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Prescribing the Jacobian in critical spaces

Pierre Bousquet * Petru Mironescu†

July 12, 2012

Abstract

We consider the Sobolev space $X = W^{s,p}(\mathbb{S}^m;\mathbb{S}^{k-1})$. We prove the existence of a robust distributional Jacobian Ju for $u \in X$ provided $sp \geq k-1$; this generalizes a result of Bourgain, Brezis and the second author [10], where the case $m = k$ is considered. In the critical case where $sp = k-1$, we identify the image of the map $X \ni u \mapsto Ju$. This extends a result of Alberti, Baldo and Orlandi [2] for $s = 1$ and $p = k-1$. We also present a new, analytical, dipole construction method.

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1 Introduction

In general, one cannot approximate manifold-valued Sobolev maps with manifold-valued smooth maps. This contrasts with the case of scalar maps. Here is a simple example, with $\mathbb{D}$ the unit disc in $\mathbb{R}^2$: the map $u_0 : \mathbb{D} \to \mathbb{S}^1$, $u_0(z) = \dfrac{z}{|z|}$, belongs to $W^{1,1}$, but cannot be approximated in $W^{1,1}$ with smooth $\mathbb{S}^1$-valued maps [37]. Beyond the approximation problem, singularities occur in the regularity theory of manifold-valued maps (developed by Schoen and Uhlenbeck [37, 38] and extended by Hardt and Lin [24]): unlike the scalar case, manifold-valued minimizers of the Dirichlet integral $\int |\nabla u|^2$ or more generally of $\int |\nabla u|^p$ are not smooth, but have a (small) singular set. In the case of sphere-valued maps, Brezis, Coron and Lieb [13] generalized a

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calculation of Ball [4] and discovered the possibility of detecting the singularities of maps *via* the distributional Jacobian. We will recall below the definition of the distributional Jacobian in the general case, but for the time being let us consider the case of maps $u : \mathbb{D} \to \mathbb{S}^1$. If $u \in C^1$, then the Jacobian determinant $\operatorname{Jac} u$ satisfies

$$\operatorname{Jac} u = 0. \tag{1.1}$$

Indeed, by differentiating the identity $|u|^2 \equiv 1$ we find that $u \cdot \partial_j u \equiv 0$, $j = 1,2$, and thus $\partial_1 u \parallel \partial_2 u$. Therefore $\operatorname{Jac} u = 0$. Ball [4] discovered that for less smooth maps the Jacobian, when defined properly, need not vanish. Indeed, consider the distribution $Ju = \frac{1}{2\pi}\partial_1(u \wedge \partial_2 u) + \frac{1}{2\pi}\partial_2(\partial_1 u \wedge u)$,[1] which makes sense if u is merely $W^{1,1} \cap L^\infty$. If $u \in C^2$, then $Ju = \frac{1}{\pi}\operatorname{Jac} u$. Therefore, if u is in addition circle-valued, then Ju vanishes as above. With more work, one may prove that the same holds if u is less smooth, for example if $u \in W^{1,2}$. However, the above u_0 belongs to $W^{1,p}$, $p < 2$, and it is easy to see that $Ju = \delta_0$ [4].[2] More generally, if $u \in W^{1,1}(\mathbb{D};\mathbb{S}^1)$ is smooth outside a finite set $A(u)$, then [13]

$$Ju = \sum_{a \in A(u)} \deg(u,a)\delta_a. \tag{1.2}$$

Here, $\deg(u,a)$ is the winding number (degree) of u computed on a small circle around a.[3] It is still possible to compute the distribution Ju for an arbitrary $u \in W^{1,1}(\mathbb{D};\mathbb{S}^1)$, but this requires additional ingredients.

1. First ingredient: the class $\mathscr{R}$ of circle-valued maps u smooth outside some finite set $A(u)$ is dense in $W^{1,1}(\mathbb{D};\mathbb{S}^1)$. This is a result of Bethuel and Zheng [8].
2. Second ingredient: for $u \in \mathscr{R}$, it is possible to rearrange the singularities of u and write

$$Ju = \sum(\delta_{P_j} - \delta_{N_j}), \quad \text{with } P_j, N_j \in \overline{\mathbb{D}} \text{ and } \sum |P_j - N_j| \le \frac{1}{2\pi}\|\nabla u\|_{L^1};$$

   this was proved by Brezis, Coron and Lieb [13].

By combining the above, a straightforward Cauchy sequences argument leads to the representation

$$Ju = \sum(\delta_{P_j} - \delta_{N_j}), \quad \text{with } P_j, N_j \in \overline{\mathbb{D}} \text{ and } \sum |P_j - N_j| \le \frac{1}{2\pi}\|\nabla u\|_{L^1}; \tag{1.3}$$

this time, the sum may contain an infinite number of terms.

A remarkable fact is that there is a "converse" to (1.3): given sequences $(P_j),(N_j) \subset \overline{\mathbb{D}}$ such that $\sum_j |P_j - N_j| < \infty$, there exists some $u \in W^{1,1}(\mathbb{D};\mathbb{S}^1)$ such that $Ju = \sum(\delta_{P_j} - \delta_{N_j})$. The ingredients of the proof of this fact are already in [13], but the result was explicitly formulated much later; see [2] and also [16].

A similar approach can be followed for maps in $W^{1,k-1}(B^k;\mathbb{S}^{k-1})$, $k \ge 2$.[4] However, for applications this functional setting is not sufficient. Indeed, the distributional Jacobian plays a crucial role e.g. in approximation problems [5, 34] or in the study of the Ginzburg-Landau equation [26, 9]. In these contexts, the natural functional setting need not be the one of $W^{1,k-1}(B^k;\mathbb{S}^{k-1})$. For example, the trace space $W^{1/2,2}(\mathbb{S}^2;\mathbb{S}^1)$ is related to the study of the Ginzburg-Landau equation [9]. The above suggests the following general program in the Sobolev space $X = W^{s,p}(\Omega;\mathbb{S}^{k-1})$, with $s > 0$, $1 \le p < \infty$ and Ω an open set in $\mathbb{R}^m$ or an m-dimensional manifold:

Task 1. Prove density in X of a suitable class $\mathscr{R}$ of $\mathbb{S}^{k-1}$-valued maps u smooth except on a small set $A(u)$.

[1] Here, $\wedge$ stands for the vector product of complex numbers: $a \wedge b = a_1 b_2 - a_2 b_1$.
[2] Here, δ_0 is the Dirac mass at the origin.
[3] By invariance of the degree by homotopy, this integer is independent of the circle.
[4] Here, B^k is the open unit ball in $\mathbb{R}^k$. The closed unit ball is denoted $\overline{B}^k$.

Task 2. Define (possibly by density starting from maps in $\mathscr{R}$) the distributional Jacobian Ju of a map $u \in X$.

Task 3. Determine the range of the map $X \ni u \mapsto Ju$.

Our paper presents new results concerning Tasks 2 and 3. Additional directions, though not discussed in the present paper, are equally interesting. An example: Bethuel [5] proved that the equality $Ju = 0$ characterizes the "absence of singularities": a map $u \in W^{1,2}(B^3;\mathbb{S}^2)$ satisfies $Ju = 0$ if and only if u is in the strong $W^{1,2}$-closure of smooth $\mathbb{S}^2$-maps; see also [7, 11, 35]. The generalization of this result to different Sobolev regularities has not been obtained in full generality.

The above program has not been completely realized; we present below the state-of-the-art in these directions. To start with, let us define properly the class $\mathscr{R}$. From the pioneering work of Bethuel [6], the good candidate is known to be the class

$$\mathscr{R} = \left\{ u \in W^{s,p}(\Omega;\mathbb{S}^{k-1}) \cap C^\infty(\Omega \setminus A(u)); \dim A(u) = m - [sp] - 1 \right\},$$

where $A(u)$ is a finite union of Lipschitz manifolds (depending on u). Bethuel [6] proved density of $\mathscr{R}$ when $s = 1$ and the target is an arbitrary compact oriented manifold.[5] Very recently, Ponce, Van Schaftingen and the first author [12] extended this result for $s = 2, 3, \ldots$ On the other hand, density of $\mathscr{R}$ (for arbitrary targets) when $s < 1$ was obtained by Brezis and the second author [15]. Another essentially known case is the one where $k-1 \le sp < k$. This case can be tackled by adapting the projection method of Federer and Fleming [20]. The idea of using this method for manifold-valued maps goes back to Hardt, Kinderlehrer and Lin [23]. The projection method was used to prove density of the class $\mathscr{R}$ in the following cases:

a) When $s = 1$, $k - 1 \le p < k$ in [8].

b) When $s = 1/2$, $p = 2$, $k = 1$, $m = 2$ in [36].

c) When $s < 1$ and $k - 1 \le sp < k$ in [9].[6]

d) When $s \ge 1$, $1 \le sp < 2$ and $k = 2$ in [11].

Though the case where $k - 1 \le sp < k$ does not appear anywhere in full generality, the subcase $s = 1$ is already in [8], the subcase $s < 1$ is contained in [15], and the subcase $s > 1$ is very likely to follow from the arguments in [11]. However, we emphasize the fact that, even though this special case is under control and even if the case of spheres is simpler than the one of arbitrary compact targets, density of $\mathscr{R}$ for sphere-valued maps and for arbitrary m, k, s and p is not known.

Task 2 concerns the possibility of defining Ju. Some cases are to be discarded from the beginning:

a) When $sp < k - 1$, smooth maps are dense in $W^{s,p}(\Omega;\mathbb{S}^{k-1})$ [28]. In this case, it is natural to let $Ju = 0$ for each u.

b) Same holds when $sp \ge m = \dim \Omega$. This goes back essentially to [37]; see also [17].

By the above, we may always assume that

$$k - 1 \le sp < m, \tag{1.4}$$

and we see that the limiting cases are the ones where $sp = k - 1$. When $m = k$, $s = 1$ and $p = k - 1$, the distributional Jacobian Ju is defined as the divergence of an L^1-vector field:

$$Ju = \operatorname{div} \frac{1}{k\omega_k} (\det(u, \partial_2 u, \ldots, \partial_k u), \det(\partial_1 u, u, \partial_3 u, \ldots, \partial_k u), \ldots, \det(\partial_1 u, \ldots, \partial_{k-1} u, u)); \tag{1.5}$$

here, ω_k is the volume of B^k. This definition can be adapted to the case where $m > k$, $s = 1$ and $p = k - 1$ [3, 25, 2]. We explain the procedure e.g. when $\Omega = \mathbb{S}^m$.[7] We consider the volume form

[5] The case of sphere-valued maps was previously known for $k - 1 \le p < k$ [8].

[6] The proof there is for special values of s and p, but works without changes in the general case.

[7] All our results are stated for $\Omega = \mathbb{S}^m$, but it will be transparent from the proofs that Ω could be any bounded open set in $\mathbb{R}^m$ or m-dimensional compact manifold.

on $\mathbb{S}^{k-1}$,

$$\omega_0 := \frac{1}{k\omega_k}\sum_{i=1}^{k}(-1)^{i-1}y_i dy_1\wedge\ldots\wedge\widehat{dy_i}\wedge\ldots\wedge dy_k \tag{1.6}$$

(with the convention that $\widehat{\ }$ denotes a missing term). For every $u\in W^{1,k-1}(\mathbb{S}^m;\mathbb{S}^{k-1})$, we define the $(m-k)$-current Ju by its action on the space $W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m)$ of Lipschitz $(m-k)$-forms in $\mathbb{S}^m$:

$$\langle Ju,\zeta\rangle := (-1)^{m-k+1}\int_{\mathbb{S}^m} d\zeta\wedge(u^\sharp\omega_0),\quad \forall\ \zeta\in W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m). \tag{1.7}$$

Here, $u^\sharp\omega_0$ stands for the pullback

$$u^\sharp\omega_0 = \frac{1}{k\omega_k}\sum_{i=1}^{k}(-1)^{i-1}u_i du_1\wedge\ldots\wedge\widehat{du_i}\wedge\ldots\wedge du_k.$$

The formula (1.7) is a generalization of (1.5) when $m>k$. It is clear that Ju is well-defined as an element of the dual of $W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m)$, and that the map

$$W^{1,k-1}(\mathbb{S}^m;\mathbb{S}^{k-1})\ni u\mapsto Ju\in(W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m))^* \tag{1.8}$$

is continuous. Similarly, the distribution Ju is continuous wrt u in the space $W^{s,p}(\mathbb{S}^m;\mathbb{S}^{k-1})$ provided $s\geq 1$ and $sp\geq k-1$. This follows from the embedding $W^{s,p}\cap L^\infty\hookrightarrow W^{1,k-1}$, valid when $s\geq 1$ and $sp\geq k-1$.

Existence of Ju is less clear when $s<1$ and $sp\geq k-1$. A first result in this direction was obtained by Hang and Lin [22]. These authors were able to define Ju for $u\in W^{s,p}(\mathbb{S}^k;\mathbb{S}^{k-1})$ provided $s\geq 1-1/k$ and $sp\geq k-1$. This result was completed in [10]; there, it is proved that there is a natural way of defining Ju in the spaces $W^{s,p}(\mathbb{S}^k;\mathbb{S}^{k-1})$ when $s<1$ and $sp=k-1$. This is obtained as follows: first, the class $W^{1,k-1}\cap W^{s,p}(\mathbb{S}^k;\mathbb{S}^{k-1})$ is dense in $W^{s,p}(\mathbb{S}^k;\mathbb{S}^{k-1})$.[8] Next, the key step in [10] consists in proving that $u\mapsto Ju$, initially defined for maps in $W^{1,k-1}\cap W^{s,p}(\mathbb{S}^k;\mathbb{S}^{k-1})$, is continuous wrt the $W^{s,p}$-norm. This gives a (unique) natural notion of Ju for $u\in W^{s,p}(\mathbb{S}^k;\mathbb{S}^{k-1})$, provided $sp\geq k-1$. Our first result is a generalization of the above to higher dimensions m.

1.1 Theorem. *There exists a (unique) map*

$$J:\bigcup_{\substack{0<s\leq 1\\ p=(k-1)/s}} W^{s,p}(\mathbb{S}^m;\mathbb{S}^{k-1})\to(W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m))^*$$

such that:

1. *For $u\in W^{1,k-1}(\mathbb{S}^m;\mathbb{S}^{k-1})$, Ju coincides with the distributional Jacobian defined by (1.7).*
2. *The restriction of J to every $W^{s,p}(\mathbb{S}^m;\mathbb{S}^{k-1})$ with $0<s<1$ and $p=(k-1)/s$ is continuous.*
3. *In addition, for s and p as above the following estimate holds:*

$$|\langle Ju,\zeta\rangle|\leq C_p|u|^p_{W^{s,p}}\|d\zeta\|_{L^\infty}. \tag{1.9}$$

Here $|\ |_{W^{s,p}}$ stands for the standard semi-norm in $W^{s,p}$; for the convenience of the reader, its formula will be recalled in Section 2.

Task 2 is completed.

We next turn to Task 3: characterize the set $\left\{Ju;\, u\in W^{s,p}(\Omega;\mathbb{S}^{k-1})\right\}$. This requires establishing the analog of (1.3) and of its "converse". A first hint is given by [13]. There, it is proved that

$$\inf\left\{\int_{\mathbb{R}^3}|\nabla u|^2;\, Ju=\sum_{j=1}^{k}(\delta_{P_j}-\delta_{N_j})\right\}=8\pi\inf_{\sigma\in S_k}\sum_{j=1}^{k}|P_j-N_{\sigma(j)}|, \tag{1.10}$$

[8] This is proved in the same way as the density of the class $\mathscr{R}$.

the sum in the right-hand side of (1.10) being the length of the shortest curve connecting the P_j's to the N_j's. It was conjectured in [13] that when, say, $m = 4$ and $k = 3$, the minimal Dirichlet energy required to create a Jacobian supported by a simple closed curve $M \subset \mathbb{R}^4$ equals

$$8\pi \min\{\mathbf{M}(N); N \text{ is a } 2-\text{rectifiable current such that } \partial N = M\}.$$

Here, $\mathbf{M}(N)$ stands for the mass of N. For polygonal lines, this conjecture[9] has been confirmed by Almgren, Browder and Lieb [3]. Far reaching extensions of this result were obtained by Alberti, Baldo and Orlandi [2]. Their results are the following: on the one hand,

$$\left\{Ju; u \in \dot{W}^{1,k-1}(\mathbb{R}^m; \mathbb{S}^{k-1})\right\} = \{\partial N; N \text{ is an } (m-k+1)-\text{dimensional rectifiable current}\}. \quad (1.11)$$

Here, $\dot{W}^{1,k-1}$ stands for the homogeneous Sobolev space. On the other hand,

$$\inf\left\{\int_{\mathbb{R}^k} |\nabla u|^k; Ju = M\right\} \sim \inf\{\mathbf{M}(N); N \text{ is rectifiable and } \partial N = M\}. \quad (1.12)$$

Very recently, by combining the arguments in [3] and [2], Molnar [31] improved the conclusion (1.12) to

$$\inf\left\{\int_{\mathbb{R}^k} |\nabla u|^k; Ju = M\right\} = (k-1)^{(k-1)/2} k\omega_k \inf\{\mathbf{M}(N); N \text{ rectifiable}, \partial N = M\}. \quad (1.13)$$

A crucial point in [13, 3, 2] is the dipole construction discovered in [13]. Let us give an example: for $\mathbb{S}^1$-valued maps in $\mathbb{R}^2$, a dipole with a positive singularity P_1 and a negative singularity N_1 is a map $u : \mathbb{R}^2 \to \mathbb{S}^1$ such that $Ju = \delta_{P_1} - \delta_{N_1}$ which is, in addition, constant outside an arbitrarily small set $B(u)$. For arbitrary s and p, one cannot control both the $W^{s,p}$-semi-norm of u and the size of $B(u)$. However, this is possible in $W^{1,1}$: for every $\varepsilon > 0$, it is possible to construct u such that $\|\nabla u\|_{L^1} \le 2\pi|P_1 - N_1| + \varepsilon$ and $|B(u)| \le \varepsilon$. An inspection of the dipole construction in [13] leads to the conclusion that this construction can be adapted to the *critical spaces* $W^{s,p}(\Omega; \mathbb{S}^{k-1})$ characterized by the equality $sp = k-1$. Our next results extend the results in [2] to the full scale of critical spaces.

1.2 Theorem. *Let $0 < s \le k-1$ and $p \ge 1$ be such that $sp = k-1$. Assume that $m \ge k$. Let $u \in W^{s,p}(\mathbb{S}^m; \mathbb{S}^{k-1})$. Then there exists an $(m-k+1)$-rectifiable current N in $\mathbb{S}^m$ such that*

1. $\partial N = Ju$.
2. $\mathbf{M}(N) \le C_p |u|^p_{W^{s,p}}$.

1.3 Theorem. *Let M be the boundary of an $(m-k+1)$-rectifiable current N in $\mathbb{S}^m$. Then there exists $u \in \bigcap_{sp=k-1} W^{s,p}(\mathbb{S}^m; \mathbb{S}^{k-1})$ such that:*

1. $Ju = M$.
2. $|u|^p_{W^{s,p}} \le C_p \mathbf{M}(N)$ *if* $sp = k-1$.

Let us discuss how Theorems 1.2 and 1.3 compare to the corresponding results in [2]. When $k \ge 3$, the critical spaces are nested: if the couples (s_1, p_1) and (s_2, p_2) are critical and $s_1 > s_2$, then we have the Gagliardo-Nirenberg type embeddings $W^{s_1,p_1}(\mathbb{S}^m; \mathbb{S}^{k-1}) \hookrightarrow W^{s_2,p_2}(\mathbb{S}^m; \mathbb{S}^{k-1})$. Therefore, Theorem 1.3 follows from the special case $s = k-1$, $p = 1$ combined with the Gagliardo-Nirenberg inequalities. In turn, we prove this special case by adapting the techniques in [2]. As explained in the course of the proof, the new technical difficulties[10] arise from the fact that we need smooth (and not merely Lipschitz) dipoles and from the fact that we work on $\mathbb{S}^m$ instead of $\mathbb{R}^m$.[11] However, the case $k = 2$ does not follow from the above strategy. This is due to the fact that the corresponding Gagliardo-Nirenberg embeddings are wrong: e.g., we do not have

[9] And its generalization to arbitrary m and k.
[10] When compared to [2].
[11] Our proof adapts also to the case of a bounded domain in $\mathbb{R}^m$.

$W^{1,1}(\mathbb{S}^m;\mathbb{S}^1) \hookrightarrow W^{1/2,2}(\mathbb{S}^m;\mathbb{S}^1)$. In this case, we present two proofs. One of them is "geometric": it relies on a dipole construction with control of several norms combined with a Cauchy sequences argument. The second one is purely "analytical": by an averaging argument reminiscent of the projection method of Federer and Fleming [20], we associate to each function u a "better" function with the same singular set.[12] We think that this second approach is of independent interest and may serve in other situations. The proof of Theorem 1.2 is very much in the spirit of Almgren, Browder and Lieb [3], who were the first to use the coarea formula in the proof of estimates for sphere-valued maps. Later, Hang and Lin [22] and Alberti, Baldo and Orlandi [2] used a similar approach in order to establish special cases of Theorem 1.2. Our approach is similar to the one in [22], but we had to face new technical difficulties. A final remark about the coarea formula setting. The content of Theorem 1.2 is that Ju is the boundary of a rectifiable current N. The main idea of [3] is to take $N = u^{-1}(a)$ for some appropriate $a \in \mathbb{S}^{k-1}$; this idea is also at the heart of [2]. This leads, in general, to a nonsmooth N.[13] It turns out that it is possible to write $Ju = \partial M$ with smooth M provided we let M live in the higher dimensional space $\mathbb{S}^m \times [0,\infty)$; this is how our construction goes.[14] We next land on $\mathbb{S}^m$ by letting N be the projection of M. Technically, this leads us to the use of the standard coarea formula, as opposed to its Sobolev spaces version required in [2].

Task 3 is thus completed in the scale of critical spaces. However, little is known about the non critical spaces. To give an example, the characterization of the space $\{Ju; u \in W^{1,p}(\mathbb{S}^3;\mathbb{S}^2\}$, with $2 < p < 3$, is not known. The situation is completely understood only for circle-valued maps. The space $\{Ju; u \in W^{s,p}(\mathbb{S}^m;\mathbb{S}^1\}$, with $1 < sp < 2$, has been characterized in [11] when $s \geq 1$ and in [30] when $s < 1$.[15] However, the proofs there rely heavily on the fact that the target space is $\mathbb{S}^1$. The case where the target is $\mathbb{S}^{k-1}$ with $k \geq 3$ is widely open.

Our paper is organized as follows: in Sections 2, 3 and 4 we recall, for the convenience of the reader, the main analytical tools required in the proofs. In Section 5, we establish Theorems 1.1 and 1.2. The proof of Theorem 1.1 relies on a technical lemma, Lemma 5.1, whose long proof is postponed to Section 7. Section 6 is devoted to the proof of Theorem 1.3.

