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Conditional Markov regime switching model applied to economic modelling.

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Abstract

In this paper we discuss the calibration issues of regime switching models built on mean-reverting and local volatility processes combined with two Markov regime switching processes. In fact, the volatility structure of this model depends on a first exogenous Markov chain whereas the drift structure depends on a conditional Markov chain with respect to the first one. The structure is also assumed to be Markovian and both structure and regime are unobserved. Regarding this construction, we extend the classical Expectation-Maximization (EM) algorithm to be applied to our regime switching model. We apply it to economic datas (Euro-Dollars foreign exchange rate and Brent oil price) to show that this modelling well identifies both mean reverting and volatility regimes switches. Moreover, it allows us to give economic interpretations of this regime classification such as some financial crisis or some economic policies.

Keywords: Markov regime switching; Expectation-Maximization algorithm; mean-reverting; local volatility; economics data.

MSC classification: 91G70, 60J05, 91G30.

JEL classification: F31, C58, C51, C01.

Introduction

The use of Hamilton's Markov switching models to study economic times series data as business cycle, economic growth or unemployment is not new. In his seminal paper [7], Hamil-

ton already noticed that Markov-switching models are able to reproduce the different phase of the business cycles and captures the cyclical behavior of the U.S. GDP growth data. More recently, Bai and Wang in [3] went one step further by allowing changes in variance and showed that their restricted model well identifies both short-run regime switches and long-run structure changes in the U.S. macroeconomic data. Janczura and Weron in [8] showed that Markov regime switching diffusion well fits market data as electricity spot prices and allows us for useful economic interpretations of the regime states. Goutte and Zou in [6] compared the results given by the well fit of different regime switching models against non regime switching diffusion on foreign exchange rates data. They proved that regime switching models with both mean reverting and local volatility structures are the best choice to fit well data. Moreover, this modelling allows them to capture well some significant economic behavior as crisis time period or change in the variance dynamic level.

Basing on the above facts that Markov switching models capture the economic cycles and regime switching, therefore, we would like to extend the model stated by Goutte and Zou in [6] with a conditional Markov chain structure as in Bai and Wang in [3]. Indeed, in [3], the authors did not take into account that the model could have a mean reverting effect and that the model could have a regime switching local volatility structure. As mentioned before, Goutte and Zou in [6] proved also that continuous time regime switching model fits better economic times series data than non-regime switching model. Hence, in this paper, we will define a mean reverting local volatility regime switching model where the volatility structure will depend on a first Markov chain and the drift structure will have a mean reverting effect which depends on a conditional Markov chain with respect to the first one. We will develop an Expectation-Maximization (EM) algorithm to apply to this class of regime switching model. Indeed, the (EM) algorithm initiated by Hamilton in [7] is a two steps algorithm: firstly an estimation procedure where we evaluate all the probabilities of the regime switching model; secondly a likelihood maximization step to estimate all the parameters of our models. Hence, in this paper, we will follow these two steps to give the procedure in our specific regime switching model case. Finally, since one of the aim of this paper is to establish a model that could capture various key features or trend of an economic times series data, such as a mean level change or a growth of the volatility ... We will use it on some economic time series data: firstly on the Euro/Dollar foreign exchange rate and secondly on the brent crude oil spot price in Euros.

Hence, the paper is structured as follow, in a first section, we will give some notations and introduce our model. Then, in a second part, we will give the (EM) algorithm which is the method to estimate all the parameters of our regime switching model. Then in the last section, we will apply this method to economic times series data. We will also give economic interpretations and thus we will show the ability of this regime switching model to capture various key features such as spikes in data or changes in the volatility level or crisis time periods.

1 The model

Let $T > 0$ be a fixed maturity time and denote by $(\Omega, \mathbb{F} := (\mathcal{F}_t)_{[0,T]}, \mathbb{P})$ an underlying probability space. We will follow, in this paper, the seminal Markov switching model introduced by Hamilton in [7]. Whereas, we will use in the sequel a generalization of this classical regime switching model. Firstly, we will use conditional Markov chain as initiated by Bai and Wang in [3]. Secondly, we will use a more global class of stochastic model using a mean-reverting local volatility regime switching diffusion instead of a basic autoregressive model.

1.1 Conditional Markov chain

We begin with the construction of our Markov regime switching model. We will classify the states of the economy into two regimes: an exogenous regime and an endogenous regime. The exogenous regime will characterize the long run structure changes and the endogenous regime will characterize the short run business cycles. The exogenous regime values will be given by a homogenous continuous time Markov chain X^2 on finite state $\mathcal{K} := \{1, 2, \dots, K\}$ and with transition matrix P^{X^2} given by

$$P^{X^2} = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1K} \\ p_{21} & p_{22} & \dots & p_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ p_{K1} & p_{K2} & \dots & p_{KK} \end{pmatrix}. \quad (1.1)$$

Remark 1.1. *The quantity p_{ij} represents the intensity of the jump from state i to state j .*

The endogenous regime values will be given also by a homogenous continuous time Markov chain X^1 on finite state $\mathcal{L} := \{1, 2, \dots, L\}$ but its transition matrix will depend on the value of the exogenous regime. Hence, the transition matrix of X^1 will be conditional on the value of the Markov chain X^2 . The endogenous economic regime thus follows a conditional Markov chain, where the Markovian property applies only after conditioning on the exogenous state. Hence, the state of the endogenous regime X^1 will be determined conditioning on the state of the exogenous regime X^2 .

To define the transitions matrix of X^1 we firstly construct a time grid partition of the time interval $[0, T]$. For this, we partition the time interval such that,

$$0 = t_0 < t_1 < \dots < t_N = T \quad \text{with} \quad \Delta_t := t_{k+1} - t_k = 1 \quad \text{for all} \quad k \in \{0, \dots, N\}. \quad (1.2)$$

For all $s \in \mathcal{K}$, we can now define the probability transition to state $i \in \mathcal{L}$ to $j \in \mathcal{L}$ with respect to the value of the Markov chain X^2 of the Markov chain X^1 as

$$p_{ij}^s = \mathbb{P}\left(X_{t_k}^1 = j | X_{t_k}^2 = s, X_{t_{k-1}}^1 = i\right), \quad \forall k \in \{1, \dots, N\}, \quad \forall s \in \mathcal{K}. \quad (1.3)$$

Hence we get K possible transition matrix $P_s^{X_1}$, $s \in \mathcal{K}$ given by

$$P_s^{X_1} = \begin{pmatrix} p_{11}^s & p_{12}^s & \cdots & p_{1L}^s \\ p_{21}^s & p_{22}^s & \cdots & p_{2L}^s \\ \vdots & \vdots & \vdots & \vdots \\ p_{L1}^s & p_{L2}^s & \cdots & p_{LL}^s \end{pmatrix}. \quad (1.4)$$

We assume in the sequel that

Assumption 1.1. 1. For all $k \in \{1, \dots, N\}$, $X_{t_k}^2$ is an exogenous Markov process. Hence, it satisfies

$$\mathbb{P}\left(X_{t_{k+1}}^2 | X_{t_k}^2, X_{t_k}^1, X_{t_{k-1}}^2, X_{t_{k-1}}^1, \dots, X_{t_0}^2, X_{t_0}^1\right) = \mathbb{P}\left(X_{t_{k+1}}^2 | X_{t_k}^2\right). \quad (1.5)$$

2. For all $k \in \{1, \dots, N\}$, $X_{t_k}^1$ is conditionally Markovian:

$$\mathbb{P}\left(X_{t_{k+1}}^1 | X_{t_{k+1}}^2, X_{t_k}^1, X_{t_{k-1}}^2, X_{t_{k-1}}^1, \dots, X_{t_0}^2, X_{t_0}^1\right) = \mathbb{P}\left(X_{t_{k+1}}^1 | X_{t_{k+1}}^2, X_{t_k}^1\right). \quad (1.6)$$

Point 2. of Assumption (1.1) means that the value at time t_k , $k \in \{1, \dots, N\}$, of the Markov chain X^1 depends both on the value of the Markov chain X^1 at time t_{k-1} and of the Markov chain X^2 at time t_{k-1} .

