\ f & \mapsto & hf \end{array}$. The loop measure μ is invariant under Doob's h-transform. The loop measure can be defined without assuming that P is a substochastic matrix. A weaker condition would be to assume that P is a positive and contractive matrix. Taking such matrices does not extend the set of loop measures. Indeed, Perron-Frobenius theorem states that if Q is a positive and irreductible matrix and if its spectral radius $\rho(Q)$ is smaller or equal to 1, then there exists a positive function h on X such that $Ph = \rho(Q)h$. The h-transform of Q is then a substochastic matrix.

1.2. **Poisson loop sets.** Let $\mathcal{DP}$ be a Poisson point process with intensity $\mathrm{Leb} \otimes \mu$ defined on $\mathbb{R}_+ \otimes \mathcal{DL}(X)$. For $\alpha \geq 0$, let $\mathcal{DL}_\alpha$ denote the projection of the set $\mathcal{DP} \cap ([0, \alpha] \times \mathcal{DL}(X))$ on $\mathcal{DL}(X)$; $(\mathcal{DL}_\alpha)_{\alpha \geq 0}$ is an increasing family of loop sets which has stationary independent increments. It coincides with the family of non-trivial discrete loop sets induced by the Poisson Point process of continuous-time loops defined in [9].

1.3. **Coalescent process.** An edge $e \in \mathcal{E}$ is said to be *open at time* α if e is traversed by at least one loop of $\mathcal{DL}_\alpha$. The set of open edges defines a subgraph $\mathcal{G}_\alpha$ with vertices X. The connected components of $\mathcal{G}_\alpha$ define a partition of X denoted by $\mathcal{C}_\alpha$. The elements of the partition $\mathcal{C}_\alpha$ are the loop clusters defined by $\mathcal{DL}_\alpha$ (an example is drawn in Figure 2). This paper is devoted to the study of $\mathcal{C}_\alpha$.

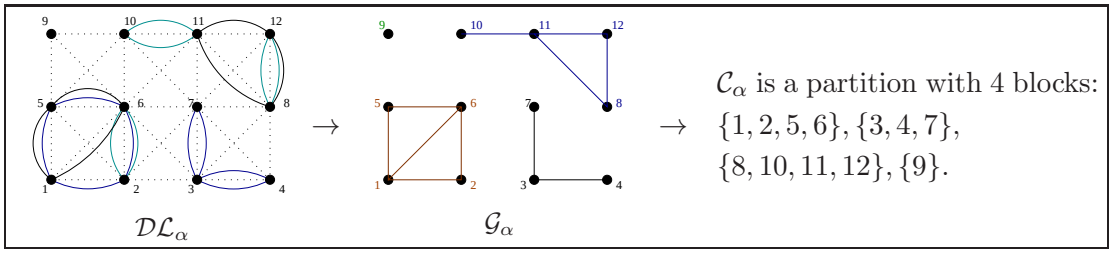


FIGURE 2. Example of a loop set $\mathcal{DL}_\alpha$ at time α on a finite graph $\mathcal{G}$, the subgraph $\mathcal{G}_\alpha$ defined by the open edges and the partition $\mathcal{C}_\alpha$ induced by its loop clusters (the dotted-lines on the left figure represent the edges of $\mathcal{G}$; $\mathcal{DL}_\alpha$ consists of three loops of length 2, two loops of length 3 and two loops of length 4.)

Remark 1.2. If A is a subset of X, let us define the $\mathcal{DL}_\alpha$-neighborhood of A:

$$\tau_\alpha(A) = A \cup \{x \in X,\ \exists \ell \in \mathcal{DL}_\alpha \text{ visiting } A \text{ and } x\}.$$

Given any $(x, y) \in X^2$, set $x \underset{\alpha}{\sim} y$ if and only if $\exists k \in \mathbb{N}^*$ such that $y \in \tau_\alpha^k(\{x\})$ (in the example drawn figure 2, $\tau_\alpha(\{10\}) = \{10, 11\}$ and for every $k \geq 2$, $\tau_\alpha^k(\{10\}) = \tau_\alpha(\{10, 11\}) = \{8, 10, 11, 12\}$ for instance). This defines an equivalence relation and the associated partition is $\mathcal{C}_\alpha$.

2. General properties of discrete loop clusters

2.1. **The distribution of the set of primitive discrete loops.** A discrete loop ξ is said to be *primitive* if it does not have a non-trivial period; Any discrete loop $\xi \in \mathcal{DL}(X)$ can be represented as some multiple $[\eta]^{M(\xi)}$ of a primitive discrete loop η denoted by $\pi\xi$. Let $\mathcal{PL}(X)$ denote the set of primitive discrete loops X of length at least 2 and let $\mathcal{PL}_\alpha$ denote the set of primitive discrete loops defined by $\mathcal{DL}_\alpha$. Clearly $\mathcal{C}_\alpha$ depends only on $\mathcal{PL}_\alpha$.

Proposition 2.1. *The distribution of $\mathcal{PL}_\alpha$ is a product measure ν on $\{0,1\}^{\mathcal{PL}(X)}$ defined by*

$$\nu(\omega,\ \omega_\eta = 1) = 1 - (1-\mu(\eta))^\alpha.$$

Proof. If $\eta_1, \dots, \eta_r$ are distinct primitive discrete loops, the sets $L_i = \{\ell \in \mathcal{DL}(X),\ \pi\ell = \eta_i\}$, $i = 1, \dots, r$ are disjoint, hence the r events $E_i = \{\exists \ell \in \mathcal{DL}_\alpha$ such that $\pi\ell = \eta_i\}$, $i = 1, \dots, r$ are independent. Therefore, the law of $\mathcal{PL}_\alpha$ is a product measure on $\{0,1\}^{\mathcal{PL}(X)}$ and for every $\eta \in \mathcal{PL}(X)$,

$$\mathbb{P}(\eta \in \mathcal{PL}_\alpha) = \mathbb{P}(\exists \ell \in \mathcal{DL}_\alpha,\ \pi\ell = \eta) = 1 - \exp(-\alpha \sum_{m=1}^{+\infty} \mu(\ell, \ell = [\eta]^m)).$$

As $\mu([\eta]^m) = \frac{1}{m}\mu(\eta)^m$ for every $m \in \mathbb{N}^*$, we deduce that

$$\mathbb{P}(\eta \in \mathcal{PL}_\alpha) = 1 - \exp(\alpha \log(1-\mu(\eta))).$$

□

It follows from Proposition 2.1 that Harris inequality (and also B-K inequality) holds on increasing events (see e.g. [15]). In particular let us say that *a subset A of X is connected at time α* if it is contained in a cluster of $\mathcal{C}_\alpha$. Then by Harris inequality,

$$\mathbb{P}(A \text{ and } B \text{ are both connected at time } \alpha) \geq \mathbb{P}(A \text{ is connected at time } \alpha)\,\mathbb{P}(B \text{ is connected at time } \alpha).$$

In a similar way, we say that *a subgraph is open at time α* if all its edges are traversed by jumps of loops in $\mathcal{DL}_\alpha$. Then the same inequality holds.
These inequalities can be extended to any number of increasing events. In particular,

- For every partition $\pi = (B_i)_{i\in I}$ of X into non-empty blocks, the probability that $\mathcal{C}_\alpha$ is a coarser partition than π satisfies:

$$\mathbb{P}(\mathcal{C}_\alpha \preceq \pi) \geq \prod_{i\in I} \mathbb{P}(B_i \text{ is connected at time } \alpha).$$

- If $\mathcal{F} \subset \mathcal{E}$, then $\mathbb{P}(\mathcal{F}$ is open at time $\alpha) \geq \prod_{e\in\mathcal{F}} \mathbb{P}(e$ is open at time $\alpha)$.

2.2. **The transition rate of the coalescent process.** The evolution in α of $\mathcal{C}_\alpha$ defines a partition-valued Markov chain. Let π be a partition of X into non-empty blocks $\{B_i,\ i\in I\}$. From state π, the only possible transitions are to a partition $\pi^{\oplus J}$ obtained by merging blocks indexed by some subset J of I to form one block $B_J=\cup_{j\in J}B_j$ and leaving all other blocks unchanged. The transition rate from π to $\pi^{\oplus J}$ is

$$\begin{aligned}\tau_{\pi,\pi^{\oplus J}} &= \mu(\ell, \forall j\in J,\ \ell \text{ intersects } B_j \text{ and } \forall u\notin J,\ \ell \text{ does not intersect } B_u)\\ &= \sum_{k\geq 2}\frac{1}{k}\sum_{(x_1,\dots,x_k)\in W_k(B_j,\ j\in J)} P_{x_1,x_2}\dots P_{x_k,x_1} \quad (2.1)\end{aligned}$$

where $W_k(B_j,\ j\in J)$ is the set of k-tuples of elements of B_J having at least one element that belongs to each block $B_j,\ j\in J$:

$$W_k(B_j,\ j\in J)=\{(x_1,\dots,x_k)\in(\cup_{j\in J}B_j)^k \text{ such that } \forall j\in J,\ x_u\in B_j \text{ for some } u\}.$$

If the graph $\mathcal{G}$ is finite, these transition rates can be expressed with determinants of Green functions of subgraphs. In order to describe the formula, let us introduce some notations; Let G denote the Green function of $\mathcal{G}$: $G=(\lambda I-C)^{-1}$ where $\lambda=(\lambda_x)_{x\in X}$, I is the identity matrix and C is the conductance matrix. For every subset F of vertices, set $F^c=X\setminus F$ and let $G^{(F)}$ denote the Green function of the subgraph $(F,\mathcal{E}_{|F\times F})$ of $\mathcal{G}$: $G^{(F)}$ is the inverse of the matrix $(\lambda I-C)_{|F\times F}$.

Proposition 2.2. *Let us assume that $\mathcal{G}=(X,\mathcal{E})$ is a finite graph.*
For every partition $\pi=\{B_i,\ i\in I\}$ of X and every subset J of I with at least two elements:

$$\tau_{\pi,\pi^{\oplus J}}=\sum_{I\subsetneq J}(-1)^{|I|}\log(\det(G^{(\cup_{u\in J\setminus I}B_u)})). \quad (2.2)$$

Proof. By the inclusion-exclusion principle,

$$\begin{aligned}\tau_{\pi,\pi^{\oplus J}} &= \mu\Big(\ell,\ \ell \text{ does not intersect } \bigcup_{j\notin J}B_j\Big)-\mu\Big(\ell,\ \exists u\in J,\ \ell \text{ does not intersect } \bigcup_{j\in J^c\cup\{u\}}B_j\Big)\\ &= \sum_{K\subsetneq J}(-1)^{|K|}\mu(\ell,\ \ell \text{ does not intersect } \bigcup_{u\in J^c\cup K}B_u).\end{aligned}$$

To conclude, we use that:

- For a subset F of vertices,
$$\mu(\ell,\ \ell \text{ does not intersect } F)=\mu(\mathcal{DL}(F^c))=\log\Big(\det(G^{(F^c)})\prod_{x\in F^c}\lambda_x\Big).$$
- For every family of reals $(a_u)_{u\in J}$ indexed by a finite subset J with at least two elements,
$$\sum_{I\subsetneq J}(-1)^{|I|}\sum_{u\in J\setminus I}a_u=0. \quad (2.3)$$

□

Example 2.3. Let us consider the complete graph K_n with unit conductances and a uniform killing measure with intensity κ. The transition matrix P verifies: $P_{x,y}=\frac{1}{n-1+\kappa}\mathbb{1}_{\{x\neq y\}}$ for every vertices x and y. Thus for every subset F of vertices, $\det(G^{(F)})=((n+\kappa)^{|F|-1}(n+\kappa-|F|))^{-1}$

(the computation of this determinant is detailed in Lemma A.3). Using (2.2) and (2.3), we obtain:

$$\tau_{\pi,\pi^{\oplus J}} = \sum_{I \subsetneq J} (-1)^{|I|+1} \log\Big(1 - \frac{1}{n+\kappa} \sum_{u \in J\setminus I} |B_u|\Big). \tag{2.4}$$

2.3. **The semigroup of the coalescent process on a finite graph.** In this section, we assume that the graph $\mathcal{G}$ is finite. The probability that $\mathcal{C}_\alpha$ is finer than a given partition of X has a simple expression:

Lemma 2.4. *Let us assume that the graph $\mathcal{G} = (X, \mathcal{E})$ is finite. Given two partitions π_0 and $\pi = (B_i)_{i\in I}$ of X,*

$$\mathbb{P}_{\pi_0}(\mathcal{C}_\alpha \succeq \pi) = \left(\frac{\prod_{i\in I} \det(G^{(B_i)})}{\det(G)}\right)^\alpha \mathbb{1}_{\{\pi_0 \succeq \pi\}}\,. \tag{2.5}$$

Proof. Let us assume that π is coarser than π_0. The event '$\mathcal{C}_\alpha$ *is finer than* π' means that every loop of $\mathcal{DL}_\alpha$ is included in a block of π. Therefore,

$$\mathbb{P}_{\pi_0}(\mathcal{C}_\alpha \succeq \pi) = \exp\Big(-\alpha\Big(\mu(\mathcal{DL}(X)) - \sum_{i\in I} \mu(\mathcal{DL}(B_i))\Big)\Big).$$

since the set of loops in each block B_i of π at time α defines independent Poisson point processes with intensity $\mu^{(B_i)}$ respectively. To conclude, we use that

$$\mu(\mathcal{DL}(X)) = \log(\det(G)\prod_{x\in X}\lambda_x) \text{ and } \mu(\mathcal{DL}(B_i)) = \log(\det(G^{(B_i)})\prod_{x\in B_i}\lambda_x).$$

□

An explicit formula for $\mathbb{P}_{\pi_0}(\mathcal{C}_\alpha = \pi)$ can be derived from Lemma 2.4. Let us first introduce some notations. For a partition π, let $|\pi|$ denote the number of non-empty blocks of π. For a subset A, let $\pi_{|A}$ denote the restriction of π to A: $\pi_{|A}$ is a partition of A, the blocks of which are the intersection of the blocks of π with A.