2 Sobolev spaces of currents and forms

Our presentation follows essentially [21, Chapters 2, 3]. Let H be a subspace of the Euclidean space $\mathbb{R}^m$, $m \geq 1$. For $0 \leq l$, we denote by $\Lambda_l H$ the vector space of l-vectors in H and by $\Lambda^l H = \Lambda_l H^*$ the vector space of l-covectors in H. The duality pairing between $\Lambda_l H$ and $\Lambda^l H$ is defined in terms of simple vectors and covectors as follows: If $h = h_1 \wedge \ldots \wedge h_l \in \Lambda_l H$, $h_j \in H$ and $\theta = \theta_1 \wedge \ldots \theta_l \in \Lambda^l H$, $\theta_i \in H^*$, then

$$\langle h, \theta \rangle = \langle \theta, h \rangle = \det(\theta_i(h_j)).$$

The inner product of vectors and covectors (which only depends on the inner product defined on $\mathbb{R}^m$) is defined as follows. Let (e_i) denote an orthonormal basis in $\mathbb{R}^m$, e_i^* be the dual basis, and set

$$I(l,m) = \{(i_1,\ldots,i_l); 1 \leq i_1 < \ldots < i_l \leq m\}.$$

If we set

$$e_I = e_{i_1} \wedge \ldots \wedge e_{i_l}\ ,\ e_I^* = e_{i_1}^* \wedge \ldots \wedge e_{i_l}^*, \quad \forall I \in I(l,m),$$

then, for $\xi = \sum_I \xi_I e_I$, $\xi' = \sum_I \xi'_I e_I$, $\alpha = \sum_I \alpha_I e_I^*$, $\alpha' = \sum_I \alpha'_I e_I^*$, we define

$$\langle \xi, \xi' \rangle = \sum_I \xi_I \xi'_I\ ,\ \langle \alpha, \alpha' \rangle = \sum_I \alpha_I \alpha'_I, \tag{2.1}$$

[12] The key result is Lemma 6.15.

[13] This is due to the fact that Ju need not be the boundary of a smooth N. See the discussion in [2, p. 277].

[14] This approach is inspired by [22].

[15] The results in [30] are described in [29].

$$|\xi| = \left(\sum_I |\xi_I|^2\right)^{1/2}, \ |\alpha| = \left(\sum_I |\alpha_I|^2\right)^{1/2}. \tag{2.2}$$

The above quantities do not depend on the choice of the orthonormal basis (e_i).

A linear map $L : H \to K$ induces the morphism

$$\Lambda_l L : \Lambda_l H \to \Lambda_l K, \ \Lambda_l H \ni v_1 \wedge \ldots \wedge v_l \xrightarrow{\Lambda_l L} Lv_1 \wedge \ldots \wedge Lv_l \in \Lambda_l K,$$

and its adjoint $\Lambda^l L : \Lambda^l K \to \Lambda^l H$.

Let $\Omega \subset \mathbb{R}^m$ be an open set. For $0 \le l \le m$, we denote by $C^\infty(\Lambda^l \Omega)$ (an abbreviation for $C^\infty(\Omega; \Lambda^l \mathbb{R}^m)$) the set of smooth l-forms in Ω. Similarly, $C_c^\infty(\Lambda^l \Omega)$ is the set of smooth l-forms compactly supported in Ω. The differential of a form ω is denoted by $d\omega$. An l-current $T \in (C_c^\infty(\Lambda^l \Omega))^*$, $l \ge 0$, is a continuous linear form on the set $C_c^\infty(\Lambda^l \Omega)$ endowed with the topology of distributions. The boundary of an $l-$current T, $l \ge 1$, is the $(l-1)$-current $\partial T \in (C_c^\infty(\Lambda^{l-1}\Omega))^*$ defined by

$$\langle \partial T, \omega \rangle = \langle T, d\omega \rangle, \quad \forall\, \omega \in C_c^\infty(\Lambda^{l-1}\Omega).$$

For $0 < s < \infty$, $1 \le p < \infty$, we denote by $W^{s,p}(\Lambda^l \Omega)$ the set of l-forms such that

$$\|\omega\|_{W^{s,p}} := \sum_{k \le [s]} \left\| D^k \omega \right\|_{L^p} + \left| D^{[s]} \omega \right|_{W^{\sigma,p}} < \infty,$$

where $[s]$ is the integer part of s, $\sigma := s - [s]$ and for $\omega = \sum \omega_I dx_I$, $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_l}$,

$$\left\| D^k \omega \right\|_{L^p} := \left\{ \sum_I \sum_{|\alpha|=k} \int_\Omega \left| \frac{\partial^\alpha \omega_I}{\partial x_\alpha}(x) \right|^p dx \right\}^{1/p}$$

$$\left| D^{[s]} \omega \right|_{W^{\sigma,p}} := \left\{ \sum_I \sum_{|\alpha|=[s]} \int_\Omega \int_\Omega \frac{\left| \frac{\partial^\alpha \omega_I}{\partial x_\alpha}(x) - \frac{\partial^\alpha \omega_I}{\partial x_\alpha}(y) \right|^p}{|x-y|^{m+\sigma p}} dx\, dy \right\}^{1/p}.$$

The definition extends, with the obvious modification, to the case $p = \infty$.

We next turn to the definition of Sobolev spaces of forms on manifolds. This requires defining the pullback of forms. Let $\varphi : \Omega_1 \subset \mathbb{R}^{m_1} \to \Omega_2 \subset \mathbb{R}^{m_2}$ be a smooth map between two open sets Ω_1 and Ω_2. The pullback $\varphi^\sharp \omega$ of a measurable l-form ω in Ω_2 is defined by

$$(\varphi^\sharp \omega)(x)(\eta_1, \ldots, \eta_l) = \omega(\varphi(x))(d\varphi(x)(\eta_1), \ldots, d\varphi(x)(\eta_l)), \quad \forall\ x \in \Omega_1,\ \eta_1, \ldots, \eta_l \in \mathbb{R}^{m_1}.$$

Then $\varphi^\sharp \omega$ is a measurable l-form on Ω_1, and, if ω is smooth, then we have the chain rule $d\varphi^\sharp \omega = \varphi^\sharp d\omega$.

Let $\varphi : \Omega_1 \subset \mathbb{R}^{m_1} \to \Omega_2 \subset \mathbb{R}^{m_2}$ be a smooth proper map between two open sets Ω_1 and Ω_2. Then we define the pushforward $\varphi_\sharp T$ of an l-current $T \in (C_c^\infty(\Lambda^l \Omega_1))^*$ as the following l-current on Ω_2 :

$$\langle \varphi_\sharp T, \omega \rangle = \langle T, \varphi^\sharp \omega \rangle, \quad \forall\, \omega \in C_c^\infty(\Lambda^l \Omega_2).$$

In the special case where $\varphi : \Omega_1 \to \Omega_2$ is a diffeomorphism, we may define two additional objects: $\varphi_\sharp \omega = (\varphi^{-1})^\sharp \omega$ and $\varphi^\sharp T = (\varphi^{-1})_\sharp T$, whenever ω is a form on Ω_1 and T is a current on Ω_2.

We are now in position to define the Sobolev spaces of forms on manifolds. Consider a smooth oriented compact m-dimensional boundaryless submanifold M of $\mathbb{R}^{m+n}$. The set $\Lambda^l M := \bigcup_{x \in M} (\{x\} \times \Lambda^l T_x M)$ has a natural differential structure. Same for $\Lambda_l M := \bigcup_{x \in M} (\{x\} \times \Lambda_l T_x M)$. The set of smooth compactly supported l-forms in M is denoted by $C_c^\infty(\Lambda^l M)$. This set reduces to the set $C^\infty(\Lambda^l M)$ of smooth l-forms on M when M is compact. In order to simplify the presentation, we further assume that either M is compact, or $M = N \times \omega$, with N compact and t-dimensional and $\omega \subset \mathbb{R}^l$ (this will indeed be the case of all the submanifolds considered later). The definition of $W^{s,p}(\Lambda^l M)$ follows from that of $W^{s,p}(\Lambda^l \Omega)$ *via* a finite covering of M by domains of local coordinates. Let (φ_j, U_j, V_j) be a finite atlas of M, where:

a) If M is compact, then $\varphi_j : \overline{U_j} \subset M \to \overline{V_j} \subset \mathbb{R}^m$ is a smooth diffeomorphism.

b) If $M = N \times \omega$, then $\varphi_j = \eta_j \otimes \mathrm{Id}$, with $\eta_j : \overline{W_j} \subset N \to \overline{X_j} \subset \mathbb{R}^t$ is a smooth diffeomorphism. Here, (W_j) is a covering of N with domains of local coordinates.

Consider maps $\theta_j : M \to [0,1]$, such that:

a) If M is compact, then (θ_j) is a partition of the unit subordinated to the covering (U_j).

b) If $M = N \times \omega$, then $\theta_j = \psi_j \otimes \mathrm{Id}$, with (ψ_j) partition of the unit subordinated to the covering (W_j) of N.

For $s \geq 0$ and $p \geq 1$, we let

$$\|\omega\|_{W^{s,p}} = \|\omega\|_{W^{s,p}(\Lambda^l M)} = \sum_j \left\|\varphi_{j\sharp}(\theta_j \omega)\right\|_{W^{s,p}(\Lambda^l V_j)}.$$

Two different choices of finite coverings lead to the same space $W^{s,p}(\Lambda^l M)$, with equivalent norms.

Let $0 < s$ and $1 \leq p \leq \infty$. When $\omega \in W^{s,p}_{loc}(\Lambda^0 M)$,[16] we define the seminorm $|\omega|_{W^{s,p}(M)} \in [0,\infty]$ as follows.

$$|\omega|^p_{W^{s,p}(M)} = \begin{cases} \displaystyle\int_M \int_M \frac{|\omega(x) - \omega(y)|^p}{d(x,y)^{m+sp}}\, dx\, dy, & \text{if } s < 1 \\ \displaystyle\int_M |d\omega|^p, & \text{if } s = 1 \\ \|d\omega\|_{W^{s-1,p}}, & \text{if } s > 1 \end{cases},$$

with the obvious modification when $p = \infty$. Here, $d(x,y)$ is the geodesic distance between x and y.

An equivalent semi-norm is given by $\left\|\omega - \fint_M \omega\right\|_{W^{s,p}}$. Alternatively, one can consider the standard norm on the quotient space $W^{s,p}/\mathbb{R}$.[17] Other (equivalent) semi-norms will be introduced in the next sections.

We next identify currents defined in M with currents defined in $\mathbb{R}^{m+n}$. This goes as follows. If φ is a smooth map defined on M with values into some other manifold, then one may define the pullback and the pushforward by φ as above. Let U be an open neighborhood of M in $\mathbb{R}^{m+n}$ and

$$i : M \to U, M \ni x \xrightarrow{i} x \in U$$

be the canonical inclusion. Given an l-form ω on U, we denote by $\omega_{|M}$ the form on M defined by $i^\sharp \omega$. We say that

1. A form $\omega \in C^\infty_c(\Lambda^l U)$ is null at M if $i^\sharp \omega = 0$,
2. An l-current $T \in (C^\infty_c(\Lambda^l U))^*$ is a current *along* M if $\langle T, \omega \rangle = 0$ for every $\omega \in C^\infty_c(\Lambda^l U)$ which is null at M.

One can easily see that $T \mapsto i_\sharp T$ is an isomorphism between the space of l-currents defined in M and the space of l-currents *along* M.[18]

3 Rectifiability, area and coarea

We follow [19, Chapters 2, 3]; see also [21, Chapter 2] and [32, Chapters 3, 4].

An m-rectifiable set $\mathscr{M} \subset \mathbb{R}^{m+n}$ is a set of the form $\mathscr{M} = \mathscr{M}_0 \cup \bigsqcup_{j=1}^{\infty} \mathscr{M}_j$, where

1. $\sqcup$ stands for a disjoint union.

[16] In this case, ω is identified to a function.

[17] We will apply this with ω an $\mathbb{R}^k$-valued map. In this case, the quotient space is $(W^{s,p}(\Lambda^0 M))^k/\mathbb{R}^k$.

[18] Here, the spaces of currents are endowed with the usual topologies.

2. $\mathcal{H}^m(\mathcal{M}_0) = 0$.
3. For $j \geq 1$, every $\mathcal{M}_j$ is a Borel subset of an m-dimensional C^1-submanifold $\mathcal{N}_j$ of $\mathbb{R}^{m+n}$.
4. The total measure $\mathcal{H}^m(\mathcal{M}) = \sum_{j=1}^{\infty} \mathcal{H}^m(\mathcal{M}_j)$ is finite.

Let $\mathcal{M}$ be m-rectifiable. An m-dimensional subspace P of $\mathbb{R}^{m+n}$ is the approximate tangent space of $\mathcal{M}$ at $x \in \mathbb{R}^{m+n}$ if

$$\lim_{\lambda \to 0^+} \int_{(\mathcal{M}-x)/\lambda} g(y)\, d\mathcal{H}^m(y) = \int_P g(y)\, d\mathcal{H}^m(y), \quad \forall\, g \in C_c^\infty(\mathbb{R}^{m+n}).$$

If it exists, such a P is unique and is denoted by $T_x\mathcal{M}$. An m-rectifiable set $\mathcal{M}$ admits $\mathcal{H}^m$-a.e. on $\mathcal{M}$ an approximate tangent space [21, Section 2.1.4], and moreover

$$T_x\mathcal{M} = T_x\mathcal{N}_j, \quad \mathcal{H}^m - a.e.\ x \in \mathcal{M}_j. \tag{3.1}$$

An m-rectifiable current[19] in the open set $U \subset \mathbb{R}^{m+n}$ is defined *via* a triple $(\mathcal{M}, \xi, \theta)$, where

1. $\mathcal{M} \subset U$ is an m-rectifiable set.
2. $\xi : \mathcal{M} \to \Lambda_m \mathbb{R}^{m+n}$ is a Borel measurable map such that $|\xi| = 1$ and $\xi(x) \in \Lambda_m T_x\mathcal{M}$ for $\mathcal{H}^m$-a.e. x.[20]
3. $\theta : \mathcal{M} \to \mathbb{Z}$ is a Borel locally $\mathcal{H}^m$-summable function.[21]

The m-rectifiable current associated to this triple acts through the formula

$$\langle T, \omega \rangle = \int_{\mathcal{M}} \langle \xi, \omega \rangle \theta\, d\mathcal{H}^m\ , \quad \forall\, \omega \in C_c^\infty(\Lambda^m U).$$

Clearly, such a T acts on the space of compactly supported m-forms with continuous coefficients, and thus it may be identified with a (vector-valued) measure. As a measure, the total variation (mass) of T is given by

$$\mathbf{M}(T) := \int_{\mathcal{M}} |\theta|\, d\mathcal{H}^m.$$

We end this section by recalling the area and coarea formulae.

Let M be a smooth m-dimensional submanifold of an Euclidean space. Let $f \in Lip_{loc}(M; \mathbb{R}^k)$. We define the Jacobian of f at $x \in M$ by

$$J_f^M(x) := \begin{cases} (\det(df(x))^*(df(x)))^{1/2}, & \text{if } k \geq m \\ (\det(df(x))(df(x))^*)^{1/2}, & \text{if } k \leq m \end{cases}.$$

Here, the differential $df(x) : T_xM \to \mathbb{R}^k$ of f at x is defined for a.e. x. The area formula and coarea formula read as follows [21, Section 2.1.5]. Let $f : M \to \mathbb{R}^k$ be locally Lipschitz. Let $A \subset M$ be an $\mathcal{H}^m$-measurable set. Then for any nonnegative Borel measurable function $u : M \to \mathbb{R}$, we have

1. (*Area formula*) If $k \geq m$, then

$$\int_A u(x) J_f^M(x)\, d\mathcal{H}^m(x) = \int_{f(A)} \left(\sum_{f^{-1}(y) \cap A} u(x) \right) d\mathcal{H}^m(y).$$

2. (*Coarea formula*) If $k < m$, then

$$\int_A u(x) J_f^M(x)\, d\mathcal{H}^m(x) = \int_{f(A)} \left(\int_{f^{-1}(y) \cap A} u\, d\mathcal{H}^{m-k}(x) \right) d\mathcal{H}^k(y).$$

[19] Sometimes refered as an *integer multiplicity* m-rectifiable current.
[20] ξ is a Borel measurable orientation.
[21] θ is the multiplicity function. By changing ξ if necessary, one may assume that θ takes only nonnegative values.

4 Interpolation inequalities

We start by recalling the following Gagliardo-Nirenberg type inequality [33, Lecture II], [14, Corollary 2].[22]

4.1 Lemma. *Let $\theta \in (0,1)$, $a > 0$, $p \geq 1$, $1 < q \leq \infty$. Let r satisfy $\dfrac{1}{r} = \dfrac{\theta}{p} + \dfrac{1-\theta}{q}$. The inequality*

$$\|u\|_{W^{\theta a,r}} \leq C\|u\|^{\theta}_{W^{a,p}}\|u\|^{1-\theta}_{L^q}, \quad \forall\, u \in C^\infty_c(\mathbb{R}^m) \tag{4.1}$$

is true in the following cases.

1. *a is not an integer.*
2. *a and θa are integers.*
3. *a is an integer and $p > 1$.*

As a consequence, we have

4.2 Lemma. *For every $p \geq 1$ and $k \geq 3$, there exists $C = C(p,m) > 0$ such that*

$$\|u\|_{W^{(k-1)/p,p}(\mathbb{S}^m)} \leq C\|u\|^{1/p}_{W^{k-1,1}(\mathbb{S}^m)}\|u\|^{1-1/p}_{L^\infty(\mathbb{S}^m)}, \quad \forall\, u \in C^\infty(\mathbb{S}^m). \tag{4.2}$$

Proof. It suffices to prove (4.2) for $u \in C^\infty_c(B^m)$. Indeed, assume that (4.2) holds in this case. Consider a covering of $\mathbb{S}^m$ with finitely many open subsets U_j such that $\overline{U_j}$ is diffeomorphic to the closed ball $\overline{B}^m$. We denote by $\varphi_j : \overline{U_j} \to \overline{B}^m$ the corresponding diffeomorphisms, and by (θ_j) a partition of the unit subordinated to the covering (U_j). Then, for every $u \in C^\infty(\mathbb{S}^m)$ we have

$$\begin{aligned}\|u\|_{W^{(k-1)/p,p}(\mathbb{S}^m)} &\sim \sum_j \left\|(\varphi_j)_\sharp(\theta_j u)\right\|_{W^{(k-1)/p,p}(B^m)} \leq C\sum_j \left\|(\varphi_j)_\sharp(\theta_j u)\right\|^{1/p}_{W^{k-1,1}(B^m)}\left\|(\varphi_j)_\sharp(\theta_j u)\right\|^{1-1/p}_{L^\infty(B^m)}\\ &\leq C\|u\|^{1-1/p}_{L^\infty(\mathbb{S}^m)}\sum_j \left\|(\varphi_j)_\sharp(\theta_j u)\right\|^{1/p}_{W^{k-1,1}(B^m)} \leq C\|u\|^{1-1/p}_{L^\infty(\mathbb{S}^m)}\|u\|^{1/p}_{W^{k-1,1}(\mathbb{S}^m)},\end{aligned}$$

as claimed.

Let $u \in C^\infty_c(B^m)$. Since (4.2) is trivially true for $p = 1$, we may assume that $p > 1$. By item *2* in Lemma 4.1, we have

$$\|u\|_{W^{j,(k-1)/j}} \leq C\|u\|^{j/(k-1)}_{W^{k-1,1}}\|u\|^{1-j/(k-1)}_{L^\infty}, \quad \forall\, j \in [\![1,\dots,k-1]\!]. \tag{4.3}$$

We claim that (4.2) is a consequence of the following inequality:

$$\|u\|_{W^{s,p}} \leq C\|u\|^{s}_{W^{1,1}}\|u\|^{1-s}_{L^q}, \quad \forall\, s \in (0,1),\ q \in (1,\infty),\ u \in C^\infty_c(\mathbb{R}^m), \tag{4.4}$$

where p is defined by $\dfrac{1}{p} = s + \dfrac{1-s}{q}$. Indeed, assume for a moment that (4.4) is true. Then we are able to complete the proof of (4.2) by considering three different cases.

Case 1. $\dfrac{k-1}{p} \in (k-2, k-1)$.

By (4.3), we have

$$\|u\|_{W^{k-2,(k-1)/(k-2)}} \leq C\|u\|^{(k-2)/(k-1)}_{W^{k-1,1}}\|u\|^{1-(k-2)/(k-1)}_{L^\infty}. \tag{4.5}$$

By combining this estimate with (4.4) applied to the map $D^{k-2}u$, to $q = \dfrac{k-1}{k-2}$ and to $s = \dfrac{k-1}{p} - (k-2)$, we obtain

$$\begin{aligned}\left\|D^{k-2}u\right\|_{W^{(k-1)/p-(k-2),p}} &\leq C\left\|D^{k-2}u\right\|^{(k-1)/p-(k-2)}_{W^{1,1}}\left\|D^{k-2}u\right\|^{1-(k-1)/p+k-2}_{L^{(k-1)/(k-2)}} \leq \text{ (by (4.5))}\\ &\leq C\left\|D^{k-2}u\right\|^{(k-1)/p-(k-2)}_{W^{1,1}}\left(\|u\|^{(k-2)/(k-1)}_{W^{k-1,1}}\|u\|^{1/(k-1)}_{L^\infty}\right)^{1-(k-1)/p+k-2}\\ &\leq C\|u\|^{1/p}_{W^{k-1,1}}\|u\|^{1-1/p}_{L^\infty}.\end{aligned} \tag{4.6}$$

[22] The results in [14] are stated only for $p > 1$, but the arguments still hold when $p = 1$.

Using Rellich's theorem, one may prove by contradiction that there exists some $C > 0$ such that the following Poincaré type inequality holds:

$$\|u\|_{W^{(k-1)/p,p}} \le C \left\| D^{k-2} u \right\|_{W^{(k-1)/p-k+2,p}}, \quad \forall\, u \in C_c^\infty(B^m). \tag{4.7}$$

We obtain (4.2) by combining (4.6) and (4.7).

Case 2. $\dfrac{k-1}{p}$ is an integer.

This case follows from item *2* in Lemma 4.1.

Case 3. $\dfrac{k-1}{p} \in (l-1, l)$ for some integer $l \in [\![1, k-2]\!]$.

In this case, we have

$$\begin{aligned}
\|u\|_{W^{(k-1)/p,p}} &\le C \|u\|_{W^{l,(k-1)/l}}^{(k-1)/(pl)} \|u\|_{L^\infty}^{1-(k-1)/(pl)} \quad \text{(by (4.1))} \\
&\le C \left(\|u\|_{W^{k-1,1}}^{l/(k-1)} \|u\|_{L^\infty}^{1-l/(k-1)} \right)^{(k-1)/(pl)} \|u\|_{L^\infty}^{1-(k-1)/(pl)} \quad \text{(by (4.3))} \\
&\le C \|u\|_{W^{k-1,1}}^{1/p} \|u\|_{L^\infty}^{1-1/p};
\end{aligned}$$

that is (4.2) holds.