Remark 1.2. In the particular case where $\mathcal{K} \equiv \mathcal{L} := \{1, 2\}$ and under Assumptions 1.1, this model can be defined by the joint distribution $Z_{t_k} = (X_{t_k}^1, X_{t_k}^2)$ in the space $\mathcal{S} := \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Hence, in this two regimes case, the transition matrix of the Markov chains X^1 and X^2 are given by:

$$P^{X_2} = \begin{pmatrix} p & 1 - q \\ 1 - p & q \end{pmatrix}$$

and

$$P_1^{X_1} = \begin{pmatrix} p_1 & 1 - q_1 \\ 1 - p_1 & q_1 \end{pmatrix}, \quad P_2^{X_1} = \begin{pmatrix} p_2 & 1 - q_2 \\ 1 - p_2 & q_2 \end{pmatrix}.$$

Moreover, we have that

$$\mathbb{P}\left(Z_{t_{k+1}} | Z_{t_k}, Z_{t_{k-1}}, Z_{t_0}\right) = \mathbb{P}\left(Z_{t_{k+1}} | Z_{t_k}\right) = \mathbb{P}\left(X_{t_{k+1}}^1 | X_{t_{k+1}}^2, X_{t_k}^1\right) \cdot \mathbb{P}\left(X_{t_{k+1}}^2 | X_{t_k}^2\right) \quad (1.7)$$

and so the 4×4 transition matrix of Z is given by

$$P^Z = \begin{pmatrix} p \cdot P_1^{X_1} & (1 - p) \cdot P_1^{X_1} \\ (1 - q) \cdot P_2^{X_1} & q \cdot P_2^{X_1} \end{pmatrix}. \quad (1.8)$$

Remark 1.3. *If we assume that for all $i, j \in \mathcal{L}$ and $s_1 \neq s_2 \in \mathcal{K}$ that $p_{ij}^{s_1} = p_{ij}^{s_2}$, then the Markov chain X^1 is no longer a conditional Markov chain. Indeed, its transition matrix no longer depends on the values of the Markov chain X^2 and so the two Markov chains X^1 and X^2 are now independent. Hence, this regime switching model becomes an independent regime switching model studied, for example, by Goutte and Zou in [6], applied to foreign exchange rate data.*

In an economic point of view, we can interpret the two states case as mentioned in Remark 1.2 as a low/high mean and a low/high variance. Hence, as soon as we know the variance level state we can know if we are in low or high mean level. Hence this model can capture a different level of mean in each level of variance. Indeed, with this modelling, an economic data can be in a high variance regime but with a low mean trend and respectively in a low variance level but with a high mean level. Thus, this conditional regime switching model allows us to differentiate these different possible states.

1.2 Regime switching diffusion

In the sequel, we will work on a discretized version of the mean-reverting, heteroskedastic process given by the following stochastic differential equation

$$dY_t = (\mu(X_t^1, X_t^2) - \beta(X_t^1, X_t^2) Y_t) dt + \sigma(X_t^2) |Y_t|^\delta dW_t, \quad \delta \in \mathbb{R}^+.$$

Thus, we will work on the following observed data process Y_{t_k} , where time $(t_k)_{k \in \{0, 1, \dots, N\}}$ are defined by the construction (1.2), given by:

Definition 1.1. *Let $(Y_{t_k})_{k \in \{0, \dots, N\}}$ be our data process (i.e. a time series) and let $(X_{t_k}^1)_{k \in \{0, \dots, N\}} \in \mathcal{L}$ and $(X_{t_k}^2)_{k \in \{0, \dots, N\}} \in \mathcal{K}$ be two Markov processes. Then our general model is given by*

$$Y_{t_k} = \mu(X_{t_k}^1, X_{t_k}^2) + (1 - \beta(X_{t_k}^1, X_{t_k}^2)) Y_{t_{k-1}} + \sigma(X_{t_k}^2) |Y_{t_{k-1}}|^\delta \epsilon_{t_k}, \quad \delta \in \mathbb{R}^+. \quad (1.9)$$

where $(\epsilon_{t_k})_{k \in \{0, \dots, N\}}$ follows a $\mathcal{N}(0, 1)$.

Remark 1.4. – *The regime switching model (1.9) is a continuous time regime switching diffusion with drift $\mu(X_{t_k}^1, X_{t_k}^2) + (1 - \beta(X_{t_k}^1, X_{t_k}^2)) Y_{t_{k-1}}$ and volatility $\sigma(X_{t_k}^2) |Y_{t_{k-1}}|^\delta$, $\delta \in \mathbb{R}^+$.*

- *The drift factor ensures mean reversion of the process towards the long run value $\frac{\mu(X_{t_k}^1, X_{t_k}^2)}{\beta(X_{t_k}^1, X_{t_k}^2)}$, with speed of adjustment governed by the parameter $\beta(X_{t_k}^1, X_{t_k}^2)$. From an economic point of view, if the value of $\beta(X_{t_k}^1, X_{t_k}^2)$ is large then the dynamic of the process Y is almost near the value of the mean, even if there is a spike at time $t \in [0, T]$. Then, for a small time period η , the value of $Y_{t+\eta}$ will be again close to the value of the mean.*

- The two Markov chains can be seen as economic impact factors. Indeed, assume that our regime switching diffusion Y models the spread of a firm A . Then, an economic interpretation of the regime switching model is that the exogenous Markov chain X^2 could be the credit rating of the firm A given by an exogenous rating company as "Standard and Poors". And the endogenous regime X^1 is then an indicator of the potentially "good health" of the firm A given the value of its credit rating (i.e. the value of the exogenous regime X^2).

The regime switching model (1.9) is so a mean reverting model with local volatility. Hence it is a regime switching mean reverting constant of elasticity variance model (CEV)¹. So our model is constructed to encompass most of the financial models stated in the literature. Indeed, we can obtain a:

- regime switching Cox-Ingersoll-Ross model (CIR) by taking $\delta = \frac{1}{2}$.
- regime switching Vasicek model by taking $\delta = 0$.
- regime switching mean reverting Geometric Brownian motion by taking $\delta = 1$.

Regarding the Remark 1.4, we have that, given $Y_{t_{k-1}}, Y_{t_k}$ has a conditional Gaussian distribution:

$$Y_{t_k} \sim \mathcal{N} \left(\mu(X_{t_k}^1, X_{t_k}^2) + (1 - \beta(X_{t_k}^1, X_{t_k}^2)) Y_{t_{k-1}}, \sigma^2(X_{t_k}^2) |Y_{t_{k-1}}|^{2\delta} \right).$$

Let denote by $\mathbf{Y}_k := \{Y_{t_0}, Y_{t_1}, \dots, Y_{t_k}\}$ the history of Y up to time $t_k, k \in \{1, \dots, N\}$. Therefore $\mathbf{Y}_n := \mathbf{Y}_T$ represents the full history of the data process Y . Assume, now, that we work with the bivariate Markov process $Z_t = (X_t^1, X_t^2)$ defined in Remark 1.2. Hence, it takes its values in the finite space $\mathcal{S} := \mathcal{K} \times \mathcal{L}$. Let denote by Θ the set of all parameters to estimate. In fact there are $K(2L + 1) + 6$ parameters in Θ .