Lemma 2.5. *Let us consider a finite graph $\mathcal{G} = (X, \mathcal{E})$. Let π_0 and π be two partitions of X. If π has k non-empty blocks denoted by $B_1 \dots, B_k$ then*

$$\mathbb{P}_{\pi_0}(\mathcal{C}_\alpha = \pi) = \sum_{\tilde\pi \succeq \pi} (-1)^{|\tilde\pi|-k} \prod_{i=1}^{k} (|\tilde\pi_{|B_i}| - 1)!\, \mathbb{P}_{\pi_0}(\mathcal{C}_\alpha \succeq \tilde\pi). \tag{2.6}$$

Proof. Let us first assume that $\pi = \{X\}$. To obtain equation (2.6), it is sufficient to prove the following identity:

$$\mathbb{1}_{\{\mathcal{C}_\alpha = \{X\}\}} = \sum_{\ell=1}^{|X|} (-1)^{\ell-1}(\ell-1)! \sum_{\pi_\ell \in \mathfrak{P}_\ell(X)} \mathbb{1}_{\{\mathcal{C}_\alpha \succeq \pi_\ell\}}\,. \tag{2.7}$$

where $\mathfrak{P}_\ell(X)$ denotes the set of partitions of X with ℓ non-empty blocks. Let us assume that $\mathcal{C}_\alpha$ is a partition with j non-empty blocks. For $\ell \le j$, we can construct a partition coarser than $\mathcal{C}_\alpha$ with ℓ non-empty blocks by choosing how to merge some blocks of $\mathcal{C}_\alpha$, that is by choosing a partition of $\{1, \dots, j\}$ with ℓ blocks. Therefore

$$\sum_{\pi_\ell \in \mathfrak{P}_\ell(X)} \mathbb{1}_{\{\mathcal{C}_\alpha \succeq \pi_\ell\}} = |\mathfrak{P}_\ell(\{1, \dots, j\})|\, \mathbb{1}_{\{\ell \le j\}}$$

and the right-hand side of (2.7) is equal to $\sum_{\ell=1}^{j}(-1)^{\ell-1}(\ell-1)!|\mathfrak{P}_\ell(\{1,\dots,j\})|$. By an identity on the Stirling numbers of the second kind (see for example [11] equation (1.30) page 22), this sum is equal to 1 if $j=1$ and 0 if $j\geq 2$.

To prove (2.6) for a partition π with k non-empty blocks $(B_1,\dots,B_k)$, we consider $\mathcal{C}_\alpha$ as a sequence of partitions on $B_1,\dots,B_k$ respectively and apply (2.7):

$$\mathbb{1}_{\{\mathcal{C}_\alpha=(B_1,\dots,B_k)\}}=\prod_{i=1}^{k}\mathbb{1}_{\{\mathcal{C}_{\alpha|B_i}=B_i\}}=\sum_{\ell_1=1}^{|B_1|}\cdots\sum_{\ell_k=1}^{|B_k|}\prod_{i=1}^{k}V_{i,\ell_i}.$$

where $V_{i,\ell}=(-1)^{\ell-1}(\ell-1)!\sum_{\tilde\pi\in\mathfrak{P}_\ell(B_i)}\mathbb{1}_{\{\mathcal{C}_{\alpha|B_i}\succeq\tilde\pi\}}$. Therefore,

$$\begin{aligned}\mathbb{1}_{\{\mathcal{C}_\alpha=(B_1,\dots,B_k)\}} &= \sum_{\ell_1=1}^{|B_1|}\cdots\sum_{\ell_k=1}^{|B_k|}(-1)^{\ell_1+\dots+\ell_k-k}(\ell_1-1)!\dots(\ell_k-1)!\sum_{\tilde\pi\in\mathfrak{P}_{\ell_1}(B_1)\times\dots\times\mathfrak{P}_{\ell_k}(B_k)}\mathbb{1}_{\{\mathcal{C}_\alpha\succeq\tilde\pi\}}\\ &= \sum_{\tilde\pi\succeq\pi}(-1)^{|\tilde\pi|-k}\prod_{i=1}^{k}(|\tilde\pi_{|B_i}|-1)!\,\mathbb{1}_{\{\mathcal{C}_\alpha\succeq\tilde\pi\}}\,.\end{aligned}$$

□

Example 2.6. Let us consider the complete graph K_n endowed with unit conductances and a uniform killing measure of intensity κ. If π is a partition of the set of vertices X with k blocks $B_1,\dots,B_k$ then

$$\mathbb{P}_{\pi_0}(\mathcal{C}_\alpha\succeq\pi)=\Big(\frac{\kappa}{\kappa+n}\Big)^\alpha\prod_{i\in I}\Big(1-\frac{|B_i|}{n+\kappa}\Big)^{-\alpha}\mathbb{1}_{\{\pi_0\succeq\pi\}}\,.$$

and

$$\mathbb{P}_{\pi_0}(\mathcal{C}_\alpha=\pi)=\Big(\frac{\kappa}{\kappa+n}\Big)^\alpha\sum_{\substack{\tilde\pi=(\tilde B_j)_{j\in J}\\ \text{s.t. }\pi_0\succeq\tilde\pi\succeq\pi}}(-1)^{|J|-k}\prod_{i=1}^{k}\Big(\operatorname{card}(\{j\in J,\,\tilde B_j\subset B_i\})-1\Big)!\prod_{j\in J}\Big(1-\frac{|\tilde B_j|}{n+\kappa}\Big)^{-\alpha}.$$

Let us note that $(1-\frac{j}{n+\kappa})^{-\alpha}$ is the j-th moment m_j of the random variable $Y=\exp(\frac{Z}{n+\kappa})$ where Z denotes a Gamma$(\alpha,1)$-distributed random variable. Thus $\mathbb{P}_{\pi_0}(\mathcal{C}_\alpha\succeq\pi)=\dfrac{m_n}{\prod_{i=1}^{k}m_{|B_i|}}\mathbb{1}_{\{\pi_0\succeq\pi\}}$. Let c_n denote the n-th cumulant of Y. Formula (2.6) and the expression of cumulants in terms of moments (see formula (1.30) in [11] for example) yield $\mathbb{P}_{\pi_0}(\mathcal{C}_\alpha=\{X\})=\frac{c_n}{m_n}$.

2.4. **Loop clusters included in a subset of X.** For a subset D of X, let $\mathcal{DL}_\alpha^{(D)}$ denote the loops of $\mathcal{DL}_\alpha$ contained in D and $\mathcal{C}_\alpha^{(D)}$ the partition of D induced by $\mathcal{DL}_\alpha^{(D)}$. In general $\mathcal{C}_\alpha^{(D)}$ is finer than the restriction of $\mathcal{C}_\alpha$ to D but coincides with it if D is a union of elements of $\mathcal{C}_\alpha$. For every partition π of X with k non-empty blocks $B_1,\dots,B_k$, the loop sets $\mathcal{DL}_\alpha^{(B_1)}$, …, $\mathcal{DL}_\alpha^{(B_k)}$, $\mathcal{DL}_\alpha\setminus(\cup_{i=1}^{k}\mathcal{DL}_\alpha^{(B_i)})$ are independent. This yields the following equality:

$$\mathbb{P}(\mathcal{C}_\alpha=\pi)=\mathbb{P}(\mathcal{C}_\alpha\succeq\pi)\prod_{i=1}^{k}\mathbb{P}(\mathcal{C}_\alpha^{(B_i)}=\{B_i\}).$$

Let us note also that if $U\subset D\subset X$ then

$$\mathbb{P}(\mathcal{C}_\alpha\succeq\{U,U^c\})=\mathbb{P}(\mathcal{C}_\alpha^{(D)}\succeq\{U,D\setminus U\})\,\mathbb{P}(\nexists\ell\in\mathcal{DL}_\alpha\ \text{ visiting }\ U\text{ and }D^c)$$

(see [5] page 2 for a related result in Schramm-Loewner Evolution (SLE) context). The second term on the right-hand side equals

$$e^{-\alpha\left(\mu(\mathcal{DL}(X))-\mu(\mathcal{DL}(U^c))-\mu(\mathcal{DL}(D))+\mu(\mathcal{DL}(D\setminus U))\right)};$$

If X is finite, it can be written as

$$\Big(\frac{\det(G^{(U^c)})\det(G^{(D)})}{\det(G)\det(G^{(D\setminus U)})}\Big)^{\alpha} \text{ or, with Jacobi's identity, as } \Big(\frac{\det(G_{|U\times U})\det(G_{|D^c\times D^c})}{\det(G_{|(U\cup D^c)\times(U\cup D^c)})}\Big)^{-\alpha}.$$

2.5. **Computation of the semigroup using exit distributions.** The probability that $\mathcal{C}_\alpha$ is finer than a partition π can also be expressed using exit distributions. The formula obtained is easier to use than (2.5) for graphs such as tori, trees ... Let us first introduce some notations. For a subset D of X, let ∂D denote the inner boundary of D:

$$\partial D = \{x \in D, C_{x,y} > 0 \text{ for some } y \in D^c\}$$

and let $H^{(D)}$ denote the exit distribution (or Poisson kernel) from D: for $x \in X$ and $y \in D^c$, $H^{(D)}_{x,y}$ is the probability that a Markov chain with transition matrix P starting from x exits from D at y.

Lemma 2.7. *Let $\pi = (B_i)_{i=1,\dots,k}$ be a partition of X. Let $B = \cup_{j=1}^k \partial B_j$ denote the union of the boundary points of the blocks of π. Let $\mathcal{H}^{(\pi)}$ denote the matrix indexed by B defined by:*

$$\mathcal{H}^{(\pi)}_{x,y} = \begin{cases} \mathbb{1}_{\{x=y\}} & \text{if } x,y \in \partial B_i \\ -H^{(B_i)}_{x,y} & \text{if } x \in \partial B_i, y \in \partial B_j \text{ and } i \neq j. \end{cases}$$

If B is finite then $\mathbb{P}(\mathcal{C}_\alpha \succeq \pi) = \det(\mathcal{H}^{(\pi)})^\alpha$.

Proof.

- First, let us assume that X is finite. Let K denote the product of the block diagonal matrix $\mathrm{diag}(G^{(B_i)}, i \in \{1,\dots,k\})$ by the matrix G^{-1}. We can rewrite the expression of $\mathbb{P}(\mathcal{C}_\alpha \succeq \pi)$ given by Lemma 2.4 as $\mathbb{P}(\mathcal{C}_\alpha \succeq \pi) = \det(K)^\alpha$. The restriction of K to $B_i \times B_i$ is the identity. The exit distribution from a subset D verifies: $H^{(D)}_{x,y} = \sum_{z\in D} G^{(D)}_{x,z} C_{z,y}$ for every $x \in D$ and $y \in D^c$. Therefore,

$$\mathbb{P}(\mathcal{C}_\alpha \succeq \pi) = \det(K)^\alpha \text{ with } K_{x,y} = \begin{cases} \mathbb{1}_{\{x=y\}} & \text{if } x,y \in B_i \\ -H^{(B_i)}_{x,y} & \text{if } x \in B_i,\ y \in B_j \text{ and } i \neq j. \end{cases} \tag{2.8}$$

  Let $(\xi_n)_n$ denote a Markov chain with transition matrix P. The trace of $(\xi_n)_n$ on B defines a Markov chain denoted by $(\tilde{\xi}_n)$ on B and thus a Poisson point process $\mathcal{D}\tilde{\mathcal{P}}$ on $\mathbb{R}_+ \times \mathcal{DL}(B)$ (see [9] chap. 7): the discrete loops set at time α is

$$\mathcal{D}\tilde{\mathcal{L}}_\alpha := \{\ell_{|B},\ \ell \in \mathcal{DL}_\alpha \setminus \mathcal{DL}^{(B^c)}_\alpha\}.$$

  Let $\tilde{\mathcal{C}}_\alpha$ be the partition of B induced by $\mathcal{D}\tilde{\mathcal{L}}_\alpha$. The non-empty subsets ∂B_i, $i \in \{1,\dots,k\}$ define a partition of B denoted by $\partial\pi$. As $\{\mathcal{C}_\alpha \succeq \pi\}$ is the event '$\forall i,j \in \{1,\dots,k\}$ *such that* $i \neq j$, $\forall x \in \partial B_i$, $y \in \partial B_j$, $\{x,y\}$ *is not crossed by a loop of* $\mathcal{DL}_\alpha$ *at time* α', it only depends on the restriction of the loops on B, hence $\mathbb{P}(\mathcal{C}_\alpha \succeq \pi) = \mathbb{P}(\tilde{\mathcal{C}}_\alpha \succeq \partial\pi)$.

For a subset F of B, let $\tilde{H}^{(F)}$ denote the exit distribution from F for $(\tilde{\xi}_n)$. It follows from formula (2.8) applied to $\tilde{\mathcal{C}_\alpha}$ that

$$\mathbb{P}(\mathcal{C}_\alpha \succeq \pi) = \det(\tilde{K})^\alpha \text{ where } \tilde{K}_{x,y} = \begin{cases} \mathbb{1}_{\{x=y\}} & \text{if } x,y \in \partial B_i \\ -\tilde{H}^{(\partial B_i)}_{x,y} & \text{if } x \in \partial B_i, y \in \partial B_j \text{ and } i \neq j. \end{cases}$$

To conclude, it remains to note that $\tilde{H}^{(\partial B_i)}_{x,y} = H^{(B_i)}_{x,y}$ for every $x \in \partial B_i$ and $y \in B \setminus \partial B_i$.

- Let us now assume that X is a countable set. Let $(X_k)_k$ be an increasing sequence of finite subsets of X such that $X = \cup_{k=1}^{+\infty} X_k$. As B is assumed to be finite, there exists an integer k_0 such that B is included in X_{k_0}. For $k \geq k_0$, a loop in $\mathcal{DL}_\alpha^{(X_k)}$ that passes through two different blocks of π, passes through two different blocks of $\pi_{|X_k}$ the restriction of π to X_k. Therefore,

$$\begin{aligned} \mathbb{P}(\mathcal{C}_\alpha \succeq \pi) &= 1 - \sup_{k \geq k_0} \mathbb{P}(\exists \ell \in \mathcal{DL}_\alpha^{(X_k)} \text{ passing through two different blocks of } \pi) \\ &= \inf_{k \geq k_0} \mathbb{P}(\mathcal{C}_\alpha^{(X_k)} \succeq \pi_{|X_k}). \end{aligned}$$

By the first part of the proof, $\mathbb{P}(\mathcal{C}_\alpha^{(X_k)} \succeq \pi_{|X_k}) = \det(\mathcal{H}^{(\pi_{|X_k})})^\alpha$. It remains to note that the matrix $\mathcal{H}^{(\pi_{|X_k})}$ coincides with $\mathcal{H}^{(\pi)}$ for every $k \geq k_0$ to deduce that $\mathbb{P}(\mathcal{C}_\alpha \succeq \pi) = \det(\mathcal{H}^{(\pi)})^\alpha$.