It remains to prove (4.4). Let $\varepsilon \in (0, s)$. By [18, Theorem 1.5], we have

$$\|u\|_{W^{s,p}} \le C \|u\|_{W^{1,1}}^{\theta_\varepsilon} \|u\|_{W^{\varepsilon,r}}^{1-\theta_\varepsilon}, \tag{4.8}$$

where $\theta_\varepsilon = \dfrac{s-\varepsilon}{1-\varepsilon}$ and r is defined by $\dfrac{1}{p} = \theta_\varepsilon + \dfrac{1-\theta_\varepsilon}{r}$.

By (4.1), we have

$$\|u\|_{W^{\varepsilon,r}} \le C \|u\|_{L^q}^{\eta_\varepsilon} \|u\|_{W^{s,p}}^{1-\eta_\varepsilon}, \tag{4.9}$$

where $\eta_\varepsilon = 1 - \dfrac{\varepsilon}{s}$ satisfies $\dfrac{1}{r} = \dfrac{\eta_\varepsilon}{q} + \dfrac{1-\eta_\varepsilon}{p}$.

We obtain (4.2) by combining (4.8) with (4.9). The proof of Lemma 4.2 is complete. □

4.3 Corollary. *For every $p \ge 1$ and $k \ge 3$, there exists $C = C(p,m) > 0$ such that*

$$|u|_{W^{(k-1)/p,p}(\mathbb{S}^m)} \le C |u|_{W^{k-1,1}(\mathbb{S}^m)}^{1/p} \|u\|_{L^\infty(\mathbb{S}^m)}^{1-1/p}, \quad \forall\, u \in C^\infty(\mathbb{S}^m). \tag{4.10}$$

Proof. Apply Lemma 4.2 to $u - \fint_{\mathbb{S}^m} u$. □

4.4 Remark. In the proof of Theorem 1.2, item *2*, it suffices to consider the case where $p > k-1$. Indeed, assume that the conclusion of Theorem 1.2 holds when $p > k-1$. Let $u \in W^{(k-1)/p,p}(\mathbb{S}^m; \mathbb{S}^{k-1})$ for some $p \le k-1$.

Assume first that $p > 1$. Let $s \in (0,1)$. By Lemma 4.1, we have $u \in W^{s,(k-1)/s}(\mathbb{S}^m; \mathbb{S}^{k-1})$ and the argument leading to Corollary 4.3 yields

$$|u|_{W^{s,(k-1)/s}(\mathbb{S}^m)} \le C |u|_{W^{(k-1)/p,p}(\mathbb{S}^m)}^{(sp)/(k-1)} \|u\|_{L^\infty(\mathbb{S}^m)}^{1-(sp)/(k-1)}.$$

By Theorem 1.2, there exists some $(m-k+1)$-rectifiable current N in $\mathbb{S}^m$ such that $Ju = \partial N$ and

$$\mathbf{M}(N) \le C |u|_{W^{s,(k-1)/s}(\mathbb{S}^m)}^{(k-1)/s} \le C |u|_{W^{(k-1)/p,p}(\mathbb{S}^m)}^{p}.$$

When $p = 1$, Lemma 4.1 combined with the argument leading to Corollary 4.3 implies

$$|u|_{W^{1,k-1}(\mathbb{S}^m)} \le C |u|_{W^{k-1,1}(\mathbb{S}^m)}^{1/(k-1)} \|u\|_{L^\infty(\mathbb{S}^m)}^{1-1/(k-1)}.$$

In this case, it suffices to invoke Theorem 1.2 in the space $W^{1,k-1}(\Omega; \mathbb{S}^{k-1})$, which was proved in [2, Theorem 3.8]. Here, $\Omega \subset \mathbb{R}^m$ is an open set, and not $\mathbb{S}^m$ as we want; however, it is easy to transfer results from open sets to spheres (or, more generally, embedded manifolds); see the proof of Theorem 1.3 at the end of the Section 6.1.

Similarly, in the proof of Theorem 1.3 it suffices to consider the cases $p = 1$ and $k \ge 3$ (by Corollary 4.3), respectively $k = 2$ and $p \ge 1$.

For further use, we note the following special case of Lemma 4.1.

4.5 Lemma. *Let $1 < p_0 < p < \infty$. Then we have*

$$\|u\|_{W^{1/p,p}} \le C\|u\|^{p_0/p}_{W^{1/p_0,p_0}}\|u\|^{1-p_0/p}_{L^\infty}, \quad \forall\, u \in C^\infty(\mathbb{S}^m)$$

and

$$|u|_{W^{1/p,p}} \le C|u|^{p_0/p}_{W^{1/p_0,p_0}}\|u\|^{1-p_0/p}_{L^\infty}, \quad \forall\, u \in C^\infty(\mathbb{S}^m).$$

5 Proof of Theorems 1.1 and 1.2

In the definition of Ju beyond the space $W^{1,k-1}(\mathbb{S}^m;\mathbb{S}^{k-1})$, we rely on the following quantitative version of the trace theory.

5.1 Lemma. *For $p > k-1$ and $u \in W^{(k-1)/p,p}\left(\mathbb{S}^m;\mathbb{R}^k\right)$, there exists a map $v : \mathbb{S}^m \times [0,\infty) \to \mathbb{R}^k$ satisfying:*

1. *$v \in C^\infty(\mathbb{S}^m \times (0,\infty))$.*
2. *For $T > 0$, we have $v \in W^{k/p,p}(\mathbb{S}^m \times (0,T);\mathbb{R}^k)$ and in addition*

$$|v|_{W^{k/p,p}(\mathbb{S}^m\times(0,T))} \le C_T|u|_{W^{(k-1)/p,p}(\mathbb{S}^m)} \tag{5.1}$$

and

$$\int_{\mathbb{S}^m} |v(x,\cdot)|^p_{W^{k/p,p}(0,\infty)}\, d\mathcal{H}^m(x) \le C|u|^p_{W^{(k-1)/p,p}(\mathbb{S}^m)}. \tag{5.2}$$

3. *$tr_{|\mathbb{S}^m\times\{0\}}v = u$.*
4. *If $u \in C^\infty$, then $v \in C^\infty(\mathbb{S}^m \times [0,\infty))$ and $v(\cdot,0) = u$.*
5. *If $|u| \le 1$ a.e. then $|v(x,t)| \le 1$ and $|dv(x,t)| \le \dfrac{C}{t}$, $\forall\,(x,t) \in \mathbb{S}^m \times (0,\infty)$.*

Finally, we may choose v such that the correspondence $u \mapsto v$ is linear and, in addition, v does not depend on p.

Here,

$$|u|^p_{W^{(k-1)/p,p}(\mathbb{S}^m)} = \int_{\mathbb{S}^m}\int_{\mathbb{S}^m} \frac{|u(x)-u(x')|^p}{d(x,x')^{m+k-1}}\, d\mathcal{H}^m(x)d\mathcal{H}^m(x'),$$

and the semi-norm $|v(x,\cdot)|^p_{W^{k/p,p}(0,\infty)}$ is given by

1. $\displaystyle\int_0^\infty\int_0^\infty \frac{|v(x,t)-v(x,t')|^p}{|t-t'|^{1+k}}\, dtdt'$, if $p > k$.
2. $\displaystyle\int_0^\infty |\partial_t v(x,t)|^p\, dt$, if $k = p$.
3. $\displaystyle\int_0^\infty\int_0^\infty \frac{\left|\partial_t v(x,t)-\partial_t v(x,t')\right|^p}{|t-t'|^{1+k-p}}\, dtdt'$, if $k-1 < p < k$.

The above result is well-known to experts, but seems difficult to find in the literature. For the convenience of the reader, we present a proof of Lemma 5.1 in Section 7.

5.2 Remark. When $p = k$, we recover the classical result that any map in $W^{1-1/k,k}$ is the trace of a map in $W^{1,k}$. More precisely, the proof of Lemma 5.1 shows that

$$|dv|_{L^k(\mathbb{S}^m\times(0,\infty))} \le C|u|_{W^{1-1/k,k}(\mathbb{S}^m)}. \tag{5.3}$$

In what follows, it will be convenient to denote by $u \mapsto Tu$ the distributional Jacobian whose existence we want to establish (to be distinguished of the already existent map $u \mapsto Ju$). Thus the conclusion of Theorem 1.1 *1* reads $Tu = Ju$ if $u \in W^{1,k-1}$, and similarly Theorems 1.2 and 1.3 can be rephrased in terms of Tu.

We now fix $p > k-1$ and $u \in W^{(k-1)/p,p}(\mathbb{S}^m;\mathbb{S}^{k-1})$ and proceed to defining Tu. Let $\Phi \in C^\infty\left(\mathbb{R}^k;\overline{B}^k\right)$ be such that

$$\Phi(X) = \frac{X}{|X|} \quad \text{if } |X| \geq \frac{1}{2}. \tag{5.4}$$

Let v be given by Lemma 5.1 and set $U := \Phi \circ v$, which is defined in $\mathbb{S}^m \times [0,\infty)$, and whose trace on $\mathbb{S}^m \times \{0\}$ is u. Clearly, $U \in C^\infty\left(\mathbb{S}^m \times (0,\infty);\overline{B}^k\right)$ and $|U(x,t)| = 1$ for every $(x,t) \in \mathscr{G}$, where

$$\mathscr{G} := \left\{(y,s) \in \mathbb{S}^m \times (0,\infty);\, |v(y,s)| > \frac{1}{2}\right\}.$$

As in the proof of (1.1), we find that $U^\sharp\gamma = 0$ on $\mathscr{G}$ for every $\gamma \in C^\infty(\Lambda^k\mathbb{R}^k)$; in particular, this holds for $\gamma = dy$. The key estimates in the proof of Theorems 1.1 and 1.2 are given by the following lemma, reminiscent of similar results in [10, Lemma 1, Lemma 2].

5.3 Lemma. *Let $\gamma \in C_c^\infty(\Lambda^k\mathbb{R}^k)$. Then*

1. *$U^\sharp\gamma \in L^1(\mathbb{S}^m \times (0,\infty))$ and $\|U^\sharp\gamma\|_{L^1(\mathbb{S}^m\times(0,\infty))} \leq C|u|^p_{W^{(k-1)/p,p}(\mathbb{S}^m)}$.*
2. *$\mathscr{H}^m\left(\left\{x;\, |v(x,t)| \leq \frac{1}{2}\right\}\right) = o\left(t^{k-1}\right)$ as $t \to 0$.*

Proof. Since $|U^\sharp\gamma| \leq C|dv|^k$, we have

$$\int_{\mathbb{S}^m\times(0,\infty)} |U^\sharp\gamma| = \int_{\mathscr{M}} |U^\sharp\gamma| \leq C\int_{\mathscr{M}} |dv|^k,$$

where $\mathscr{M} := (\mathbb{S}^m \times (0,\infty)) \setminus \mathscr{G}$.

For $x \in \mathbb{S}^m$, we define

$$d(x) := \inf\left\{t \in (0,\infty);\, |v(x,t)| \leq \frac{1}{2}\right\}. \tag{5.5}$$

In particular, we have $\mathscr{M} \subset \{(x,t);\, t \geq d(x)\}$ and, if $d(x) < \infty$, then $|v(x,d(x))| = \frac{1}{2}$.

By the above and Lemma 5.1 *5*, we find that

$$\int_{\mathbb{S}^m\times(0,\infty)} |U^\sharp\gamma| \leq C\int_{\mathbb{S}^m}\left(\int_{t=d(x)}^{+\infty} |dv(x,t)|^k\,dt\right)d\mathscr{H}^m(x) \leq C\int_{\mathbb{S}^m}\left(\int_{t=d(x)}^{+\infty}\frac{dt}{t^k}\right)d\mathscr{H}^m(x),$$

and thus

$$\int_{\mathbb{S}^m\times(0,\infty)} |U^\sharp\gamma| \leq C\int_{\mathbb{S}^m}\frac{1}{d(x)^{k-1}}\,d\mathscr{H}^m(x). \tag{5.6}$$

For almost every $x \in \mathbb{S}^m$ and for every $t > 0$ such that $|v(x,t)| \leq \frac{1}{2}$ we have

1. $|v(x,t) - u(x)| \geq \frac{1}{2}$.
2. $v(x,\cdot) \in W^{k/p,p}(0,\infty) \subset C^{0,(k-1)/p}(0,\infty)$.
3. $\mathrm{tr}_{|t=0}v(x,\cdot) = u(x)$.

On the other hand, by the Morrey embeddings we have

$$|v(x,t)-u(x)| \le |v(x,\cdot)|_{C^{0,(k-1)/p}(0,t)} t^{(k-1)/p} \le C|v(x,\cdot)|_{W^{k/p,p}(0,t)} t^{(k-1)/p}. \tag{5.7}$$

Thus, for a.e. $x \in \mathbb{S}^m$ we have

$$\frac{1}{t^{k-1}} \le C|v(x,\cdot)|^p_{W^{k/p,p}(0,t)} \quad \text{if } t \ge d(x). \tag{5.8}$$

By taking $t = d(x)$ in (5.8) and using Lemma 5.1 *2*, this gives

$$\int_{\mathbb{S}^m} \frac{1}{d(x)^{k-1}}\, d\mathscr{H}^m(x) \le C \int_{\mathbb{S}^m} |v(x,\cdot)|^p_{W^{k/p,p}(0,\infty)} \le C|u|^p_{W^{(k-1)/p,p}(\mathbb{S}^m)}. \tag{5.9}$$

Lemma 5.3 *1* is now a consequence of (5.6) and (5.9).

Finally, (5.7) implies

$$\mathscr{H}^m\left(\left\{x;\, |v(x,t)| \le \frac{1}{2}\right\}\right) \le C \int_{\mathbb{S}^m} |v(x,t)-u(x)|^p\, d\mathscr{H}^m(x) \le C t^{k-1} |v|^p_{W^{k/p,p}(\mathbb{S}^m\times(0,t))}.$$

This implies Lemma 5.3 *2*. □

By Lemma 5.3, the following quantity is well-defined for every $\varsigma \in W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m \times [0,\infty))$:

$$\langle Iu, \varsigma\rangle = \frac{(-1)^{m+1}}{\omega_k} \int_{\mathbb{S}^m\times(0,\infty)} (d\varsigma)\wedge U^\sharp dy. \tag{5.10}$$

The final step in defining Tu is provided by the following lemma.

5.4 Lemma. *Let $\zeta \in W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m)$. Let i denote the canonical inclusion map*

$$i : \mathbb{S}^m \to \mathbb{S}^m \times [0,\infty), \quad \mathbb{S}^m \ni x \xrightarrow{i} (x,0) \in \mathbb{S}^m \times [0,\infty).$$

Let $\varsigma_1, \varsigma_2 \in W^{1,\infty}(\Lambda^{m-k}(\mathbb{S}^m \times [0,\infty))$ satisfy $i^\sharp\varsigma_1 = i^\sharp\varsigma_2$. Then $\langle Iu,\varsigma_1\rangle = \langle Iu,\varsigma_2\rangle$.

Therefore, the map

$$\langle Tu,\zeta\rangle := \langle Iu,\varsigma\rangle, \quad \forall \zeta \in W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m),\ \forall \varsigma \in W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m\times[0,\infty)) \text{ such that } i^\sharp\varsigma = \zeta \tag{5.11}$$

is well-defined.

Proof. *Via* a convenient partition of the unit, we may always assume that the forms we consider are supported in Ω or $\Omega\times[0,\infty)$, where Ω is a domain of local coordinates on $\mathbb{S}^m$. We denote the coordinates on $\Omega\times\mathbb{R}$ by $(y_1,\dots,y_m,t)$. If ς is an $(m-k)$-form supported in $\Omega\times\mathbb{R}$, then we may decompose it as $\varsigma = \varsigma^h + \varsigma^v$, where ς^h represents the horizontal part (which does not contain dt), the part ς^v representing the remaining, vertical, part. The hypothesis $i^\sharp\varsigma_1 = i^\sharp\varsigma_2$ is equivalent to the fact that, restricted to $\mathbb{S}^m\times\{0\}$, the horizontal parts ς_1^h and ς_2^h coincide. Therefore, the difference $\varsigma := \varsigma_1 - \varsigma_2$ can be approximated by a sequence $(\eta_j) \subset C^\infty(\Lambda^{m-k}\mathbb{S}^m\times[0,\infty))$, supported in $\Omega\times[0,\infty)$, such that (η_j) converges uniformly to η, $\|d\eta_j\|_{L^\infty} \le C$, $(d\eta_j)$ converges to $d\varsigma$ a.e. on $\mathbb{S}^m\times(0,\infty)$ and η_j^h vanishes in a neighborhood of $\mathbb{S}^m\times\{0\}$.

Fix some $j \ge 1$. Since, by the chain rule, we have $d(U^\sharp dy) = (U^\sharp d^2 y) = 0$ in $\mathbb{S}^m\times(0,\infty)$, and since $\eta_j^h = 0$ on $\mathbb{S}^m\times[0,\varepsilon]$ for small ε, we have by Stokes' formula:

$$\int_{\mathbb{S}^m\times(0,\infty)} (d\eta_j)\wedge(U^\sharp dy) = \int_{\mathbb{S}^m\times(\varepsilon,\infty)} (d\eta_j)\wedge(U^\sharp dy) = (-1)^{m-k+1}\int_{\mathbb{S}^m\times(\varepsilon,\infty)} \eta_j\wedge d(U^\sharp dy) = 0. \tag{5.12}$$

We then let $j \to +\infty$ and find, by dominated convergence, that

$$0 = \int_{\mathbb{S}^m\times(0,\infty)} (d\varsigma)\wedge(U^\sharp dy) = \langle Iu,\varsigma_1\rangle - \langle Iu,\varsigma_2\rangle.$$ □

Proof of Theorem 1.1 3. When $0<s<1$ and $p=\dfrac{k-1}{s}$, it follows from Lemma 5.3 *1* and Lemma 5.4 that

$$|\langle Tu,\zeta\rangle| \le C|u|^p_{W^{s,p}}\|d\zeta\|_{L^\infty}, \quad \forall\,\zeta\in W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m).$$

Since we will establish later that $Tu=Ju$ when $u\in W^{1,k-1}(\mathbb{S}^m;\mathbb{S}^{k-1})$, the above is exactly the conclusion of Theorem 1.1 *3* when $0<s<1$. The case $s=1$ is clear from the definition of Ju. □

Proof of Theorem 1.1 2. We have to establish the following fact: for $k-1<p<\infty$, the map

$$T: W^{(k-1)/p,p}(\mathbb{S}^m;\mathbb{S}^{k-1})\to (W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m))^*, \quad u\xrightarrow{T} Tu$$

is continuous.

Let $(u_j)\subset W^{(k-1)/p,p}(\mathbb{S}^m;\mathbb{S}^{k-1})$ be a sequence converging in $W^{(k-1)/p,p}$ to some u. For $j\ge 1$, let v_j, $U_j=\Phi(v_j)$ and $d_j(x)$ be associated to u_j as above, and similarly let v, $U=\Phi(v)$ and $d(x)$ be associated to u.

It suffices to prove that $U_j^\sharp dy$ converges to $U^\sharp dy$ in $L^1(\mathbb{S}^m\times(0,\infty))$. By Lemma 5.1 *2*, there exists $K\in L^1(\mathbb{S}^m)$ such that up, to a subsequence, we have

$$|v_j(x,\cdot)|^p_{W^{k/p,p}(0,\infty)}\le K(x), \quad \forall\, j\ge 1,\ \mathscr{H}^m \text{ a.e. } x\in\mathbb{S}^m. \tag{5.13}$$

As in the proof of Lemma 5.3, (5.13) leads to

$$\frac{1}{d_j(x)^{k-1}}\le C|v_j(x,\cdot)|^p_{W^{k/p,p}(0,\infty)}\le CK(x). \tag{5.14}$$

Since (u_j) converges to u in $L^p(\mathbb{S}^m)$, the proof of Lemma 5.1 implies that (dv_j) converges to dv a.e., and therefore $U_j^\sharp dy$ converges to $U^\sharp dy$ a.e.

By (5.14), we have

$$|U_j^\sharp dy|(x,t)\le\begin{cases}0, & \text{if } t\le d_j(x)\\ C_0t^{-k}, & \text{if } t>d_j(x)\end{cases}\le f(x,t):=\begin{cases}0, & \text{if } t\le (CK(x))^{-1/(k-1)}\\ C_0t^{-k}, & \text{if } t>(CK(x))^{-1/(k-1)}.\end{cases}$$

The function f belongs to $L^1(\mathbb{S}^m\times(0,\infty))$, since $K\in L^1(\mathbb{S}^m\times(0,\infty))$. By dominated convergence, we find that $U_j^\sharp dy\to U^\sharp dy$ in $L^1(\mathbb{S}^m\times(0,\infty))$, possibly up to a subsequence. Uniqueness of the limit implies that convergence occurs for the full sequence. This completes the proof of Theorem 1.1 *2*. □

Proof of Theorem 1.1 1. We start by noting that Tu and Ju make sense for some classes of not necessarily $\mathbb{S}^{k-1}$-valued maps. To start with, Ju is defined for $u\in W^{1,k-1}\cap L^\infty(\mathbb{S}^m;\mathbb{R}^k)$, and $u\mapsto Ju$ is continuous wrt u in the following sense: if $u_j\to u$ in $W^{1,k-1}$ and $\|u_j\|_{L^\infty}\le C$, then $Ju_j\to Ju$ in $(W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m))^*$. On the other hand, Iu is defined for $u\in W^{1-1/k,k}(\mathbb{S}^m;\mathbb{R}^k)$ and continuous from $W^{1-1/k,k}(\mathbb{S}^m;\mathbb{R}^k)$ into $(W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m\times[0,\infty))^*$. This follows from the fact that by (5.3) we have

$$|U^\sharp dy|\le C|dU|^k\le C|dv|^k\in L^1(\mathbb{S}^m\times(0,\infty)).$$

The key observation is the following.

5.5 Lemma. *Tu is well-defined and the mapping $u\mapsto Tu$ is continuous from $W^{1-1/k,k}(\mathbb{S}^m;\mathbb{R}^k)$ into $(W^{1,\infty}(\Lambda^{m-k}(\mathbb{S}^m))^*$.*

In addition, for $u\in W^{1,k-1}\cap L^\infty\cap W^{1-1/k,k}(\mathbb{S}^m;\mathbb{R}^k)$, we have

$$Tu=Ju. \tag{5.15}$$

Proof. To start with, let $u \in C^\infty$. With ω_0 the canonical volume form on $\mathbb{S}^{k-1}$ given by (1.6) and ζ, ς as in Lemma 5.4, we have $d\omega_0 = \dfrac{1}{\omega_k} dy$, and thus

$$\begin{aligned}\langle Iu, \varsigma\rangle &= \frac{(-1)^{m+1}}{\omega_k} \int_{\mathbb{S}^m \times (0,\infty)} d\varsigma \wedge U^\sharp dy = (-1)^{m+1} \int_{\mathbb{S}^m \times (0,\infty)} d\varsigma \wedge U^\sharp d\omega_0 \\ &= (-1)^k \int_{\mathbb{S}^m \times (0,\infty)} d(d\varsigma \wedge U^\sharp \omega_0) = (-1)^{m-k+1} \int_{\mathbb{S}^m} d\zeta \wedge u^\sharp \omega_0 = \langle Ju, \zeta\rangle.\end{aligned}$$

In the last line, we have used Stokes' formula. In particular, this proves that $\langle Iu, \varsigma\rangle$ does not depend on ς, and thus Tu is well-defined and $Tu = Ju$. By a standard approximation procedure, (5.15) remains true for a general u.