Remark 1.5. If $\mathcal{K} = \mathcal{L} = \{1, 2\}$, then Θ contains 16 parameters to estimate:

$$\Theta := \{\mu(1, 1), \mu(1, 2), \mu(2, 1), \mu(2, 2), \beta(1, 1), \beta(1, 2), \beta(2, 1), \beta(2, 2), \sigma(1), \sigma(2), p, q, p_1, q_1, p_2, q_2\}.$$

Given the data process history information, we have that the probability distribution function (pdf) of Y_{t_k} is given by

$$f \left(Y_{t_k} | Z_{t_k} = (i, j); \mathbf{Y}_{k-1}; \Theta^{(n)} \right) = \frac{1}{\sqrt{2\pi}\sigma(j)|Y_{t_{k-1}}|^\delta} \exp \left\{ -\frac{[Y_{t_k} - (1 - \beta(i, j))Y_{t_{k-1}} - \mu(i, j)]^2}{2\sigma^2(j)|Y_{t_{k-1}}|^{2\delta}} \right\} \quad (1.10)$$

with $X_{t_k}^1 = i, i \in \mathcal{L}$ and $X_{t_k}^2 = j$ for $j \in \mathcal{K}$.

¹This model was developed by Cox, J. in "Notes on Option Pricing I: Constant Elasticity of Diffusions." Unpublished draft, Stanford University, 1975.

2 The estimation procedure

As we said in the introduction, we will use the Expectation-Maximization (EM) algorithm initiated by Hamilton in [7]. Indeed, we will extend this algorithm to cover our regime switching model (1.9). This algorithm starts with an arbitrarily chosen vector of initial parameters $\Theta^{(0)}$. Then, firstly, in the Expectation step (**E-step**), the probabilities relative on the bivariate Markov chain are calculated. Hence, we evaluate the so-called smoothed and filtered probabilities². Secondly, in the Maximization step (**M-step**), we evaluate the new maximum likelihood estimates of the parameter vector Θ based on the probabilities evaluated in the (**E-step**). Finally, we repeat this two steps until the maximum of the likelihood function is reached.

2.1 The expectation step (E-step)

Assume that $\Theta^{(n)}$ is the parameter vector calculated in the **M-step** during the previous iteration ($n \in \mathbb{N}$). Recall that for $k \in \{0, 1, \dots, N\}$, $\mathbf{Y}_k := \{Y_{t_0}, Y_{t_1}, \dots, Y_{t_k}\}$ is the information available at time t_k . Then the filtered and smoothed probabilities of our model are given by the following procedure based on the standard formulas given by Kim and Nelson (1999) in [10].

Assume that we are in the iteration $n \in \mathbb{N}$ of the estimation procedure. Then, we can evaluate:

- **The Filtered Probabilities:**

Based on the Bayes rule, for $k = 1, \dots, N$, iterate on equations:

$$P\left(Z_{t_k} | \mathbf{Y}_k; \Theta^{(n)}\right) = \frac{f(Y_{t_k} | Z_{t_k}; \mathbf{Y}_{k-1}; \Theta^{(n)}) \cdot P\left(Z_{t_k} | \mathbf{Y}_{k-1}; \Theta^{(n)}\right)}{\sum_{Z_{t_k}} f(Y_{t_k} | Z_{t_k}; \mathbf{Y}_{k-1}; \Theta^{(n)}) \cdot P\left(Z_{t_k} | \mathbf{Y}_{k-1}; \Theta^{(n)}\right)}, \quad (2.11)$$

where $\sum_{Z_{t_k}}$ means the sum over all the possible states of the bivariate Markov chain Z and

$$\begin{aligned} P\left(Z_{t_k} | \mathbf{Y}_{k-1}; \Theta^{(n)}\right) &= \sum_{Z_{t_{k-1}}} P\left(Z_{t_k}, Z_{t_{k-1}} | \mathbf{Y}_{k-1}; \Theta^{(n)}\right), \\ &= \sum_{Z_{t_{k-1}}} P(Z_{t_k} | Z_{t_{k-1}}) \cdot P\left(Z_{t_{k-1}} | \mathbf{Y}_{k-1}; \Theta^{(n)}\right), \end{aligned}$$

until $P\left(Z_{t_n} | \mathbf{Y}_n; \Theta^{(n)}\right) := P\left(Z_T | \mathbf{Y}_T; \Theta^{(n)}\right)$ is calculated. We recall that the model definition (1.9) implies that the probability distribution function $f(Y_{t_k} | Z_{t_k}; \mathbf{Y}_{k-1}; \Theta^{(n)})$ is given by (1.10).

²The smoothed probability is the conditional probability that the Markov chain Z_{t_k} is in state (i, j) at time t_k with respect to $\mathbf{Y}_n := \mathbf{Y}_T$ and the filtered probability is with respect to \mathbf{Y}_k , $k \in \{0, 1, \dots, n\}$

• **The Smoothed Probabilities:**

For $k = N - 1, N - 2, \dots, 1$ iterate on

$$\begin{aligned} P\left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{n}}; \Theta^{(n)}\right) &= \sum_{Z_{t_{k+1}}} P\left(Z_{t_k} = (i, j), Z_{t_{k+1}} | \mathbf{Y}_{\mathbf{n}}; \Theta^{(n)}\right), \\ &= \sum_{Z_{t_{k+1}}} \frac{P\left(Z_{t_{k+1}} | \mathbf{Y}_{\mathbf{n}}; \Theta^{(n)}\right) \cdot P\left(Z_{t_{k+1}} | Z_{t_k} = (i, j)\right) \cdot P\left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{t}_k}; \Theta^{(n)}\right)}{P\left(Z_{t_{k+1}} | \mathbf{Y}_{\mathbf{t}_k}; \Theta^{(n)}\right)}. \end{aligned} \quad (2.12)$$

2.2 The maximization step (M-step)

In the second step of the EM algorithm, new maximum likelihood (ML) estimates $\Theta^{(n+1)}$, for all parameters of the model, are calculated.

Remark 2.6. In a standard maximum likelihood estimation, the log-likelihood function given by

$$\sum_{k=0}^N \log(f(Y_{t_k}, \Theta^{(n)}))$$

is maximized. Here, each component of this sum has to be weighted with the corresponding smoothed probabilities. Thus, our log-likelihood function becomes

$$L\left(\Theta^{(n)}; \mathbf{Y}_{\mathbf{T}}\right) = \sum_{k=0}^N \sum_{Z_{t_k}} \log\left(f(Y_{t_k} | Z_{t_k}, \mathbf{Y}_{\mathbf{k}-1}; \Theta^{(n)})\right) \cdot P\left(Z_{t_k} | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right). \quad (2.13)$$

Proposition 2.1. The (ML) estimates for all the parameters of the model defined by (1.9) are given, for $\delta \in \mathbb{R}^+$, $i \in \mathcal{L}$ and $j \in \mathcal{K}$, by the following formulas:

$$\mu(i, j)^{(n+1)} = \frac{\sum_{k=1}^N \left[P\left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right) |Y_{t_{k-1}}|^{-2\delta} \left(Y_{t_k} - (1 - \beta(i, j)^{(n+1)})Y_{t_{k-1}}\right) \right]}{\sum_{k=1}^N \left[P\left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right) |Y_{t_{k-1}}|^{-2\delta} \right]},$$

$$\beta(i, j)^{(n+1)} = \frac{\sum_{k=1}^N \left[P\left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right) |Y_{t_{k-1}}|^{-2\delta} Y_{t_{k-1}} B_1 \right]}{\sum_{k=1}^N \left[P\left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right) |Y_{t_{k-1}}|^{-2\delta} Y_{t_{k-1}} B_2 \right]},$$

$$\left(\sigma(j)^{(n+1)}\right)^2 = \frac{\sum_{k=1}^N \sum_{i \in \mathcal{L}} \left[P\left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right) |Y_{t_{k-1}}|^{-2\delta} \left(Y_{t_k} - \alpha(i, j)^{(n+1)} - (1 - \beta(i, j)^{(n+1)})Y_{t_{k-1}}\right)^2 \right]}{\sum_{k=1}^N \left[P\left(X_t^2 = j | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right) \right]},$$

with

$$\begin{aligned}
B_1 &= Y_{t_k} - Y_{t_{k-1}} - \frac{\sum_{k=1}^N [P(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}) |Y_{t_{k-1}}|^{-2\delta} (Y_{t_k} - Y_{t_{k-1}})]}{\sum_{k=1}^N [P(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}) |Y_{t_{k-1}}|^{-2\delta}]}, \\
B_2 &= \frac{\sum_{k=1}^N [P(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}) |Y_{t_{k-1}}|^{-2\delta} Y_{t_{k-1}}]}{\sum_{k=1}^N [P(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}) |Y_{t_{k-1}}|^{-2\delta}]} - Y_{t_{k-1}}.
\end{aligned}$$