□

2.6. **Closed edges in a finite graph.** An edge is said to be *closed* at time α if it is not crossed by a loop of $\mathcal{DL}_\alpha$. More generally, a family of edges E is said to be closed at time α if every edge of E is closed at time α.

Let us assume that $\mathcal{G}$ is a finite graph. To compute the probability for a family of edges E to be closed, we modify the conductances and the killing measure so that the conductance of every edge in E is 0 and the measure λ is unchanged: $\tilde{C}_{x,y} = C_{x,y}\, \mathbb{1}_{\{\{x,y\} \notin E\}}\ \forall \{x,y\} \in \mathcal{E}$ and $\tilde{\kappa}_x = \kappa_x + \sum_{y,\ \{x,y\} \in E} C_{x,y}\ \forall x \in X$. Let G_E denote the Green function associated with $\{\tilde{C}_e, e \in \mathcal{E},\ \tilde{\kappa}_x, x \in X\}$.

Lemma 2.8. *Let us assume that $\mathcal{G} = (X, \mathcal{E})$ is a finite graph. The probability for a family of edges E to be closed at time α is $\left(\frac{\det(G_E)}{\det(G)}\right)^\alpha$.*

Proof. For a loop ℓ and an edge e, let $N_e(\ell)$ denote the number of jumps of ℓ across e. Set $N_E(\ell) = \sum_{e \in E} N_e(\ell)$.

$$\mathbb{P}(E \text{ is closed at time } \alpha) = \exp\Big(- \alpha\mu(\mathcal{DL}(X)) + \alpha\mu(\ell, N_E(\ell) = 0)\Big).$$

Recall that $\mu(\mathcal{DL}(X)) = -\log(\det(I - P))$. For $s \in [0,1]$, let $P_E(s)$ denote a perturbation of P defined by

$$[P_E(s)]_{x,y} = \begin{cases} sP_{x,y} & \text{if } \{x,y\} \in E \\ P_{x,y} & \text{if } \{x,y\} \notin E. \end{cases}$$

Similarly, $\mu(s^{N_E}) = -\log(\det(I - P_E(s)))$. In particular,

$$\mu(\ell,\ N_E(\ell) = 0) = -\log(\det(I - P_E(0))).$$

Lemma 2.8 follows since $P_E(0)$ is the transition matrix associated with the $\{\tilde{C}_e, e \in \mathcal{E},\ \tilde{\kappa}_x, x \in X\}$. □

We can deduce from Lemmas 2.8 and 2.4 that if E is a set of edges of a finite graph with extremities in different blocks of a partition $\pi = \{B_i,\ i \in I\}$ then

$$\mathbb{P}(\mathcal{C}_\alpha \succeq \pi \mid E \text{ is closed at time } \alpha) = \left(\frac{\prod_{i\in I} \det(G^{(B_i)})}{\det(G_E)}\right)^\alpha$$

3. Renewal processes

On the graph $\mathbb{Z}$, the clusters $\mathcal{C}_\alpha$ are intervals between closed edges at time α (namely edges which are not crossed by any loop of $\mathcal{DL}_\alpha$). The Poisson loop sets induced by a simple random walk killed at constant rate κ have the following properties:

Proposition 3.1. *Let us consider the graph $\mathbb{Z}$ endowed with unit conductances and a uniform killing measure with intensity κ. Set $\rho^{(\kappa)} = \log(1 + \frac{\kappa}{2} + \sqrt{\kappa + \frac{\kappa^2}{4}})$.*

(i) *The midpoints of the closed edges at time α form a renewal process. Moreover,*
- *the probability that $\{n, n+1\}$ is closed at time α is equal to $(1-e^{-2\rho^{(\kappa)}})^\alpha$ for every $n \in \mathbb{Z}$;*
- *given that $\{0,1\}$ is closed at time α, the probability that $\{n, n+1\}$ is also closed is equal to $\left(\frac{1-e^{-2\rho^{(\kappa)}}}{1-e^{-2\rho^{(\kappa)}(n+1)}}\right)^\alpha$ for every $n \in \mathbb{N}$.*

(ii) *Assume that $\alpha \in]0,1[$. Let $\nu^{(\kappa)}$ denote the law of this renewal process i.e. the law of the distance between two consecutive closed edges at time α. For $\varepsilon > 0$ let $(Y_{\varepsilon,i})_{i\in\mathbb{N}^*}$ denote a sequence of independent random variables with distribution $\nu^{(\kappa\varepsilon)}$.*

For every $t > 0$, as ε converges to 0, $\sqrt{\varepsilon} \sum_{i=1}^{[\varepsilon^{-\frac{1-\alpha}{2}} t]} Y_{\varepsilon,i}$ converges in law to the value at t of a subordinator with potential density $(\frac{2\sqrt{\kappa}}{1-e^{-2u\sqrt{\kappa}}})^\alpha$.

Remark 3.2. This subordinator is associated with the Poisson covering (cf. [1], chapter 7) defined by the infinite measure on $\mathbb{R}^+$ with density $u \mapsto -\alpha \frac{d^2}{du^2} \log(1 - e^{-2u\sqrt{\kappa}}) = \frac{4\kappa\alpha e^{-2u\sqrt{\kappa}}}{(1-e^{-2u\sqrt{\kappa}})^2}$. These covering intervals can be viewed as images of the "loop soup" of intensity α associated with the Brownain motion killed at constant rate κ.

Proof of Proposition 3.1.

(i) An edge $\{x-1, x\}$ is closed at time α if and only if $\mathcal{DL}_\alpha = \mathcal{DL}_\alpha^{(x+\mathbb{N})} \cup \mathcal{DL}_\alpha^{(x-1-\mathbb{N})}$. The next closed edge is the first edge which is not crossed by any loop of $\mathcal{DL}_\alpha^{(x+\mathbb{N})}$, and previous closed edges are defined by $\mathcal{DL}_\alpha^{(x-1-\mathbb{N})}$ which is independent of $\mathcal{DL}_\alpha^{(x+\mathbb{N})}$. Stationarity is obvious as $\mathcal{DL}_\alpha^{(x+\mathbb{N})} - x$ is distributed like $\mathcal{DL}_\alpha^{(\mathbb{N})}$. For $n \in \mathbb{N}$, let $r_n^{(\kappa)}$ denote the probability that $\{n, n+1\}$ is closed at time α. By Lemma 2.7, $r_n^{(\kappa)} = (1 - H_{n+1,n}^{(n+1+\mathbb{N})} H_{n,n+1}^{(n-\mathbb{N})})^\alpha$ where $H_{x,y}^{(D)}$ denote the probability that a simple random walk killed at constant rate κ and starting at x exits D at point y. The functions $x \mapsto H_{x,y}^{(D)}$ can be computed from the solutions of equation

$$(2+\kappa)u(x) - u(x+1) - u(x-1) = 0,$$

which are linear combinations of $\exp(\rho^{(\kappa)}x)$ and $\exp(-\rho^{(\kappa)}x)$.
We obtain $H^{(n+1+\mathbb{N})}_{n+k,n} = H^{(n-\mathbb{N})}_{n+1-k,n+1} = \exp(-\rho^{(\kappa)}k)$ for every $k \in \mathbb{N}$.
For $n \in \mathbb{N}^*$, let $q^{(\kappa)}(n)$ denote the probability that $\{n, n+1\}$ is closed at time α given that $\{0,1\}$ is closed at time α. To compute $q^{(\kappa)}(n)$, we consider a simple random walk on $\mathbb{N}$ killed at rate κ and at point 0; we denote it $(\zeta_k)_k$. As $(\mathcal{DL}^{(\mathbb{N})}_\alpha)_\alpha$ is the Poisson loop sets associated with $(\zeta_k)_k$, by Lemma 2.7 we obtain: $q^{(\kappa)}(n) = (1 - h^{(\{0,\dots,n\})}_n h^{(n+1+\mathbb{N})}_{n+1})^\alpha$ where $h^{(D)}$ is the exit distribution from a subset D of $\mathbb{N}$ for $(\zeta_k)_k$:

$$h^{(\{0,\dots,n\})}_k = \frac{\sinh(\rho^{(\kappa)}k)}{\sinh(\rho^{(\kappa)}(n+1))} \text{ for } 1 \le k \le n+1 \text{ and } h^{(n+1+\mathbb{N})}_k = e^{\rho^{(\kappa)}(n-k)} \text{ for } k \ge n.$$

Therefore $q^{(\kappa)}(n) = (1 - \frac{\sinh(\rho^{(\kappa)}n)}{\sinh(\rho^{(\kappa)}(n+1))}e^{-\rho^{(\kappa)}})^\alpha = (\frac{1-e^{-2\rho^{(\kappa)}}}{1-e^{-2\rho^{(\kappa)}(n+1)}})^\alpha$.

(ii) Let I_κ denote the Laplace transform of the function $u \mapsto (\frac{2\sqrt{\kappa}}{1-e^{-2\sqrt{\kappa}u}})^\alpha$:

$$I_\kappa(s) = \int_0^\infty (\frac{2\sqrt{\kappa}}{1-e^{-2\sqrt{\kappa}u}})^\alpha e^{-su}du \text{ for } s > 0.$$

Let $\widehat{\nu^{(\kappa)}}$ denote the Laplace transform of $\nu^{(\kappa)}$. We shall prove that for every $s > 0$, $-\varepsilon^{\frac{\alpha-1}{2}}\log(\widehat{\nu^{(\kappa\varepsilon)}}(s\sqrt{\varepsilon}))$ converges towards $1/I_\kappa(s)$ as ε tends to 0 which yields (ii).
As $q^{(\kappa)}(n) = \sum_{k=1}^\infty (\nu^{(\kappa)})^{*k}(n)$ for every $n \in \mathbb{N}^*$, we have the Laplace transforms identity: $\widehat{\nu^{(\kappa)}} = \frac{\widehat{q^{(\kappa)}}}{1+\widehat{q^{(\kappa)}}}$. Therefore, it is sufficient to show that $\varepsilon^{(1-\alpha)/2}\widehat{q^{(\kappa\varepsilon)}}(s\sqrt{\varepsilon})$ converges to $I_\kappa(s)$ as ε tends to 0. Let κ_1 and κ_2 two positive reals such that $0 < \kappa_1 < \kappa < \kappa_2$. For ε small enough, $\sqrt{\varepsilon\kappa_1} \le \rho^{(\kappa\varepsilon)} \le \sqrt{\varepsilon\kappa_2}$ and $2\sqrt{\varepsilon\kappa_1} \le 1 - e^{-2\rho^{(\kappa\varepsilon)}} \le 2\sqrt{\varepsilon\kappa_2}$. For $a > 0$ let f_a be the function defined by $f_a(x) = (1-e^{-2ax})^{-\alpha}$ for $x > 0$. Therefore, for ε small enough,

$$(2\sqrt{\varepsilon\kappa_1})^\alpha f_{\kappa_2}(\sqrt{\varepsilon}(n+1)) \le q^{(\kappa\varepsilon)}(n) \le (2\sqrt{\varepsilon\kappa_2})^\alpha f_{\kappa_1}(\sqrt{\varepsilon}(n+1)) \text{ for every } n \in \mathbb{N}\,.$$

By Lebesgue's dominated convergence theorem applied to the function $g_{a,s,\varepsilon}$ defined by $g_{a,s,\varepsilon}(x) = f_a(\sqrt{\varepsilon}\lceil\frac{x}{\sqrt{\varepsilon}}\rceil)e^{-s\sqrt{\varepsilon}\lfloor\frac{x}{\sqrt{\varepsilon}}\rfloor}\ \forall x > 0$, we obtain:

$$\sqrt{\varepsilon}\sum_{n=0}^{+\infty} f_a(\sqrt{\varepsilon}(n+1))e^{-s\sqrt{\varepsilon}n} \underset{\varepsilon\to 0}{\to} \int_0^{+\infty} f_a(x)e^{-sx}dx$$

(for $\varepsilon < \frac{1}{2a}$, $g_{a,s,\varepsilon}$ is dominated by $x \mapsto (\frac{2}{ax})^\alpha \mathbb{1}_{]0,\frac{1}{2a}]} + \frac{e^{-sx}}{(1-e^{-1/2})^\alpha}\mathbb{1}_{]\frac{1}{2a},+\infty[}(x)$). This implies that for every $s > 0$

$$\int_0^{+\infty}\Big(\frac{2\sqrt{\kappa_1}}{1-e^{-2\sqrt{\kappa_2}u}}\Big)^\alpha e^{-su}du \le \underline{\lim}\,\varepsilon^{(1-\alpha)/2}\widehat{q^{(\kappa\varepsilon)}}(s\sqrt{\varepsilon})$$
$$\le \overline{\lim}\,\varepsilon^{(1-\alpha)/2}\widehat{q^{(\kappa\varepsilon)}}(s\sqrt{\varepsilon}) \le \int_0^{+\infty}\Big(\frac{2\sqrt{\kappa_2}}{1-e^{-2\sqrt{\kappa_1}u}}\Big)^\alpha e^{-su}du$$

These inequalities hold for every $0 < \kappa_1 < \kappa < \kappa_2$. Therefore for every $s > 0$, $\varepsilon^{(1-\alpha)/2}\widehat{q^{(\kappa\varepsilon)}}(s\sqrt{\varepsilon})$ converges to $I_\kappa(s)$ which ends the proof of (ii).