Finally, the continuity of $u \mapsto Tu$ is clear. □

Proof of Theorem 1.1 1 completed: the case $k \geq 3$. The Gagliardo-Nirenberg embedding

$$W^{1,k-1} \cap L^\infty \subset W^{(k-1)/q,q}, \quad \forall\, q \geq k-1 \geq 2$$

implies that

$$\begin{aligned} W^{1,k-1} \cap W^{(k-1)/p,p}(\mathbb{S}^m;\mathbb{S}^{k-1}) &= W^{1,k-1}(\mathbb{S}^m;\mathbb{S}^{k-1}) \\ &= W^{1,k-1} \cap W^{1-1/k,k}(\mathbb{S}^m;\mathbb{S}^{k-1}), \quad \forall\, p > k-1. \end{aligned} \tag{5.16}$$

The preceding lemma combined with (5.16) proves that the equality $Tu = Ju$ holds for $u \in W^{1,k-1}(\mathbb{S}^m;\mathbb{S}^{k-1})$. □

Proof of Theorem 1.1 1 completed: the case $k = 2$. This follows from Lemma 5.5 and Lemma 5.6 below. □

5.6 Lemma. *Let $p > 1$. If $u \in W^{1,1} \cap W^{1/p,p}(\mathbb{S}^m;\mathbb{S}^1)$, then there exists a sequence*

$$(u_j) \subset W^{1,1} \cap W^{1/2,2} \cap W^{1/p,p}(\mathbb{S}^m;\mathbb{S}^1)$$

converging to u in $W^{1,1} \cap W^{1/p,p}$.

More generally, we may pick u_j such that $u_j \in W^{1,q}(\mathbb{S}^m;\mathbb{S}^1)$, $\forall\, 1 < q < 2$.

Proof. The result is obtained by combining the arguments in [8, Section IV] and [9, Lemma 23, Remark 5.1]. For the convenience of the reader, we sketch the proof. For $a \in \mathbb{D}$, let π_a be the projection onto $\mathbb{S}^1$ with vertex at a. That is, $\pi_a(x)$ is the intersection with $\mathbb{S}^1$ of the half-line from a to x.

Consider a regularized family (u_ε), so that $u_\varepsilon \to u$ in $W^{1,1} \cap W^{1/p,p}$ and $u_\varepsilon \in C^\infty(\mathbb{S}^m;\overline{\mathbb{D}})$. For a generic $a \in \mathbb{D}$, the set $u_\varepsilon^{-1}(a)$ is a smooth manifold of dimension $m-2$ and consists of regular points of u_ε. For any such a, the map $v_{a,\varepsilon} := \pi_a(u_\varepsilon)$ belongs to $W^{1,q}(\mathbb{S}^m;\mathbb{S}^1)$, $\forall\, 1 \leq q < 2$; see [9, Remark 5.1].

By [9, Lemma 23, Step 4], there exists a measurable set $A_\varepsilon \subset B(0,1/10) \subset \mathbb{R}^2$ such that $|A_\varepsilon| > |B(0,1/10)|/2$ and for every $a_\varepsilon \in A_\varepsilon$ we have $v_{a_\varepsilon,\varepsilon} \to u$ in $W^{1/p,p}(\mathbb{S}^m;\mathbb{S}^1)$.[23] On the other hand, we may pick a measurable set $B_\varepsilon \subset B(0,1/10)$ such that $|B_\varepsilon| > |B(0,1/10)|/2$ and for every $a_\varepsilon \in B_\varepsilon$ we have $v_{a_\varepsilon,\varepsilon} \to u$ in $W^{1,1}(\mathbb{S}^m;\mathbb{S}^1)$ [8, Proof of Theorem 4].[24] It then suffices to let $u_j = v_{a_{1/j},j}$ with $a_j \in A_{1/j} \cap B_{1/j}$. □

[23] The proof there is for $p = 2$, but the argument applies with no change to every $p > 1$.

[24] The map π_a in [8] is not the same as ours, but this is not relevant for the proof.

We now turn to the proof of Theorem 1.2.

Let $u \in W^{(k-1)/p,p}(\mathbb{S}^m;\mathbb{S}^{k-1})$, and associate to u the maps v and U as above. By the coarea formula, we have

$$\int_{B^k} \mathscr{H}^{m-k+1}(U^{-1}(a))\,d\mathscr{H}^k(a) = \int_{\mathbb{S}^m\times(0,\infty)} |U^\sharp dy|\,d\mathscr{H}^{m+1}.$$

Lemma 5.3 then implies

$$\int_{B^k} \mathscr{H}^{m-k+1}(U^{-1}(a))\,d\mathscr{H}^k(a) \le C|u|^p_{W^{(k-1)/p,p}(\mathbb{S}^m)}.$$

Hence, for almost every $a \in B^k$ we have $\mathscr{H}^{m-k+1}(U^{-1}(a)) < \infty$ and we can find a large set of a's such that

$$\mathscr{H}^{m-k+1}(U^{-1}(a)) \le C|u|^p_{W^{(k-1)/p,p}}. \tag{5.17}$$

We next establish a version of the coarea formula for differential forms.

5.7 Lemma. *For a convenient orientation of $U^{-1}(y)$ and for every $\varpi \in C^0 \cap L^\infty(\Lambda^{m-k+1}\mathbb{S}^m \times [0,\infty))$ we have*

$$\int_{\mathbb{S}^m\times(0,\infty)} \varpi \wedge (U^\sharp dy) = \int_{B^k}\left(\int_{U^{-1}(y)} \varpi\right) d\mathscr{H}^k(y). \tag{5.18}$$

Proof. We first define the orientation of $U^{-1}(y)$. This is done according to the following general principle. Let f be a linear map between two finite dimensional oriented vector spaces E and F. We want to define the orientation of Ker f. Consider a positively oriented basis $(\varepsilon_1,\dots,\varepsilon_r)$ of F. Let G be any supplement of Ker f and let $S := f_{|G} : G \to F$. By convention, we say that a basis $(e_1,\dots,e_l)$ of Ker f is positively oriented if $(e_1,\dots,e_l,S^{-1}\varepsilon_1,\dots,S^{-1}\varepsilon_r)$ is a positively oriented basis of E. It is easy to see that the definition of positively oriented bases of Ker f does not depend on the choice of $(\varepsilon_1,\dots,\varepsilon_r)$ or G. This procedure allows us to define an orientation on $U^{-1}(y)$. Indeed, if $x \in U^{-1}(y)$ is a regular point of U, then we have $T_xU^{-1}(y) = \text{Ker } dU(x)$.

Let $\tau \in C^\infty(\Lambda_{m+1}\mathbb{S}^m \times (0,\infty))$ be the oriented unit $(m+1)$-vector on $\mathbb{S}^m \times (0,\infty)$ and set

$$f(x) := \begin{cases} \left\langle \varpi(x) \wedge \dfrac{U^\sharp dy}{|U^\sharp dy|}(x), \tau(x)\right\rangle, & \text{if } |U^\sharp dy| \neq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Let $y \in \mathbb{R}^k$ be a regular value of U and $x \in U^{-1}(y)$. Let $(e_1,\dots,e_{m-k+1})$ be a positively oriented orthonormal basis of $T_xU^{-1}(y)$ that we complete to a positively oriented orthonormal basis $(e_1,\dots,e_{m+1})$ of $T_x\mathbb{S}^m \times (0,\infty)$. By the convention detailed above, this means that

$$(dU(x)(e_{m-k+2}),\dots,dU(x)(e_{m+1}))$$

is a positively oriented basis of $\mathbb{R}^k$. In particular, $\det\left(\left(dU^i(x)(e_j)\right)_{i\in[\![1,k]\!]}^{j\in[\![m-k+2,m+1]\!]}\right) > 0$.

Moreover, we have $\tau(x) = e_1 \wedge \dots \wedge e_{m+1}$, while $\sigma(x) := e_1 \wedge \dots \wedge e_{m+1-k}$ is the unit $(m+1-k)$-vector orienting $T_xU^{-1}(y)$. We write

$$\varpi(x) = \sum_{1\le i_1<\dots<i_{m+1-k}\le m+1} \varpi_{i_1,\dots,i_{m+1-k}}(x) e^*_{i_1} \wedge \dots \wedge e^*_{i_{m+1-k}}.$$

We have

$$(U^\sharp dy)(x) = (dU^1 \wedge \dots \wedge dU^k)(x) = \det\left(\left(dU^i(x)(e_j)\right)_{i\in[\![1,k]\!]}^{j\in[\![m-k+2,m+1]\!]}\right) e^*_{m-k+2} \wedge \dots \wedge e^*_{m+1}.$$

This gives

$$\begin{aligned}\left\langle \varpi(x) \wedge \frac{U^\sharp dy}{|U^\sharp dy|}(x), \tau(x)\right\rangle &= \varpi_{1,\dots,m-k+1} \operatorname{sgn}\det\left(\left(dU^i(x)(e_j)\right)_{i\in[\![1,k]\!]}^{j\in[\![m-k+2,m+1]\!]}\right) \\ &= \varpi_{1,\dots,m-k+1} = \langle \varpi(x), \sigma(x)\rangle.\end{aligned}$$

Therefore, we have $f(x) = \langle \varpi(x), \sigma(x)\rangle$. By the coarea formula, we find that

$$\begin{aligned}\int_{\mathbb{S}^m\times(0,\infty)} \varpi\wedge(U^\sharp dy) &= \int_{\mathbb{S}^m\times(0,\infty)} f(x) J_U^{\mathbb{S}^m\times(0,\infty)}\, d\mathscr{H}^{m+1}(x)\\ &= \int_{B^k}\left(\int_{U^{-1}(y)} \langle \varpi(x),\sigma(x)\rangle\, d\mathscr{H}^{m+1-k}(x)\right) d\mathscr{H}^k(y) = \int_{B^k}\left(\int_{U^{-1}(y)} \varpi\right) d\mathscr{H}^k(y).\end{aligned}$$

□

We will also need the following

5.8 Lemma. *Let $\gamma \in C_c^\infty(\Lambda^{k-1}B^k)$. Then for every $\varsigma \in C^1\cap W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m\times[0,\infty))$ we have*

$$\int_{\mathbb{S}^m\times(0,\infty)} d\varsigma\wedge U^\sharp d\gamma = 0.$$

Proof. Let $t>0$. Using the fact that γ is compactly supported in B^k, we find that $U^\sharp\gamma$ vanishes in the open set $\{x\in\mathbb{S}^m; |v(x,t)|>1/2\}$, which is contained in the set $\{x\in\mathbb{S}^m; |U(x,t)|=1\}$. As in the proof of (5.12), we obtain

$$\begin{aligned}\left|\int_{\mathbb{S}^m\times(t,\infty)} d\varsigma\wedge U^\sharp d\gamma\right| &= \left|\int_{\mathbb{S}^m\times\{t\}} d\varsigma\wedge U^\sharp\gamma\right|\\ &= \left|\int_{\{x\in\mathbb{S}^m;\,|v(x,t)|\le 1/2\}\times\{t\}} d\varsigma\wedge U^\sharp\gamma\right| \le \frac{C}{t^{k-1}}\mathscr{H}^m\left(\left\{x\in\mathbb{S}^m;\,|v(x,t)|\le\frac{1}{2}\right\}\right).\end{aligned}$$

This last quantity goes to 0 as $t\to 0$ by Lemma 5.3 *2*. This implies

$$\int_{\mathbb{S}^m\times(0,\infty)} d\varsigma\wedge U^\sharp d\gamma = \lim_{t\to 0}\int_{\mathbb{S}^m\times(t,\infty)} d\varsigma\wedge U^\sharp d\gamma = 0.$$

□

We continue by presenting some consequences of Lemma 5.8. Let $\rho\in C^\infty\left(\overline{B}^k\right)$ be such that $\int_{B^k}\rho = \omega_k$ and $\rho=1$ in a neighborhood of $\mathbb{S}^{k-1}$. Then there exists $\gamma\in C_c^\infty(\Lambda^{k-1}B^k)$ such that $d\gamma = (1-\rho)dy$ [27, Lemma 7.5]. Hence, we can apply the above lemma to γ, which yields for every $\varsigma\in C^1\cap W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m\times[0,\infty))$

$$\int_{\mathbb{S}^m\times(0,\infty)} d\varsigma\wedge U^\sharp d\gamma = 0. \tag{5.19}$$

Therefore, if we set, for every ρ as above and for $\varsigma\in C^1\cap W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m\times[0,\infty))$,

$$I(\rho) = I(\rho,u,\varsigma) := \int_{B^k}\rho(y)\left(\int_{U^{-1}(y)} d\varsigma\right) d\mathscr{H}^k(y),$$

then Lemma 5.7 applied to the form $(1-\rho\circ U)d\varsigma$ combined with (5.19) implies

$$I(1)-I(\rho) = \int_{\mathbb{S}^m\times(0,\infty)}((1-\rho\circ U)d\varsigma)\wedge U^\sharp dy = \int_{\mathbb{S}^m\times(0,\infty)} d\varsigma\wedge U^\sharp((1-\rho)dy) = \int_{\mathbb{S}^m\times(0,\infty)} d\varsigma\wedge U^\sharp d\gamma = 0.$$

We have thus proved

5.9 Lemma. *For $\varsigma\in W^{1,\infty}\cap C^1(\Lambda^{m-k}\mathbb{S}^m\times[0,\infty))$, the quantity $I(\rho)$ does not depend on the function $\rho\in C^\infty\left(\overline{B}^k\right)$ such that $\int_{B^k}\rho=\omega_k$ and $\rho=1$ in a neighborhood of $\mathbb{S}^{k-1}$.*

With ς fixed, let us denote by f the map $y \mapsto \int_{U^{-1}(y)} d\varsigma$. By applying Lemma 5.9 with $\rho = 1-\sigma$, we find that, for every $\sigma \in C_c^\infty(B^k)$ such that $\int_{B^k} \sigma = 0$, we have $\int_{B^k} \sigma(y) f(y) d\mathscr{H}^k(y) = 0$. This implies that the function f is constant almost everywhere. Hence, for every ς, we have by Lemma 5.7 with $\varpi = d\varsigma$

$$\int_{\mathbb{S}^m \times (0,\infty)} d\varsigma \wedge U^\sharp dy = \omega_k \int_{U^{-1}(y)} d\varsigma \quad \text{for a.e. } y \in B^k. \tag{5.20}$$

By the above, (5.17) and the separability of the space $C^1 \cap W^{1,\infty}(\mathbb{S}^m \times [0,\infty))$, we obtain the following result.

5.10 Lemma. *Let $p > k-1$ and $u \in W^{(k-1)/p,p}(\mathbb{S}^m;\mathbb{S}^{k-1})$. Then we can find some $a \in B^k$ such that*

1. *$U^{-1}(a)$ is a smooth $(m-k+1)$-submanifold of $\mathbb{S}^m \times (0,\infty)$.*
2. *We have*

$$\mathscr{H}^{m-k+1}(U^{-1}(a)) \le C|u|^p_{W^{(k-1)/p,p}}. \tag{5.21}$$

3. *For a.e. $y \in B^k$ and for every $\varsigma \in C^1 \cap W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m \times [0,\infty))$ we have*

$$\int_{U^{-1}(a)} d\varsigma = \int_{U^{-1}(y)} d\varsigma. \tag{5.22}$$

Proof of Theorem 1.2. By Remark 4.4, it suffices to establish items *1* and *2* for $p > k-1$ and $u \in W^{(k-1)/p,p}(\mathbb{S}^m;\mathbb{S}^{k-1})$. We have to find an $(m-k+1)$-rectifiable current N such that $Tu = \partial N$. We start by defining such a current living in $\mathbb{S}^m \times (0,\infty)$ (this is G given by (5.23)). This construction is inspired by [22]. We next project G on $\mathbb{S}^m$; the projection given by (5.25) has all the required properties.

Consider a as in Lemma 5.10. Set $\mathscr{M} := U^{-1}(a)$, oriented as in the proof of Lemma 5.7, and let

$$\langle G, \varpi \rangle := (-1)^{m+1} \int_{\mathscr{M}} \varpi, \quad \forall \varpi \in C^0 \cap L^\infty(\Lambda^{m-k+1}\mathbb{S}^m \times [0,\infty)). \tag{5.23}$$

Then G is an $(m-k+1)$-rectifiable current in $\mathbb{S}^m \times (0,\infty)$.

Next, let $\zeta \in C^1(\Lambda^{m-k}\mathbb{S}^m)$ and let $\varsigma \in C^1 \cap W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m \times [0,\infty))$ be a Lipschitz extension of ζ. By Lemma 5.7 and Lemma 5.10 item *3*, we obtain

$$\begin{aligned} \langle Tu, \zeta \rangle &= \frac{(-1)^{m+1}}{\omega_k} \int_{\mathbb{S}^m \times (0,\infty)} d\varsigma \wedge U^\sharp dy = \frac{(-1)^{m+1}}{\omega_k} \int_{B^k} \left(\int_{U^{-1}(y)} d\varsigma \right) d\mathscr{H}^k(y) \\ &= (-1)^{m+1} \int_{U^{-1}(a)} d\varsigma = \langle G, d\varsigma \rangle = \langle \partial G, \varsigma \rangle. \end{aligned} \tag{5.24}$$

That is, as mentioned in the introduction, Tu is the boundary of a smooth $(m-k+1)$-rectifiable current living in $\mathbb{S}^m \times (0,\infty)$.

Next, let π be the projection

$$\pi : \mathbb{S}^m \times [0,\infty) \to \mathbb{S}^m, \ \mathbb{S}^m \times [0,\infty) \ni (x,t) \xrightarrow{\pi} x \in \mathbb{S}^m$$

and set

$$N := \pi_\sharp G. \tag{5.25}$$

The conclusion of Theorem 1.2 follows by combining Lemma 5.10 *2* with the definition of G and with the following result. □

5.11 Lemma. *N is an $(m-k+1)$-rectifiable current in $\mathbb{S}^m$. Moreover, we have*

$$\mathbf{M}(N) \le C\mathbf{M}(G) = C\mathcal{H}^{m-k+1}(\mathcal{M}) \tag{5.26}$$

and

$$\partial N = Tu. \tag{5.27}$$

Though the above lemma essentially follows from [21, p. 149, Section 2.4], we found useful to present a compete proof clarifying some points only sketched in [21].

Proof. As it is easy to guess, we will prove that the support of N is contained in $\pi(\mathcal{M})$. In the line of Section 3, in the first part of the proof of Lemma 5.11 we establish the fact that $\pi(\mathcal{M})$ is rectifiable, and give the orientation ξ and the multiplicity function θ on $\pi(\mathcal{M})$. We complete the proof of the lemma by establishing (5.27).

We start by proving that $\pi(\mathcal{M})$ is $(m-k+1)$-rectifiable. Since $\mathcal{H}^{m-k+1}(\mathcal{M}) < \infty$ and π is globally Lipschitz, we have

$$\mathcal{H}^{m-k+1}(\pi(\mathcal{M})) \le C\mathcal{H}^{m-k+1}(\mathcal{M}) < \infty.$$

Let $\mathcal{M}_0 := \{x \in \mathcal{M}; J_\pi^{\mathcal{M}}(x) = 0\}$. By the area formula, we have $\mathcal{H}^{m-k+1}(\pi(\mathcal{M}_0)) = 0$. For every $x \in \mathcal{M} \setminus \mathcal{M}_0$, there exists an open neighborhood $U_x \subset \mathcal{M}$ of x in $\mathcal{M}$ such that π is a C^1 embedding from U_x into $\mathbb{S}^m$. It follows that $\pi(U_x)$ is a C^1 manifold of dimension $m-k+1$ in $\mathbb{S}^m$. There exists a countable covering of $\mathcal{M} \setminus \mathcal{M}_0$ with such open sets U_{x_j}, $j \in \mathbb{N}$. This proves that $\pi(\mathcal{M}) = \pi(\mathcal{M}_0) \cup \bigcup_j \pi(U_{x_j})$ is $(m-k+1)$-rectifiable. In order to obtain a disjoint covering, we define

$$A_0 := \pi(U_{x_0}),\ A_j := \pi(U_{x_j}) \setminus \cup_{i<j}\pi(U_{x_i}),\ \forall j \ge 1,$$

and then we have $\pi(\mathcal{M} \setminus \mathcal{M}_0) = \sqcup A_j$.

We next define an orientation ξ of $\pi(\mathcal{M})$; since $\pi(\mathcal{M}_0)$ is an $\mathcal{H}^{m-k+1}$-null set, it suffices to define ξ $\mathcal{H}^{m-k+1}$-a.e. in $\pi(\mathcal{M} \setminus \mathcal{M}_0)$. Let $\chi \in C^\infty(\Lambda_{m+1-k}\mathcal{M})$ be the smooth orientation of $\mathcal{M}$ defined in the proof of Lemma 5.7. For every $x \in U_{x_j}$, $d\pi(x)$ maps $T_x\mathcal{M}$ onto $T_{\pi(x)}\pi(U_{x_j})$. Then $\Lambda^{m-k+1}d\pi(x)(\chi(x))$ is an $(m-k+1)$-vector in $T_{\pi(x)}\pi(U_{x_j})$. The unit $(m-k+1)$-vector $\dfrac{\Lambda^{m-k+1}d\pi(x)(\chi(x))}{|\Lambda^{m-k+1}d\pi(x)(\chi(x))|}$ gives an orientation of $\pi(U_{x_j})$. However, it may happen that for some $y \in \pi(\mathcal{M} \setminus \mathcal{M}_0)$, there exist two antecedents $z_1, z_2 \in \pi^{-1}(y)$ such that the corresponding unit vectors $\dfrac{\Lambda^{m-k+1}d\pi(z_j)(\chi(z_j))}{|\Lambda^{m-k+1}d\pi(z_j)(\chi(z_j))|}$, $j = 1,2$, do not define the same orientation. In order to define an orientation on $\pi(\mathcal{M} \setminus \mathcal{M}_0)$, for $y \in A_j$ we define the unit $(m-k+1)$-vector $\xi(y) = \dfrac{\Lambda^{m-k+1}d\pi(z)(\chi(z))}{|\Lambda^{m-k+1}d\pi(z)(\chi(z))|}$, where $\{z\} = \pi^{-1}(y) \cap U_{x_j}$. Then ξ is a Borel measurable map with values into unit $(m-k+1)$-vectors on $\pi(\mathcal{M})$.

For further use, let us note that the Cauchy-Binet formula implies that

$$\left|(\Lambda^{m-k+1}d\pi(x))\chi(x)\right| = J_\pi^{\mathcal{M}}(x), \quad \forall x \in \mathcal{M} \tag{5.28}$$

(where $|\cdot|$ denotes the Euclidean norm on the set of $(m-k+1)$-vectors, see (2.2)).