Proof. Let Z_{t_k} be in state (i, j) with $i \in \mathcal{L}$ and $j \in \mathcal{K}$ then the (i, j) -th regime weighted log-likelihood function is given by

$$\begin{aligned}
\log [L(\Theta^{(n+1)})] &= - \sum_{k=1}^N P(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}) \left[\log \left(\sqrt{2\pi(\sigma(j)^{(n+1)})^2 |Y_{t_{k-1}}|^{2\delta}} \right) \right. \\
&\quad \left. + \left(\frac{[Y_{t_k} - (1 - \beta(i, j)^{(n+1)})Y_{t_{k-1}} - \mu(i, j)^{(n+1)}]^2}{2(\sigma(j)^{(n+1)})^2 |Y_{t_{k-1}}|^{2\delta}} \right) \right].
\end{aligned}$$

So in order to find the (ML) estimates, the partial derivatives of the previous expression are set to zero. This leads to

$$\begin{aligned}
\frac{\partial \log L}{\partial \mu(i, j)^{(n+1)}} &= - \sum_{k=1}^N P(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}) \frac{-2 [Y_{t_k} - (1 - \beta(i, j)^{(n+1)})Y_{t_{k-1}} - \mu(i, j)^{(n+1)}]}{2(\sigma(j)^{(n+1)})^2 |Y_{t_{k-1}}|^{2\delta}}, \\
&= \sum_{k=1}^N \frac{P(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}) [Y_{t_k} - (1 - \beta(i, j)^{(n+1)})Y_{t_{k-1}}]}{(\sigma(j)^{(n+1)})^2 |Y_{t_{k-1}}|^{2\delta}} \\
&\quad - \sum_{k=1}^N \frac{P(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}) \mu(i, j)^{(n+1)}}{(\sigma(j)^{(n+1)})^2 |Y_{t_{k-1}}|^{2\delta}}, \\
&= \sum_{k=1}^N P(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}) [Y_{t_k} - (1 - \beta(i, j)^{(n+1)})Y_{t_{k-1}}] (\sigma(j)^{(n+1)})^{-2} |Y_{t_{k-1}}|^{-2\delta} \\
&\quad - \sum_{k=1}^N P(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}) \mu(i, j)^{(n+1)} (\sigma(j)^{(n+1)})^{-2} |Y_{t_{k-1}}|^{-2\delta}.
\end{aligned}$$

Hence set $\frac{\partial \log L}{\partial \mu(i, j)^{(n+1)}} = 0$, we obtain

$$\mu(i, j)^{(n+1)} = \frac{\sum_{k=1}^N P(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}) [Y_{t_k} - (1 - \beta(i, j)^{(n+1)})Y_{t_{k-1}}] |Y_{t_{k-1}}|^{-2\delta}}{\sum_{k=1}^N P(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}) |Y_{t_{k-1}}|^{-2\delta}}.$$

Similarly straightforward calculus applied to solve $\frac{\partial \log L}{\partial \beta(i, j)^{(n+1)}} = 0$ allow us to obtain the expression of $\beta(i, j)^{(n+1)}$. Finally, estimate $(\sigma(j)^{(n+1)})^2$ is quite different since it depends only on

the exogenous Markov chain X^2 . Thus, we get

$$\begin{aligned}
\frac{\partial \log L}{\partial (\sigma(j)^{(n+1)})^2} &= - \sum_{k=1}^N \sum_{i \in \mathcal{L}} P \left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)} \right) \left[\frac{1}{2(\sigma(j)^{(n+1)})^2} \right. \\
&\quad \left. - \frac{[Y_{t_k} - (1 - \beta(i, j)^{(n+1)})Y_{t_{k-1}} - \mu(i, j)^{(n+1)}]^2 2|Y_{t_{k-1}}|^{2\delta}}{(2(\sigma(j)^{(n+1)})^2 |Y_{t_{k-1}}|^{2\delta})^2} \right], \\
&= - \sum_{k=1}^N \sum_{i \in \mathcal{L}} P \left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)} \right) \left[\frac{1}{2(\sigma(j)^{(n+1)})^2} \right. \\
&\quad \left. - \frac{[Y_{t_k} - (1 - \beta(i, j)^{(n+1)})Y_{t_{k-1}} - \mu(i, j)^{(n+1)}]^2}{2(\sigma(j)^{(n+1)})^4 |Y_{t_{k-1}}|^{2\delta}} \right], \\
&= - \sum_{k=1}^N \sum_{i \in \mathcal{L}} \frac{P \left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)} \right)}{2(\sigma(j)^{(n+1)})^2} \\
&\quad + \sum_{k=1}^N \sum_{i \in \mathcal{L}} \frac{P \left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)} \right) [Y_{t_k} - (1 - \beta(i, j)^{(n+1)})Y_{t_{k-1}} - \mu(i, j)^{(n+1)}]^2}{2(\sigma(j)^{(n+1)})^4 |Y_{t_{k-1}}|^{2\delta}}.
\end{aligned}$$

Hence set again $\frac{\partial \log L}{\partial (\sigma(j)^{(n+1)})^2} = 0$, we obtain

$$\begin{aligned}
(\sigma(j)^{(n+1)})^2 \sum_{k=1}^N \sum_{i \in \mathcal{L}} P \left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)} \right) &= \sum_{k=1}^N \sum_{i \in \mathcal{L}} P \left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)} \right) \\
&\quad \times [Y_{t_k} - (1 - \beta(i, j)^{(n+1)})Y_{t_{k-1}} - \mu(i, j)^{(n+1)}]^2 |Y_{t_{k-1}}|^{-2\delta}, \\
(\sigma(j)^{(n+1)})^2 &= \frac{\sum_{k=1}^N \sum_{i \in \mathcal{L}} P \left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)} \right) |Y_{t_{k-1}}|^{-2\delta} [Y_{t_k} - (1 - \beta(i, j)^{(n+1)})Y_{t_{k-1}} - \mu(i, j)^{(n+1)}]^2}{\sum_{k=1}^N \sum_{i \in \mathcal{L}} P \left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)} \right)}.
\end{aligned}$$

Moreover, we obtain the expected result using the fact that

$$\sum_{k=1}^N \sum_{i \in \mathcal{L}} P \left(Z_{t_k} = (i, j) | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)} \right) := \sum_{k=1}^N P \left(X_{t_k}^2 = j | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)} \right).$$

□

Finally, in the last part of the **M-step**, the transition probabilities appearing in (1.1) and (1.4) need to be estimated. Following formulae in Bai and Wang (2011) [3], we get for the Markov chain X^2 :

$$p_{ij} = \frac{\sum_{k=2}^N P \left(X_{t_{k-1}}^2 = i, X_{t_k}^2 = j | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)} \right)}{\sum_{X_{t_k}^2} \sum_{k=2}^N P \left(X_{t_{k-1}}^2 = i, X_{t_k}^2 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)} \right)}, \quad i, j \in \mathcal{K}. \quad (2.14)$$