□

Remark 3.3. In the case of the simple random walk on $\mathbb{N}$ killed at 0, a similar argument (detailed in [10]) shows that:

- For $0 < \alpha < 1$, the midpoints of the closed edges at time α form a renewal process with holding times $(Y_n^{(\alpha)})_{n\geq 1}$; The generating function of $Y_n^{(\alpha)}$ is $1 - \frac{s}{\mathrm{Li}_\alpha(s)}$ where Li denotes the polylogarithm: $\forall\ |s| < 1$, $\mathrm{Li}_\alpha(s) = \sum_{k=1}^{+\infty} \frac{s^k}{k^\alpha}$. Set $S_n^{(\alpha)} = \sum_{i=1}^n Y_i^{(\alpha)}$ for $n \geq 1$. As ε tends to 0, $(\varepsilon S^{(\alpha)}_{\lfloor \varepsilon^{\alpha-1} t\rfloor}, t \geq 0)$ converges in law towards a stable subordinator $(S_t^{(\alpha)}, t \geq 0)$ with index $1-\alpha$. In the case of a finite interval $[0, L]$, we obtain a renewal process conditioned to jump at point L.
- For $\alpha > 1$, there are only a finite number of clusters. In particular, $\mathbb{P}(S_1^{(\alpha)} = \infty) = \frac{1}{\zeta(\alpha)}$.

4. Bernoulli percolation and loop percolation

4.1. **Bernoulli percolation.** Let $s = \{s_e,\ e \in \mathcal{E}\}$ be a family of coefficients in $[0, 1]$. In Bernoulli percolation model of parameter s on the graph $\mathcal{G} = (X, \mathcal{E})$, every edge e is, independently of each other, called 'open' with probability s_e and 'closed' with probability $1 - s_e$. Vertices connected by open paths define a partition of X denoted by $\mathcal{P}(s)$. We can compare $\mathcal{P}(s)$ to the partition induced by the set of primitive discrete loops of length 2 for a finite graph $(X, \mathcal{E})$.

Lemma 4.1. *Let us consider a finite graph $(X, \mathcal{E})$. For $\alpha > 0$ and $e = \{x, y\} \in \mathcal{E}$, set $s_{\alpha,e} = 1 - (1 - P_{x,y}P_{y,x})^\alpha$.*
The partition induced by the set of primitive discrete loops of length 2 at time α has the same law as $\mathcal{P}(\{s_{\alpha,e},\ e \in \mathcal{E}\})$.

Proof. Let $\mathcal{PA}_\alpha$ denote the set of primitive discrete loops of length 2 at time α. The law $\mathcal{PA}_\alpha$ for every α is a product measure ν on $\{0, 1\}^{\mathcal{E}}$: for every edge $\{x, y\} \in \mathcal{E}$, let $\eta_{x,y}$ denote the class of the based loop (x, y). The probability that a loop in $\mathcal{DL}_\alpha$ has $\{x, y\}$ as support is

$$\nu(\omega,\ \omega_{\{x,y\}} = 1) = 1 - \exp\big(-\alpha\mu(\ell,\ \pi\ell = \eta_{x,y})\big).$$

and

$$\mu(\ell,\ \pi\ell = \eta_{x,y}) = \sum_{k=1}^{+\infty} \frac{1}{k}(P_{x,y}P_{y,x})^k = -\log(1 - P_{x,y}P_{y,x}).$$

□

Bernoulli percolation clusters on a graph appear to be a limiting case of partitions defined by loop clusters in which only two points loops contribute asymptotically.

Proposition 4.2 ([10])**.** *Let us consider a finite graph $\mathcal{G} = (X, \mathcal{E})$ endowed with unit conductances and a uniform killing measure with intensity $\kappa > 0$. Let $\mathcal{C}_\alpha^{(\kappa)}$ be the partition induced by the Poisson loop set on $\mathcal{G}$ at time α. Fix $u > 0$.*
If κ and α tend to $+\infty$ such that α is equivalent to $u\kappa^2$, then $\mathcal{C}_\alpha^{(\kappa)}$ converges in law towards the Bernoulli percolation of parameter $1 - e^{-u}$.

Proof. For a partition $\pi = \{B_i,\ i \in I\}$ of X, let $L(\pi)$ denote the set of edges of $\mathcal{E}$ linking different blocks of π.
The law of Bernoulli percolation of parameter $1-e^{-u}$ denoted by $\mathcal{P}(1-e^{-u})$ is characterized by the identities:

$$\mathbb{P}(\mathcal{P}(1-e^{-u}) \succeq \pi) = e^{-u|L(\pi)|} \text{ for every partition } \pi \text{ of } X.$$

To prove the convergence of $\mathbb{P}(\mathcal{C}_\alpha^{(\kappa)} \succeq \pi)$, we apply Lemma 2.7: $\mathbb{P}(\mathcal{C}_\alpha^{(\kappa)} \succeq \pi) = \det(\mathcal{H}^{(\pi)})^\alpha$ where $\mathcal{H}^{(\pi)})$ is defined as in Lemma 2.7. We note then that $\mathcal{H}^{(\pi)}_{x,y}$ is equivalent to κ^{-1} if $\{x,y\}$ belongs to $L(\pi)$ and of order less or equal to κ^{-2} otherwise. Indeed, if $x \in \partial B_i$ and $y \in \partial B_j$ for $i \neq j$, $H^{(B_i)}_{x,y} = \sum_{z \in B_i} P_{x,z} H^{(B_i)}_{z,y} + P_{x,y}$ and $P_{i,j} \leq \frac{1}{\kappa}$ for all $(i,j) \in X \times X$.
A second-order Taylor expansion shows that $\log(\det(\mathcal{H}^{(\pi)})) = Tr(\log(\mathcal{H}^{(\pi)}))$ is equivalent to $-\frac{1}{2}Tr(Q^2)$, with $Q_{x,y} = \kappa^{-1}\, \mathbb{1}_{\{\{x,y\} \in L(\pi)\}}$. □

4.2. **Loop percolation on $\mathbb{Z}^d$ with $d \geq 2$.** Let us consider the Poisson loop process induced by the simple random walk on $\mathbb{Z}^d$, $d \geq 2$, killed at a constant rate $\kappa > 0$: $P_{x,x+u} = \frac{1}{2d+\kappa}$ for every $x \in \mathbb{Z}^d$ and $u \in \{\pm 1\}^d$. Let $\theta(\alpha,\kappa)$ denote the probability of percolation at time α *i.e.* the probability of any fixed point to be connected to infinity by an open path at time α. The following Proposition presents some properties of the function $(\alpha,\kappa) \mapsto \theta(\alpha,\kappa)$:

Proposition 4.3. *Let p_c denote the critical probability for bond percolation on $\mathbb{Z}^d$ ($d \geq 2$).*

(i) *$\theta(\alpha,\kappa)$ is an increasing function of α and a decreasing function of κ.*
(ii) *$\theta(\alpha,\kappa) > 0$ for every $\alpha > 0$ and $\kappa > 0$ such that $(1 - \frac{1}{(2d+\kappa)^2})^\alpha < 1 - p_c$.*
(iii) *For any $\alpha > 0$, $\theta(\alpha,\kappa)$ vanishes for κ large enough.*

Proof.

(i) $\theta(\alpha,\kappa)$ is an increasing function of α since $\alpha \mapsto \mathcal{DL}_\alpha^{(\kappa)}$ is increasing.
To show that $\theta(\alpha,\kappa)$ is a decreasing function of κ, we use an independent thinning procedure. Let $\kappa_1 > \kappa_2 > 0$. The corresponding measures on based loops satisfy: $\dot{\mu}^{(\kappa_j)}(\ell = (x_1,\dots,x_k)) = \frac{1}{k}(\frac{1}{2d+\kappa_j})^k\, \mathbb{1}_{\{x_{i+1}-x_i \in \{\pm 1\}^d\ \forall i \in \{1,\dots,k\}\}}$ for $j \in \{1,2\}$, $k \geq 2$ and $x_1,\dots,x_k \in \mathbb{Z}^d$ with the convention $x_{k+1} = x_1$. By erasing independently each based loop $\ell \in \mathcal{D}\dot{\mathcal{L}}_\alpha^{(\kappa_2)}$ of length $k \geq 2$ with probability $1-(\frac{2d+\kappa_2}{2d+\kappa_1})^k$, we obtain a discrete loop set having the same distribution as $\mathcal{DL}_\alpha^{(\kappa_1)}$.

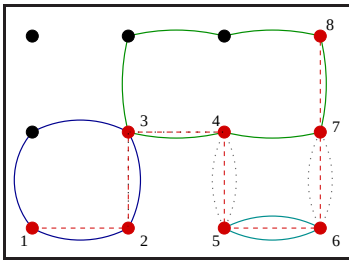
(ii) By Lemma 4.1, the partition induced by the set of primitive discrete loops of length 2 at time α has the same law as the Bernoulli percolation with parameter $1-(1-\frac{1}{(2d+\kappa)^2})^\alpha$. It follows from Bernoulli percolation on $\mathbb{Z}^d$ that if $1-(1-\frac{1}{(2d+\kappa)^2})^\alpha > p_c$ then $\theta(\alpha,\kappa) > 0$.

(iii) To prove that $\theta(\alpha,\kappa)$ vanishes for κ large enough, we show that there exists a finite real $C_\alpha > 0$ such that any self-avoiding path $x = (x_1, x_2, ..., x_L)$ of length $L \in 2\,\mathbb{N}^*$ is open at time α with probability less than $(\frac{C_\alpha}{\kappa})^L$. We can then conclude with the usual path-counting argument: for every $L \in 2\,\mathbb{N}^*$, $\theta(\alpha,\kappa)$ is bounded above by the probability that there exists an open self-avoiding path of length L at time α starting from the origin, hence $\theta(\alpha,\kappa) \leq \limsup_{L\to+\infty}(2d)^L(\frac{C_\alpha}{\kappa})^L = 0$ for $\kappa > 2dC_\alpha$.
Let us first introduce some notations.

– Let $\mathfrak{P}(2,\{1,\dots,L\})$ consist of partitions of $\{1,\dots,L\}$ in which all blocks have at least two elements (the number of blocks of such a partition π is denoted by $|\pi|$ and blocks are denoted by π_1, π_2, ...).
– For a vertex v and a loop $\ell \in \mathcal{DL}(\mathbb{Z}^d)$, let $N_v(\ell)$ denote the number of times ℓ passes through v: $N_v(\ell) = |\{1 \le i \le k,\ u_i = v\}|$ if ℓ is the class of the based loop $(u_1,\dots,u_k)$.

Let us consider a self-avoiding path of length $L \in 2\,\mathbb{N}^*$ denoted by $x = (x_1, x_2, \dots, x_L)$ and let $\mathcal{E}_x$ denote the set of edges $\{x_{2i-1}, x_{2i}\},\ 1 \le i \le L/2$.
To be open, the edges of x have to be covered by the edges of $N \le L$ loops of $\mathcal{DL}_\alpha$. Among these loops, those that cover at least one edge $e \in \mathcal{E}_x$ can be used to define a partition $\pi \in \mathfrak{P}(2,\{1,\dots,L\})$ as follows: let ℓ_1 be a loop in $\mathcal{DL}_\alpha$ covering edge $\{x_1, x_2\}$. The first block π_1 of π consists of the indices of the endpoints of $e \in \mathcal{E}_x$ covered by ℓ_1. If $\pi_1 \neq \{1,\dots,L\}$, let j be the smallest integer i such that x_i is not in π_1 and let ℓ_2 be a loop covering $\{x_j, x_{j+1}\}$. The second block π_2 of π is defined as the set of indices of endpoints of $e \in \mathcal{E}_x$ covered by ℓ_2 that are not in π_1, and so on until all elements of $\{1,\dots,L\}$ belong to a block written down (an example is presented in Figure 3). Therefore,

$$\mathbb{1}_{\{x \text{ open at time } \alpha\}} \le \sum_{\pi\in\mathfrak{P}(2,\{1,\dots,L\})} \Big(\sum_{\substack{\ell_1,\dots,\ell_{|\pi|}\in\mathcal{DL}_\alpha \\ \text{pairwise distinct}}} \prod_{j=1}^{|\pi|} \prod_{i\in\pi_j} \mathbb{1}_{\{N_{x_i}(\ell_j)>0\}} \Big)$$
$$\le \sum_{\pi\in\mathfrak{P}(2,\{1,\dots,L\})} \Big(\sum_{\substack{\ell_1,\dots,\ell_{|\pi|}\in\mathcal{DL}_\alpha \\ \text{pairwise distinct}}} \prod_{j=1}^{|\pi|} \prod_{i\in\pi_j} N_{x_i}(\ell_j) \Big).$$



Example of an open path $x = (1,2,3,4,5,6,7,8)$ with five loops covering its edges. Three of them (drawn in solid lines) cover the set of edges $\mathcal{E}_x$; we associate to these three loops, the partition $\pi \in \mathfrak{P}(2,\{1,\dots,8\})$ with three blocks $\pi_1 = \{1,2\}$, $\pi_2 = \{3,4,7,8\}$ and $\pi_3 = \{5,6\}$.

FIGURE 3.

By Campbell formula (equation (4.2) page 36 in [9]), for every $k \in \mathbb{N}^*$ and for every positive functions $F_1,\dots,F_k$

$$\mathbb{E}\Big(\sum_{\substack{\ell_1,\dots,\ell_k\in\mathcal{DL}_\alpha, \\ \text{pairwise distinct}}} \prod_{i=1}^{k} F_i(\ell_i) \Big) = \prod_{i=1}^{k} \Big(\int F_i(\ell)\alpha d\mu(\ell) \Big).$$

Therefore,

$$\mathbb{P}(x \text{ is open at time } \alpha) \le \sum_{\pi\in\mathfrak{P}(2,\{1,\dots,L\})} \prod_{j=1}^{|\pi|} \int \Big(\prod_{i\in\pi_j} N_{x_i}(\ell) \Big) \alpha\mu(d\ell).$$

For a finite set A, let $\mathfrak{S}_A$ denote the set of permutations of elements of A. By Proposition 5 page 20 in [9], for every $k \geq 2$ and for every vertices $y_1, \dots, y_k$,

$$\int \prod_{i=1}^{k} N_{y_i}(\ell) d\mu(\ell) = \frac{1}{k}\Big(\prod_{i=1}^{k} \lambda_{y_i}\Big) \sum_{\sigma \in \mathfrak{S}_{\{1,\dots,k\}}} G_{y_{\sigma(1)},y_{\sigma(2)}} G_{y_{\sigma(2)},y_{\sigma(3)}} \dots G_{y_{\sigma(k)},y_{\sigma(1)}}$$

Therefore, $\mathbb{P}(x \text{ is open at time } \alpha) \leq \Big(\prod_{i=1}^{L} \lambda_{x_i}\Big) S_\alpha(x)$ where

$$S_\alpha(x) = \sum_{\pi \in \mathfrak{P}(2,\{1,\dots,L\})} \alpha^{|\pi|} \prod_{j=1}^{|\pi|} \Big(\frac{1}{|\pi_j|} \sum_{\sigma \in \mathfrak{S}_{\pi_j}} G_{x_{\sigma(1)},x_{\sigma(2)}} G_{x_{\sigma(2)},x_{\sigma(3)}} \dots G_{x_{\sigma(|\pi_j|)},x_{\sigma(1)}} \Big). \tag{4.1}$$

The blocks of a partition $\pi \in \mathfrak{P}(2, \{1, \dots, L\})$ in (4.1) can be seen as the orbits of a permutation without fixed point. Since a circular order on k integers corresponds to k different permutations of these integers, $S_\alpha(x)$ can be rewritten as follows:

$$S_\alpha(x) = \sum_{\sigma \in \mathfrak{S}^0_{\{1,\dots,L\}}} \alpha^{m(\sigma)} G_{x_1,x_{\sigma(1)}} \dots G_{x_L,x_{\sigma(L)}}. \tag{4.2}$$

where

- $m(\sigma)$ denotes the number of cycles in a permutation σ,
- $\mathfrak{S}^0_{\{1,\dots,L\}}$ denotes the set of permutations of $\{1, \dots, L\}$ without fixed point (such a permutation corresponds to a configuration without isolated points).