We next define the multiplicity function θ. By (3.1), for $\mathcal{H}^{m-k+1}$-a.e. $y \in \pi(\mathcal{M} \setminus \mathcal{M}_0)$ and for every $x \in \pi^{-1}(y) \cap (\mathcal{M} \setminus \mathcal{M}_0)$, there exists some $\varepsilon(x) \in \{\pm 1\}$ such that

$$\varepsilon(x)\xi(y) = \frac{\Lambda^{m-k+1}d\pi(x)(\chi(x))}{|\Lambda^{m-k+1}d\pi(x)(\chi(x))|}. \tag{5.29}$$

For such y, we let $\theta(y) := (-1)^{m+1} \sum_{x \in \pi^{-1}(y) \cap (\mathcal{M} \setminus \mathcal{M}_0)} \varepsilon(x)$. We start by noting that

$$|\theta(y)| \le \#[\pi^{-1}(y) \cap (\mathcal{M} \setminus \mathcal{M}_0)].$$

By the area formula, we have

$$\int_{\pi(\mathscr{M})} \#[\pi^{-1}(y)\cap(\mathscr{M}\setminus\mathscr{M}_0)]\,d\mathscr{H}^{m-k+1}(y) = \int_{\mathscr{M}} J_\pi^{\mathscr{M}}(x)\,d\mathscr{H}^{m-k+1}(x) \le C\mathscr{H}^{m-k+1}(\mathscr{M}). \quad (5.30)$$

This implies that θ is finite almost everywhere. We can write θ in an obvious way as a sum of countably many Borel maps taking their values in $\{-1,0,1\}$ and the above inequality also shows that this sum converges almost everywhere. Hence, θ is Borel measurable as well. In particular, the triple $(\pi(\mathscr{M}),\xi,\theta)$ defines an $(m-k+1)$-rectifiable current. In view of (5.30), the estimate (5.26) follows if we prove that this current coincides with the current N given by (5.25).

If $v\in C^\infty(\Lambda^{m-k+1}\mathbb{S}^m)$, then

$$\langle(\pi^\sharp v)(x),\chi(x)\rangle = \langle v(\pi(x)),\Lambda_{m+1-k}d\pi(x)\chi(x)\rangle.$$

Since by definition we have

$$\langle N,v\rangle = \langle G,\pi^\sharp v\rangle, \quad \forall\, v\in C^\infty(\Lambda^{m-k+1}\mathbb{S}^m),$$

we obtain, by combining (5.28), (5.29) and the area formula:

$$\begin{aligned}\langle N,\omega\rangle &= (-1)^{m+1}\int_{\mathscr{M}\setminus\mathscr{M}_0}\left\langle\omega(\pi(x)),\frac{(\Lambda_{m-k+1}d\pi(x))\chi(x)}{|(\Lambda_{m-k+1}d\pi(x))\chi(x)|}\right\rangle J_\pi^{\mathscr{M}}(x)\,d\mathscr{H}^{m-k+1}(x)\\ &= (-1)^{m+1}\int_{\pi(\mathscr{M})}\left\langle\omega(y),\sum_{x\in\pi^{-1}(y)\cap(\mathscr{M}\setminus\mathscr{M}_0)}\frac{(\Lambda_{m-k+1}d\pi(x))\chi(x)}{|(\Lambda_{m-k+1}d\pi(x))\chi(x)|}\right\rangle d\mathscr{H}^{m-k+1}(y)\\ &= \int_{\pi(\mathscr{M})}\langle\omega,\xi\rangle\theta\,d\mathscr{H}^{m-k+1};\end{aligned}$$

that is, N is the current defined by the triple $(\pi(\mathscr{M}),\xi,\theta)$.

Finally, we establish (5.27). For every $\zeta\in C^1(\Lambda^{m-k}\mathbb{S}^m)$ we have

$$\langle\partial N,\zeta\rangle = \langle\partial(\pi_\sharp G),\zeta\rangle = \langle\pi_\sharp G,d\zeta\rangle = \langle G,\pi^\sharp d\zeta\rangle = \langle G,d(\pi^\sharp\zeta)\rangle = \langle\partial G,\pi^\sharp\zeta\rangle = \langle Tu,\zeta\rangle.$$

Here, we rely on (5.24) and on the fact that $\varsigma=\pi^\sharp\zeta$ belongs to $C^1\cap W^{1,\infty}(\Lambda^{m-k}\mathbb{S}^m\times[0,\infty))$ and satisfies (with the notations in Lemma 5.4)

$$i^\sharp\varsigma = \zeta. \quad (5.31)$$

Here is a proof of (5.31). Since $\varsigma=\pi^\sharp\zeta$, we have $i^\sharp\varsigma = i^\sharp\left(\pi^\sharp\zeta\right) = (\pi\circ i)^\sharp\zeta$. It then suffices to note that $\pi\circ i = \mathrm{id}_{|\mathbb{S}^m}$, and thus $(\pi\circ i)^\sharp\zeta=\zeta$. This completes the proof Lemma 5.11 and thus of Theorem 1.2. □

5.12 Remark. An inspection of the proofs in this section shows that Theorems 1.1 and 1.2 still hold in bounded open subsets Ω of $\mathbb{R}^m$. It suffices to replace integration over $\mathbb{S}^m$ by integration over Ω.

5.13 Remark. Let $p=k-1$ and $s=1$. For these values of s and p, one of the results in [2] asserts that $Ju=\partial N$, with $\mathbf{M}(N)\le C_p|u|^p_{W^{s,p}}$. In addition, the proof there shows that we may build N from the rectifiable set $\mathscr{N}=u^{-1}(y)$ for an appropriate $y\in\mathbb{S}^{k-1}$.[25] By the Gagliardo-Nirenberg inequalities, the same holds when $p\in[1,k-1)$, with $s=\dfrac{k-1}{p}$. We do not know whether it is possible to take N as above when $p>k-1$ and $s=\dfrac{k-1}{p}$.

6 Proof of Theorem 1.3

In view of Remark 4.4, it suffices to consider the cases $p=1$ and $k=2$.

[25] In the same way that G is built from $\mathscr{M}$ in the proof of Lemma 5.11.

6.1 The case $p = 1$

In this case, we adapt the strategy of Alberti, Baldo and Orlandi [2]. Their approach consists in combining the dipole construction of Brezis, Coron and Lieb [13] with Federer's theorem on the approximation of integral currents with polyhedral chains [19, Theorem 4.2.20].

We first prove Theorem 1.3 when $\mathbb{S}^m$ is replaced by a bounded open subset Ω of $\mathbb{R}^m$. By Remark 5.12, Ju makes sense for maps defined in Ω.

In Lemma 6.1 below, we describe the basic tool in the proof of Theorem 1.3: the dipole construction. This construction was first introduced in [13] and was subsequently used in [2] that we closely follow here. Since we need a dipole which is smooth (and not merely Lipschitz continuous) outside its singular set, we present, for the convenience of the reader, a detailed proof.

We say that a set $E \subset \mathbb{R}^m$ is an oriented $(m-k+1)$-disc when E is an orientation preserving isometric linear embedding of the unit ball $B^{m-k+1} \times \{0_{k-1}\} \subset \mathbb{R}^m$, endowed with the canonical orientation. We will often write $|E| = \mathscr{H}^{m-k+1}(E)$.

6.1 Lemma. *Let $E \subset \mathbb{R}^m$ be an oriented $(m-k+1)$-disc with center x and radius r. Let $\sigma > 0$. Then there exists $u \in W^{k-1,1}_{loc}(\mathbb{R}^m;\mathbb{S}^{k-1}) \cap C^\infty(\mathbb{R}^m \setminus \partial E)$ such that*

1. *$Ju = \partial E$.*
2. *There exists $Y \in \mathbb{S}^{k-1}$ such that $u = Y$ in $\mathbb{R}^m \setminus \overline{B}(x,r)$ and*
$$|\{y \in \mathbb{R}^m : u(y) \neq Y\}| \leq \sigma. \tag{6.1}$$
3. *There exists $C_\sigma > 0$ such that*
$$\left|D^l u(y)\right| \leq \frac{C_\sigma}{\operatorname{dist}(y,\partial E)^l}, \quad \forall\, y \in \mathbb{R}^m \setminus \partial E,\ \forall\, l \in \mathbb{N}. \tag{6.2}$$
4. *There exists $C > 0$ such that*
$$\left\|D^l u\right\|_{L^1(\mathbb{R}^m)} \leq C|E|^{(m-l)/(m-k+1)}, \quad \forall\, l \in [\![1,k-1]\!]. \tag{6.3}$$

Here, ∂E is the boundary of E considered as a manifold with boundary.

Proof. We start with a remark in the spirit of [2, Section 7.2]: if $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth diffeomorphism, and if $u \in W^{1,k-1} \cap L^\infty$, then $\varphi^\sharp Ju = J\left(\varphi^\sharp u\right)$. Indeed, for every $\zeta \in C^\infty_c(\Lambda^{m-k}\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m} d(\varphi_\sharp \zeta) \wedge u^\sharp \omega_0 = \int_{\mathbb{R}^m} \varphi_\sharp\left(d\zeta \wedge \varphi^\sharp\left(u^\sharp \omega_0\right)\right) = \int_{\mathbb{R}^m} d\zeta \wedge \left(\left(\varphi^\sharp u\right)^\sharp \omega_0\right),$$

which is equivalent to $\langle \varphi^\sharp Ju, \zeta\rangle = \langle J\left(\varphi^\sharp u\right), \zeta\rangle$ and implies that $\varphi^\sharp Ju = J\left(\varphi^\sharp u\right)$.

By taking φ an affine homothety, we see that in the proof of Lemma 6.1 we may assume that $E = B^{m-k+1} \times \{0_{k-1}\} \subset \mathbb{R}^m$.

Let $P : B^{k-1} \to \mathbb{S}^{k-1}$ be a smooth map of degree $(-1)^{m-k+1}$ such that P is constant near $\mathbb{S}^{k-2} = \partial B^{k-1}$. For instance, with $|\ |$ standing for the Euclidean norm and $\mu \in (0, 1/10)$ a fixed parameter, a suitable P is given by

$$P(x) := \left(f(\sin \pi|x|)\frac{x}{|x|}, (-1)^m h(|x|)\cos(\pi|x|)\right),$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a smooth nondecreasing function which vanishes on $(0,\mu)$ and is equal to the identity on $(2\mu,\infty)$ and for $r \in [0,1]$

$$h(r) := \begin{cases} \left(\dfrac{1-(f(\sin \pi r))^2}{\cos^2 \pi r}\right)^{1/2}, & \text{if } r \neq \dfrac{1}{2} \\ 1, & \text{if } r = \dfrac{1}{2} \end{cases}.$$

Note that when $|x|$ is close to 0, the first coordinate of P is 0, while the second one is equal to $(-1)^m$. Hence P is smooth despite the singularity of $x \mapsto |x|$ at the origin.

We claim that $\deg P = (-1)^{m-k+1}$. Indeed, let $x = (t, 0, \dots, 0) \in B^{k-1}$ be such that $\sin \pi t > 2\mu$. Then $P^{-1}(P(x)) = \{x\}$, and thus $\deg P$ is given by the orientation of $d_x P$. Since the orientation of $\mathbb{S}^{k-1}$ is given by ω_0 and the one of $\mathbb{R}^{k-1}$ by $dx_1 \wedge \dots \wedge dx_{k-1}$, and in addition we have (with x as above)

$$P^\sharp \omega_0(x) = \frac{(-1)^{m-k+1}(\sin \pi t)^{k-2}}{k\omega_k t^{k-2}} \pi dx_1 \wedge \dots \wedge dx_{k-1},$$

we find that $\deg P = (-1)^{m-k+1}$, as claimed.

When $|x|$ is close to 1, the first coordinate of P vanishes and the second one is equal to $(-1)^{m+1}$. Thus, P is constant on a neighborhood of $\mathbb{S}^{k-2}$, where it takes the value $Y := \left(0_{k-1}, (-1)^{m+1}\right)$. Let $\eta > 0$ be such that $P(x) = Y$ when $|x| > 1 - \eta$.

Let $\delta \in (0, 1/2)$ to be specified later and let $q : [0,1] \to [0, \delta]$ be a smooth nonincreasing function such that $q(t) = \delta$ on $[0, 1-2\delta]$, $q(t) = 1 - t$ on $[1 - \delta/2, 1]$, $q(t) \le 1 - t$ on $[0,1]$. Moreover, we require that the derivatives of $q^{(l)}$ can be bounded independently of δ.

For every $x \in \mathbb{R}^m$, we write $x = (x', x'') \in \mathbb{R}^{m-k+1} \times \mathbb{R}^{k-1}$. We define

$$u(x) := \begin{cases} P\left(\dfrac{x''}{q(|x'|)}\right), & \text{if } |x'| < 1 \text{ and } |x''| < q(|x'|) \\ Y, & \text{otherwise} \end{cases}.$$

The map u is smooth in

$$\mathbb{R}^m \setminus \partial E = \mathbb{R}^m \setminus \left(\mathbb{S}^{m-k} \times \{0_{k-1}\}\right)$$

and we have $u = Y$ in the open set

$$V := \{(x', x''); |x'| < 1, |x''| > (1-\eta) q(|x'|)\} \cup \{(x', x''); |x'| > 1\}, \tag{6.4}$$

which is a neighborhood of $\mathbb{S}^{m-1} \setminus \left(\mathbb{S}^{m-k} \times \{0_{k-1}\}\right)$. Moreover, we have

$$|\{x; u(x) \neq Y\}| \le |\{x = (x', x''); |x''| < q(|x'|), |x'| < 1\}| \le C_0 \delta^{k-1}.$$

We choose $\delta > 0$ such that $C_0 \delta^{k-1} < \sigma$. This implies (6.1).

Next, for every $l \ge 0$ there exists $C_l > 0$ independent of δ such that

$$\left|D^l u(x)\right| \le \frac{C_l}{q(|x'|)^l}, \quad \forall x \notin \mathbb{S}^{m-k} \times \{0_{k-1}\}. \tag{6.5}$$

In view of the definition of q, this implies the item *3* and also the fact that $u \in W^{k-1,1}_{loc}(\mathbb{R}^m; \mathbb{S}^{k-1})$. As a consequence of (6.5), we have for every $l \in [\![1, k-1]\!]$

$$\int_{\mathbb{R}^m} \left|D^l u(x)\right| dx \le C_l \int_{B^{m-k+1}} \left(\int_{|x''| < q(|x'|)} dx'' \right) \frac{dx'}{q(|x'|)^l} \le C'_l \delta^{k-l-1}, \tag{6.6}$$

which implies item *4*.

It remains to prove item *1* which follows from the following explicit calculation.[26] We introduce, for $\varepsilon \in (0, \delta/2)$, the open set

$$T(\varepsilon) := \left\{(x', x'') \in \mathbb{R}^{m-k+1} \times \mathbb{R}^{k-1}; |1 - |x'|| < \varepsilon, |x''| < 2\varepsilon\right\}.$$

Let $\zeta \in C^\infty_c(\Lambda^{m-k}\mathbb{R}^m)$. By dominated convergence, we have

$$\langle Ju, \zeta \rangle = (-1)^{m-k+1} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^m \setminus T(\varepsilon)} d\zeta \wedge u^\sharp \omega_0.$$

[26] Alternatively, we could have invoked Jerrard and Soner [25, Theorem 1].

In $\mathbb{R}^m \setminus T(\varepsilon)$, u is smooth and $\mathbb{S}^{k-1}$-valued, so that $d(u^\sharp\omega_0) = 0$. Stokes' formula then implies

$$\langle Ju, \zeta\rangle = (-1)^{m-k} \lim_{\varepsilon\to 0} \int_{\partial T(\varepsilon)} \zeta \wedge u^\sharp\omega_0. \tag{6.7}$$

Since u is constant in the open set V given by (6.4), we have

$$\int_{\partial T(\varepsilon)} \zeta \wedge u^\sharp\omega_0 = \int_{\partial T(\varepsilon)\setminus V} \zeta \wedge u^\sharp\omega_0. \tag{6.8}$$

Let, in the remaining part of the proof, $\overline{B}^l(x,r)$ denote the closed ball $\overline{B}(x,r) \subset \mathbb{R}^l$ and set $\overline{B}^l_r = \overline{B}^l(0,r)$; we use similar notation for the open balls. Observe that

$$\partial T(\varepsilon) \setminus V \subset \left\{((1-\varepsilon)x', x''); x' \in \mathbb{S}^{m-k}, x'' \in \overline{B}^{k-1}_{2\varepsilon}\right\} \tag{6.9}$$

and that, on $\partial T(\varepsilon) \setminus V$, u depends only on x''. Using (6.5) and the properties of q, we find that

$$\|u^\sharp\omega_0\|_{L^\infty(\partial T(\varepsilon)\setminus V)} \le C/\varepsilon^{k-1}. \tag{6.10}$$

Recall that, if $M \subset U$ (with M manifold and U open set), then we denote by $\zeta_{|M}$ the pullback $i^\sharp\zeta$ of a form on U; here, $i : M \to U$ is the inclusion. With this notation and using the fact that, on $\partial T(\varepsilon) \setminus V$, u depends only on x'', for every $((1-\varepsilon)x', x'')$ as in (6.9) we have

$$(\zeta \wedge u^\sharp\omega_0)|_{\partial T(\varepsilon)\setminus V}((1-\varepsilon)x', x'') = \zeta_{|(1-\varepsilon)\mathbb{S}^{m-k}\times\{x''\}} \wedge \left(u^\sharp\omega_0\right)_{|\{(1-\varepsilon)x'\}\times\overline{B}^{k-1}_{2\varepsilon}}((1-\varepsilon)x', x'').$$

Assume for a moment that $\zeta_{|\mathbb{S}^{m-k}\times\{0\}} = 0$. By combining the estimate $\mathscr{H}^{m-k}(\partial T(\varepsilon) \setminus V) \le C\varepsilon^{k-1}$ with (6.10), we find that

$$\left|\int_{\partial T(\varepsilon)\setminus V} \zeta \wedge u^\sharp\omega_0\right| \le C \max_{\substack{|x'|=1\\ |x''|\le 2\varepsilon}} \left|\zeta_{|(1-\varepsilon)\mathbb{S}^{m-k}\times\{x''\}}(x', x'')\right|,$$

which implies

$$\lim_{\varepsilon\to 0} \int_{\partial T(\varepsilon)\setminus V} \zeta \wedge u^\sharp\omega_0 = 0. \tag{6.11}$$

We now turn to a general $\zeta \in C^\infty_c(\Lambda^{m-k}\mathbb{R}^m)$. If p is the projection

$$p : \left(\mathbb{R}^{m-k+1} \setminus \{0\}\right) \times \mathbb{R}^{k-1} \to \mathbb{S}^{m-k}, \ \left(\mathbb{R}^{m-k+1} \setminus \{0\}\right) \times \mathbb{R}^{k-1} \ni x = (x', x'') \xrightarrow{p} \frac{x'}{|x'|} \in \mathbb{S}^{m-k},$$

then we have

$$\left(\zeta - p^\sharp\left(\zeta_{|\mathbb{S}^{m-k}\times\{0_{k-1}\}}\right)\right)_{|\mathbb{S}^{m-k}\times\{0_{k-1}\}} = 0. \tag{6.12}$$

In view of (6.11) and (6.12), we have

$$\lim_{\varepsilon\to 0}\left(\int_{\partial T(\varepsilon)\setminus V} \zeta \wedge u^\sharp\omega_0 - \int_{\partial T(\varepsilon)\setminus V} p^\sharp\left(\zeta_{|\mathbb{S}^{m-k}\times\{0_{k-1}\}}\right) \wedge u^\sharp\omega_0\right) = 0.$$

From the above, we may thus assume that ζ is of the form $p^\sharp\widetilde{\zeta}$ with $\widetilde{\zeta} \in C^\infty(\Lambda^{m-k}\mathbb{S}^{m-k})$, which we henceforth do.

Let us introduce the diffeomorphism $\psi : \left[(1-\varepsilon)\mathbb{S}^{m-k}\right] \times \mathbb{R}^{k-1} \to \mathbb{S}^{m-k} \times \mathbb{R}^{k-1}$,

$$\left[(1-\varepsilon)\mathbb{S}^{m-k}\right] \times \mathbb{R}^{k-1} \ni x = (x', x'') \xrightarrow{\psi} \left(\frac{x'}{|x'|}, x''\right) \in \mathbb{S}^{m-k} \times \mathbb{R}^{k-1},$$

and the projection

$$\pi : \mathbb{S}^{m-k} \times \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}, \ \mathbb{S}^{m-k} \times \mathbb{R}^{k-1} \ni x = (x', x'') \xrightarrow{\pi} x'' \in \mathbb{R}^{k-1}.$$

Observe that the orientation of $(1-\varepsilon)\mathbb{S}^{m-k}\times\{0_{k-1}\}$ is defined by the orientation of $\partial T(\varepsilon)$, while $\mathbb{S}^{m-k}$ is endowed with its standard orientation. Hence, the maps p and π preserve the orientation whereas ψ does not.

Let $v(x'')=P\left(\frac{x''}{\varepsilon}\right)$. Observe that

$$\int_{\mathbb{R}^{k-1}} v^\sharp\omega_0=(-1)^{m-k+1}. \tag{6.13}$$

Moreover, we have $\psi_\sharp u=\pi^\sharp v$ and $\psi_\sharp\zeta=\left(p^\sharp\widetilde{\zeta}\right)_{|\mathbb{S}^{m-k}\times\mathbb{R}^{k-1}}$.

By (6.13), the fact that $u^\sharp\omega_0=0$ in V and Fubini's theorem for differential forms [27, Lemma 6.32] we have

$$\begin{aligned}\int_{\partial T(\varepsilon)\setminus V}\zeta\wedge u^\sharp\omega_0&=\int_{[(1-\varepsilon)\mathbb{S}^{m-k}]\times\mathbb{R}^{k-1}}\zeta\wedge u^\sharp\omega_0=-\int_{\mathbb{S}^{m-k}\times\mathbb{R}^{k-1}}\psi_\sharp\left(\zeta\wedge\left(u^\sharp\omega_0\right)\right)\\&=-\int_{\mathbb{S}^{m-k}\times\mathbb{R}^{k-1}}p^\sharp\tilde{\zeta}\wedge\pi^\sharp\left(v^\sharp\omega_0\right)=-\int_{\mathbb{S}^{m-k}}\left(\int_{\mathbb{R}^{k-1}}v^\sharp\omega_0\right)\tilde{\zeta}=(-1)^{m-k}\int_{\mathbb{S}^{m-k}\times\{0_{k-1}\}}\zeta.\end{aligned}$$

In view of (6.7) and (6.8), this completes the proof of Lemma 6.1. □

In order to proceed with the proof of Theorem 1.3, let us recall the following quantitative version of Federer's approximation theorem, devised by Alberti, Baldo and Orlandi [2, Corollary 7.13].