And for the Markov chain X^1 :

$$p_{ij}^s = \frac{\sum_{k=2}^N P\left(X_{t_k}^2 = s, X_{t_k}^1 = j, X_{t_{k-1}}^1 = i | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}{\sum_{k=2}^N \sum_{X_{t_k}^1} P\left(X_{t_k}^2 = s, X_{t_k}^1, X_{t_{k-1}}^1 = i | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}, \quad s \in \mathcal{K} \quad \text{and} \quad i, j \in \mathcal{L}. \quad (2.15)$$

Remark 2.7. In the specific case with $\mathcal{K} \equiv \mathcal{L} := \{1, 2\}$ and with the notation of the Remark 1.2, we obtain

$$\begin{aligned} p &= \frac{\sum_{k=2}^N P\left(X_{t_{k-1}}^2 = 1, X_{t_k}^2 = 1 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}{\sum_{k=2}^N P\left(X_{t_{k-1}}^2 = 1, X_{t_k}^2 = 1 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right) + \sum_{k=2}^N P\left(X_{t_{k-1}}^2 = 1, X_{t_k}^2 = 2 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}, \\ q &= \frac{\sum_{k=2}^N P\left(X_{t_{k-1}}^2 = 2, X_{t_k}^2 = 2 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}{\sum_{k=2}^N P\left(X_{t_{k-1}}^2 = 2, X_{t_k}^2 = 2 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right) + \sum_{k=2}^N P\left(X_{t_{k-1}}^2 = 2, X_{t_k}^2 = 1 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}, \\ p_1 &= \frac{\sum_{k=2}^N P\left(X_{t_k}^2 = 1, X_{t_k}^1 = 1, X_{t_{k-1}}^1 = 1 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}{\sum_{k=2}^N \sum_{X_{t_k}^1} P\left(X_{t_k}^2 = 1, X_{t_k}^1, X_{t_{k-1}}^1 = 1 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}, \\ q_1 &= \frac{\sum_{t=2}^T P\left(X_{t_k}^2 = 1, X_{t_k}^1 = 2, X_{t_{k-1}}^1 = 2 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}{\sum_{k=2}^N \sum_{X_{t_k}^1} P\left(X_{t_k}^2 = 1, X_{t_k}^1, X_{t_{k-1}}^1 = 2 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}, \\ p_2 &= \frac{\sum_{k=2}^N P\left(X_{t_k}^2 = 2, X_{t_k}^1 = 1, X_{t_{k-1}}^1 = 1 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}{\sum_{k=2}^N \sum_{X_{t_k}^1} P\left(X_{t_k}^2 = 2, X_{t_k}^1, X_{t_{k-1}}^1 = 1 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}, \\ q_2 &= \frac{\sum_{k=2}^N P\left(X_{t_k}^2 = 2, X_{t_k}^1 = 2, X_{t_{k-1}}^1 = 2 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}{\sum_{k=2}^N \sum_{X_{t_k}^1} P\left(X_{t_k}^2 = 2, X_{t_k}^1, X_{t_{k-1}}^1 = 2 | \mathbf{Y}_{\mathbf{T}}; \Theta^{(n)}\right)}. \end{aligned}$$

3 Applications to economic datas

We run the estimation procedure in the specific case where each Markov chain admits two regimes, so $\mathcal{K} \equiv \mathcal{L} = \{1, 2\}$. Thus, regarding the Remark 1.2, the bivariate Markov chain Z takes values in a four states space \mathcal{S} . In economic sense, this means that they are a high and low variance regimes; and for each variance regime, they are again a high and low mean regimes.

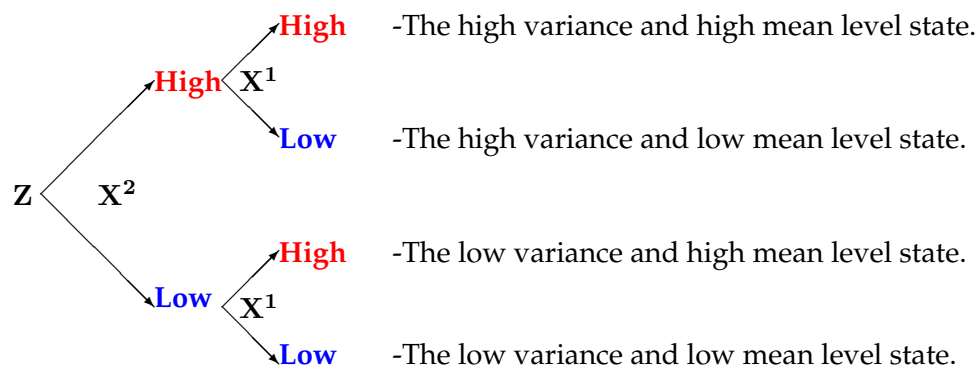


Figure 1: Construction of the states of the bivariate process Z .

3.1 Euro/Dollar exchange rate

Our first data set corresponds to the foreign exchange rate between Euro and Dollars on the time period between January 2000 and May 2012³. We begin by giving in Table 1 some general



Figure 2: Price of 1 Euro in Dollars between Jan. 2000 and May 2012.

descriptive statistics.

³ The data are taken on the web site <http://fxtop.com/fr/>

| | Minimum | Maximum | Mean | Std. Dev. | Skewness | Kurtosis |
|------|----------|----------|----------|-----------|-----------|----------|
| Data | 0.832400 | 1.584900 | 1.218614 | 0.196548 | -0.003278 | 0.003184 |

Table 1: Summary Statistics

3.1.1 Good fit and classification measures

An ideal model is that classifying regimes sharply and having smoothed probabilities which are either close to zero or one. In order to measure the quality of the regime classification, we propose two measures:

1. The regime classification measure (RCM) introduced by Ang and Bekaert (2002) in [1] and generalized for multiple states by Baele (2005) in [2].
2. The Smoothed probability indicator.

These two measures are defined such that:

1. **Regime classification measure:** Let $K(> 0)$ be the number of regimes, the RCM statistic is then given by

$$\text{RCM}(K) = 100. \left(1 - \frac{K}{K-1} \frac{1}{T} \sum_{k=1}^N \sum_{Z_{t_k}} \left(P(Z_{t_k} | \mathbf{Y}_T; \Theta^{(n)}) - \frac{1}{K} \right)^2 \right), \quad (3.16)$$

where the quantity $P(Z_{t_k} | \mathbf{Y}_T; \Theta^{(n)})$ is the smoothed probability given in (2.12) and $\Theta^{(n)}$ is the vector parameter estimation result. The constant serves to normalize the statistic to be between 0 and 100. Good regime classification is then associated with low RCM statistic value: a value of 0 means perfect regime classification and a value of 100 implies that no information about regimes is revealed.

2. **Smoothed probability indicator:** A good classification for data can be also seen when the smoothed probability is less than p or great than $1 - p$ with $p \in [0, 1]$. Indeed, this means that the data at time $k \in \{1, \dots, N\}$ is with a probability higher than $(100 - 2p)\%$ in one of the regimes for the $2p\%$ error. We will call this percentage as the smoothed probability indicator with $p\%$ error and we will denote here by $P^p\%$.

Furthermore, we calculate also the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) which are given by

$$\text{AIC} = -2 \ln(L(\Theta^{(n)})) + 2 * k \quad \text{and} \quad \text{BIC} = -2 * \ln(L(\Theta^{(n)})) + k \ln(n), \quad (3.17)$$

where $L(\Theta^{(n)})$ is the log-likelihood value obtained with the estimated parameters $\Theta^{(n)}$ founded by the (EM) procedure, k is the degree of freedom of each models and n the number of observations. We recall that the preferred model is the one with the minimum AIC or BIC value.