(See Lemma A.1 in the appendix for details.) The right-hand side of equality (4.2) is nothing but $\mathrm{Per}^0_\alpha(G_{x_i,x_j}, 1 \leq i, j \leq L)$ defined in [9] page 41.
To conclude we use that

$$\mathrm{Per}^0_1(G_{x_i,x_j}, 1 \leq i, j \leq L) \leq \prod_{i=1}^{L} \Big(\sum_{1 \leq j \leq L,\ j \neq i} G_{x_i,x_j} \Big) \leq \Big(\sum_{y \in \mathbb{Z}^d \setminus \{0\}} G_{0,y} \Big)^L$$

since the vertices x_i are pairwise distinct. As $(P1)_u = \frac{2d}{2d+\kappa}$ for every $u \in \mathbb{Z}^d$,

$$\sum_{y \in \mathbb{Z}^d \setminus \{0\}} \lambda_y G_{0,y} = \sum_{k=1}^{+\infty} (P^k 1)_0 = \frac{2d}{\kappa}.$$

Thus $\mathbb{P}(x \text{ is open at time } \alpha) \leq (\frac{2d}{\kappa} \max(\alpha, 1))^L$ which ends the proof.

□

Remark 4.4.

(i) It follows from Proposition 4.3 that for every $\alpha > 0$, there is a finite value of κ, we can denote $\kappa^c(\alpha)$, above which $\theta(\alpha, \kappa)$ vanishes; Morever κ^c is an increasing function that converges to $+\infty$ as $\alpha \to \infty$.

(ii) As the simple random walk in $\mathbb{Z}^2$ is recurrent, for $d = 2$ the probability that a fixed edge is open at time $\alpha > 0$ converges to 1 as κ tends to 0. Indeed, let us first note that for every $x \in \mathbb{Z}^2$, $\mathbb{P}(N_x^{(\alpha)} = 0) = \Big(\frac{1}{\lambda_x G_{x,x}}\Big)^\alpha$ (this equality has been proven for a finite

graph[1], so we apply it to the restriction of the random walk to $\mathbb{Z}^2 \cap [-M, M]^2$ and let M tend to $+\infty$). Therefore $\mathbb{P}(N_x^{(\alpha)} > 0)$ converges to 1 as κ tends to 0.
Fix $x \in \mathbb{Z}^2$ and $u \in \{\pm 1\}^2$. By symmetry,

$$\mathbb{P}(N_x^{(\alpha)} > 0) = \sum_{v \in \{\pm 1\}^2} \mathbb{P}(N_{x,x+v}^{(\alpha)} > 0) = 4\,\mathbb{P}(N_{x,x+u}^{(\alpha)} > 0).$$

Since $\alpha \mapsto \mathcal{DL}_\alpha$ has independent stationary increments, $\mathbb{P}(N_{x,x+u}^{(\alpha)} = 0) = \mathbb{P}(N_{x,x+u}^{(\alpha/n)} = 0)^n$ for every $n > 0$. Therefore,

$$\mathbb{P}(N_{x,x+u}^{(\alpha)} > 0) = 1 - \mathbb{P}(N_{x,x+u}^{(\alpha/n)} = 0)^n \geq 1 - (1 - \frac{1}{4}\,\mathbb{P}(N_x^{(\alpha/n)} > 0))^n. \tag{4.3}$$

For a fixed $\varepsilon > 0$, let us choose n such that $(1 - \frac{1}{8})^n \leq \varepsilon$. There exists κ_ε such that for $\kappa \leq \kappa_\varepsilon$, $\mathbb{P}(N_x^{(\alpha/n)} = 0) \leq \frac{1}{2}$. It follows from (4.3) that $\mathbb{P}(N_{x,x+u}^{(\alpha)} > 0) \geq 1 - \varepsilon$.

5. Complete graph

This section is devoted to the study of Poisson loop sets on the complete graph K_n endowed with unit conductances and a uniform killing measure. The set of vertices is identified with $\{1, \dots, n\}$ and the set of partitions of $\{1, \dots, n\}$ is denoted by $\mathfrak{P}(\{1, \dots, n\})$. The intensity of the killing measure is denoted by $\kappa_n = n\varepsilon_n$ with $\varepsilon_n > 0$, hence the coefficients of the transition matrix P are: $P_{x,y} = \frac{\mathbf{1}_{\{x \neq y\}}}{\lambda^{(n)}}$ with $\lambda^{(n)} = n - 1 + n\varepsilon_n$ for every $x, y \in \{1, \dots, n\}$. As n will vary, the loop set and the partition defined by the loop clusters at time α will be denoted by $\mathcal{DL}_\alpha^{(n)}$ and $\mathcal{C}_\alpha^{(n)}$ respectively. In a first part, n is fixed; We present another construction of the coalesecent process $(\mathcal{C}_\alpha^{(n)})_{\alpha \geq 0}$ and use this construction to define a similar coalescent process on the interval $[0, 1]$. In a second part, we let n tend to $+\infty$ and describe the distribution of the first time when $\mathcal{C}_\alpha^{(n)}$ has no block of size one (cover time) and the first time when $(\mathcal{C}_\alpha^{(n)})_\alpha$ has only one block (coalescent time).

5.1. **Another construction of the coalescent process.**

Proposition 5.1. *From state $\pi = \{B_i,\ i \in I\} \in \mathfrak{P}(\{1, \dots, n\})$, the only possible transitions of $(\mathcal{C}_\alpha^{(n)})_{\alpha \geq 0}$ are to partitions $\pi^{\oplus J}$ where $J = \{j_1, \dots, j_L\}$ is a subset of I with $L \geq 2$ elements. Its transition rate from π to $\pi^{\oplus J}$ is equal to:*

$$\tau_{\pi, \pi^{\oplus J}}^{(n)} = \sum_{k \geq L} \frac{1}{k n^k (\varepsilon_n + 1)^k} \sum_{(i_1, \dots, i_k) \in W_k(J)} \prod_{u=1}^{k} |B_{i_u}| \tag{5.1}$$

$$= \sum_{k \geq L} \frac{1}{k n^k (\varepsilon_n + 1)^k} \sum_{\substack{(k_1, \dots, k_L) \in (\mathbb{N}^*)^L, \\ k_1 + \dots + k_L = k}} \binom{k}{k_1, \dots, k_L} \prod_{u=1}^{L} |B_{j_u}|^{k_u}. \tag{5.2}$$

where $W_k(J)$ is the set of k-tuples of J in which each element of J appears.

[1]For a finite graph $\mathcal{G} = (X, \mathcal{E})$, $\mathbb{P}(N_x^{(\alpha)} = 0) = \exp\Big(-\alpha\Big(\mu(\mathcal{DL}(X)) - \mu(\mathcal{DL}(X \setminus \{x\}))\Big)\Big) = \Big(\frac{\det(G^{(X \setminus \{x\})})}{\det(G)\lambda_x}\Big)^\alpha = (\lambda_x G_{x,x})^{-\alpha}$.

Proof of Proposition 5.1. Let us recall the expression of $\tau^{(n)}_{\pi,\pi^{\oplus J}}$ obtained in Example 2.3:

$$\tau^{(n)}_{\pi,\pi^{\oplus J}} = \sum_{I \subsetneq J} (-1)^{|I|+1} \log\Big(1 - \frac{1}{\lambda^{(n)}+1} \sum_{u \in J\setminus I} |B_u|\Big).$$

By expanding the logarithm, we obtain:

$$\tau^{(n)}_{\pi,\pi^{\oplus J}} = \sum_{k\geq 1} \frac{1}{k(\lambda^{(n)}+1)^k} \sum_{I \subsetneq J} (-1)^{|I|} \Big(\sum_{u \in J\setminus I} |B_u| \Big)^k$$

To establish

$$\sum_{I, I \subsetneq J} (-1)^{|I|} \Big(\sum_{u \in J\setminus I} |B_u| \Big)^k = \sum_{(i_1,\dots,i_k)\in W_k(J)} \prod_{u=1}^{k} |B_{i_u}| \tag{5.3}$$

which ends the proof of (5.1), we make use of the exclusion-inclusion principle. Let us consider the following random experiment: *'we choose uniformly at random k points in the set $B := \cup_{j\in J} B_j$'* and for $j \in J$, let E_j denote the event *'at least one of the k points falls into B_j'*. By the inclusion-exclusion principle, the coefficient $\sum_{I \subsetneq J} (-1)^{|I|} \Big(\sum_{u\in J\setminus I} |B_u| \Big)^k$ divided by $|B|^k$ is equal to $\mathbb{P}(\cap_{j\in J} E_j)$. The probability $\mathbb{P}(\cap_{j\in J} E_j)$ can also be decomposed by introducing the events $A_{i,j}$ *'the i-th point falls into B_j'* for every $i \in \{1,\dots,k\}$ and $j \in J$: $\mathbb{P}(\cap_{j\in J} E_j) = \sum_{(i_1,\dots,i_k)\in W_k(J)} \mathbb{P}(\cap_{u=1}^k A_{u,i_u})$. This corresponds to the right-hand side of (5.3) divided by $|B|^k$. Equation (5.2) is obtained by rearranging the terms of the product $\prod_{u=1}^k |B_{i_u}|$. □

Formula (5.1) of the transition rate has a simple interpretation: if we choose R points uniformly at random in $\{1,\dots,n\}$ where R has the logarithmic series distribution[2] with parameter $\frac{1}{\varepsilon_n+1}$, the probability that at least one point falls into each block $\{B_i,\, i\in J\}$ and none outside of $\cup_{j\in J} B_j$ is equal to $(-\log(1-\frac{1}{\varepsilon_n+1}))^{-1} \tau^{(n)}_{\pi,\pi^{\oplus J}}$. From this remark, we derive a simpler construction of $(\mathcal{C}^{(n)}_\alpha)_{\alpha\geq 0}$:

Proposition 5.2. *Let us define a $\mathfrak{P}(\{1,\dots,n\})$-valued sequence $(Y_k)_k$ iteratively as follows:*

- *Y_0 is a random variable with values in $\mathfrak{P}(\{1,\dots,n\})$;*
- *Let $k \in \mathbb{N}$. Given Y_k,*
  - *we choose an integer $R \geq 2$ independent of $Y_0,\dots,Y_n$, following the distribution $\nu = \sum_{k\geq 2} \frac{1}{\beta_{\varepsilon_n} k(\varepsilon_n+1)^k} \delta_k$ where $\beta_{\varepsilon_n} = -\log(1-\frac{1}{\varepsilon_n+1}) - \frac{1}{\varepsilon_n+1}$.*
  - *we choose R points $U_1,\dots,U_R$ uniformly at random on $\{1,\dots,n\}$ and independently of $Y_0,\dots,Y_k, R$.*
  - *Y_{k+1} is obtained from Y_k by merging blocks of Y_k that contain at least one of the R points $U_1,\dots,U_R$.*

Let $(Z_t)_{t\geq 0}$ be a Poisson process with intensity β_{ε_n} and independent of $(Y_k)_k$.
The process $(\Pi^{(n)}_t)_{t\geq 0}$ defined by $\Pi^{(n)}_t = Y_{Z_t}$ for every $t \geq 0$ is a continuous Markov chain:

[2]The logarithmic series distribution with parameter $0 < a < 1$ is defined as the measure $\nu_n = \frac{1}{-\log(1-a)} \sum_{k=1}^{+\infty} \frac{a^k}{k} \delta_k$.

- *The only possible transitions from state $\pi = (B_i)_{i\in I}$ are to partitions $\pi^{\oplus J}$ where J is a subset of I with at least two elements and its transition rate from π to $\pi^{\oplus J}$ is $\tau^{(n)}_{\pi,\pi^{\oplus J}}$ defined in Proposition 5.1.*
- *If π is a partition of $\{1,\dots,n\}$ with k non-empty blocks $(B_1,\dots,B_k)$ then*

$$\mathbb{P}_{\pi_0}(\Pi^{(n)}_t \succeq \pi) = (\frac{\varepsilon_n}{\varepsilon_n+1})^t \prod_{i=1}^{k}(1-\frac{|B_i|}{n(1+\varepsilon_n)})^{-t}\,\mathbb{1}_{\{\pi_0\succeq\pi\}}\,.$$

5.2. **A similar coalescent process on the interval $[0,1]$.** The algorithm described in Proposition 5.2 can be adapted to define a coalescent process $(\Pi_t)_{t\geq 0}$ on the interval $[0,1]$ such that for every partition π of $[0,1]$ and every $t\geq 0$, $\mathbb{P}(\Pi^{(n)}_t \succeq \pi^{(n)})$ converges to $\mathbb{P}(\Pi_t\succeq\pi)$ if ε_n converges to $\varepsilon>0$ and the partition $\pi^{(n)}$ of $\{1,\dots,n\}$ converges to π as n tends to $+\infty$.