6.2 Lemma. *Let R be an $(m-k+1)$-rectifiable current in an open set $\Omega\subset\mathbb{R}^m$, and ρ a function in Ω which is positive $\mathscr{H}^{m-k+1}$-a.e. Then there exist finitely many oriented $(m-k+1)$-discs E_j with centers x_j and radii $r_j<\rho(x_j)$ and an $(m-k+1)$-rectifiable current T such that*

1. $\partial R=\sum\partial E_j+\partial T$.
2. $\sum|E_j|\le\frac{1}{2^{m-k}}\mathbf{M}(R)$.
3. $\mathbf{M}(T)\le\left(1-\frac{1}{2^{m-k+2}}\right)\mathbf{M}(R)$.
4. *The balls $\overline{B}^m(x_j,2r_j)$ are pairwise disjoint and contained in Ω.*

6.3 Corollary. *Let N be an $(m-k+1)$-rectifiable current in $\Omega\subset\mathbb{R}^m$. Then there exists a sequence of finite families of oriented $(m-k+1)$-discs $(E_{j,l})_{j\in[\![1,N(l)]\!]}$, of centers $x_{j,l}$ and radii $r_{j,l}$, such that:*

1. $\partial N=\sum_l\sum_j\partial E_{j,l}$.
2. $\sum_j|E_{j,l}|\le\frac{1}{2^{m-k}}\left(1-\frac{1}{2^{m-k+2}}\right)^{l-1}\mathbf{M}(N)$, $\forall\, l\ge1$. *In particular,* $\sum_l\sum_j|E_{j,l}|\le4\mathbf{M}(N)$.
3. *For a fixed l, the balls $\overline{B}^m(x_{j,l},2r_{j,l})$ are pairwise disjoint and contained in Ω.*

Proof. We first apply Lemma 6.2 with $R=N$. This yields a family $(E_{j,1})$ and a reminder T_1. We next apply the lemma with $R=T_1$ and obtain a family $(E_{j,2})$, and so on. It is straightforward that the families obtained by this procedure have all the required properties. □

We obtain our next result by combining Lemma 6.1 and Lemma 6.2 and by adapting the proof of [2, Theorem 5.6].

6.4 Lemma. *Let R be an $(m-k+1)$-rectifiable current in a bounded open set $\Omega\subset\mathbb{R}^m$. Let $u\in W^{k-1,1}(\Omega;\mathbb{S}^{k-1})$ be smooth in the complement of a closed set $S\subset\Omega$ such that $\mathscr{H}^{m-k}(S)<\infty$. Then there exist an $(m-k+1)$-rectifiable current T in Ω and a map $v\in W^{k-1,1}(\Omega;\mathbb{S}^{k-1})$, smooth in the complement of a closed set $S'\subset\Omega$ such that $\mathscr{H}^{m-k}(S')<\infty$, satisfying:*

1. $Jv=Ju+\partial R-\partial T$.

2. $\|D^l u - D^l v\|_{L^1(\Omega)} \le C\mathbf{M}(R)$, $\forall\, l \in [\![0, k-1]\!]$.

3. $\mathbf{M}(T) \le \left(1 - \dfrac{1}{2^{m-k+2}}\right)\mathbf{M}(R)$.

Proof. For the convenience of the reader, we explain how the argument in [2, Proof of Theorem 5.6] adapts to our case. We divide the proof into several steps.

Step 1. Choice of ρ.
Since S is closed, for every $x \in \Omega \setminus S$ we can find $r = r(x) > 0$ such that $\overline{B}^m(x,r) \subset \Omega \setminus S$. Since u is smooth on the complement of S, $D^l u$ is bounded in $B^m(x,r)$ for every integer l. We let $\rho : \Omega \to [0, +\infty)$ be defined by $\rho(x) = 0$ for $x \in S$ and by

$$0 < \rho(x) \le \min\left\{\frac{r}{2}, \left(\frac{1}{\omega_{m-k+1}}\right)^{1/(m-k+1)}, \min_{1\le l\le k-1} \frac{1}{8\,\|D^l u\|^{1/l}_{L^\infty(B^m(x,r))}}\right\}, \quad \forall\ x \in \Omega \setminus S. \tag{6.14}$$

We next apply Lemma 6.2 with the above ρ and obtain finitely many discs E_j, with centers $x_j \in \Omega \setminus S$ and radii $r_j < \rho(x_j)$, and an $(m-k+1)$-rectifiable current T in Ω such that:

$$\partial R = \sum \partial E_j + \partial T, \tag{6.15}$$

$$\sum_j |E_j| \le \frac{1}{2^{m-k}}\mathbf{M}(R), \tag{6.16}$$

$$\mathbf{M}(T) \le \left(1 - \frac{1}{2^{m-k+2}}\right)\mathbf{M}(R), \tag{6.17}$$

and the balls $\overline{B}^m(x_j, 2r_j)$ are pairwise disjoint and contained in Ω.

Step 2. Construction of v.
For every j, we construct a dipole u_j associated to E_j as in Lemma 6.1 with the parameter $\sigma = 1$. Thus $Ju_j = \partial E_j$, there exists $Y_j \in \mathbb{S}^{k-1}$ such that $u_j = Y_j$ in $\mathbb{R}^m \setminus B^m(x_j, r_j)$, and

$$\left|D^l u_j(x)\right| \le \frac{C_l}{\operatorname{dist}(x, \partial E_j)^l}, \quad \forall x \in \mathbb{R}^m \setminus \partial E_j,\ \forall\, l \in \mathbb{N}. \tag{6.18}$$

Note that the choice of Y_j is not relevant in the above construction. Therefore, we may assume that $Y_j = u(y_j)$ for some $y_j \in B^m(x_j, 2r_j)$. Let $\theta \in C_c^\infty\left(B_2^m\right)$ be such that $0 \le \theta \le 1$ and $\theta \equiv 1$ in $\overline{B}_1^m$. Then we define $\theta_j(x) := \theta\left(\dfrac{x - x_j}{r_j}\right)$ and

$$W := \begin{cases} \theta_j u_j + (1-\theta_j)u & \text{in } B^m(x_j, 2r_j) \\ u & \text{in } \Omega \setminus \cup_j B^m(x_j, 2r_j) \end{cases}.$$

We have

$$\|u_j - u\|_{L^\infty(B^m(x_j,2r_j)\setminus B^m(x_j,r_j))} \le 4r_j \|Du\|_{L^\infty(B^m(x_j,2r_j))} < \frac{1}{2}. \tag{6.19}$$

Here, we have used the fact that $u_j \equiv Y_j$ in $B^m(x_j, 2r_j) \setminus B^m(x_j, r_j)$ and that $u(y_j) = Y_j$ for some $y_j \in B^m(x_j, 2r_j)$, and also the inequality

$$r_j < \frac{1}{8\|Du\|_{L^\infty(B^m(x_j,2r_j))}}.$$

In view of (6.19) and of the properties of θ, we have $|W| \ge 1/2$ and thus we may define the $\mathbb{S}^{k-1}$-valued map $v := \dfrac{W}{|W|}$.

Step 3. The map v has all the required properties.
On the one hand, v agrees with u_j in $B^m(x_j,r_j)$; on the other hand, we have $u \in W^{k-1,1}(\Omega;\mathbb{S}^{k-1})$. Therefore, the map v belongs to $W^{k-1,1}(\Omega;\mathbb{S}^{k-1})$. Next, v is smooth outside $S' := S \cup \cup_j \partial E_j$. Moreover, since the Jacobian of u is supported in S, and since $S \cap \cup_j \partial E_j = \emptyset$, we have

$$Jv = Ju + \sum_j Ju_j = Ju + \sum \partial E_j. \tag{6.20}$$

The first equality in (6.20) is easily obtained using (1.7) and a partition of the unit. We obtain items *1* and *3* by combining (6.15), (6.17) and (6.20).

It remains to prove item *2*. For $l \in [\![0,k-1]\!]$ we have

$$\left\|D^l u - D^l v\right\|_{L^1(\Omega)} = \sum_j \left\|D^l u - D^l v\right\|_{L^1(B^m(x_j,2r_j))} \le \sum_j \left(\left\|D^l u\right\|_{L^1(B^m(x_j,2r_j))} + \left\|D^l v\right\|_{L^1(B^m(x_j,2r_j))}\right).$$

When $l=0$, we have $|v(x)| = |u(x)| = 1$ a.e., and therefore

$$\|u-v\|_{L^1(\Omega)} \le C\sum_j r_j^m \le C\sum_j |E_j|^{m/(m-k+1)} \le C\sum |E_j| \le C\mathbf{M}(R). \tag{6.21}$$

Here, we have used (6.16) and the fact that (by (6.14)) we have $|E_j| \le 1$.

We now assume that $1 \le l \le k-1$. On the one hand, we have

$$\left\|D^l u\right\|_{L^1(B^m(x_j,2r_j))} \le C r_j^m \left\|D^l u\right\|_{L^\infty(B^m(x_j,2r_j))} \le C r_j^{m-l} \le C|E_j|^{(m-l)/(m-k+1)},$$

by definition of ρ. As above, we find that $\left\|D^l u\right\|_{L^1(B^m(x_j,2r_j))} \le C|E_j|$, and thus

$$\sum_j \left\|D^l u\right\|_{L^1(B^m(x_j,2r_j))} \le C\mathbf{M}(R). \tag{6.22}$$

On the other hand, by the Faà-Di Bruno formula,

$$\left|D^l v(x)\right| \le C\sum_{s=1}^{l} \sum_{t_1+\ldots+t_s=l} \left|D^{t_1}W(x)\right| \ldots \left|D^{t_s}W(x)\right|. \tag{6.23}$$

For every $t \in [\![1,l]\!]$ and $x \in B^m(x_j,2r_j)$, the Leibniz formula implies

$$\left|D^t W(x)\right| \le \left|D^t u(x)\right| + \left|D^t[\theta_j(u_j-u)](x)\right| \le \left|D^t u(x)\right| + C\sum_{h=0}^{t} \left|D^{t-h}\theta_j(x)\right| \left[\left|D^h u_j(x)\right| + \left|D^h u(x)\right|\right].$$

In view of the definitions of θ_j and ρ we obtain, *via* (6.18):

$$\left|D^t W(x)\right| \le C\sum_{h=0}^{t} \frac{1}{r_j^{t-h}} \left[\frac{1}{r_j^h} + \left|D^h u_j(x)\right|\right] \le C\sum_{h=0}^{t} \frac{1}{r_j^{t-h}} \frac{1}{\operatorname{dist}(x,\partial E_j)^h}.$$

Inserting this inequality into (6.23), we obtain

$$\left|D^l v(x)\right| \le C\sum_{s=1}^{l} \sum_{t_1+\ldots+t_s=l} \sum_{\substack{0\le h_p\le t_p \\ \forall p\in[\![1,s]\!]}} \frac{1}{r_j^{l-h_1-\ldots-h_s}} \frac{1}{\operatorname{dist}(x,\partial E_j)^{h_1+\ldots+h_s}}, \quad \forall x \in B^m(x_j,2r_j).$$

By scaling, we easily find that

$$\int_{B^m(x_j,2r_j)} \frac{dx}{\operatorname{dist}(x,\partial E_j)^{h_1+\ldots+h_s}} \le C r_j^{m-h_1-\ldots-h_s},$$

and thus

$$\int_{B^m(x_j,2r_j)} \left|D^l v(x)\right| dx \le C r_j^{m-l} \le C|E_j|^{(m-l)/(m-k+1)} \le C|E_j|.$$

Finally, with the help of (6.16) we have proved that

$$\left\|D^l u - D^l v\right\|_{L^1(\Omega)} \le C\sum_j |E_j| \le C\mathbf{M}(R). \tag{6.24}$$

Item *2* follows by combining (6.21) and (6.24). This completes the proof of Lemma 6.4. □

Using Lemma 6.4, we obtain the flat[27] version of Theorem 1.3.

6.5 Lemma. *Let Ω be a bounded open subset in $\mathbb{R}^m$ and let M be the boundary of an $(m-k+1)$-rectifiable current N in Ω. Then there exists $u \in W^{k-1,1}(\Omega;\mathbb{S}^{k-1})$ such that $Ju = M$ and $|u|_{W^{k-1,1}} \leq C\mathbf{M}(N)$.*

This follows from Lemma 6.4 and an iterative procedure which is exactly the same as the one in [2, proof of Theorem 5.6]; we omit the proof.

Proof of Theorem 1.3 when $p = 1$. In what follows, we denote by S^m_r the m-dimensional sphere $S(0,r) \subset \mathbb{R}^{m+1}$.

Let $\Omega := \left\{x \in \mathbb{R}^{m+1}; 1/2 < |x| < 3/2\right\}$. For every $l \geq 1$, $\varpi \in C^\infty_c(\Lambda^l\Omega)$, and $r \in (1/2,3/2)$, we define $\varpi_{>S^m_r} \in C^\infty(\Lambda^{l-1}\mathbb{S}^m)$ as follows:

$$\varpi_{>S^m_r}(x)(\tau_1,\dots,\tau_{l-1}) = \varpi(rx)(\tau_1,\dots,\tau_{l-1},x), \quad \forall x \in \mathbb{S}^m,\ \tau_j \in T_x\mathbb{S}^m,\ j \in [\![1,l-1]\!].$$

If T is an $(l-1)$-current in $\mathbb{S}^m$, then we define the l-current $T \times_r \left(\frac{1}{2},\frac{3}{2}\right)$ in Ω through the formula

$$\left\langle T \times_r \left(\frac{1}{2},\frac{3}{2}\right), \varpi \right\rangle := \int_{1/2}^{3/2} r^{l-1} \left\langle T, \varpi_{>S^m_r} \right\rangle dr, \quad \forall \varpi \in C^\infty_c(\Lambda^l\Omega).$$

Fix $\rho \in C^\infty_c(1/2,3/2)$ such that $\int \rho = 1$. We introduce $X(x) = \frac{x}{|x|}$, $\forall x \in \mathbb{R}^{m+1} \setminus \{0\}$. For every $\zeta \in C^\infty(\Lambda^{m-k}\mathbb{S}^m)$, we define $\alpha = \alpha(\zeta) \in C^\infty_c(\Lambda^{m-k+1}\Omega)$ by

$$\alpha(rx) = \rho(r)\widetilde{\zeta}(rx) \wedge dr, \quad \forall x \in \mathbb{S}^m,\ r \in (1/2,3/2),$$

where $\widetilde{\zeta} = X^\sharp\zeta$. Then we claim that

$$(d\alpha)_{>S^m_r} = \frac{1}{r^{m-k+1}}\rho(r)d\zeta. \tag{6.25}$$

Here is a proof of this fact. We start from the identity $d\alpha = \frac{1}{r^{m-1}}\rho(r)X^\sharp d\zeta \wedge dr$. If $\tau_1,\dots,\tau_{m-k+1} \in T_x\mathbb{S}^m$, then

$$\begin{aligned} d\alpha(rx)(\tau_1,\dots,\tau_{m-k+1},x) &= \rho(r)(d\widetilde{\zeta}(rx) \wedge dr)(\tau_1,\dots,\tau_{m-k+1},x) \\ &= \rho(r)d\widetilde{\zeta}(rx)(\tau_1,\dots,\tau_{m-k+1})dr(x) \\ &= \frac{1}{r^{m-k+1}}\rho(r)d\zeta(x)(\tau_1,\dots,\tau_{m-k+1}). \end{aligned} \tag{6.26}$$

Indeed, in (6.26), the next to the last equality relies on the fact that $dr(\tau_j) = 0$, $\forall j \in [\![1,m-k+1]\!]$. In order to justify the last equality in (6.26), observe that $dX(rx)(\tau_j) = \frac{\tau_j}{r}$ (since $\langle \tau_j, x\rangle = 0$). This proves (6.25).

Let N be an $(m-k+1)$-rectifiable current in $\mathbb{S}^m$. By Lemma 6.6 below, $K := N \times_r \left(\frac{1}{2},\frac{3}{2}\right)$ is an $(m-k+2)$-rectifiable current in Ω, of mass comparable to the one of N. We apply Lemma 6.5 in Ω to the current $(-1)^{m-k+2}K$. Then there exists $v \in W^{k-1,1}(\Omega;\mathbb{S}^{k-1})$ such that $|v|_{W^{k-1,1}(\Omega)} \leq C\mathbf{M}(N)$ and such that for every $\varpi \in C^\infty_c(\Lambda^{m-k+1}\Omega)$,

$$\int_\Omega d\varpi \wedge v^\sharp\omega_0 = \int_{1/2}^{3/2} r^{m-k+1} \left\langle N, (d\varpi)_{>S^m_r} \right\rangle dr. \tag{6.27}$$

In the case where $\varpi = \alpha$, (6.25) and (6.27) give

$$\int_\Omega d\alpha \wedge v^\sharp\omega_0 = \langle N, d\zeta\rangle. \tag{6.28}$$

[27] I.e., with $\mathbb{S}^m$ replaced by Ω.

We next note the following identity:[28] if we let

$$h_r(x) = rx,\ v_r(x) = v(rx) = h_r^\sharp v(x), \quad \forall\, x \in \mathbb{S}^m,\, r \in \left(\frac{1}{2}, \frac{3}{2}\right),$$

then

$$d\alpha \wedge v^\sharp \omega_0(rx) = (-1)^{m-k+1} \rho(r) dr \wedge \left(d\widetilde{\zeta} \wedge v^\sharp \omega_0\right)_{|\mathbb{S}_r^m}(rx). \tag{6.29}$$

Moreover,

$$\int_{\mathbb{S}_r^m} d\widetilde{\zeta} \wedge v^\sharp \omega_0 = \int_{\mathbb{S}^m} h_r^\sharp (d\widetilde{\zeta} \wedge v^\sharp \omega_0) = \int_{\mathbb{S}^m} d\zeta \wedge v_r^\sharp \omega_0. \tag{6.30}$$

By (6.28), (6.29) and the Fubini theorem for forms, we have

$$\langle N, d\zeta \rangle = (-1)^{m-k+1} \int_{1/2}^{3/2} \rho(r) \left(\int_{\mathbb{S}^m} d\zeta \wedge \left((v_r)^\sharp \omega_0 \right) \right) dr. \tag{6.31}$$

The left-hand side of (6.31) does not depend on ρ. This proves that the function

$$(1/2, 3/2) \ni r \mapsto \int_{\mathbb{S}^m} d\zeta \wedge \left((v_r)^\sharp \omega_0 \right)$$

is constant a.e.

On the other hand, for a.e. $r \in (1/2, 3/2)$, $v(r\,\cdot)$ belongs to $W^{k-1,1}(\mathbb{S}^m; \mathbb{S}^{k-1})$, and in addition we may find a positive measure subset $A \subset (1/2, 3/2)$ such that $|v_r|_{W^{k-1,1}(\mathbb{S}^m)} \le C\mathbf{M}(N)$ for every $r \in A$. By the above and the separability of $C^1(\Lambda^{m-k}\mathbb{S}^m)$, one can find $r \in A$ such that $u := v(r\,\cdot)$ satisfy

$$(-1)^{m-k+1} \int_{\mathbb{S}^m} d\zeta \wedge (u^\sharp \omega_0) = \langle N, d\zeta \rangle \quad , \quad \forall\, \zeta \in C^\infty(\Lambda^{m-k}\mathbb{S}^m).$$

This u has all the required properties. □

6.6 Lemma. *Assume that N is an $(l-1)$-rectifiable current in $\mathbb{S}^m$. Then $K := N \times_r \left(\frac{1}{2}, \frac{3}{2}\right)$ is an l-rectifiable current in Ω and $\mathbf{M}(K) \sim \mathbf{M}(N)$.*

Proof. Let $(\mathcal{M}, \xi, \theta)$ be the triple defining N. Define $\widetilde{\mathcal{M}} = \{rx; x \in \mathcal{M}, 1/2 < r < 3/2\}$. Then clearly $\widetilde{\mathcal{M}}$ is an l-rectifiable part of Ω, and for $\mathcal{H}^{l-1}$-a.e. $x \in \mathcal{M}$ and for every $r \in (1/2, 3/2)$

$$T_{rx}\widetilde{\mathcal{M}} \text{ is spanned by the orthogonal spaces } T_x\mathcal{M} \text{ and } \mathbb{R}x. \tag{6.32}$$

We also set, with r and x as above, $\widetilde{\xi}(rx) = \xi(x)$, $\widetilde{\theta}(rx) = \theta(x)$ and $X(rx) = \dfrac{x}{|x|}$.

With this notation, we have for $\varpi \in C_c^\infty(\Lambda^l \Omega)$:

$$\langle K, \varpi \rangle = \int_{1/2}^{3/2} r^{l-1} \int_{\mathcal{M}} \langle \widetilde{\xi}(rx), \varpi(rx)(\cdot, x) \rangle \theta(x)\, d\mathcal{H}^{l-1}(x) dr = \int_{\widetilde{\mathcal{M}}} \langle \widetilde{\xi} \wedge X, \varpi \rangle \widetilde{\theta}\, d\mathcal{H}^l.$$

In view of (6.32), the l-current K is given by the triple $\left(\widetilde{\mathcal{M}}, \widetilde{\xi} \wedge X, \widetilde{\theta}\right)$, and we also have $\mathbf{M}(K) \sim \mathbf{M}(N)$. □

[28] Whose proof is similar to the one of (6.25) and left to the reader.

6.2 The case $k = 2$: geometrical approach

Given an $(m-1)$-rectifiable current N in $\mathbb{S}^m$, we have to construct a map $u \in \cap_{p\geq 1} W^{1/p,p}(\mathbb{S}^m;\mathbb{S}^1)$ such that $Ju = \partial N$ and

$$|u|^p_{W^{1/p,p}} \leq C_p \mathbf{M}(N), \quad \forall\, p \geq 1. \tag{6.33}$$

This has to hold for every $p \geq 1$. In view of Lemma 4.5, if $u \in W^{1/p,p}(\mathbb{S}^m;\mathbb{S}^1)$ for some $p > 1$ and if (6.33) holds for this p, then for every $q > p$ we have $u \in W^{1/q,q}(\mathbb{S}^m;\mathbb{S}^1)$ and (6.33) holds. Therefore, it suffices to fix some $p_1 \in (1,2)$ and to construct u such that $u \in \cap_{1\leq p\leq p_1} W^{1/p,p}(\mathbb{S}^m;\mathbb{S}^1)$ and

$$|u|^p_{W^{1/p,p}} \leq C_p \mathbf{M}(N), \quad \forall\, p \in [1,p_1]. \tag{6.34}$$

We start by establishing the appropriate variant of Lemma 6.1.