The maximum likelihood estimates found by the (EM) algorithm are stated in Table 5 in Appendix. We give now the log-likelihood, RCM, AIC and BIC values obtained by our estimation procedure for different regime switching models.

| δ | LogL | AIC | BIC | RCM(K=4) | $P^{10\%}$ | $P^{5\%}$ |
|----------|--------|---------|---------|----------|------------|-----------|
| 0 | 1128.0 | -2287.9 | -2158.0 | 33.15 | 70.70% | 60.96% |
| 0.5 | 1150.1 | -2332.2 | -2202.3 | 22.63 | 82.32% | 78.25% |
| 1 | 1172.4 | -2376.8 | -2246.8 | 38.53 | 65.36% | 58.04% |
| 1.5 | 1193.0 | -2417.9 | -2288.0 | 56.61 | 52.70% | 42.40% |

Table 2: Log likelihood value, AIC, BIC, RCM statistics and smoothed probability indicator given by the (EM) procedure for different value of δ .

Let us now check which one is the better model. To make a choice, we have to take into account two things:

1. The log likelihood value given by the model. Indeed, higher is this value higher the fit of the model on data is good.
2. But, we have to weight these values with the values given by the Regime classification measure (RCM) and the Smoothed probability indicator. Effectively, they measure the good classification of the data.

Thus, even if a model has a higher log likelihood value, it is important that its RCM to be close to zero to insure to have regimes significantly different. All the results are stated in Table (2). If we look only the log likelihood value, we can see that the higher value is obtained for the regime switching model with parameter $\delta = 1.5$. But if we look also the RCM values or the smoothed probabilities indicators, we can see that this model gives a very bad classification of the data. Indeed, we can show that the model with $\delta = 0.5$ obtains a RCM equal to 22.63 while the model with $\delta = 1.5$ obtains only a RCM of 56.61. Moreover, this model classifies well only 52.70% of the data while the model with $\delta = 0.5$ classifies well 82.32% of it.

Moreover, we can see, in the Table 2, in the case where there is no local volatility contribution (i.e. model with $\delta = 0$), that this model gives less good results than models with a local

volatility effect and particularly the regime switching model with $\delta = 0.5$. Hence, this justifies well the adding of a local volatility component in our regime switching model (1.9).

To conclude, the choice of the regime switching model with $\delta = 0.5$ seems to be a good choice to fit this data since it obtains a log likelihood value close to the best model and it obtains significantly better results in the state classification of the data than other else.

3.1.2 Economic interpretations

To interpret the results given by the (EM) algorithm, we classify firstly the data in two clusters. The first one will be called "smoothed low mean" cluster. It corresponds to the lowest values of the mean level of the model. In fact, for each two states of the variance Markov chain X^2 , there are two states for the conditional drift Markov chain X^1 . Thus, in this smoothed low mean cluster, we regroup the data where the model is in one of these two lowest values of the mean for each variance state. And we put the two highest mean level values in the smoothed high mean cluster. If we look the Graph 3, the smoothed low mean states are given by the two states: high variance low mean and low variance low mean level states.

Figure 4 shows the corresponding values of each smoothed probabilities. Regarding our conclusions in the section 3.1.1, we interpret the results based on the regime switching model with parameter $\delta = 0.5$. Figure 3 shows the classification result obtained for the time period considered. To complete this model choice, we give in Figure 5 the corresponding result for the model with $\delta = 1.5$. We can see that the interpretation of the regime classification seems to be, in this case, very hard or impossible since as we saw in Table 2 the regime classification measures give bad results for this model.

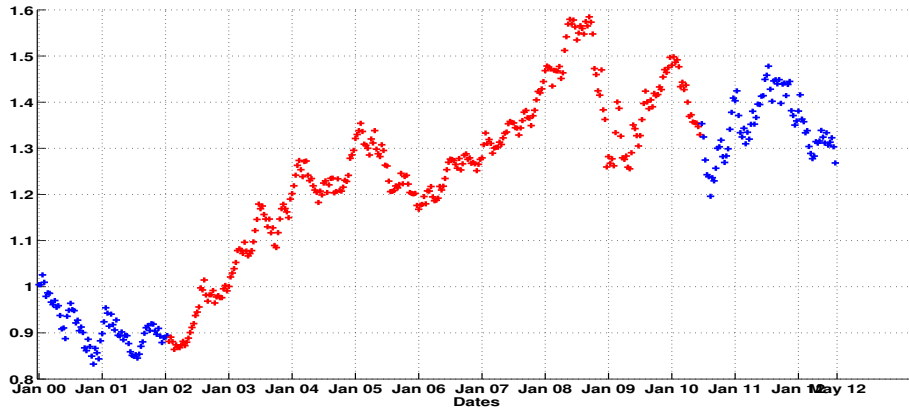


Figure 3: Classification result given by the regime switching model with $\delta = 0.5$. The high mean regime is in red and the low mean is in blue.

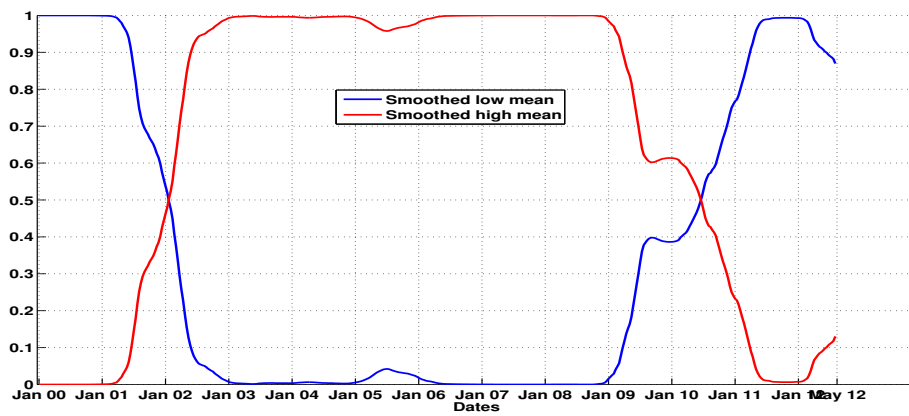


Figure 4: Smoothed probability given by the regime switching model with $\delta = 0.5$.

Therefore, we can see in Figures 3 and 4 that in the time period January 2000-December 2001 and May 2010-May 2012, we are in the smoothed low mean regime. And so, in the time period January 2002-April 2010, we are in the smoothed high mean period. The fact that we are in the smoothed high mean period can be easily see graphically in Figure 3. Indeed, this red time period corresponds to an increasing period for the Euro/Dollar exchange rate. Effectively, the foreign exchange increases in this time period to reach a higher mean level corresponding to the high mean regime. We can remark also that the smoothed low mean regime period after May 2010 corresponds to the after world financial crisis. And before this economic crisis, we were in a world economic increasing period. Hence, the regime switching model captures well this economic crisis behavior.

Moreover, if we look the Figures 6 and 7, we can differentiate this classification in term of the variance level (i.e. the value of the exogenous Markov chain X^2). This gives us very interesting economic and financial interpretations. Indeed, we can see that during the time period just before the world economic crisis in 2010, we were in the smoothed high mean level regime but in fact from september 2008, we switched into a high variance level. Hence, just before the crisis, our regime switching model switched between a low to a high volatility level. This could leave to imply the economic crisis start in 2010.