Proposition 5.3. *For $k\in\mathbb{N}^*$, let $\mathfrak{P}_k([0,1])$ denote the set of partitions on $[0,1]$ with k blocks having a positive Lebesgue measure. Set $\mathfrak{P}([0,1]) = \cup_{k\in\mathbb{N}^*}\mathfrak{P}_k([0,1])$. For a partition $\pi=(b_i)_{i\in I}\in\mathfrak{P}([0,1])$ and a subset J of I, let $\pi^{\oplus J}$ denote the partition obtained from π by merging blocks b_i for $i\in J$. Let ε be a positive real. Let us define a $\mathfrak{P}([0,1])$-valued sequence $(Y_n)_n$ iteratively as follows:*

- *Y_0 is a random variable with values in $\mathfrak{P}([0,1])$;*
- *Let $n\in\mathbb{N}$. Given Y_n,*
  - *we choose an integer $R\geq 2$ independent of $Y_0,\dots,Y_n$ following the distribution $\nu=\sum_{k\geq 2}\frac{1}{\beta_\varepsilon k(\varepsilon+1)^k}\delta_k$ where $\beta_\varepsilon = -\log(1-\frac{1}{\varepsilon+1})-\frac{1}{\varepsilon+1}$.*
  - *we choose R points $U_1,\dots,U_R$ uniformly at random in the interval $[0,1]$ and independently of $Y_0,\dots,Y_n,R$.*
  - *Y_{n+1} is obtained from Y_n by merging blocks of Y_n that contain at least one of the R points $U_1,\dots,U_R$.*

Let $(Z_t)_{t\geq 0}$ be a Poisson process with intensity β_ε and independent of $(Y_n)_n$.
The process $(\Pi_t)_{t\geq 0}$ defined by $\Pi_t = Y_{Z_t}$ for every $t\geq 0$ is a continuous Markov chain:

- *its positive transition rates are from a partition $\pi=(b_i)_{i\in I}$ to a partition $\pi^{\oplus J}$ with $J\subset I$ having at least two elements; the value of such a transition rate is*

$$\tau_{\pi,\pi^{\oplus J}} := \sum_{k\geq|J|}\frac{1}{k(\varepsilon+1)^k}\sum_{\substack{(k_1,\dots,k_{|J|})\in(\mathbb{N}*)^{|J|},\\ k_1+\dots+k_{|J|}=k}}\binom{k}{k_1,\dots,k_{|J|}}\prod_{u=1}^{|J|}\mathrm{Leb}(b_u)^{k_u}.$$

- *if π is a partition of $[0,1]$ with k non-empty blocks $(b_1,\dots,b_k)$ then*

$$\mathbb{P}_{\pi_0}(\Pi_t\succeq\pi) = (\frac{\varepsilon}{\varepsilon+1})^t\prod_{i=1}^{k}(1-\frac{\mathrm{Leb}(b_i)}{1+\varepsilon})^{-t}\,\mathbb{1}_{\{\pi_0\succeq\pi\}}\,.$$

Proof. By construction $(Y_n)_n$ is a $\mathfrak{P}([0,1])$-valued Markov chain such that for every $n\in\mathbb{N}$, $Y_{n+1}=Y_n$ or Y_{n+1} is a coarser partition of $[0,1]$ than Y_n obtained by merging several blocks of Y_n in one block.
Let $\pi=(b_i)_{i\in I}$ be a partition of $[0,1]$ and let J be a subset of I with at least two elements. Set $b_J =: \cup_{i\in J}b_i$. Given the event $\{Y_n=\pi\}$, Y_{n+1} is equal to $\pi^{\oplus J}$ if among the R points $U_1,\dots,U_R$

uniformly distributed in the interval $[0,1]$ at least one point falls in every block b_i for $i \in J$ and none falls in $[0,1] \setminus b_J$. Therefore,

$$\begin{aligned} P(Y_{n+1} = \pi^{\oplus J} | Y_n = \pi) &= \sum_{k \geq |J|}^{+\infty} \nu(\{k\}) \sum_{(i_1,\dots,i_k) \in W_k(J)} \prod_{u=1}^{k} \mathrm{Leb}(b_{i_u}) \\ &= \sum_{k \geq |J|}^{+\infty} \frac{1}{\beta_\varepsilon k(\varepsilon+1)^k} \sum_{\substack{(k_1,\dots,k_{|J|}) \in (\mathbb{N}^*)^{|J|}, \\ \sum_i k_i = k}} \binom{k}{k_1,\dots,k_{|J|}} \prod_{i \in J} \mathrm{Leb}(b_i)^{k_i}. \end{aligned}$$

Given the event $\{Y_n = \pi\}$, Y_{n+1} is equal to π if $U_1, \dots, U_R$ fall in one block of the partition π. Therefore,

$$P(Y_{n+1} = \pi | Y_n = \pi) = \sum_{k \geq 2}^{+\infty} \nu(\{k\}) \Big(\sum_{i \in I} \mathrm{Leb}(b_i)^k \Big) = -\frac{1}{\beta_\varepsilon} \Big(\frac{1}{\varepsilon+1} + \sum_{i \in I} \log(1 - \frac{\mathrm{Leb}(b_i)}{\varepsilon+1}) \Big).$$

The generator of $(\Pi_t)_{t \geq 0}$ is then $A = \beta_\varepsilon(Q - I)$ where Q is the transition matrix of $(Y_n)_n$.
Let us now compute $\mathbb{P}_{\pi_0}(\Pi_t \succeq \pi)$ where π is a partition of $[0,1]$ with k non-empty blocks $(b_1, \dots, b_k)$ coarser than or equal to π_0. $\{\Pi_t \succeq \pi\}$ means that all points that are chosen simultaneously in the interval $[0,1]$ before time t belong to the same block of π. Therefore, by decomposing $\{\Pi_t \succeq \pi\}$ according to the value of Z_t and the number of points falling at the same time, we obtain:

$$\mathbb{P}_{\pi_0}(\Pi_t \succeq \pi) = \exp(-\beta_\varepsilon t) \Big(1 + \sum_{n=1}^{+\infty} \frac{(\beta_\varepsilon t)^n}{n!} \sum_{r_1,\dots,r_n \geq 2} \nu(\{r_1\}) \dots \nu(\{r_n\}) \prod_{i=1}^{n} (\mathrm{Leb}(b_1)^{r_i} + \dots + \mathrm{Leb}(b_k)^{r_i}) \Big)$$

We expand the product

$$\prod_{i=1}^{n} (\mathrm{Leb}(b_1)^{r_i} + \dots + \mathrm{Leb}(b_k)^{r_i}) = \sum_{(u_1,\dots,u_n) \in \{1,\dots,k\}^n} \mathrm{Leb}(b_{u_1})^{r_1} \dots \mathrm{Leb}(b_{u_n})^{r_n}$$

and use that

$$\sum_{r \geq 2} \nu(\{r\}) \, \mathrm{Leb}(b_u)^r = \frac{1}{\beta_\varepsilon} \Big(-\log(1 - \frac{\mathrm{Leb}(b_u)}{\varepsilon+1}) - \frac{\mathrm{Leb}(b_u)}{\varepsilon+1} \Big)$$

to obtain:

$$\mathbb{P}_{\pi_0}(\Pi_t \succeq \pi) = \exp(-\beta_\varepsilon t) \Big(1 + \sum_{n=1}^{+\infty} \frac{t^n}{n!} \sum_{(u_1,\dots,u_n) \in \{1,\dots,k\}^n} \prod_{i=1}^{n} (-\log(1 - \frac{\mathrm{Leb}(b_{u_i})}{\varepsilon+1}) - \frac{\mathrm{Leb}(b_{u_i})}{\varepsilon+1}) \Big)$$

The second sum is also equal to:

$$\sum_{\substack{(n_1,\dots,n_k) \in \mathbb{N}^k \\ n_1 + \dots + n_k = n}} \binom{n}{n_1,\dots,n_k} \prod_{i=1}^{k} \Big(-\log(1 - \frac{\mathrm{Leb}(b_i)}{\varepsilon+1}) - \frac{\mathrm{Leb}(b_i)}{\varepsilon+1} \Big)^{n_i}.$$

This yields

$$\begin{aligned}\mathbb{P}_{\pi_0}(\Pi_t \succeq \pi) &= \exp(-\beta_\varepsilon t)\Big(1+\sum_{n=1}^{+\infty}\sum_{\substack{(n_1,\dots,n_k)\in\mathbb{N}^k\\ n_1+\dots+n_k=n}}\prod_{i=1}^{k}\frac{t^{n_i}}{n_i!}\Big(-\log(1-\frac{\mathrm{Leb}(b_i)}{\varepsilon+1})-\frac{\mathrm{Leb}(b_i)}{\varepsilon+1}\Big)^{n_i}\Big)\\ &= \exp\Big(-\beta_\varepsilon t + t\sum_{i=1}^{k}(-\log(1-\frac{\mathrm{Leb}(b_i)}{\varepsilon+1})-\frac{\mathrm{Leb}(b_i)}{\varepsilon+1})\Big)\\ &= \exp\Big(t\log(1-\frac{1}{\varepsilon+1})-t\sum_{i=1}^{k}\log(1-\frac{\mathrm{Leb}(b_i)}{\varepsilon+1})\Big).\end{aligned}$$

□

5.3. **Asymptotics for cover time.** A vertex x is said to be *isolated at time* α if no loop $\ell \in \mathcal{DL}_\alpha^{(n)}$ passes through x. We call *'cover time'* the smallest α such that K_n has no isolated vertex at time α and we denote it T_n.
Let us assume that the intensity of the killing measure is proportional to the size of the graph: $\kappa_n = n\varepsilon$ with $\varepsilon > 0$. If we use the algorithm described in Proposition 5.2 to define $\mathcal{C}_\alpha^{(n)}$, then we need to choose uniformly at random an average of $\alpha\beta_\varepsilon E(R) = \frac{\alpha}{\varepsilon(\varepsilon+1)}$ points in $\{1,\dots,n\}$. If the points are drawn one by one and not by packs of random sizes, then the solution of the classical coupon collector's problem provides that the values of T_n is around $\varepsilon(1+\varepsilon)n\log(n)$ for large n. This analogy holds:

Proposition 5.4. *Assume that the intensity of the killing measure on K_n is $\kappa_n = n\varepsilon$ with $\varepsilon > 0$.*
For every $a \in \mathbb{R}$, the number of isolated vertices at time $\alpha_n = \varepsilon(1+\varepsilon)n(\log(n)+a+o(1))$ converges in law to the Poisson distribution with parameter e^{-a}. In particular, $\frac{T_n}{n\varepsilon(1+\varepsilon)} - \log(n)$ converges in law to the Gumbel distribution[3].

Proof. For each vertex x, let $I_{n,x}$ denote the indicator of the event 'x *is isolated at time* α_n'. The number of isolated vertices at time α_n is $S_n = \sum_{x=1}^n I_{n,x}$. For every $k \in \mathbb{N}^*$, its k-th factorial moment $E(S_n)_k := E(S_n(S_n-1)\dots(S_n-k+1))$ is the sum of $P(I_{n,x_1} = \dots = I_{n,x_k} = 1)$ over all (ordered) k-tuples of distinct vertices $(x_1,\dots,x_k)$. As $P(I_{n,x_1} = \dots = I_{n,x_k} = 1)$ is the probability that no loop in $\mathcal{DL}_{\alpha_n}$ intersects the subset $F = \{x_1,\dots,x_k\}$,

$$P(I_{n,x_1} = \dots = I_{n,x_k} = 1) = \exp\Big(-\alpha_n \mu(\ell,\ \ell \text{ intersects } F)\Big)$$

and

$$\begin{aligned}\mu(\ell,\ \ell \text{ intersects } F) &= \mu(\mathcal{DL}(X)) - \mu(\mathcal{DL}(F^c))\\ &= \log\Big(\det(G)\prod_{x\in\{1,\dots,n\}}\lambda_x\Big) - \log\Big(\det(G^{(F^c)})\prod_{x\in F^c}\lambda_x\Big)\end{aligned}$$

Therefore, $E(S_n)_k = k!\sum_{F\in\mathcal{P}_k(\{1,\dots,n\})}\Big(\frac{\det(G^{(F^c)})}{\det(G)\prod_{x\in F}\lambda_x}\Big)^{\alpha_n}$.
In our setting, for every $x \in \{1,\dots,n\}$, $\lambda_x = \lambda^{(n)} = n(\varepsilon+1)-1$ and for every $D \subset \{1,\dots,n\}$,

[3]The cumulative distribution function of the Gumbel distribution is $x \mapsto \exp(-\exp(-x))$.

$\det(G^{(D)})^{-1} = (\lambda^{(n)}+1)^{|D|-1}(\lambda^{(n)}+1-|D|)$. Thus

$$E(S_n)_k = n(n-1)\dots(n-k+1)\Big(1-\frac{1}{n(1+\varepsilon)}\Big)^{-k\alpha_n}\Big(1+\frac{k}{n\varepsilon}\Big)^{-\alpha_n}.$$

$$= \prod_{i=1}^{k}(1-\frac{i}{n})\exp\Big(k(\log(n)-\frac{\alpha_n}{n\varepsilon(1+\varepsilon)})+k\frac{\alpha_n}{2n^2}(\frac{1}{(1+\varepsilon)^2}+\frac{k}{\varepsilon^2})+\alpha_n O(\frac{1}{n^3\varepsilon^3})\Big).$$

As $\alpha_n = n\varepsilon(1+\varepsilon)(\log(n)+a+o(1))$, we deduce that $E(S_n)_k$ converges to $\exp(-ka)$ for every $k\in\mathbb{N}^*$. The convergence to the Poisson distribution with parameter e^{-a} follows from the theory of moments. □

Remark 5.5.