6.7 Lemma. *Let $E \subset \mathbb{R}^m$ be an oriented $(m-1)$-disc with center x and radius r. Let $\sigma > 0$. Then there exists $u : \mathbb{R}^m \to \mathbb{S}^1$ such that:*

1. $u - 1 \in \cap_{p\geq 1} W^{1/p,p}(\mathbb{R}^m) \cap C^\infty(\mathbb{R}^m \setminus \partial E)$.
2. $Ju = \partial E$.
3. $u = 1$ *in* $\mathbb{R}^m \setminus \overline{B}(x,r)$ *and*

$$|\{y \in \mathbb{R}^m : u(y) \neq 1\}| \leq \sigma. \tag{6.35}$$

4. *We have*

$$|u|^p_{W^{1/p,p}} \leq C_p |E|, \ \forall\, p \geq 1. \tag{6.36}$$

Proof. In view of Lemma 6.1 and of its proof, it suffices to prove that the map u constructed in that lemma (with $Y = 1$) satisfies item *4* when $p > 1$. By scaling, we may assume that the diameter of E is 2, and thus $u - 1$ is supported in $\overline{B}(x,1)$. Therefore, for $q \geq 1$ we have

$$\|u-1\|_{W^{1,q}} \leq C_q \|\nabla u\|_{L^q}. \tag{6.37}$$

Fix $p > 1$ and consider $q \in (1, \min\{p,2\})$. Define $r \in (q,\infty)$ by the formula

$$\frac{p-1}{r} + \frac{1}{q} = 1. \tag{6.38}$$

By Lemma 4.1 *3* combined with (6.37), we have

$$|u|^p_{W^{1/p,p}} \leq C\|u-1\|_{W^{1,q}} \|u-1\|^{p-1}_{L^r} \leq CC_q \|\nabla u\|_{L^q} \|u-1\|^{p-1}_{L^r}. \tag{6.39}$$

Item *3* implies that

$$\|u-1\|_{L^r} \leq C\sigma^{1/r}. \tag{6.40}$$

On the other hand, estimate (6.5) with $l = 1$ combined with the fact that $q < 2$ leads to[29]

$$\|\nabla u\|^q_{L^q} \leq C\sigma^{1-q}. \tag{6.41}$$

We obtain item *4* by combining (6.38)-(6.41). □

We next state and prove the remaining technical ingredients required to glue dipoles by keeping control of the $W^{1/p,p}$ norm.

6.8 Lemma. *Let $u, v : \mathbb{R}^m \to \mathbb{S}^1$ be such that $u - 1, v - 1 \in W^{1,1}$. Then $J(uv) = Ju + Jv$.*

[29] The argument is the same as the one used in the proof of (6.6).

Proof. This follows by combining the definition (1.7) with the identity

$$(uv)^\sharp \omega_0 = u^\sharp \omega_0 + v^\sharp \omega_0. \qquad \square$$

6.9 Lemma. *Let $u, v : \mathbb{R}^m \to \mathbb{S}^1$ be such that $u-1, v-1 \in W^{1,1} \cap W^{1/p,p}(\mathbb{R}^m)$ for some $p > 1$. Then*

1. $|uv - u|_{W^{1,1}} \le |v|_{W^{1,1}} + 2|u|_{W^{1,1}([v\neq 1])}$.
2. $|uv - u|_{W^{1/p,p}} \le |v|_{W^{1/p,p}} + 2\left(\iint_{v(x)\neq 1} \frac{|u(x)-u(y)|^p}{|x-y|^{m+1}}\,dxdy\right)^{1/p}$.

Proof. Item *1* follows from

$$|\nabla(uv-u)| \le |\nabla v| + |v-1||\nabla u| \le |\nabla v| + 2\mathbb{1}_{[v\neq 1]}|\nabla u|.$$

Item *2* is a consequence of

$$\begin{aligned}|(u(x)v(x)-u(x)) - (u(y)v(y)-u(y))| &= |u(y)(v(x)-v(y)) + (u(x)-u(y))(v(x)-1)| \\ &\le |v(x)-v(y)| + |v(x)-1||u(x)-u(y)| \\ &\le |v(x)-v(y)| + 2\mathbb{1}_{[v(x)\neq 1]}|u(x)-u(y)|.\end{aligned} \qquad \square$$

6.10 Lemma. *Let $1 < q < 2$ and $u : \mathbb{R}^m \to \mathbb{S}^1$ be such that $u - 1 \in W^{1,q}(\mathbb{R}^m)$. Consider the L^q-modulus of continuity of ∇u:*

$$\omega(\delta) := \sup\left\{\|\nabla u\|_{L^q(A)};\, A \subset \mathbb{R}^m, |A| \le \delta\right\}.$$

Then there are constants C_p, $1 < p \le q$, independent of u, such that

$$I_p(A) := \iint_{x\in A} \frac{|u(x)-u(y)|^p}{|x-y|^{m+1}}\,dxdy \le C_p\left(|A| + [\omega(|A|)]^p\,|A|^{(q-p)/q}\right), \quad \forall A \subset \mathbb{R}^m. \tag{6.42}$$

In particular, given $\varepsilon > 0$ there exists some $\delta > 0$ such that, if $|A| \le \delta$, then $I_p(A) \le C_p\varepsilon$, $\forall p \in (1, q]$.

Proof. By a standard approximation argument, we may assume u smooth and bounded by 1 (but not necessarily circle-valued).

We split $I_p(A) = J_p(A) + K_p(A)$, where

$$J_p(A) = \iint_{x\in A, |y-x|\ge 1} \ldots,\ K_p(A) = \iint_{x\in A, |y-x|<1} \ldots$$

On the one hand, we have

$$J_p(A) \le 2^p|A| \int_{|z|\ge 1} \frac{1}{|z|^{m+1}}\,dz = C_p|A|. \tag{6.43}$$

On the other hand, the inequality

$$|u(x+z) - u(x)|^p \le |z|^p \int_0^1 |\nabla u(x+tz)|^p\,dt$$

leads to

$$\begin{aligned}K_p(A) &\le \int_0^1 \int_{x\in A} \int_{z\in B^m} \frac{|\nabla u(x+tz)|^p}{|z|^{m-p+1}}\,dzdxdt \\ &\le \int_0^1 t^{1-p} \int_{w\in B^m} \frac{1}{|w|^{m-p+1}} \int_{x\in w+A} |\nabla u(x)|^p\,dxdzdt \\ &\le C_p \sup_w \|\nabla u\|^p_{L^p(w+A)} \le C_p\,[\omega(|A|)]^p\,|A|^{(q-p)/q}.\end{aligned} \tag{6.44}$$

We obtain (6.42) by combining (6.43) with (6.44). $\square$

6.11 Lemma. *Let $1<q<2$. Consider a finite collection (E_j) of oriented $(m-1)$-discs of centers x_j and radii r_j in $\mathbb{R}^m$, such that the balls $\overline{B}^m(x_j,2r_j)$ are pairwise disjoint. Let $u:\mathbb{R}^m\to\mathbb{S}^1$ be such that $u-1\in W^{1,q}(\mathbb{R}^m)$. Let $\varepsilon>0$. Then there exists a map $v:\mathbb{R}^m\to\mathbb{S}^1$ such that:*

1. $v-1\in W^{1,r}(\mathbb{R}^m)$, $\forall\, r\in[1,2)$.
2. $Jv=\sum\partial E_j$.
3. $|v|^p_{W^{1/p,p}}\le C_p\sum|E_j|$, $\forall\, p\in[1,2]$.
4. $|uv-u|_{W^{1/p,p}}\le|v|_{W^{1/p,p}}+C_p\varepsilon^{1/p}$, $\forall\, p\in[1,q]$.

Proof. Associate to each j and $\sigma>0$ a map $v_{j,\sigma}$ as in Lemma 6.7. We will take $v=\prod v_{j,\sigma}$ for a convenient σ. Note that, by the proof of Lemma 6.7, v satisfies item *1*. On the other hand, item *2* follows by combining Lemma 6.7 *2* with Lemma 6.8.

We next turn to *3*. Let $A_{j,\sigma}\subset B(x_j,r_j)$ denote the set of measure $\le\sigma$ in which $v_{j,\sigma}\neq 1$. Set $B_j:=B(x_j,2r_j)$. Let $C:=\mathbb{R}^m\setminus\cup_j B_j$. Set also $w_p(x,y)=\dfrac{|v(x)-v(y)|^p}{|x-y|^{m+1}}$ and $f(x,y)=\dfrac{1}{|x-y|^{m+1}}$. The identity

$$|v|^p_{W^{1/p,p}}=\sum_j\iint_{B_j\times(B_j\cup C)}w_p+\sum_j\iint_{A_{j,\sigma}\times(\cup_{k\neq j}B_k)}w_p+\sum_j\iint_{(B_j\setminus A_{j,\sigma})\times(\cup_{k\neq j}A_{k,\sigma})}w_p$$
$$+\iint_{C\times(\cup_j A_{j,\sigma})}w_p$$

leads to the estimate

$$|v|^p_{W^{1/p,p}}\le\sum_j|v_{j,\sigma}|^p_{W^{1/p,p}}+2^{p+1}\sum_j\underbrace{\iint_{A_{j,\sigma}\times(C\cup\cup_{k\neq j}B_k)}f}_{L_{j,\sigma}}.$$

We obtain item *3 via* Lemma 6.7 *4* combined with the fact that $L_{j,\sigma}\to 0$ as $\sigma\to 0$.

Finally, item *4* with $p>1$ follows from Lemma 6.10 combined with Lemma 6.9 *2* and with the fact that the set $[v\neq 1]$ can be chosen arbitrarily small. The case $p=1$ follows immediately from Lemma 6.9 *1*. □

First proof of Theorem 1.3 in $\mathbb{R}^m$ for $k=2$. As explained at the beginning of this section, we fix $p_1\in(1,2)$. Consider the oriented $(m-1)$-discs $E_{j,l}$ as in Corollary 6.3. Let $u_0\equiv 1$. By Lemma 6.11, for every sequence (ε_l) of positive numbers we may construct inductively maps u_l such that $u_l-1\in W^{1,r}(\mathbb{R}^m)$, $\forall\, 1\le r<2$, $Ju_l=\sum_j\partial E_{j,l}$, $|u_l|^p_{W^{1/p,p}}\le C_p\sum_j|E_{j,l}|$, $\forall\, p\in[1,2]$, and

$$|u_0\dots u_l-u_0\dots u_{l-1}|_{W^{1/p,p}}\le|u_l|_{W^{1/p,p}}+C_p\varepsilon_l^{1/p},\quad\forall\, p\in[1,p_1].$$

Moreover,

$$\|u_0\dots u_l-1\|^p_{L^p(\mathbb{R}^m)}\le C\sum_l\sum_j|E_{j,l}|^{m/(m-1)}\le C(\sum_l\sum_j|E_{j,l}|)^{m/(m-1)}.\tag{6.45}$$

Using Corollary 6.3 *2* and choosing $\varepsilon_l=2^{-l}\mathbf{M}(N)$, we find that the product $\prod_j u_j$ converges in $W^{1/p,p}_{loc}(\mathbb{R}^m;\mathbb{S}^1)$, $\forall\, p\in[1,p_1]$, to some u such that $u-1\in W^{1/p,p}(\mathbb{R}^m)$, $Ju=\partial N$ and $|u|^p_{W^{1/p,p}}\le C_p\mathbf{M}(N)$. □

Proof of Theorem 1.3 for $k=2$. Let N be an $(m-1)$-rectifiable current in $\mathbb{S}^m$. Let $\Omega=\{x\in\mathbb{R}^{m+1};1/2<|x|<3/2\}$ and $K=N\times_r\left(\frac{1}{2},\frac{3}{2}\right)$. By the above, there exists a map $v\in\cap_{p\ge1}W^{1/p,p}(\Omega;\mathbb{S}^1)$ such that $Jv=(-1)^m\partial K$ and

$$|v|^p_{W^{1/p,p}}\le C_p\mathbf{M}(K)\le C'_p\mathbf{M}(N),\quad\forall\, p\ge 1.\tag{6.46}$$

Let $v_r = v(r\cdot)$, $1/2 < r < 3/2$. We have the following inequality:[30]

$$\int_{1/2}^{3/2} |v_r|^p_{W^{s,p}(\mathbb{S}^m)}\, dr \le C_{s,p} |v|^p_{W^{s,p}(\Omega)}, \quad \forall s \ge 0, \forall p \in [1,\infty). \tag{6.47}$$

We will also use the following elementary fact: if $(f_n)_{n\ge 0}$ is a sequence of non-negative integrable functions in (a,b), then there exists a set $A \subset (a,b)$ of measure $\ge (b-a)/2$ such that

$$f_n(t) \le 2^{n+2} \fint_a^b f_n, \quad \forall n \in \mathbb{N}, \forall t \in A. \tag{6.48}$$

The proof of Theorem 1.3 for $p = 1$ implies that

$$Jv_r = \partial N \quad \text{for a.e. } r \in (1/2, 3/2). \tag{6.49}$$

Let $p_0 = 1$ and consider a sequence $(p_n)_{n\ge 1} \subset (1,\infty)$ such that $p_n \searrow 1$. By (6.46)-(6.49),[31] there exists some r such that $u := v_r$ satisfies:

1. $u \in \cap_{n\ge 0} W^{1/p_n, p_n}(\mathbb{S}^m; \mathbb{S}^1)$.
2. $Ju = \partial N$.
3. $|u|^{p_n}_{W^{1/p_n,p_n}} \le K_n \mathbf{M}(N)$, $\forall n \in \mathbb{N}$.

By combining item *3* with Lemma 4.5, we find that u has all the properties required by Theorem 1.3. □

6.3 The case $k = 2$: analytical approach

As explained in the previous section, it suffices to establish Theorem 1.3 in a bounded domain Ω in an Euclidean space. The case of maps defined in $\mathbb{S}^m$ follows from this special case. Let $\Omega \subset \mathbb{R}^m$ be smooth bounded. We consider a slightly larger smooth bounded open set, U, whose role is to enable the definition of a convenient class of almost smooth maps. Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. By reflections across $\partial\Omega$, we may extend u to U, with norm control. Using, in U, the density result of Bethuel and Zheng [8, proof of Theorem 1], we obtain the following: the class

$$\mathscr{R} = \left\{ u \in C^\infty(\Omega \setminus \Sigma(u)) \cap W^{1,1}(\Omega; \mathbb{S}^1);\ \left| D^l u(x) \right| \le \frac{C_l(u)}{[\delta_u(x)]^l}, \forall l \in \mathbb{N} \right\}$$

is dense in $W^{1,1}(\Omega)$. Here, $\Sigma(u)$ is a smooth $(m-2)$-submanifold of U, and δ_u denotes the distance to $\Sigma(u)$.

It will be convenient to consider the more general class $\widetilde{\mathscr{R}}$, defined as $\mathscr{R}$, except that we allow $\Sigma(u)$ to be a finite union of $(m-2)$-submanifolds of U. It is clear that $\widetilde{\mathscr{R}}$ is an algebra; in particular, the product of two maps in $\mathscr{R}$ belongs to $\widetilde{\mathscr{R}}$. Using the density of the class $\mathscr{R}$ and a straightforward Cauchy sequences argument, we obtain the following result.

6.12 Lemma. *Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. Then there exists a sequence $(v_j)_{j\ge 0} \subset \widetilde{\mathscr{R}}$ such that:*

1. *$v_1 \dots v_j \to u$ in $W^{1,1}$ as $j \to \infty$.*
2. *$\|\nabla v_j\|_{L^1} \le 2^{-j} \|\nabla u\|_{L^1}$, $\forall j \ge 0$.*

By the above, we have $Ju = \sum Jv_j$, the series being convergent in $(W^{1,\infty}(\Lambda^{m-2}\Omega))^*$. We may now explain our strategy for proving Theorem 1.3. Let N be a rectifiable $(m-1)$-current in Ω. By [2, Theorem 5.6], we may find some $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ such that $Ju = \partial N$ and $\|\nabla u\|_{L^1} \le C\mathbf{M}(N)$. Now comes the main idea (Lemma 6.15 below): we replace v_j by some w_j such that $Jw_j = Jv_j$ and such that the product $\prod_j w_j$ converges in $W^{1/p,p}$ for every $p \ge 1$, with norm control. Note the difference with Lemma 6.12 *2*, which provides control only for the $W^{1,1}$ norm. Once Lemma 6.15 is established, we prove that the map $w = \prod_j w_j$ satisfies $Jw = \partial N$ and all the required estimates.

We now proceed to the proofs. We start by recalling the following result [2, Corollary 3.9].

[30] For the flat case, see [1, Example 7.27], [1, Lemma 7.44]. The argument there adapts to manifolds.

[31] Inequality (6.48) is applied with $(a,b) = (1/2, 3/2)$ and $f_n(r) = |v_r|^{p_n}_{W^{1/p_n,p_n}(\mathbb{S}^m)}$.

6.13 Lemma. *Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be Lipschitz. Let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$. Then $J(\varphi \circ u) = \deg \varphi \, Ju$.*

For $\varepsilon \in (0,2\pi)$ and $\theta \in \mathbb{R}$, we let

$$\varphi_{\varepsilon,\theta} : \mathbb{S}^1 \to \mathbb{S}^1, \ \varphi_{\varepsilon,\theta}\left(e^{\imath\psi}\right) = \begin{cases} e^{2\imath\pi(\psi-\theta)/\varepsilon}, & \text{if } \psi \in [\theta,\theta+\varepsilon] \\ 1, & \text{if } \psi \in [\theta+\varepsilon,\theta+2\pi] \end{cases}.$$

We collect some straightforward properties of $\varphi_{\varepsilon,\theta}$.

6.14 Lemma. *We have:*

1. *$\varphi_{\varepsilon,\theta}$ is Lipschitz of constant $2\pi/\varepsilon$ and of degree 1.*
2. *If $z_1, z_2 \in \mathbb{S}^1$ and $p \in [1,\infty)$, then*

$$\int_0^{2\pi} |\varphi_{\varepsilon,\theta}(z_1) - \varphi_{\varepsilon,\theta}(z_2)|^p \, d\theta \le C_p \min\left\{\varepsilon, \frac{|z_1 - z_2|^p}{\varepsilon^{p-1}}\right\}. \tag{6.50}$$

Proof. Only (6.50) requires a proof. Let $I_\varepsilon(z) = \{w \in \mathbb{S}^1; d(z,w) < \varepsilon\}$; here, d denotes the geodesic distance in $\mathbb{S}^1$. If $e^{\imath\theta} \notin I_\varepsilon(z_1) \cup I_\varepsilon(z_2)$, then $\varphi_{\varepsilon,\theta}(z_1) = \varphi_{\varepsilon,\theta}(z_2) = 1$; therefore, the integral in (6.50) has to be considered only over $I_\varepsilon(z_1) \cup I_\varepsilon(z_2)$. On the other hand, we have

$$|\varphi_{\varepsilon,\theta}(z_1) - \varphi_{\varepsilon,\theta}(z_2)|^p \le \min\left\{2^p, \frac{2^p \pi^p d^p(z_1,z_2)}{\varepsilon^p}\right\}. \tag{6.51}$$

We obtain the conclusion by integrating (6.51) over $I_\varepsilon(z_1) \cup I_\varepsilon(z_2)$. □

6.15 Lemma. *Let $v \in \widetilde{\mathscr{R}}$ and $\delta > 0$. Then there are constants C_p, $\forall\, p \in [1,\infty)$, independent of v and δ, and a Lipschitz map $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ of degree 1 such that $w := \varphi \circ v$ satisfies:*

1. $|w|^p_{W^{1/p,p}} \le C_p \|\nabla v\|_{L^1}$.
2. $|\{w \ne 1\}| \le \delta$.

A word about the conclusion of Lemma 6.15. Unlike the case $k \ge 3$, one cannot rely on the Gagliardo-Nirenberg inequalities. Indeed, the inclusion $W^{1,1} \cap L^\infty \subset W^{1/p,p}$ is wrong for every $p > 1$.[32] Lemma 6.15 essentially asserts the following unexpected result: let $v : \Omega \to \mathbb{S}^1$ be slightly better then $W^{1,1}$. Then we may rearrange the values of v (that is, consider $w = \varphi(v)$, with φ a smooth homeomorphism of $\mathbb{S}^1$ into itself) such that the new map w is controlled in $W^{1/p,p}$ for every $p > 1$, with a control depending only on p and $\|\nabla v\|_{L^1}$, but not on higher order norms of v.[33] By Lemma 6.13 and Lemma 6.14, we have $Jw = Jv$. Though we state and prove Lemma 6.15 for circle-valued maps, it will be transparent from the proof that the target could be any sphere.

Proof. Let $\varepsilon \in (0,2\pi)$ and $w_{\varepsilon,\theta} := \varphi_{\varepsilon,\theta} \circ v$. We will establish, for sufficiently small ε, the following estimates:

$$\int_0^{2\pi} |\{w_{\varepsilon,\theta} \ne 1\}| \, d\theta = \varepsilon |\Omega|. \tag{6.52}$$

$$\int_0^{2\pi} |w_{\varepsilon,\theta}|^p_{W^{1/p,p}} \, d\theta \le K_p \left(C(v)\sqrt{\varepsilon} + \|\nabla v\|_{L^1}\right), \quad \forall\, p \in [1,\infty). \tag{6.53}$$

[32] This is well-known to the experts. Here is an explicit example. Let $\rho = \mathbb{1}_{[-1/2,1/2]}$. If $A \subset \mathbb{R}$ is an interval, let $\widetilde{A}$ define the interval with the same center as A and twice longer than A. Consider, in $(0,1)$ and for sufficiently large j, intervals A_j such that the $\widetilde{A_j}$'s are mutually disjoint and $|A_j| = 1/j^2$. Let $\varepsilon_j = \frac{1}{j^2} e^{-e^j}$. Then $u := \sum \frac{1}{j^2} \mathbb{1}_{A_j} * \rho_{\varepsilon_j}$ belongs to $W^{1,1}(0,1)$, but to none of the spaces $W^{1/p,p}(0,1)$ with $p > 1$.

[33] The map φ considered in the proof of Lemma 6.15 is not a diffeomorphism, but it is easy to slightly modify it and transform it into a diffeomorphism.

Taking these two estimates for granted, we proceed as follows: for small ε we have $C(v)\sqrt{\varepsilon} \leq \|\nabla v\|_{L^1}$ and $\varepsilon < \pi\delta/|\Omega|$. Consider such an ε. As at the end of the Section 6.2, estimate (6.52) and estimate (6.53) applied along a sequence $p_n \searrow 1$ combined with estimate (6.48) and with the Gagliardo-Nirenberg inequalities imply that the conclusions of Lemma 6.15 hold with $w = \varphi_{\varepsilon,\theta}$ for appropriate ε and θ.