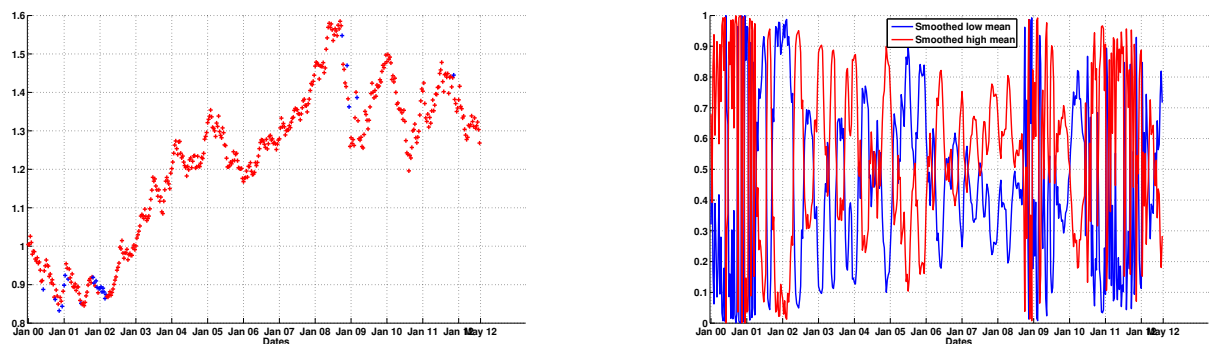


Figure 5: Classification result and Smoothed probability given by the regime switching model with $\delta = 1.5$.

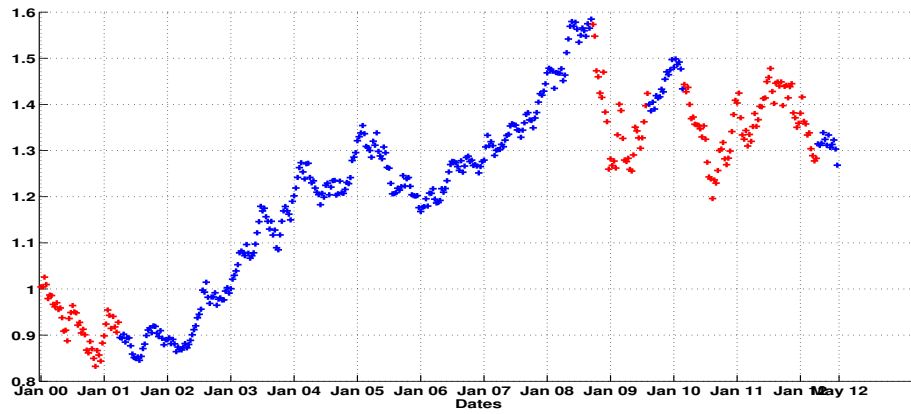


Figure 6: Smoothed probability given by the regime switching model with $\delta = 0.5$ with respect to the variance levels. The high variance regime is in red and the low variance is in blue.

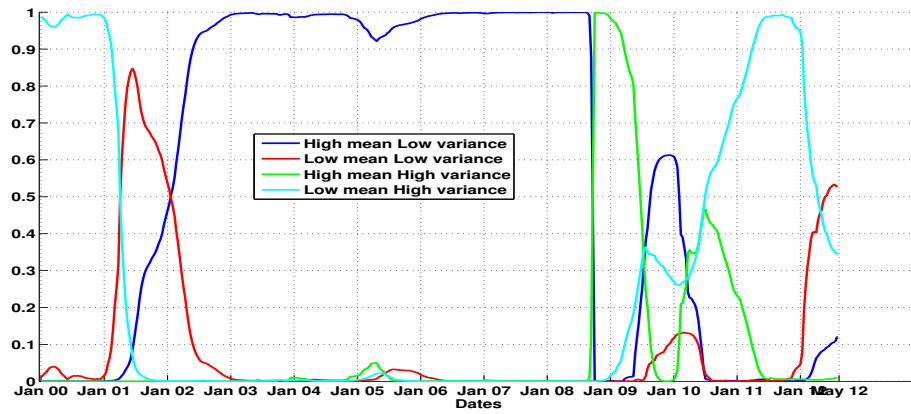


Figure 7: Smoothed probability given by the regime switching model with $\delta = 0.5$ with respect to each level of mean.

3.2 Brent oil spot price

Our second data set corresponds to the spot price in Euro of the crude brent oil on the time period between January 1990 and August 2012⁴. We begin, also, by giving in Table 3 some

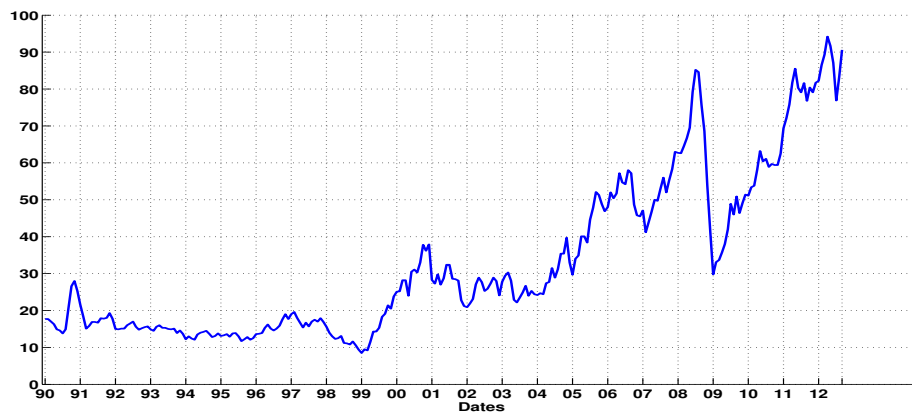


Figure 8: Price in Euro of a oil Brent between January 1990 and August 2012.

general descriptive statistics.

| Datas | Minimum | Maximum | Mean | Std. Dev. | Skewness | Kurtosis |
|-------|---------|---------|-------|-----------|----------|-----------|
| Data | 8.50 | 94.20 | 33.05 | 21.74 | 10937.49 | 688345.87 |

Table 3: Summary Statistics

⁴ The data are taken on the web site <http://www.indexmundi.com/>

We can observe on the Figure 8 that there are some spikes and some changes on the level of the volatility in the price which we hope to capture in our different regime states.

3.2.1 (EM) procedure results

| δ | LogL | AIC | BIC | RCM(K=4) | $P^{10\%}$ | $P^{5\%}$ |
|----------|--------|--------|--------|----------|------------|-----------|
| 0 | -590.9 | 1.2 | 1.3 | 19.8 | 83.36% | 80.79% |
| 0.5 | -361.9 | 691.8 | 813.5 | 25.82 | 77.94% | 71.97% |
| 1 | -142.4 | 252.9 | 374.6 | 44.53 | 60.20% | 50.73% |
| 1.5 | 85.0 | -202.0 | -80.2 | 6.52 | 94.85% | 92.28% |
| 1.8 | 205.3 | -442.7 | -320.9 | 7.80 | 95.40% | 93.11% |
| 2 | 288.7 | -609.3 | -487.6 | 6.55 | 96.51% | 94.03% |
| 2.2 | 373.3 | -778.5 | -656.8 | 7.89 | 95.04% | 90.90% |

Table 4: Log likelihood value, AIC, BIC, RCM statistics and percentage given by the smoothed probability indicator given by the (EM) procedure for different value of δ .

Regarding the results stated in Table 4, the best choice of model seems to be the model with parameter $\delta = 2$. Indeed, this regime switching model obtains a good likelihood value regarding the others and gives a regime classification measure close to the best one with 6.55. Moreover, this model obtains the best percentage of classification with 94.03% of well classification.

3.2.2 Economic interpretations

We proceed with the same construction of classification of each regime as in the previous dataset (see section 3.1.2 and graph 3).