(i) The distribution of the number of isolated vertices at a time α can also be obtained by the inclusion-exclusion principle; the probability that there exists r isolated vertices at time α is

$$\sum_{j=0}^{n-r}(-1)^j\frac{n!}{r!j!(n-r-j)!}(1-\frac{1}{n(1+\varepsilon)})^{-\alpha(r+j)}(1+\frac{r+j}{n\varepsilon})^{-\alpha}.$$

(ii) In the statement of Proposition 5.4, we assume that the intensity of the killing measure κ_n is proportional to the number of vertices n: $\kappa_n = n\varepsilon$. In fact the same proof shows that Proposition 5.4 also holds if we only assume that $\frac{\kappa_n}{\log(n)}$ converges to $+\infty$ and replace ε by $\varepsilon_n = \frac{\kappa_n}{n}$ in the statement. Let us note that if $\frac{\kappa_n}{\log(n)}$ converges to a constant $c>0$ then $E(S_n)_k$ converges to $m_k = \exp(-ka+\frac{k^2}{2c})$ for every $k\in\mathbb{N}^*$ and $\frac{t^k m_k}{k!}$ tends to $+\infty$ for any $t>0$.

5.4. **Asymptotics for coalescence time.** Let τ_n denote the coalescence time, *i.e.* the first time at which $\mathcal{C}_\alpha^{(n)}$ has only one block. The following theorem shows that the cover time and the coalescent time have the same asymptotic distribution:

Theorem 5.6. *Let us assume that $\kappa_n = n\varepsilon$ with $\varepsilon>0$. Set $\alpha_n = \varepsilon(1+\varepsilon)n(\log(n)+a+o(1))$ where a is a fixed real.*
For every $k\in\mathbb{N}$, the probability that $\mathcal{DL}_{\alpha_n}^{(n)}$ consists only of a component of size greater or equal to 2 and k isolated points converges to $\exp(-e^{-a})\frac{e^{-ak}}{k!}$ as n tends to $+\infty$.
In particular, $\frac{\tau_n}{n\varepsilon(1+\varepsilon)} - \log(n)$ converges in distribution to the Gumbel distribution.

Remark 5.7.

(i) Theorem 5.6 shows that $\varepsilon(\varepsilon+1)n\log(n)$ is a sharp threshold function for the connectedness of the random graph process $(\mathcal{G}_\alpha^{(n)})_{\alpha\geq 0}$ defined by the loop sets $(\mathcal{DL}_\alpha^{(n)})_\alpha$.

(ii) Such properties have been proven for the first time by Erdös and Rényi in [4] for the random graph model they have introduced. To facilitate comparison, the following theorem states some of their results in a slightly different way from that used in [4]:

Theorem (Erdös and Rényi, [4]). *Let $G(n,N)$ denote a random graph obtained by forming N links between n labelled vertices, each of the $\binom{N}{\binom{n}{2}}$ graphs being equally likely.*
For every $c\in\mathbb{R}$ and every $k\in\mathbb{N}$, the probability that $G(n,\lfloor\frac{n}{2}(\log(n)+c)\rfloor)$ contains a connected component of size $n-k$ and k isolated points converges to $\exp(-e^{-c})\frac{e^{-ck}}{k!}$ as $n\to$ tends to $+\infty$.

(iii) Let us note that the ratio of $\frac{\mu(\ell,\ \ell \text{ passes through exactly two vertices})}{\mu(\mathcal{DL}(\{1,\dots,n\}))}$ converges to $-\frac{1}{2}(\varepsilon+1+(\varepsilon+1)^2\log(1-\frac{1}{\varepsilon+1}))^{-1}$ as n tends to $+\infty$. The limit is an increasing function of ε that converges to 1 as ε tends to $+\infty$. Therefore, it is not surprising to obtain properties similar to the Erdös-Rényi model for large ε.
In fact, we can show using the same proof that Theorem 5.6 holds if we replace ε by a positive sequence $(\varepsilon_n)_n$ such that $\liminf_{n\to+\infty}\log(n)\varepsilon_n>0$.
By contrast, if we let ε converge to 0 as n tends to $+\infty$, the loop sets can have very large loops: a result of Y. Chang in [3] implies that if $\varepsilon\underset{n\to+\infty}{\sim}n^{-d}$ with $d>1$ then

$$\mathbb{P}(\exists\ell\in\mathcal{DL}^{(n)}_{\frac{1}{\log(n)}}\text{ covering }\{1,\dots,n\})\underset{n\to+\infty}{\to}1-e^{-(d-1)}.$$

The rest of the section is devoted to the proof of Theorem 5.6.
Let $A_{n,k}$ denote the event '$\mathcal{DL}^{(n)}_{\alpha_n}$ *consists only of a component of size greater or equal to 2 and k isolated points*', let V_n be the number of isolated vertices in $\mathcal{DL}^{(n)}_{\alpha_n}$ and let B_n be the event '$\mathcal{DL}^{(n)}_{\alpha_n}$ *has at least two components of size greater or equal to* 2'. We have

$$\mathbb{P}(A_{n,k})=P(V_n=k)-P(\{V_n=k\}\cap B_n).$$

By Proposition 5.4, $\mathbb{P}(V_n=k)$ converges to $e^{-e^{-a}}\frac{e^{-ka}}{k!}$. We shall prove that $\mathbb{P}(B_n)$ converges to 0. For a subset F, let $q_{F,n}$ denote the probability that F is a block at time α_n.

$$\mathbb{P}(B_n)\le\sum_{r=2}^{\lfloor n/2\rfloor}\sum_{F\in\mathcal{P}_r(\{1,\dots,n\})}q_{F,n}.$$

As $\mathcal{DL}^{(n,F)}_{\alpha_n}$ is independent of $\mathcal{DL}^{(n)}_{\alpha}\setminus\mathcal{DL}^{(n,F)}_{\alpha_n}$, we have

$$q_{F,n}=\mathbb{P}(\mathcal{DL}^{(n,F)}_{\alpha_n}\text{ is connected})\,\mathbb{P}(\text{no loop in }\mathcal{DL}^{(n)}_{\alpha_n}\text{ intersects }F\text{ and }F^c).$$

For a sufficiently large set F, we shall simply bound $q_{F,n}$ from above by

$$\mathbb{P}(\text{no loop in }\mathcal{DL}^{(n)}_{\alpha_n}\text{ intersects }F\text{ and }F^c).$$

For small F, the probability $\mathbb{P}(\mathcal{DL}^{(n,F)}_{\alpha_n}$ is connected) is small. Instead of computing it, we shall consider its upper bound by the probability that the total length of non-trivial loops on $\mathcal{DL}^{(n,F)}_{\alpha_n}$ are greater or equal to $|F|$ that is $\mathbb{P}(\sum_{x\in F}N_x^{(\alpha_n,F)}\ge|F|)$ where $N_x^{(\alpha,F)}$ denotes the number of crossings of a vertex x by the set of non-trivial loops included in F at time α. Therefore for a small set F, we shall use that

$$q_{F,n}\le\mathbb{P}(\sum_{x\in F}N_x^{(\alpha_n,F)}\ge|F|)\,\mathbb{P}(\text{no loop in }\mathcal{DL}_{\alpha_n}\text{ intersects }F\text{ and }F^c).$$

We start by stating two lemmas: Lemma 5.8 provides an upper bound for the probability that no loop in $\mathcal{DL}^{(n)}_{\alpha_n}$ intersects F and F^c. Lemma 5.9 gives an exponential inequality for $\mathbb{P}(\sum_{x\in F} N_x^{(\alpha_n,F)} \geq |F|)$ based on Markov's inequality. To shorten the notations let $\mathcal{P}(n)$ denote the power set of $\{1,\dots,n\}$ and let $\mathcal{P}_r(n)$ consist of subsets of $\{1,\dots,n\}$ of cardinality r.

Lemma 5.8. *For every nonempty proper subset F of $\{1,\dots,n\}$, let $A_{n,F}$ denote the event 'no loop in $\mathcal{DL}^{(n)}_{\alpha_n}$ passes through both F and F^c' with $\alpha_n = \varepsilon(\varepsilon+1)n(\log(n)+a+o(1))$.*
For every $\delta \in]0,1[$ there exists $n_\delta > 0$ such that for every $n > n_\delta$ and for every $n^{1-\delta} \leq r \leq \frac{n}{2}$, $\sum_{F\in\mathcal{P}_r(n)} \mathbb{P}(A_{n,F}) \leq \frac{1}{\sqrt{r}} \exp(-\frac{1-\delta}{2} n^{1-\delta}\log(n))$.

Proof. Let $r \in \{1,\dots,n/2\}$ and let F be a subset of $\{1,\dots,n\}$ with r elements.

$$\mathbb{P}(A_{n,F}) = \exp\Big(-\alpha_n \mu(\{\ell,\ \ell \text{ intersects } F \text{ and } F^c\})\Big) = \Big(\frac{\det(G^{(F)})\det(G^{(F^c)})}{\det(G)}\Big)^{\alpha_n}.$$

In our setting, $\det(G^{(D)}) = (n(1+\varepsilon))^{-|D|+1}(n(1+\varepsilon)-|D|)^{-1}$ for every subset D. Therefore, $\mathbb{P}(A_{n,F}) = (1+\frac{1}{\varepsilon(\varepsilon+1)}\frac{|F|}{n}(1-\frac{|F|}{n}))^{-\alpha_n}$. Using that $\binom{n}{r} \leq \frac{1}{\sqrt{2\pi r}\sqrt{1-\frac{r}{n}}}(\frac{n}{r})^r(1-\frac{r}{n})^{-(n-r)}$ (see for example [2], formula 1.5 page 4), we obtain: $\sum_{F\in\mathcal{P}_r(n)} \mathbb{P}(A_{n,F}) \leq \frac{1}{\sqrt{r}} \exp(-n f_n(\frac{r}{n}))$ where f_n is the function on $]0,1/2]$ defined by:

$$f_n(x) = x\log(x) + (1-x)\log(1-x) + u_n\varepsilon(1+\varepsilon)\log(1+\frac{x(1-x)}{\varepsilon(\varepsilon+1)}) \text{ for } x \in]0,1/2],$$

with $u_n = \log(n) + a + o(1)$.
The function f_n is of class C^2 in the interval $]0,1/2]$ and the first two derivatives of f_n are:

- $f_n'(x) = \log(x) - \log(1-x) + u_n \frac{1-2x}{1+\frac{x(1-x)}{\varepsilon(\varepsilon+1)}}$
- $f_n''(x) = \frac{Q_n(x(1-x))}{x(1-x)(1+\frac{x(1-x)}{\varepsilon(\varepsilon+1)})^2}$ where Q_n is the polynomial function of second order defined by: $Q_n(y) = 1 + y(\frac{2}{\varepsilon(1+\varepsilon)} - (2+\frac{1}{\varepsilon(1+\varepsilon)})u_n) + y^2\frac{1}{\varepsilon(1+\varepsilon)}(\frac{1}{\varepsilon(1+\varepsilon)} + 2u_n)$ for every y.

Thus f_n has the following properties:

(i) $\lim_{x\to 0^+} f_n(x) = 0$, $f_n(1/2) > 0$,
(ii) $\lim_{x\to 0^+} f'(x) < 0$, $f'(1/2) = 0$,
(iii) $f_n'' = 0$ has at most two solutions in $]0,1/2]$.

A C^2 function in the interval $]0,1/2]$ that verifies conditions (i), (ii), (iii) has the following property: for every $c \in]0,1/2]$ such that $f_n(c)$ is positive the minimum of f_n on the interval $[c,1/2]$ lies on c or $\frac{1}{2}$ (see Lemma A.2 in the appendix). In fact, $(f_n(1/2))_n$ converges to $+\infty$, whence for sufficiently large n, $\inf_{x\in[c,1/2]} f_n(x) = f_n(c)$.

To conclude, let us consider f_n at point $\frac{1}{n^\delta}$ for some $\delta \in]0,1[$.

$$f_n(\frac{1}{n^\delta}) = -\frac{\delta\log(n)}{n^\delta} + (1-\frac{1}{n^\delta})\log(1-\frac{1}{n^\delta}) + u_n\varepsilon(1+\varepsilon)\log\Big(1+\frac{n^{-\delta}(1-n^{-\delta})}{\varepsilon(1+\varepsilon)}\Big).$$

Therefore $f_n(\frac{1}{n^\delta})$ is equivalent to $\frac{1-\delta}{n^\delta}\log(n)$. We deduce that for sufficiently large values of n and for every $x \in [\frac{1}{n^\delta}, 1/2]$, $f_n(x) \geq f_n(\frac{1}{n^\delta}) \geq \frac{1-\delta}{2n^\delta}\log(n)$ which ends the proof. □

Lemma 5.9. *Let δ and $\bar\delta$ be two positive reals such that $0<\bar\delta<\delta<1$. Assume that $\kappa_n = n\varepsilon$ with $\varepsilon>0$ and set $\alpha_n = \varepsilon(1+\varepsilon)n(\log(n)+a+o(1))$ with $a\in\mathbb{R}$ for every $n\in\mathbb{N}$.*
There exists $n_{\delta,\bar\delta}>0$ such that for every $n\geq n_{\delta,\bar\delta}$, and $F\in\mathcal{P}(n)$ with $2\leq|F|\leq n^{1-\delta}$,

$$\mathbb{P}\Big(\sum_{x\in F} N_x^{(\alpha_n,F)}\geq|F|\Big)\leq n^{-\frac{\bar\delta}{2}|F|}.$$

Proof. To prove Lemma 5.9, we shall apply the Markov inequality to the random variable $\exp(\theta\sum_{x\in F}N_x^{(\alpha_n,F)})$ for a well-chosen positive real θ.