Step 1. Proof of (6.52). For $z \in \mathbb{S}^1$, the set $\{\theta; w_{\varepsilon,\theta}(z) \neq 1\}$ has measure ε, and thus

$$\int_0^{2\pi} |\{w_{\varepsilon,\theta} \neq 1\}|\, d\theta = \int_\Omega |\{\theta; w_{\varepsilon,\theta}(v(x)) \neq 1\}|\, dx = \varepsilon|\Omega|.$$

Step 2. Proof of (6.53) *when* $p = 1$. We will actually prove that the term $C(v)\sqrt{\varepsilon}$ is not needed for $p = 1$. As in Step 1, using the fact that $|\varphi'_{\varepsilon,\theta}(z)| = 2\pi/\varepsilon$ if $\varphi_{\varepsilon,\theta}(z) \neq 1$ and that $\varphi'_{\varepsilon,\theta}(z) = 0$ a.e. otherwise, we find that

$$\int_0^{2\pi} \|\nabla w_{\varepsilon,\theta}\|_{L^1}\, d\theta = 2\pi\|\nabla v\|_{L^1}. \tag{6.54}$$

Step 3. Proof of (6.53) *when* $p > 1$. Using a second order Taylor expansion, it is easy to see that the following inequality holds uniformly in x, y and $\varepsilon < 1/4$:

$$|v(x) - v(y)| \leq |\nabla v(x)||x-y| + \frac{K(v)}{\delta_v^2(x)}|x-y|^2 \quad \text{if } |y-x| \leq \sqrt{\varepsilon}\delta_v(x). \tag{6.55}$$

On the other hand, we clearly have

$$\int_\Omega \frac{1}{[\delta_v(x)]^q}\, dx < \infty, \quad \forall\, q \in [1,2); \tag{6.56}$$

here we see why we have to replace Ω by U in the definition of $\widetilde{\mathscr{R}}$. In particular, (6.56) combined with the definition of $\widetilde{\mathscr{R}}$ implies that

$$w_{\varepsilon,\theta} \in W^{1,q}(\Omega), \quad \forall\, q \in [1,2),\ \varepsilon,\ \theta \in (0,2\pi). \tag{6.57}$$

We split

$$I := \int_0^{2\pi} |w_{\varepsilon,\theta}|^p_{W^{1/p,p}}\, d\theta = \int_0^{2\pi}\int_\Omega\int_\Omega \frac{|w_{\varepsilon,\theta}(x) - w_{\varepsilon,\theta}(y)|^p}{|x-y|^{m+1}}\, dx dy d\theta = I_1 + \ldots + I_4, \tag{6.58}$$

where I_j corresponds to the integration over A_j, $\forall\, j \in [\![1,4]\!]$, and:

$$A_1 = \left\{(x,y) \in \Omega^2; |x-y| \geq \sqrt{\varepsilon}\delta_v(x)\right\},$$

$$A_2 = \left\{(x,y) \in \Omega^2; \frac{\varepsilon}{|\nabla v(x)|} < |x-y| < \sqrt{\varepsilon}\delta_v(x)\right\},$$

$$A_3 = \left\{(x,y) \in \Omega^2; \sqrt{\frac{\varepsilon}{K(v)}}\delta_v(x) < |x-y| < \sqrt{\varepsilon}\delta_v(x)\right\},$$

$$A_4 = \Omega^2 \setminus (A_1 \cup A_2 \cup A_3).$$

Step 4. Estimate of I_1 *and* I_3. By (6.50) and (6.56), we have

$$I_1 \leq C_p\varepsilon \iint_{A_1} \frac{1}{|y-x|^{m+1}}\, dy dx \leq C_p\sqrt{\varepsilon}\int_\Omega \frac{1}{\delta_v(x)}\, dx \leq C_p C(v)\sqrt{\varepsilon}. \tag{6.59}$$

By the same argument, we obtain

$$I_3 \leq C_p C(v)\sqrt{\varepsilon K(v)}. \tag{6.60}$$

Step 5. Estimate of I_2. By (6.50), we have

$$I_2 \le C_p \varepsilon \iint_{A_2} \frac{1}{|y-x|^{m+1}}\, dy dx \le C_p \|\nabla v\|_{L^1}. \tag{6.61}$$

Step 6. Estimate of I_4. By combining (6.50) with (6.55), we find that

$$I_4 \le \frac{C_p}{\varepsilon^{p-1}} \underbrace{\iint_{A_4} \frac{|\nabla v(x)|^p}{|y-x|^{m+1-p}}\, dy dx}_{I_5} + \frac{C_p K^p(v)}{\varepsilon^{p-1}} \underbrace{\iint_{A_4} \frac{1}{\delta_v^{2p}(x)|y-x|^{m+1-2p}}\, dy dx}_{I_6}. \tag{6.62}$$

Since $|y-x| \le \dfrac{\varepsilon}{|\nabla v(x)|}$ when $(x,y) \in A_4$, we find that

$$I_5 \le C_p \varepsilon^{p-1} \|\nabla v\|_{L^1}. \tag{6.63}$$

Similarly, the fact that $|y-x| \le \sqrt{\dfrac{\varepsilon}{K(v)}}\delta_v(x)$ in A_4 leads to

$$I_6 \le C_p \left(\frac{\varepsilon}{K(v)}\right)^{p-1/2} \int_\Omega \frac{1}{\delta_v(x)}\, dx. \tag{6.64}$$

We complete the proof of (6.53) when $p > 1$ by combining (6.58)-(6.64). □

6.16 Remark. The above construction is indeed, as stated in the introduction, a dipole construction. Indeed, the support of $w-1$ can be chosen arbitrarily small, while Jw is prescribed.

Lemma 6.15 leads to the following straightforward variant of Lemma 6.11, whose proof is left to the reader.

6.17 Lemma. *Let $1 < q < 2$ and let $u \in W^{1,q}(\Omega)$. Let $\varepsilon > 0$ and $v \in \widetilde{\mathscr{R}}$. Then there exists $w \in \widetilde{\mathscr{R}}$ such that:*

1. $Jw = Jv$.
2. $|w|^p_{W^{1/p,p}} \le C_p \|\nabla v\|_{L^1}$, $\forall\, p \in [1,\infty)$.
3. $|uw - u|_{W^{1/p,p}} \le |w|_{W^{1/p,p}} + C_p \varepsilon^{1/p}$, $\forall\, p \in [1,q]$.

Second proof of Theorem 1.3 when $\Omega \subset \mathbb{R}^m$ is bounded. Let N be an $(m-1)$-rectifiable current in Ω. Let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ be such that $Ju = \partial N$ and $\|\nabla u\|_{L^1} \le C\mathbf{M}(N)$. Let v_j be as in Lemma 6.12. By applying inductively Lemma 6.17,[34] we construct a sequence (w_j) such that $Jw_j = Jv_j$, $|w_j|^p_{W^{1/p,p}} \le C_p 2^{-j}\mathbf{M}(N)$ and

$$|w_1 \dots w_{j+1} - w_1 \dots w_j|_{W^{1/p,p}} \le |w_{j+1}|_{W^{1/p,p}} + C_p (2^{-j}\mathbf{M}(N))^{1/p}, \quad \forall\, p \in [1,3/2].$$

By combining the above with Lemma 4.5, it is easy to see that the product $\prod_j w_j$ converges in $W^{1/p,p}(\Omega;\mathbb{S}^1)$, $\forall\, p \in [1,\infty)$, to some w that has all the properties required in Theorem 1.3. □

[34] Here, we use the fact that $\widetilde{\mathscr{R}} \subset W^{1,q}(\Omega)$, $\forall\, q \in [1,2)$, *cf* (6.57).

7 Proof of Lemma 5.1

Let $(U_j)_{1\le j\le 2}$ be an open covering of $\mathbb{S}^m$ by domains of local charts, and let $\varphi_j : \overline{U_j} \to \overline{B}^m$ be the corresponding smooth diffeomorphisms. We denote by $\overline{\varphi_j}$ the corresponding charts for $\mathbb{S}^m \times [0,\infty)$: $\overline{\varphi_j}(x,t) = (\varphi_j(x),t) \in \mathbb{R}^m \times \mathbb{R}$. Let $(\theta_j)_{1\le j\le 2}$ be a partition of the unit subordinate to the covering $(U_j)_{1\le j\le 2}$ of $\mathbb{S}^m$. Thus θ_j is compactly supported in U_j and $\theta_1+\theta_2 = 1$. For $1 \le j \le 2$, we define $h_j := \varphi_{j\sharp}(\theta_j u)$, which is compactly supported in B^m with values into $\overline{B}^k$. Assume for the moment that we can construct a map $w_j : \mathbb{R}^m \times [0,\infty) \to \overline{B}^k$ which satisfies all the properties enumerated in Lemma 5.1 when u is replaced by h_j and $\mathbb{S}^m$ by B^m. Let $\eta \in C_c^\infty(B^m)$ such that $\eta = 1$ on $\cup_j \varphi_j(\text{ supp } \theta_j)$. We then define $v(x,t) := \sum_j \overline{\varphi_j}^\sharp(\eta w_j) = \sum_j \eta(\varphi_j(x)) w_j(\varphi_j(x),t)$. Clearly, $v \in C^\infty(\mathbb{S}^m \times (0,\infty))$. If, in addition, u is smooth, then

$$v(x,0) = \sum_j \eta(\varphi_j(x)) w_j(\varphi_j(x),0) = \sum_j \eta(\varphi_j(x)) h_j(\varphi_j(x)) = \sum_j \eta(\varphi_j(x))\theta_j(x)u(x).$$

Since $\eta(\varphi_j(x)) = 1$ when $\theta_j(x) \neq 0$, we find that $v(x,0) = u(x)$. We shall prove that for $j = 1,2$ and $T > 0$:

$$\|w_j\|_{L^p(\mathbb{R}^m\times(0,T))} \le C_T \|h_j\|_{L^p(\mathbb{R}^m)}\ ,\ |w_j|_{W^{k/p,p}(\mathbb{R}^m\times(0,\infty))} \le C|h_j|_{W^{(k-1)/p,p}(\mathbb{R}^m)}. \tag{7.1}$$

Moreover, we will see that if $\|u\|_{L^\infty} \le 1$, then

$$|w_j(x,t)| + |d_x w_j(x,t)| + |\partial_t w_j(x,t)| \le \frac{C}{t}. \tag{7.2}$$

The chain rule easily implies that this v satisfies estimate (5.1) when the seminorms are replaced by norms. By applying the argument leading to Corollary 4.3, we finds (5.1) holds also in its original form (with seminorms).[35]

If (7.1) holds true, then we obtain (5.2) as follows.

$$\begin{aligned}\int_{\mathbb{S}^m} |v(x,\cdot)|^p_{W^{k/p,p}((0,\infty))} &\le C\sum_j \int_{\mathbb{R}^m} |w_j(x,\cdot)|^p_{W^{k/p,p}((0,\infty))} \\ &\le C\sum_j |h_j|^p_{W^{(k-1)/p,p}(\mathbb{R}^m)} \le C\|u\|^p_{W^{(k-1)/p,p}(\mathbb{S}^m)}.\end{aligned} \tag{7.3}$$

We obtain (5.2) by applying (7.3) to $u - \fint_{\mathbb{S}^m} u$.

Finally, Lemma 5.1 item *5* is a direct consequence of (7.2) above.

In conclusion of the above discussion, we only need to prove Lemma 5.1 when $\mathbb{S}^m$ is replaced by B^m and u is compactly supported. In this case, we will prove that the standard extension v of u has all the required properties, namely satisfy Lemma 5.1 items *1*, *3* and *4* as well as (7.1) and (7.2) (with w_j replaced by v). Such an extension is defined as follows. We let ρ be a standard mollifier[36] and set

$$v(x,t) = \begin{cases} u * \rho_t(x), & \text{if } t > 0 \\ u(x), & \text{if } t = 0 \end{cases}. \tag{7.4}$$

Clearly, items *1* and *4* in Lemma 5.1 hold true for this v. On the other hand, (7.2) is straightforward (when $|u| \le 1$). In addition, the mapping $u \mapsto v$ is linear. By standard trace theory, this implies that items *2* and *4* yield item *3*. In conclusion, we only need to prove (7.1). Before proceeding to the proof, we note that, by a standard limiting procedure, we may also assume u smooth and compactly supported in B^m.

The proof of Lemma 5.1 relies on Hardy's inequality [39, Lemma 3.14] that we recall here.

[35] Indeed, an inspection of the proof below shows that when the map u in Lemma 5.1 is replaced by $u - \fint_{\mathbb{S}^m} u$, then the corresponding map v is replaced by $v - \fint_{\mathbb{S}^m} u$.

[36] That is, $\rho \in C_c^\infty(B^m)$, $\rho \ge 0$ and $\int \rho = 1$.

7.1 Lemma. *Let $p \geq 1$ and $r > 1$. If $g \in W^{1,1}_{loc}([0,\infty))$, then*

$$\int_0^\infty \frac{|g(s)-g(0)|^p}{s^r}\,ds \leq \left(\frac{p}{r-1}\right)^p \int_0^\infty \frac{|g'(s)|^p}{s^{r-p}}\,ds. \tag{7.5}$$

7.2 Corollary. *Let $p > 1$ and $r > m$. If $w \in C^1(\mathbb{R}^m)$, then*

$$\int_{|y-x|<\rho} \frac{|w(y)-w(x)|^p}{|y-x|^r}\,dy \leq C \int_{|y-x|<\rho} \frac{|\nabla w(y)|^p}{|y-x|^{r-p}}\,dy, \quad \forall x \in \mathbb{R}^m, \tag{7.6}$$

where $C = C(p,r,m)$.

Proof. By replacing g' with $g'\mathbb{1}_{[0,\rho]}$, we see that (7.5) still holds when $[0,\infty)$ is replaced by $[0,\rho]$.
We have

$$\begin{aligned}\int_{|y-x|<\rho} \frac{|w(y)-w(x)|^p}{|y-x|^r}\,dy &= \int_{\mathbb{S}^{m-1}} \left(\int_0^\rho t^{m-r-1}|w(x+t\theta)-w(x)|^p\,dt\right) d\mathscr{H}^{m-1} \\ &\leq C \int_{|y-x|<\rho} \frac{|\nabla w(y)|^p}{|y-x|^{r-p}}\,dy,\end{aligned}$$

the last inequality following from (7.5) applied in $[0,\rho]$ to $s \mapsto w(x+s\theta)$. □

Proof of (7.1) *when* $u \in C_c^\infty\left(B^m;\mathbb{R}^k\right)$. Let v be given by (7.4).
Step 1. Pointwise estimates for the derivatives. The identity

$$\partial_j v(x,t) = \frac{1}{t} u * (\eta_j)_t(x), \quad \forall x \in \mathbb{R}^m,\ t > 0,\ j \in [\![1, m+1]\!],$$

where

$$\eta_j = \partial_j \rho,\ \forall j \in [\![1,m]\!],\ \eta_{m+1}(x) = -\mathrm{div}(x\rho)(x),$$

combined with the fact that $\int \eta_j = 0,\ \forall j \in [\![1,m+1]\!]$, leads to

$$|\nabla v(x,t)| \leq \frac{C}{t^{m+1}} \int_{|y-x|<t} |u(y)-u(x)|\,dy \tag{7.7}$$

and thus

$$|\nabla v(x,t)|^p \leq \frac{C}{t^{m+p}} \int_{|y-x|<t} |u(y)-u(x)|^p\,dy. \tag{7.8}$$

Similarly, for the second order derivatives we have

$$\left|D^2 v(x,t)\right| \leq \frac{C}{t^{m+2}} \int_{|x-y|<t} |u(y)-u(x)|\,dy \tag{7.9}$$

and

$$\left|D^2 v(x,t)\right|^p \leq \frac{C}{t^{m+2p}} \int_{|y-x|<t} |u(y)-u(x)|^p\,dy. \tag{7.10}$$

Step 2. L^p estimate. By (7.4) and Hölder's inequality, we have

$$|v(x,t)-u(x)|^p \leq \int_{|y-x|<t} |u(y)-u(x)|^p \rho_t(x-y)\,dy \leq \frac{C}{t^m} \int_{|y-x|<t} |u(y)-u(x)|^p\,dy. \tag{7.11}$$

This implies

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^m} |v(x,t)-u(x)|^p \, dx\, dt &\le C \int_0^\infty \frac{1}{t^m} \int\int_{|y-x|<t} |u(y)-u(x)|^p \, dy dx dt \\ &\le C \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} |u(y)-u(x)|^p \int_{|y-x|}^\infty \frac{dt}{t^m} \, dx dy \\ &= C \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{|u(y)-u(x)|^p}{|y-x|^{m-1}} \, dx dy \le C|u|^p_{W^{(k-1)/p,p}};\end{aligned} \tag{7.12}$$

the last estimate relies on Poincaré's inequality $\|u\|_{W^{(k-1)/p,p}} \le C|u|_{W^{(k-1)/p,p}}$.

Hence, for every $T>0$ we have

$$\|v\|_{L^p(\mathbb{R}^m\times(0,T))} \le C_T |u|_{W^{(k-1)/p,p}}. \tag{7.13}$$

Step 3. $W^{1,p}$ estimate. As in the previous step, when $k \ge p$ (7.7) combined with Hölder's inequality implies

$$\begin{aligned}\|\nabla v\|^p_{L^p} &\le C \int_0^\infty \int_{\mathbb{R}^m} \left(\int_{|y-x|<t} |u(y)-u(x)| \, dy \right)^p dx \, \frac{dt}{t^{(m+1)p}} \\ &\le C \int_0^\infty \int\int_{|y-x|<t} |u(y)-u(x)|^p \, dy dx \, \frac{dt}{t^{m+p}} = C \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{|u(y)-u(x)|^p}{|y-x|^{m+p-1}} \, dy dx.\end{aligned}$$

Using again Poincaré's inequality, we find that

$$\|\nabla v\|^p_{L^p} \le C|u|^p_{W^{(k-1)/p,p}}. \tag{7.14}$$

In particular, we have (by combining (7.13) with (7.14))

$$\|v\|_{W^{1,p}(\mathbb{R}^m\times(0,T))} \le C_T |u|_{W^{(k-1)/p,p}}, \tag{7.15}$$

and the case where $k=p$ is completely settled.

For further use, let us note that the proof of (7.14) leads to

$$\int_0^\infty \int_{\mathbb{R}^m} \frac{|\nabla v(x,t)|^p}{t^{k-p}} \, dx dt \le C|u|^p_{W^{(k-1)/p,p}}. \tag{7.16}$$

In order to estimate $|v|_{W^{k/p,p}}$ when $k \ne p$, we consider two different cases: $p>k$ and $p \in (k-1,k)$.

Step 4. Estimate of $|v|_{W^{k/p,p}}$ when $p>k$. We start from the following semi-norms equivalence, valid for $1 \le p < \infty$ and $0<\sigma<1$:[37]

$$\begin{aligned}|v|^p_{W^{\sigma,p}(\mathbb{R}^m\times(0,\infty))} \sim &\int_0^\infty \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{|v(y,t)-v(x,t)|^p}{|y-x|^{m+\sigma p}} \, dy dx dt \\ &+ \int_{\mathbb{R}^m}\int_0^\infty\int_0^\infty \frac{|v(x,s+t)-v(x,t)|^p}{s^{1+\sigma p}} \, ds dt dx.\end{aligned} \tag{7.17}$$

This leads to

$$|v|^p_{W^{k/p,p}} \sim I + J + K, \tag{7.18}$$

where

$$I := \int_0^\infty \int_{\mathbb{R}^m}\int_{|y-x|<t} \frac{|v(y,t)-v(x,t)|^p}{|y-x|^{m+k}} \, dy dx dt,$$

$$J := \int_0^\infty \int_{\mathbb{R}^m}\int_{|y-x|\ge t} \frac{|v(y,t)-v(x,t)|^p}{|y-x|^{m+k}} \, dy dx dt,$$

[37] As for (6.47), this is obtained by combining [1, Example 7.27] with [1, Lemma 7.44].

$$K := \int_{\mathbb{R}^m} \int_0^\infty \int_0^\infty \frac{|v(x,s+t)-v(x,t)|^p}{s^{1+k}}\, ds dt dx.$$

Using successively Corollary 7.2 and (7.8), we find that

$$\begin{aligned} I &\le C \int_0^\infty \int_{\mathbb{R}^m} \int_{|y-x|<t} \frac{|\nabla v(y,t)|^p}{|y-x|^{m+k-p}}\, dy dx dt \\ &\le C \int_0^\infty \frac{1}{t^{m+p}} \int_{\mathbb{R}^m} \int_{|y-x|<t} \frac{1}{|y-x|^{m+k-p}} \int_{|z-y|<t} |u(z)-u(y)|^p\, dz dy dx dt \\ &= C \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{|z-y|}^\infty \frac{1}{t^{m+p}} \int_{|x-y|<t} \frac{1}{|x-y|^{m+k-p}}\, dx dt\, |u(z)-u(y)|^p\, dz dy \\ &= C \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|u(z)-u(y)|^p}{|z-y|^{m+k-1}}\, dz dy = |u|^p_{W^{(k-1)/p,p}}. \end{aligned} \tag{7.19}$$

The last two lines use Fubini's theorem. Similar calculations appear in the estimates of J and K and will not be detailed.

We next estimate J. Starting from

$$|v(y,t)-v(x,t)| \le |v(y,t)-u(y)| + |u(y)-u(x)| + |v(x,t)-u(x)|$$

and using (7.11), we find that

$$J \le C(J_1 + J_2), \tag{7.20}$$

where

$$J_1 = \int_0^\infty \int_{\mathbb{R}^m} \int_{|y-x|\ge t} \frac{|u(y)-u(x)|^p}{|y-x|^{m+k}}\, dy dx dt = |u|^p_{W^{(k-1)/p,p}} \tag{7.21}$$

and

$$J_2 = \int_0^\infty \frac{1}{t^m} \int_{\mathbb{R}^m} \int_{|y-x|\ge t} \frac{1}{|y-x|^{m+k}} \int_{|z-x|<t} |u(z)-u(x)|^p\, dz dy dx dt = C|u|^p_{W^{(k-1)/p,p}}. \tag{7.22}$$

Finally, we estimate K by using successively (7.5) applied with $g(s) = v(x,s+t)$ and (7.8). We find that

$$\begin{aligned} K &\le C \int_{\mathbb{R}^m} \int_0^\infty \int_0^\infty \frac{|\nabla v(x,s+t)|^p}{s^{1+k-p}}\, ds dt dx \\ &\le C \int_{\mathbb{R}^m} \int_0^\infty \frac{1}{s^{1+k-p}} \int_0^\infty \frac{1}{(s+t)^{m+p}} \int_{|y-x|<s+t} |u(y)-u(x)|^p\, dy ds dt dx = C|u|^p_{W^{(k-1)/p,p}}. \end{aligned} \tag{7.23}$$

We complete the case where $p > k$ by combining (7.18)-(7.23).

Step 5. Estimate of $|v|_{W^{k/p,p}}$ when $p \in (k-1,k)$. Starting from $|v|_{W^{k/p,p}} = |\nabla v|_{W^{k/p-1,p}}$ and (7.17), we find that

$$|v|^p_{W^{k/p,p}} \sim \widetilde{I} + \widetilde{J} + \widetilde{K}, \tag{7.24}$$

where

$$\widetilde{I} := \int_0^\infty \int_{\mathbb{R}^m} \int_{|y-x|<t} \frac{|\nabla v(y,t)-\nabla v(x,t)|^p}{|y-x|^{m+k-p}}\, dy dx dt,$$

$$\widetilde{J} := \int_0^\infty \int_{\mathbb{R}^m} \int_{|y-x|\ge t} \frac{|\nabla v(y,t)-\nabla v(x,t)|^p}{|y-x|^{m+k-p}}\, dy dx dt,$$

$$\widetilde{K} := \int_{\mathbb{R}^m} \int_0^\infty \int_0^\infty \frac{|\nabla v(x,s+t)-\nabla v(x,t)|^p}{s^{1+k-p}}\, ds dt dx.$$

In order to estimate the quantities $\widetilde{I}$ and $\widetilde{K}$, we proceed as for I and K, but use (7.10) instead of (7.8). Finally, using (7.16) we obtain

$$\widetilde{J} \le C \int_0^\infty \int_{\mathbb{R}^m} \int_{|y-x|\ge t} \frac{|\nabla v(x,t)|^p}{|y-x|^{m+k-p}}\, dy dx dt = C \int_0^\infty \int_{\mathbb{R}^m} \frac{|\nabla v(x,t)|^p}{t^{k-p}}\, dx dt \le C|u|^p_{W^{(k-1)/p,p}}.$$

This completes the proof of Lemma 5.1. □

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