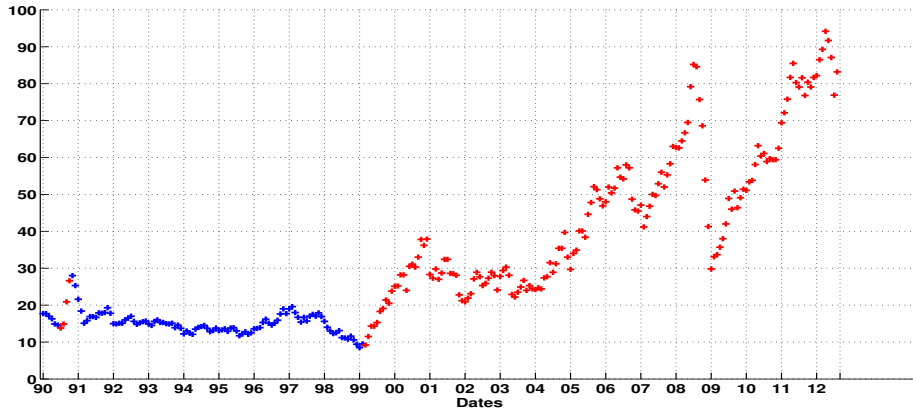


Figure 9: Classification result given by the regime switching model with $\delta = 2$. The high mean regime is in red and the low mean is in blue.

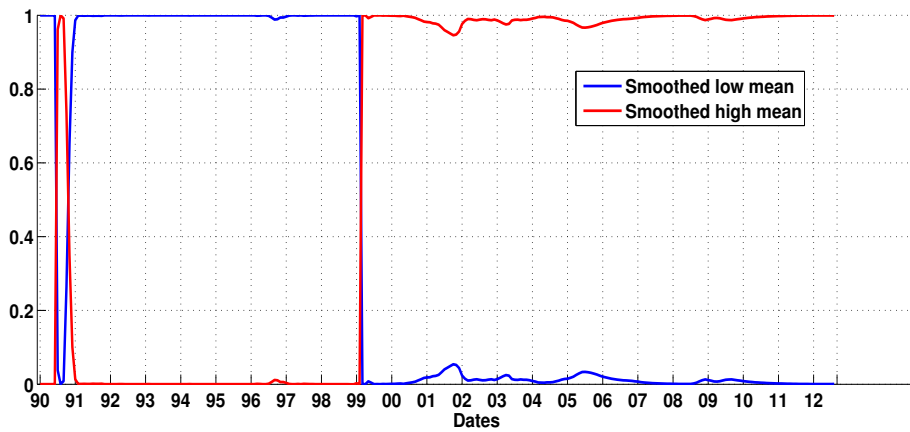


Figure 10: Smoothed probability given by the regime switching model with $\delta = 2$.

In term of mean level, there are clearly two different periods: the first one before March 1999 which corresponds to a smoothed low mean price period and the second one after March 1999 which corresponds to a high mean price state. Effectively, this date of March 1999 is important since we know that in March 1999, the oil’s producer countries members of the Organization of Petroleum Exporting Countries (OPEC) and some others countries as Oman or Russia decided to reduce their productions. Hence, the brent oil price had a huge increasing price in a short period of time. So, our model well captures this political economic decision.

We can see, also, that our regime switching model captures well a change in the trend of price during the Gulf war in 1991; since our model switches to a high mean regime state.

Moreover, if we look the Figure 11, we can see that during the high mean price level period (i.e. after March 1999), we are not always in the same level of variance. And one thing very interesting in an economic point of view is that our regime switching model captures well the effect of the world financial crisis. Indeed, it switches to a high regime of variance during the period 2008-2010. Hence during this financial crisis our model is in a high regime of variance which corresponds to high level of market volatility. This is an expected financial result in a crisis time period that the volatility to be higher than in other economic period. So, these prove that the use of two different Markov chains allows us to highlight different level of variance in a same level a mean or respectively different level of mean in a same level of variance. Our model switches also in a high variance level during the first Gulf war period.

To conclude, all these regime changes prove that the use of this class of regime switching model allows us to capture well economic behaviors and political effects.

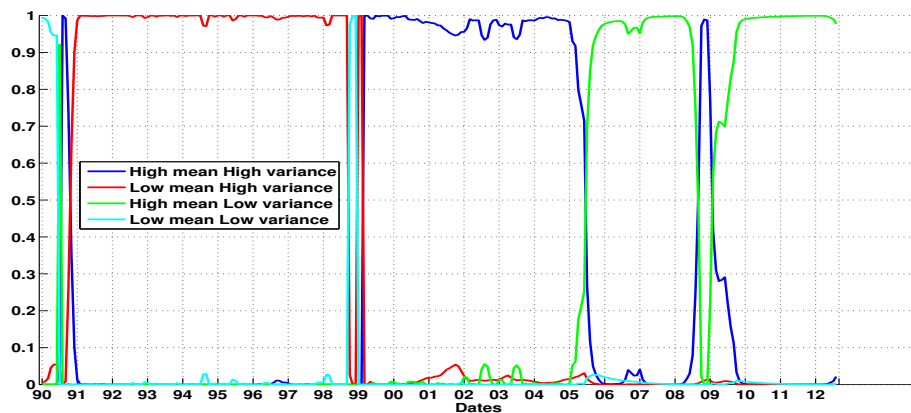


Figure 11: Smoothed probability given by the regime switching model with $\delta = 2$ with respect to each level of mean.

4 Conclusion

In conclusion, we obtained an explicit procedure to estimate all the parameters of a mean-reverting local volatility hidden conditional Markov switching model. We compared the results given by this procedure to different regime switching models. We applied this procedure on Euro/Dollars foreign exchange rate data and on Brent oil price. And we proved that this regime switching model allows us to capture well various key features of an economic time series data, such as a change in the mean level or a growth of the volatility level.

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Appendix

| δ | 0 | 0.5 | 1 | 1.5 |
|---------------|---------------------|---------------------|---------------------|---------------------|
| $\mu(1, 1)$ | 0.0093504 (0.0114) | 0.0097232 (0.0072) | 0.017365 (0.0072) | 0.021487 (0.0073) |
| $\mu(2, 1)$ | -0.003066 (0.0061) | 0.0097864 (0.0124) | 0.0066598 (0.0077) | 0.00047157 (0.0063) |
| $\mu(1, 2)$ | 0.37078 (0.0969) | 0.27175 (0.0692) | 0.29222 (0.0825) | 0.12336 (0.0137) |
| $\mu(2, 2)$ | 0.1678 (0.0425) | 0.00086963 (0.0126) | 0.0013942 (0.0123) | -0.059146 (0.0106) |
| $\beta(1, 1)$ | -0.0032391 (0.0097) | 0.0056428 (0.0059) | 0.009184 (0.0060) | 0.011879 (0.0063) |
| $\beta(2, 1)$ | -0.0013493 (0.0053) | 0.011437 (0.0121) | 0.0092859 (0.0068) | 0.0026665 (0.0056) |
| $\beta(1, 2)$ | 0.29711 (0.0722) | 0.20669 (0.0512) | 0.221 (0.0612) | 0.11096 (0.0124) |
| $\beta(2, 2)$ | 0.11932 (0.0308) | 0.0015137 (0.0107) | 0.0022205 (0.0111) | -0.052292 (0.0096) |
| $\sigma(1)$ | 0.00025699 (0.0000) | 0.00022653 (0.0000) | 0.00017183 (0.0000) | 0.00013645 (0.0000) |
| $\sigma(2)$ | 0.00061963 (0.0001) | 0.0006012 (0.0001) | 0.00052111 (0.0001) | 0.00027119 (0.0000) |
| p_1 | 0.61594 | 1 | 0.9206 | 0.86499 |
| q_1 | 0.88722 | 0.96908 | 0.8728 | 0.86558 |
| p_2 | 0.72039 | 0.97304 | 0.99427 | 0.39898 |
| q_2 | 0.92579 | 1 | 1 | 0.70019 |
| p | 0.9968 | 0.9929 | 0.99219 | 0.98647 |
| q | 1 | 0.98253 | 0.97812 | 0.97089 |

Table 5: Parameters estimated for the regime switching models (standard deviations into parenthesis obtained by taking the square root of the inverse of the Hessian matrix).