The generating function of the vector $(N_x^{(\alpha,F)}, x\in F)$ has been computed for any finite graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ in [9][4], page 37:

$$\mathbb{E}\Big(\prod_{x\in F}s_x^{N_x^{(\alpha,F)}}\Big)=\Big(\det\Big(\Big[s_x\delta_{x,y}+\lambda_x(1-s_x)G^{(F)}_{x,y}\Big]_{x,y\in F}\Big)\Big)^{-\alpha}\quad\forall\ (s_x)_{x\in\mathcal{V}}\in]0,1]^{\mathcal{V}}.$$

In our setting, the generating function of $\sum_{x\in F}N_x^{(\alpha,F)}$ satisfies:

$$\mathbb{E}\Big(s^{\sum_{x\in F}N_x^{(\alpha,F)}}\Big)=\Big(\det(sI_{|F|}+\lambda^{(n)}(1-s)G^{(F)})\Big)^{-\alpha}\ \forall\, 0<s\leq 1$$

where $\lambda^{(n)}=n(\varepsilon+1)-1$ and $G^{(F)}_{x,y}=\frac{1}{\lambda^{(n)}+1}(\mathbb{1}_{\{x=y\}}+\frac{1}{\lambda^{(n)}+1-|F|})$ for $x,y\in F$. The computation of that determinant is detailed in the Appendix (Lemma A.3). We obtain:

$$\det(sI_{|F|}+\lambda^{(n)}(1-s)G^{(F)})=(1-\frac{1-s}{\lambda^{(n)}+1})^{|F|-1}\Big(1+(1-s)\frac{|F|-1}{\lambda^{(n)}+1-|F|}\Big).\tag{5.4}$$

Equality (5.4) can be extended to $s\in\mathbb{R}$ and the determinant is positive for every $0\leq s<\frac{\lambda^{(n)}}{|F|-1}$. By the Markov inequality, for every $\theta>0$,

$$\mathbb{P}\Big(\sum_{x\in F}N_x^{(\alpha_n,F)}\geq|F|\Big)\leq\exp(-\psi_{F,n}(\theta))\tag{5.5}$$

where $\psi_{F,n}(\theta)=-\log\Big(\mathbb{E}(\exp(-\theta|F|+\theta\sum_{x\in F}N_x^{(\alpha_n,F)}))\Big)$. The expression of the generating function of $\sum_{x\in F}N_x^{(\alpha_n,F)}$ shows that the value of $\psi_{F,n}(\theta)$ depends on F only via $|F|$; we denote it $\psi_{|F|,n}(\theta)$. The value of $\psi_{r,n}(\theta)$ is

$$\psi_{r,n}(\theta)=\theta r+\alpha_n\Big((r-1)\log(1+\frac{e^\theta-1}{\lambda^{(n)}+1})+\log(1-(e^\theta-1)\frac{r-1}{\lambda^{(n)}+1-r})\Big)$$

for every $\theta>0$ such that $e^\theta\leq\frac{\lambda^{(n)}}{r-1}$.
Set $\beta_{r,n}=\frac{\lambda^{(n)}+1}{\sqrt{\alpha_n(r-1)}}$ and $\theta_{r,n}=\log(\beta_{r,n})$ for every integer $2\leq r\leq n$. We shall use inequality (5.5) for $\theta=\theta_{r,n}$. Before, let us note that $\min(\beta_{r,n},\, r\in\{2,\dots,n^{1-\delta}\})$ converges to $+\infty$ and $\max(\frac{\beta_{r,n}(r-1)}{\lambda^{(n)}},\, r\in\{2,\dots,n^{1-\delta}\})$ converges to 0. Therefore, for sufficiently large values of n and $r\in\{2,\dots,n^{1-\delta}\}$, the two conditions $\theta_{r,n}>0$ and $\exp(\theta_{r,n})\leq\frac{\lambda^{(n)}}{r-1}$ are satisfied.

[4]Formula (4.3) page 37 in [9] contains a misprint: the exponent 1 in the left-hand side term of the equation has to be replaced by α.

It remains to bound from below $\psi_{r,n}(\theta_{r,n})$.

$$\psi_{r,n}(\theta_{r,n}) = r\Big(\log(\lambda^{(n)}+1) - \frac{1}{2}\log(\alpha_n) - \frac{1}{2}\log(r-1)\Big) \\ + \alpha_n\Big((r-1)\log(1+\frac{\beta_{r,n}-1}{\lambda^{(n)}+1}) + \log(1-(r-1)\frac{\beta_{r,n}-1}{\lambda^{(n)}+1-r})\Big).$$

Using the classical lower bounds

$$\log(1-t) \geq -t-t^2 \text{ for } 0<t<1/2 \text{ and } \log(1+t) > t - \frac{1}{2}t^2 \text{ for } t>0$$

along with $\max\Big((r-1)\frac{\beta_{r,n}}{\lambda^{(n)}+1},\, r\in\{2,\dots,n^{1-\delta}\}\Big) \underset{n\to+\infty}{\to} 0$, we obtain

$$\psi_{r,n}(\theta_{r,n}) \geq \frac{r}{2}(\log(n) - \log(r-1) - I_{n,1}) - I_{n,2} - I_{n,3}$$

where, for sufficiently large values of n

- $I_{n,1} = \log(1+\frac{\varepsilon}{1-\varepsilon}) + \log(\log(n)+a+o(1)) \leq 2\log(\log(n))$;
- $I_{n,2} = \alpha_n(r-1)r\frac{\beta_{r,n}-1}{(\lambda^{(n)}+1)(\lambda^{(n)}+1-r)} \leq 2r\sqrt{\frac{r-1}{n}(\log(n)+a+o(1))}$;
- $I_{n,3} = \alpha_n(r-1)(\beta_{r,n}-1)^2\Big(\frac{1}{2(\lambda^{(n)}+1)^2} + \frac{r-1}{(\lambda^{(n)}+1-r)^2}\Big) \leq \frac{1}{2} + 2(r-1)$.

Therefore there exists a constant M such that for sufficiently large values of n and for every $2\leq r\leq \frac{n}{2}$, $\psi_{r,n}(\theta_{r,n}) \geq \frac{r}{2}\big(\log(n) - \log(r) - 2\log(\log(n)) - M\big)$. In particular for $2\leq r\leq n^{1-\delta}$, $\psi_{r,n}(\theta_{r,n}) \geq \delta\frac{r}{2}\log(n) - 2\log(\log(n)) - M)$. This shows that for every $0<\bar{\delta}<\delta$ and sufficiently large values of n, $\min_{r\in\{2,\dots,n^{1-\delta}\}}\psi_{r,n}(\theta_{r,n}) \geq \frac{\bar{\delta}}{2}r\log(n)$ which ends the proof.
Let us note that a study of $\psi_{r,n}$ shows that for every $r\in\{2,\dots,n^{1-\delta}\}$, and sufficiently large values of n, $\psi_{r,n}$ has a maximum at a point which is equivalent to $\theta_{r,n}$ as n tends to $+\infty$. □

We can now complete the proof of Theorem 5.6. Set $S_{r,n} = \sum_{F\in\mathcal{P}_r(n)} q_{F,n}$. We shall prove that the upper bound of $\mathbb{P}(B_n)$, $\sum_{r=2}^{\lfloor n/2\rfloor} S_{r,n}$ converges to 0 as n tends to $+\infty$.
By Lemma 5.8,

$$\sum_{r=n^{1-\delta}}^{\frac{n}{2}} S_{r,n} \leq n\exp(-\frac{1-\delta}{2}n^{1-\delta}\log(n)).$$

Let us consider the case of subsets F with $r\leq n^{1-\delta}$ elements. Using the notations introduced in Lemmas 5.8 and 5.9, we have $S_{r,n} \leq \frac{1}{\sqrt{r}}\exp(-nf_n(\frac{r}{n}) - \psi_{r,n}(\theta_{r,n}))$. By Lemma 5.9, for every $\bar{\delta}\in]0,\delta[$, sufficiently large n and $r\in\{2,\dots,n^{1-\delta}\}$,

$$S_{r,n} \leq \frac{1}{\sqrt{r}}\exp\Big(-n(f_n(\frac{r}{n}) + \frac{\bar{\delta}}{2}\frac{r}{n}\log(n))\Big).$$

Let us study the function $\bar{f}_n(x) = f_n(x) + \frac{\bar{\delta}}{2}x\log(n)$. By computations we obtain that $\bar{f}_n(2/n)$ is equivalent to $\frac{\bar{\delta}}{2n}\log(n)$ as n tends to $+\infty$. The study of f_n made in the proof of Lemma 5.8 shows that, for sufficiently large values of n, $\bar{f}_n$ is greater than $\bar{f}_n(2/n)$ in $[2/n, 1/2]$. Let us introduce a real $\bar{\bar{\delta}}$ such that $0<\bar{\bar{\delta}}<\bar{\delta}$. We have shown that for n large enough and every $r\in\{2,\dots,n^{1-\delta}\}$, $S_{r,n}\leq\frac{1}{\sqrt{r}}n^{-\frac{\bar{\bar{\delta}}}{2}}$ and thus $\sum_{r=2}^{n^{1-\delta}} S_{r,n} \leq n^{1-\delta-\bar{\bar{\delta}}/2}$.
By taking $\delta = 3/4$ and $\bar{\bar{\delta}} = 2/3$ for instance we obtain that $P(B_n) \leq ne^{-\frac{1}{8}n^{1/4}\log(n)} + n^{-1/12}$ for sufficiently large values of n. This ends the proof of Theorem 5.6.

Appendix A.

Lemma A.1. *For a finite set E, let $\mathfrak{S}_E$ denote the set of permutations of E and let $\mathfrak{S}_E^0$ consist of permutations of E without fixed point. For two integers $r \geq 2$ and $1 \leq k \leq r/2$, let $\mathfrak{P}_k(2,r)$ denote the partitions of $\{1,\dots,r\}$ with k blocks, each of them having at least two elements. For a $r \times r$ matrix A and a real α, set $\mathrm{Per}_\alpha^0(A) = \sum_{\sigma \in \mathfrak{S}^0_{\{1,\dots,r\}}} \alpha^{m(\sigma)} A_{1,\sigma(1)} \dots A_{r,\sigma(r)}$ where $m(\sigma)$ denotes the number of cycles in a permutation σ ($\mathrm{Per}_\alpha^0(A)$ introduced in [9] page 41, can also be defined as the α-permanent of A with the diagonal elements of the matrix set to zero). Another expression of $\mathrm{Per}_r^0(A)$ is*

$$\mathrm{Per}_r^0(A) = \sum_{k=1}^{r/2} \alpha^k \sum_{\pi=(\pi_1,\dots,\pi_k)\in\mathfrak{P}_k(2,r)} \prod_{j=1}^{k} \Big(\frac{1}{|\pi_j|} \sum_{\sigma\in\mathfrak{S}_{\pi_j}} A_{\sigma(1),\sigma(2)} A_{\sigma(2),\sigma(3)} \dots A_{\sigma(|\pi_j|),\sigma(1)} \Big).$$

Proof. For a finite set B, let $\mathfrak{S}_B^c$ be the subset of $\mathfrak{S}_B$ which consists of cycles of length $|B|$. The decomposition of permutations into disjoint cycles entails that

$$\mathrm{Per}_r^0(A) = \sum_{k=1}^{r/2} \alpha^k \sum_{\pi=(\pi_1,\dots,\pi_k)\in\mathfrak{P}_k(2,r)} \prod_{j=1}^{k} \Big(\sum_{\sigma_j\in\mathfrak{S}^c_{\pi_j}} \prod_{u\in\pi_j} A_{u,\sigma_j(u)} \Big).$$

Therefore, it remains to prove that for every $k \geq 2$

$$\sum_{\sigma\in\mathfrak{S}^c_{\{1,\dots,k\}}} \prod_{u=1}^{k} A_{u,\sigma(u)} = \frac{1}{k} \sum_{\nu\in\mathfrak{S}_{\{1,\dots,k\}}} A_{\nu(1),\nu(2)} A_{\nu(2),\nu(3)} \dots A_{\nu(k),\nu(1)}. \tag{A.1}$$

Starting from a permutation $\nu \in \mathfrak{S}_{\{1,\dots,k\}}$, we define a cycle of length k, $F(\nu)$ such that $\prod_{i=1}^k A_{i,F(\nu)(i)} = A_{\nu(1),\nu(2)} A_{\nu(2),\nu(3)} \dots A_{\nu(k),\nu(1)}$ by setting

$$F(\nu)(u) = \begin{cases} \nu(1) & \text{if } \nu^{-1}(u) = k \\ \nu(\nu^{-1}(u)+1) & \text{if } \nu^{-1}(u) \leq k-1. \end{cases}$$

Conversely, starting from a cycle σ of length k, we can construct exactly k different permutations $\nu_1,\dots,\nu_k$ such that $F(\nu_1) = \dots = F(\nu_k) = \sigma$ by setting

$$\nu_i(j) = \begin{cases} i & \text{if } j = 1 \\ \sigma^{j-1}(i) & \text{if } j \in \{2,\dots,k\} \end{cases}$$

This shows equality (A.1), completing the proof. □

Lemma A.2. *Let $a < b$ be two reals. Let f be a real function of class C^2 in the interval $]a,b]$. Assume that:*

- $\lim_{x\to a^+} f(x) = 0$, $f(b) > 0$,
- $\lim_{x\to a^+} f'(x) < 0$, $f'(b) \leq 0$,
- $f'' = 0$ *has at most two solutions in* $]a,b]$.

Then for every $c \in]a,b]$ such that $f(c) > 0$, the minimum of f on the interval $[c,b]$ lies on c or b.

Proof. The assumptions on f ensure that the only possible configurations are:

(i) there exists $\rho \in]a,b[$ such that f is a decreasing function on $]a,\rho[$ and an increasing function on $[\rho,b]$;
(ii) there exists two reals $\rho_- < \rho_+$ in $]a,b[$ such that f is a decreasing function on $]a,\rho_-[$, an increasing function on $[\rho_-,\rho_+]$ and a decreasing function on $[\rho_+,b]$.

Since $\lim_{x\to a^+} f(x) = 0$, for every $x \in]a,b[$ such that $f(x) > 0$, we have in both cases $\inf_{u\in[x,b]} f(u) = \min(f(x), f(b))$. □

Lemma A.3. *Let a and b be two reals and let n be a positive integer. Let I_n denote the identity matrix of size n and let J_n denote the $n \times n$ matrix with all entries equal to one. The determinant of the matrix $M_n = bJ_n + (a-b)I_n$ is $(a-b)^{n-1}(a+(n-1)b)$.*

Proof. First, $\det(M_1) = a$. Let $n \geq 2$. If we substract the column $n-1$ to the column n and expand the determinant along the column n, we obtain:
$\det(M_n) = (a-b)\det(M_{n-1}) + (a-b)\det(R_{n-1})$ where R_n denotes the matrix defined by

$$R_n = \begin{cases} b & \text{if } n = 1 \\ bJ_n + \operatorname{diag}(a-b,\dots,a-b,0) & \text{if } n \geq 2. \end{cases}$$

By applying the same transformations to R_n, we obtain

$$\det(R_n) = (a-b)\det(R_{n-1}) = (a-b)^{n-1}b.$$

The expression of $\det(M_n)$ follows by induction on n. □

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